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Stability Properties of Infection Diffusion Dynamics over Directed Networks

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Abstract— **We analyze the stability properties of a susceptible-infected-susceptible diffusion model over directed networks. Similar to the majority of infection spread dynamics, this model exhibits a threshold phenomenon. When the curing rates in the network are high, the all-healthy state is globally asymptotically stable (GAS). Otherwise, an endemic state arises and the entire network could become infected. Using notions from positive systems theory, we prove that the endemic state is GAS in strongly connected networks. When the graph is weakly connected, we provide conditions for the existence, uniqueness, and global asymptotic stability of weak and strong endemic states. Several simulations demonstrate our results.**

I. Introduction

Various epidemiological models have been proposed in the literature to capture infection diffusion over networks [1]–[3]. Besides modelling disease spread among humans, many of these models are motivated by engineering applications such as the spread of viruses in computer networks [3]–[6]. In order to design efficient disease (or information) spread control schemes, it is important to first understand the stability properties of the disease spread model under study. A survey of stability results for various epidemiological models can be found in [7]. The main focus of this paper is to study the stability properties of the so-called n-intertwined Markov model [6], which is a susceptible-infected-susceptible (SIS) model based on a mean-field approximation.

Similar to the majority of epidemiological models, the n-intertwined Markov model exhibits a threshold phenomenon. When the curing rates across the network are high, the infection levels converge to zero exponentially fast. However, when the curing rates are low, an endemic state emerges and a residual infection persists in the network. Having studied the stability properties of this model over undirected graphs in [8], our focus here will be stability over directed networks. In general, communication and interaction among people, animals, and computers are not necessarily symmetric, which highlights the importance of studying epidemics over directed graphs.

Statement of Contributions

The main contributions of this paper are twofold. First, we use tools from positive systems theory to fully characterize the stability properties of the n-intertwined Markov model over strongly connected digraphs. In particular, we show that the all-healthy state is GAS when the curing rates in the network are high. When the curing rates are low, we show that the endemic state is GAS, and that it is locally exponentially stable. Second, we study the existence, uniqueness, and stability properties of the endemic state over weakly connected digraphs. We show that a weak endemic state can emerge over weakly connected digraphs, in which a subset of the nodes will be healthy and the rest will be infected.

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Organization

Section II contains some mathematical preliminaries. We review the n-intertwined Markov model in Section III. The stability of the all-healthy state equilibrium is studied in Section IV. Section V establishes the existence, uniqueness, and stability of the endemic state equilibrium over weakly and strongly connected digraphs. Numerical studies are provided in Section VI. We collect our conclusions and ideas for future research in Section VII.

II. Mathematical Preliminaries

We start with some terminology and notational conventions. All the matrices and vectors in this paper are real valued. For a set of $n \in \mathbb{Z}_{\geq 1}$ elements, we use the combinatorial notation $[n]$ to denote $\{1, \ldots, n\}$. The (i,j)-th entry of a matrix $X \in \mathbb{R}^{n\times m}$, $n, m \in \mathbb{Z}_{\geq 1}$ is denoted by x_{ij}. For two real vectors $x, y \in \mathbb{R}^n$, $n \in \mathbb{Z}_{\geq 1}$, we write $x \gg y$ if $x_i > y_i$ for all i, $x \succ y$ if $x_i \geq y_i$ for all i but $x \neq y$, and $x \succeq y$ if $x_i \geq y_i$ for all i. The Euclidean norm of a vector is denoted by $\|.\|_2$, and the absolute value of a scalar variable is denoted $|.|$. When the eigenvalues of a matrix $X \in \mathbb{R}^{n\times n}$ are real, we denote the largest eigenvalue by $\lambda_1(X)$ and the smallest eigenvalue by $\lambda_n(X)$. The set of eigenvalues of a matrix X is denoted by $\sigma(X)$. The spectral radius of a matrix X is given by $\rho(X) = \max_{\lambda\in\sigma(X)} |\lambda|$, and its abscissa is given by $\mu(X) = \max_{\lambda\in\sigma(X)} \mathrm{Re}(\lambda)$.

We use the operator diag(.) for two purposes. When applied to a matrix, $\mathrm{diag}(X)$ returns a column vector that contains the diagonal elements of X. For vectors, $X = \mathrm{diag}(x)$ is a diagonal matrix with $X_{ii} = x_i$. When a diagonal matrix has positive diagonal elements, we call it a positive diagonal matrix. The identity matrix is denoted by I, and the all-ones vector is denoted by $\mathbf{1}$. Given a real continuously differentiable vector function $f : \mathbb{R}^n \to \mathbb{R}$, the Jacobian matrix of the dynamical system $\dot{x} = f(x)$ is denoted by $J(x) = \frac{\partial}{\partial x} f(x)$.

Matrix Theory

We call two matrices $X, Y \in \mathbb{R}^{n\times n}$ similar if there exists a nonsingular matrix $T \in \mathbb{R}^{n\times n}$ such that $Y = T^{-1}XT$. An important property of similar matrices is that they share the same set of eigenvalues. A real square matrix X is called a Metzler matrix if its off-diagonal entries are nonnegative. We say that a matrix $X \in \mathbb{R}^{n\times n}$ is reducible if there exists a permutation matrix T such that

$$T^{-1}XT = \begin{bmatrix} Y & Z \\ 0 & W \end{bmatrix},$$

where Y and W are square matrices, or if $n = 1$ and $X = 0$ [9]. A real square matrix is called irreducible if it is not reducible. A survey on the properties of Metzler matrices and their stability properties can be found in [10], [11]. Some of our results rely on properties of Metzler matrices, which we briefly recall below.

It is well-known that Hurwitz Metzler matrices are diagonally stable [12]. In particular, we have the following equivalent statements.

Proposition 1 ([13]): For a Metzler matrix $X \in \mathbb{R}^{n\times n}$, the following statements are equivalent:

a) The matrix X is Hurwitz.
b) There exists a vector $\xi \gg 0$ such that $X\xi \ll 0$.
c) There exists a vector $\nu \gg 0$ such that $\nu^T X \ll 0$.
d) There exists a positive diagonal matrix Q such that

$$X^TQ + QX = -K, \tag{1}$$

where K is a positive definite matrix.

The Perron-Frobenius (PF) theorem characterizes some of the properties of the spectra of nonnegative matrices [14, Theorem 8.2.11].

Theorem 1 (PF – Nonnegative Irreducible Case): Let $X \in \mathbb{R}^{n\times n}$ be a nonnegative irreducible matrix. Then:

a) $\rho(X) > 0$.
b) $\rho(X)$ is an algebraically simple eigenvalue of X.
c) If $Xv = \rho(X)v$, then $v \gg 0$.

When a Metzler matrix is irreducible, the following result applies [11, Theorem 17].

Theorem 2 (PF – Metzler Irreducible Case): Let $X \in \mathbb{R}^{n\times n}$ be an irreducible Metzler matrix. Then:

a) $\mu(X)$ is an algebraically simple eigenvalue of X.
b) Let v_F be such that $Xv_F = \mu(X)v_F$. Then v_F is unique (up to scalar multiple) and $v_F \gg 0$.
c) If $v > 0$ is an eigenvector of X, then $Xv = \mu(X)v$, and hence, v is a scalar multiple of v_F.

Graph Theory

A *directed graph*, or *digraph*, is a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of nodes and $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ is the set of edges. Given $\mathcal{G}$, we denote an edge from node $i \in \mathcal{V}$ to node $j \in \mathcal{V}$ by (i, j). We say node $i \in \mathcal{V}$ is a neighbor of node $j \in \mathcal{V}$ if and only if $(i, j) \in \mathcal{E}$. When $(i, j) \in \mathcal{E}$ if and only if $(j, i) \in \mathcal{E}$, we call the graph *undirected*. For a graph with $n \in \mathbb{Z}_{\geq 1}$ nodes, we associate an adjacency matrix $A \in \mathbb{R}^{n\times n}$ with entries $a_{ij} \in \mathbb{R}_{\geq 0}$, where $a_{ij} = 0$ if and only if $(i, j) \notin \mathcal{E}$.

In a digraph, a directed path is a collection of nodes $\{i_1, \ldots, i_\ell\} \subseteq \mathcal{V}$, $\ell \in \mathbb{Z}_{>1}$, such that $(i_k, i_{k+1}) \in \mathcal{E}$ for all $k \in [\ell - 1]$. A digraph is *strongly connected* if there exists a directed path between any two nodes in $\mathcal{V}$. A strongly connected component (SCC) of a graph is a subgraph which itself is strongly connected. A path in an undirected graph is defined in a similar manner. We call an undirected graph *connected* if it contains a path between any two nodes in $\mathcal{V}$. A digraph is called *weakly connected* if when every edge in $\mathcal{E}$ is viewed as an undirected edge, the resulting graph is a connected undirected graph. We call a graph, whether it is directed or undirected, *disconnected* if it contains at least two isolated subgraphs. Throughout this paper, when the graph $\mathcal{G}$ is directed, we assume that it is either strongly or weakly connected. When $\mathcal{G}$ is undirected, we assume that it is connected. A directed acyclic graph (DAG) is a digraph with no directed cycles.

III. The n-Intertwined Markov Model

In this section, we recall the heterogeneous n-intertwined Markov model that was recently proposed [6], [15]. This model is related to the multi-group SIS model that was proposed earlier in [16]; see also [7], [17]. Throughout this section, we prescribe the infection model over a directed network; this network consists of n nodes and is described by a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of vertices, and $\mathcal{E}$ is the set of edges.

The n-intertwined Markov model is based on viewing each node in the network as a Markov chain with two states: infected or cured. The curing and infection of a given node in the network are described by two independent Poisson processes with rates δ_i and β_i, respectively. Throughout the paper, we assume that $\delta_i > 0$ and $\beta_i > 0$. The transition rates between the healthy and infected states of given a node's Markov chain depend on the infection levels of its neighbors. A mean-field approximation is then introduced to "average" the effect of the neighbors on the infection level of a given node. This approximation yields an ODE that describes the evolution of the probability of infection of node i. Let $p_i(t)$ be the infection probability of node i at time $t \in \mathbb{R}_{\geq 0}$, and let $p = [p_1, \ldots, p_n]^T$. Also, let $D = \mathrm{diag}(\delta_1, \ldots, \delta_n)$, $P = \mathrm{diag}(p_1, \ldots, p_n)$, and $B = \mathrm{diag}(\beta_1, \ldots, \beta_n)$. The n-intertwined Markov model is given by

$$\dot{p}(t) = (A^TB - D)p(t) - P(t)A^TBp(t). \tag{2}$$

Since the above dynamics describe the evolution of a Markov chain, it follows that $p \in [0, 1]^n$, for all $t \in \mathbb{R}_{\geq 0}$ [6], [15].

We next focus on characterizing the set of equilibria of the dynamical system (2). We give this characterization using the so-called *basic reproduction number*, denoted by $\mathcal{R}_o$, which is defined as the expected number of infected nodes produced in a completely susceptible population due to the infection of a neighboring node [18]. For the n-intertwined Markov model, the basic reproduction number was found in [15], where it was called the "critical threshold", to be equal to

$$\mathcal{R}_o = \rho(D^{-1}A^TB).$$

For connected undirected graphs, it was shown in [15] that the all-healthy state is the unique equilibrium for the n-intertwined Markov model when $\mathcal{R}_o \leq 1$. When $\mathcal{R}_o > 1$, in addition to the all-healthy equilibrium, an endemic equilibrium, denoted by $p^\star$, emerges. In fact, it has been shown that $p^\star \gg 0$. We call a strictly positive endemic state *strong*. When $p^\star \succ 0$, we call it a *weak* endemic state.

A recursive expression was provided for the endemic state $p^\star$ in [15], which was shown to depend on the problem parameters only: A, δ_i, β_i. To arrive at this expression, consider the steady-state equation

$$0 = (A^T B - D)p - PA^T Bp. \tag{3}$$

Define $\xi_i := \sum_{j \neq i} a_{ji}\beta_j p_j$ and $\xi_i^\star := \sum_{j \neq i} a_{ji}\beta_j p_j^\star$. We can then write $p_i^\star$ as

$$p_i^\star = \frac{\xi_i^\star}{\delta_i + \xi_i^\star} = 1 - \frac{\delta_i}{\delta_i + \xi_i^\star}. \tag{4}$$

Since we assumed that $\delta_i > 0$, we conclude that $p_i^\star < 1$, for all i. We can then re-write (3) in the following form:

$$A^T Bp^\star = (I - P^\star)^{-1} Dp^\star. \tag{5}$$

In the following sections, we will study the stability of the n-intertwined Markov model over strongly and weakly connected digraphs.

IV. Stability of the All-Healthy State in Digraphs

In [19], and using the bound $\dot{p} \leq (A^T B - D)p$ and the comparison lemma, a sufficient condition for the stability of the origin in a digraph was provided; because the upper bound is linear in p, a sufficient condition for stability is to have $\mu(A^T B - D) < 0$. Here, using the theory of positive systems, we provide an alternative proof for the exponential stability of the origin when $\mathcal{R}_o \leq 1$, and we show that this condition is also necessary for stability.

Proposition 2: Suppose $\mathcal{G}$ is an arbitrary digraph. The origin is globally exponentially stable if and only if $\mathcal{R}_o \leq 1$.

Proof: Due to space limitation, we will show sufficiency only for the case when $\mathcal{R}_o < 1$; the complete proof can be found in [20]. Note that the matrix $A^T B - D$ is Metzler, because the entries of $A^T B$ are nonnegative. Using the convergent regular splitting property of Metzler matrices, it can be shown that $\mathcal{R}_o < 1$ if and only if $\mu(A^T B - D) < 0$, and $\mathcal{R}_o = 1$ if and only if $\mu(A^T B - D) = 0$ [9, Theorem 2.3]. Hence, when $\mathcal{R}_o < 1$, we have $\lambda_1(A^T B - D) < 0$, and the matrix $A^T B - D$ is Hurwitz. Since it is also Metzler, by Proposition 1, there exists a diagonal matrix Q with positive diagonal elements satisfying (1). Consider the Lyapunov function

$$\begin{aligned} \dot{V}(p) &= p^T((A^T B - D)^T Q + Q(A^T B - D))p \\ &- 2p^T QPA^T Bp \\ &\leq p^T((A^T B - D)^T Q + Q(A^T B - D))p \\ &= -p^T Kp \leq \lambda_1(-K)\|p\|_2^2 < 0, \quad p \neq 0, \end{aligned}$$

where the first inequality follows because $p^T QPA^T Bp \geq 0$, for all $p \in [0,1]^n$, the second equality follows from (1), and the last strict inequality follows because K is positive definite. This proves the sufficiency part. We will show necessity by proving the contrapositive. The Jacobian matrix of the vector field in (2) evaluated at the origin is given by $J(0) = A^T B - D$. If $\mathcal{R}_o > 1$, we have $\lambda_1(A^T B - D) > 0$, and the original nonlinear system is not stable. This proves that $\mathcal{R}_o \leq 1$ is necessary for asymptotic stability. ■

V. Existence and Stability of an Endemic State

Many of the works on epidemics have focused on studying the stability of the all-healthy state. However, studying the existence, uniqueness, and stability of the endemic state is more relevant while being more challenging. In this section, we provide a non-trivial extension of our stability results for undirected graph in [8] to the directed case. In particular, we study the stability of the n-intertwined Markov model for both strongly and weakly connected graphs.

A. Strongly Connected Digraphs

The existence and uniqueness of an endemic state was shown in [7] for compartmental SIS models. After appropriate relabelling, we can show that the system we are studying here is the same as the one studied in [7]. In particular, our D, $A^T B$ matrices are called $-D$, B, respectively, in [7]. We will re-state this result here for completeness.

Theorem 3: Let $\mathcal{G}$ be a strongly connected graph. Then, a unique strongly positive endemic state exists if and only if $\mathcal{R}_o > 1$.

The next result establishes the local stability of $p^\star$. Note that the Jacobian matrix of the vector field in (2) at $p^\star$ is

$$J(p^\star) = -(I - P^\star)^{-1} D + (I - P^\star)A^T B, \tag{6}$$

which is Metzler, since $A^T B$ is nonnegative.

Theorem 4: Suppose that $\mathcal{G}$ is a strongly connected graph and that $\mathcal{R}_o > 1$. Then, the endemic state $p^\star$ is locally exponentially stable.

Proof: We invoke Lyapunov's indirect method. Since the graph is strongly connected and $\mathcal{R}_o > 1$, the endemic state $p^\star$ is strictly positive. Also, the graph being strongly connected implies that A is irreducible. From (5), we deduce that $Dp^\star = (I - P^\star)A^T Bp^\star$. We can then write

$$\begin{aligned} J(p^\star)p^\star &= -A^T Bp^\star + (I - P^\star)A^T Bp^\star \\ &= -P^\star A^T Bp^\star \ll 0, \end{aligned}$$

where the last strict inequality follows because $p^\star \gg 0$, B is a positive diagonal matrix, and A is irreducible. Using Proposition 1(b), we conclude that $J(p^\star)$ is Hurwitz. ■

The following theorem is one of the main results of this paper. A proof for a weaker statement appeared in [7]. However, the proof in [7] relies on the fact that, given $p(0) \neq 0$, there exists a time t by which $p(t) \in (0,1]^n$, but no rigorous proof was provided for this claim. Our proof does not rely on this fact. An alternative proof that utilizes a logarithmic Lyapunov function has recently appeared in [17]. The novelty of our proof lies in the utilization of notions from positive systems theory to construct a quadratic Lyapunov function. In addition to the useful characteristics of using a quadratic

Lyapunov function for studying additional properties such as convergence rates, our proof allows for establishing the stability properties of the equilibrium points over weakly connected digraphs in the next section.

Theorem 5: Let $\mathcal{G}$ be strongly connected and $p(0) \neq 0$. If $\mathcal{R}_o > 1$, then the metastable state $p^\star$ is GAS.

Proof: Recall that $p(t) \in [0,1]^n$ for all $t \in \mathbb{R}_{\geq 0}$. When $\mathcal{R}_o > 1$, Proposition 2 implies that the origin is unstable. Therefore, under this condition, the set $W = [0,1]^n \backslash \{0\}$ is invariant under the evolutions of (2). The dynamics of $\tilde{p}_i$ are given by

$$\begin{aligned}\dot{\tilde{p}}_i &= \dot{p}_i = -\delta_i(\tilde{p}_i + p_i^\star) + (1 - (\tilde{p}_i + p_i^\star)(\tilde{\xi}_i + \xi_i^\star), \\ &= -\delta_i \tilde{p}_i + (1 - p_i^\star)\tilde{\xi}_i - \tilde{p}_i(\tilde{\xi}_i + \xi_i^\star),\end{aligned}$$

which in terms of the Jacobian at $p^\star$ defined in (6) can be re-written as

$$\dot{\tilde{p}}_i = J_{ii}(p^\star)\tilde{p}_i + \sum_{j \neq i} J_{ij}(p^\star)\tilde{p}_j - \tilde{p}_i\tilde{\xi}_i,$$

or in matrix form $\dot{\tilde{p}} = J(p^\star)\tilde{p} - \tilde{P}A^T B\tilde{p}$, where $\tilde{P} = \text{diag}(\tilde{p}_1, \ldots, \tilde{p}_n)$. Consider now the quadratic Lyapunov function $V(\tilde{p}) = \frac{1}{2}\tilde{p}^T\tilde{p}$. We can then write

$$\begin{aligned}\dot{V}(\tilde{p}) &= \tilde{p}^T J(p^\star)\tilde{p} - \tilde{p}^T\tilde{P}A^T B\tilde{p} \\ &= \tilde{p}^T J(p^\star)\tilde{p} + \tilde{p}^T\tilde{P}A^T Bp^\star - \tilde{p}^T\tilde{P}A^T Bp.\end{aligned}$$

Evaluating (3) at $p^\star$ yields

$$A^T Bp^\star = (I - P^\star)^{-1}Dp^\star. \tag{7}$$

Using this, and the fact $\tilde{P}p^\star = P^\star\tilde{p}$, we have

$$\tilde{p}^T\tilde{P}A^T Bp^\star = \tilde{p}^T(I - P^\star)^{-1}DP^\star\tilde{p}.$$

Define $\Sigma := (I - P^\star)^{-1}DP^\star$, and consider

$$\begin{aligned}&J(p^\star) + \Sigma = (I - P^\star)(-(I - P^\star)^{-2}D + A^T B) \\ &+(I - P^\star)^{-1}DP^\star \\ &= -D + (I - P^\star)A^T B.\end{aligned}$$

With $\Lambda := J(p^\star) + \Sigma$, we can re-write the dynamics in (2) as $\dot{\tilde{p}} = \Lambda\tilde{p} - \tilde{P}A^T Bp$. The off-diagonal elements of Λ are nonnegative, whereas its diagonal elements are negative. Hence, Λ is a Metzler matrix. Since $\mathcal{G}$ is strongly connected, the matrix Λ is also irreducible. From (5), it follows that $\Lambda p^\star = 0$. Since $p^\star$ is strictly positive, it follows from Theorem 2 that $\mu(\Lambda) = 0$. Since $\sigma(\Lambda) = \sigma(\Lambda^T)$, we have $\mu(A^T) = 0$. Using Theorem 2, there exists a vector $\xi \gg 0$ such that $\Lambda^T\xi = 0$.

Define the positive diagonal matrix Q with $Q_{ii} = \xi_i/p_i^\star$, and consider the Lyapunov function $V(\tilde{p}) = \tilde{p}^T Q\tilde{p}$. Note

$$\dot{V}(\tilde{p}) = \tilde{p}^T(\Lambda^T Q + Q\Lambda)\tilde{p} - 2\tilde{p}^T\tilde{P}QA^T Bp,$$

where we have used the fact that $\tilde{P}$ and Q commute, since they are both diagonal matrices. Multiplying Λ by Q from the left does not change the signs of the entries of Λ because Q is a diagonal positive matrix. Hence, the matrix $Q\Lambda$ is Metzler. Also, multiplying by Q only changes the weights of the links of the digraph corresponding to Λ; this multiplication does not remove any links from the underlying graph, since Q is a diagonal positive matrix. Hence, because Λ is irreducible, the graph corresponding to $Q\Lambda$ is strongly connected, and therefore $Q\Lambda$ is irreducible. It then follows that $\Lambda^T Q$ is also an irreducible Metzler matrix. Since sum of two Metzler matrices is Metzler, the matrix $\Lambda^T Q + Q\Lambda$ is Metzler. Also, because both $Q\Lambda$ and $\Lambda^T Q$ are Metzler and irreducible, the underlying digraph of $\Lambda^T Q + Q\Lambda$ is strongly connected. Hence, $\Lambda^T Q + Q\Lambda$ is also irreducible. Further, by construction, we have $(\Lambda^T Q + Q\Lambda)p^\star = \Lambda^T Qp^\star = \Lambda^T\xi = 0$. Since $\Lambda^T Q + Q\Lambda$ is symmetric, it has real eigenvalues, and since $p^\star$ is strictly positive, it follows from Theorem 2 that $\Lambda^T Q + Q\Lambda$ is negative semidefinite. We therefore have $\dot{V}(\tilde{p}) \leq -2\tilde{p}^T\tilde{P}QA^T Bp \leq 0$. Since the set W is invariant under (2), we have that $\dot{V}(\tilde{p}) = 0$ if and only if $p = p^\star$. ■

B. Weakly Connected Digraphs

Here, we consider the n-intertwined Markov model on the class of weakly connected graphs. This class is of great importance, since it is conceivable that in many practical scenarios there exist connected components that collectively serve as an infection source, but are not affected by the rest of the nodes. Studying epidemiological models over weakly connected digraphs can yield new epidemic behaviors that do not emerge over strongly connected digraphs. For example, as we will show by providing an example, the general belief that the endemic state is always strictly positive does not hold for weakly connected graphs, and weak endemic states can emerge.

We start by introducing some notations. When $\mathcal{G}$ is weakly connected, its adjacency matrix can be transformed into an upper triangular form using an appropriate labeling of vertices. Assuming that $\mathcal{G}$ contains N strongly connected components, we can write

$$A = \begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1N} \\ 0 & A_{22} & A_{23} & \ldots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \ldots & 0 & A_{NN} \end{bmatrix},$$

where A_{ii} are irreducible, and hence, correspond to SCCs in $\mathcal{G}$ [9]. For notational simplicity, we will use A_i instead of A_{ii}. The matrices A_{ij}, $j \neq i$ are not necessarily irreducible. We denote an SCC in $\mathcal{G}$ by $\mathcal{G}_i = (\mathcal{V}_i, \mathcal{E}_i)$. The diagonal matrices D_i, B_i contain, respectively, the curing and infection rates of the nodes in $\mathcal{V}_i$ along their diagonals. We introduce the partial order '$\prec$' among SCCs, and we write $\mathcal{G}_i \prec \mathcal{G}_j$ if there is a directed path from $\mathcal{G}_i$ to $\mathcal{G}_j$ but not vice versa.

We denote the state of the nodes in $\mathcal{G}_i$ by q_i and the state of the j-th node in $\mathcal{G}_i$ by $q_{i,j}$. We then have $p = [q_1^T, \ldots, q_N^T]$. Let $Q_i = \text{diag}(q_i)$ and $c_i = \sum_{j \neq i} A_{ji}^T B_j q_j$. We can now write the dynamics of the nodes in $\mathcal{G}_i$ as follows:

$$\dot{q}_i = (A_i^T B_i - D_i)q_i - Q_i A_i^T B_i q_i + (I - Q_i)c_i, \tag{8}$$

where c_i can be viewed as the input infection from the nodes outside $\mathcal{G}_i$. When an SCC comprises a single node, $A_i^T B_i - D_i$ collapses to $-\delta_i$. When an endemic state $p^\star$ emerges over $\mathcal{G}$, we call the steady-state of q_i an endemic state of $\mathcal{G}_i$ and denote it by $q_i^\star$. Hence, the endemic state emerging over the entire network is given by $p^\star = [q_1^{\star T}, \ldots, q_N^{\star T}]^T$. We start by proving the following result for DAGs.

Proposition 3: Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a DAG and suppose that $\delta_i > 0$ for all $i \in \mathcal{V}$. Then the origin is the unique equilibrium. Moreover, this equilibrium is GAS.

We proved this result using the comparison lemma and the well-known fact that the state of an exponentially stable linear system converges to zero when its input converges to zero—see [20] for the complete proof.

In what follows, we wish to obtain the counterparts of the result of Proposition 3 for weakly connected graphs consisting of multiple SCCs. We will therefore rely on the fact that a weakly connected digraph can be partitioned into a DAG of SCCs. We begin by studying the existence and uniqueness of an equilibrium and its stability properties over a weakly connected digraph consisting of two SCCs; the generalization to multiple SCCs follows along similar lines.

Let $\mathcal{R}_o^i := \rho(D_i^{-1} A_i^T B_i)$ be the basic reproduction number corresponding to $\mathcal{G}_i$. We then have the following existence and uniqueness result.

Theorem 6: Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a weakly connected digraph consisting of two SCCs $\mathcal{G}_1$, $\mathcal{G}_2$ such that $\mathcal{G}_1 \prec \mathcal{G}_2$. Assume that $q_i(0) \neq 0$ for all $i \in [2]$. Then the following statements hold:

(i) If $\mathcal{R}_o^1 > 1$, and $\mathcal{R}_o^2$ is arbitrary, then $p = 0$ and $p^\star = [q_1^{\star T}, q_2^{\star T}]^T$ are the only possible equilibrium points over $\mathcal{G}$, where $q_1^\star$ and $q_2^\star$ are unique strong endemic equilibrium points over $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively.

(ii) If $\mathcal{R}_o^1 \leq 1$ and $\mathcal{R}_o^2 > 1$, then $p = 0$ and $p^\star = [0^T, q_2^{\star T}]^T$ are the only possible equilibrium points over $\mathcal{G}$, where $q_2^\star$ is the unique strong endemic equilibrium point over $\mathcal{G}_2$.

(iii) If $\mathcal{R}_o^i \leq 1$, $i \in [2]$, then $p = 0$ is the only possible equilibrium over $\mathcal{G}$.

The complete proof of this theorem can be found in [20]. The proof relies on observing that the states of the nodes in $\mathcal{G}_1$ are not affected by the nodes in $\mathcal{G}_2$[1]. Then, by using the results of Proposition 2 and Theorem 5, we can characterize the equilibria over $\mathcal{G}_1$. Characterizing the equilibria over $\mathcal{G}_2$ requires more analysis especially when an endemic state emerges over $\mathcal{G}_1$, in which case we have used Brouwer's fixed-point theorem to prove the result.

This theorem shows that strong and weak endemic states may emerge over weakly connected digraphs. In particular, from (2), we conclude that a weak endemic state could emerge. A strong endemic state will emerge in cases (1) and (3), and the all-healthy state occurs in case (1). It is important to note that the endemic states $q_2^\star$ resulting in cases (1), (2), and (3) are not necessarily the same. Having shown existence and uniqueness of weak and strong endemic equilibria, the next theorem characterizes their stability properties—for the complete proof, see [20].

Theorem 7: Let $\mathcal{G}$ be a weakly connected digraph consisting of two SCCs $\mathcal{G}_1$, $\mathcal{G}_2$ such that $\mathcal{G}_1 \prec \mathcal{G}_2$. Let $p(0) \neq 0$. Then, when $\mathcal{R}_o^2 \leq 1$, $\mathcal{G}_2$ is input-to-state stable (ISS). Further, the equilibrium over $\mathcal{G}$ is GAS.

To prove the result, we first established that $\mathcal{G}_2$ is ISS when $\mathcal{R}_o^1$ is arbitrary and $\mathcal{R}_o^2 \leq 1$. Then, we relied on the fact that, in the absence of input from $\mathcal{G}_1$, the all-healthy state is GAS over $\mathcal{G}_2$ when $\mathcal{R}_o^2 \leq 1$ as per Proposition 2. As in the proof of Theorem 6, the fact that the dynamics of $\mathcal{G}_1$ are not affected by $\mathcal{G}_2$ is crucial for this proof.

The following corollary is immediate.

Corollary 1: Let $\mathcal{G}$ be a weakly connected digraph consisting of N SCCs ordered as $\mathcal{G}_1 \prec \ldots \prec \mathcal{G}_N$. Let $p(0) \neq 0$.

1) If $\mathcal{R}_o^i < 1$, $i \in [N]$, then the all-healthy state is the only equilibrium over $\mathcal{G}$ and it is GAS.
2) If $\mathcal{R}_o^1 > 1$ and $\mathcal{R}_o^i < 1$, $i \in \{2, \ldots, N\}$, then there exists a unique strong endemic equilibrium over $\mathcal{G}$ which is GAS.

What this leaves is the characterization of the case where $\mathcal{R}_o^i > 1$ for some i and $\mathcal{G}_i$ takes input from other SCCs, which we address in [20]; we will provide some discussion on this case in the simulations section below.

VI. Numerical Studies

We will demonstrate the emergence of a weak endemic state over the Pajek GD99c directed network [21], which is shown in Fig. 1. The network consists of 105 nodes and it contains 66 SCCs. The nodes marked "red" in Fig. 1 constitute an SCC, which we refer to as $\mathcal{G}_1$. We will select the curing rates over $\mathcal{G}_1$ to be low to make $\mathcal{R}_o^1 > 1$. For the remaining nodes, we will set $\delta_i = \sum_{j \neq i} a_{ji}\beta_j + 0.5$, which is a sufficient condition to ensure $\mathcal{R}_o^i < 1$, for all i [8]. The infection rates β_i and the weights a_{ij} are all selected to be equal to 1. There are only 4 vertices for which there is no directed path from $\mathcal{G}_1$, and they are marked "black" in Fig. 1. The initial infection profile is selected at random.

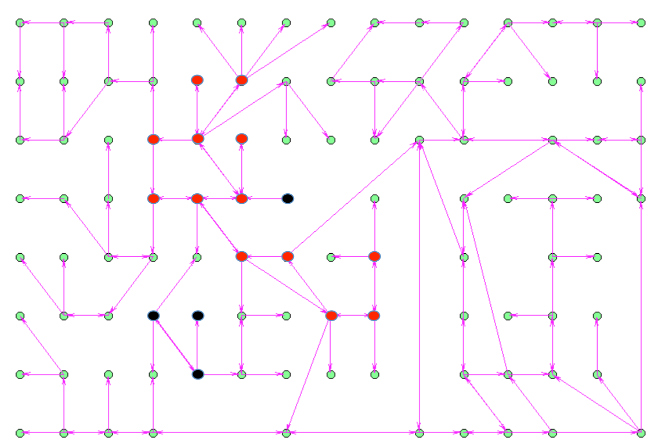

Fig. 1: The Pajek GD99c network. The "red" nodes belong to $\mathcal{G}_1$ for which $\mathcal{R}_o^1 > 1$. The "black" nodes are the only ones with no direct path from $\mathcal{G}_1$.

Fig. 2 plots the state trajectories. By examining the histogram of the values to which the state converges, we notice that there are 13 nodes with high infection levels, and those are the nodes comprising $\mathcal{G}_1$. Note that $\mathcal{G}_1$ is

[1]In computer networks, this corresponds to networks where servers (residing in $\mathcal{G}_1$) can send messages to clients (residing in $\mathcal{G}_2$), but not vice versa.

asymptotically stable even though it takes input from other SCCs and $\mathcal{R}_o^1 > 1$; as discussed at the end of Section V-B, this case was not covered by Theorem 7 and Corollary 1, and this simulation demonstrates that global asymptotic stability could also be achieved in this case. There are 4 nodes that become healthy, and those are the "black" nodes which are not reached by a directed path from $\mathcal{G}_1$. The remaining nodes all have positive infection probabilities with varying levels depending on their distance from $\mathcal{G}_1$, with the nodes that are farthest from $\mathcal{G}_1$ enjoying the lowest infection probabilities.

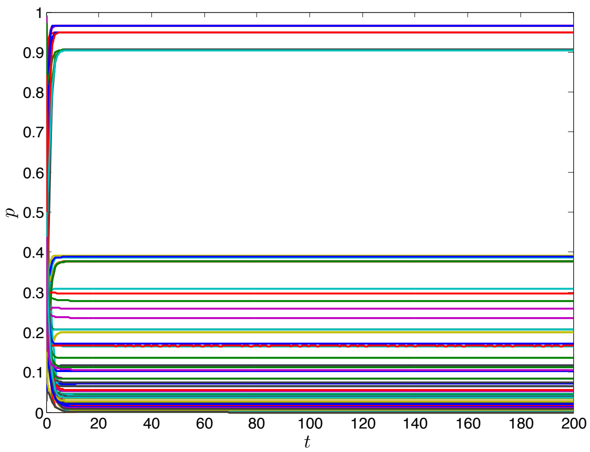

Fig. 2: Infection probabilities of the nodes.

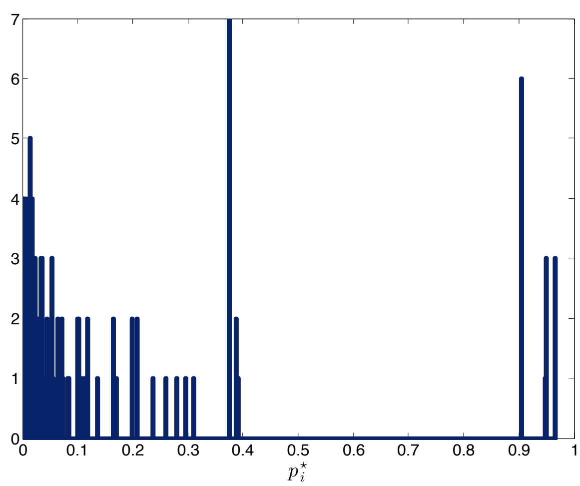

Fig. 3: A histogram of the endemic state value across the network.

VII. Conclusion

We have utilized tools from positive systems theory to establish the stability properties of the n-intertwined Markov model over digraphs. For strongly connected graphs, we have shown that when the basic reproduction number is less than one, the all-healthy state is GAS. When the basic reproduction number is greater than one, we have shown that the endemic state is GAS, and that locally around this equilibrium, the convergence is exponentially fast. Furthermore, we have studied the stability properties of weakly connected graphs. By viewing an arbitrary weakly connected graph as a DAG of SCCs, we were able to establish the existence and uniqueness of weak and strong endemic states over such graphs. Finally, we have studied the stability properties of weakly connected graphs using input-to-state stability. Future work will focus on studying the stability properties of the SIS dynamics over time-varying networks and designing optimal curing mechanisms.

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