

# Stability Properties of Infected Networks with Low Curing Rates

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**Abstract**—In this work, we analyze the stability properties of a recently proposed dynamical system that describes the evolution of the probability of infection in a network. We show that this model can be viewed as a concave game among the nodes. This characterization allows us to provide a simple condition, that can be checked in a distributed fashion, for stabilizing the origin. When the curing rates at the nodes are low, a residual infection stays within the network. Using properties of Hurwitz Mertzler matrices, we show that the residual epidemic state is locally exponentially stable. We also demonstrate that this state is globally asymptotically stable. Furthermore, we investigate the problem of stabilizing the network when the curing rates of a limited number of nodes can be controlled. In particular, we characterize the number of controllers required for a class of undirected graphs. Several simulations demonstrate our results.

## I. INTRODUCTION

Viruses, misinformation, and rumors can diffuse rapidly through a network via local interactions. Modeling the spread of misinformation in networks as well as the control of such phenomena have received wide interest in the literature [1]–[4]. Issues such as detecting a rumor source [5], allocating curing rates [6], and influence limitation [7] have also been studied. The typical approach for modeling the spread of infection has been via describing the local interactions among individuals within the network. An example of such models is the so-called  $n$ -intertwined Markov model [1], which belongs to the susceptible-infected-susceptible (SIS) class where each node can either be healthy or infected.

The flow that prescribes the evolution of the  $n$ -intertwined model is nonlinear. When the curing rate is high, the states of the individual nodes are provably convergent to the healthy state. When the curing rates are low, however, a strictly positive equilibrium point arises in the  $n$ -intertwined dynamics, which is referred to as the “metastable” state in the literature. At this stage, a residual infection will persist in the network. One main focus of this paper is characterization of the stability properties of this equilibrium. Some earlier and parallel attempts toward establishing stability of the metastable equilibrium can be found in [8] and [9]. By formulating a decentralized control system, we also study the case where the curing rates at a limited number of nodes can be controlled.

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## Statement of Contributions

The main contributions of this paper are threefold. First, we propose a generic model for the local interactions in an infected network using noncooperative games. This model provides a set of dynamical systems which describe the propagation of infection over networks. In particular, we show that the  $n$ -intertwined model prescribes the best-response dynamics of a concave game [10]. This allows us to provide a new condition for the stability of the origin, which can be checked collectively by the nodes. Next, using properties of Hurwitz Mertzler matrices, we show that the metastable state of the  $n$ -intertwined model is locally exponentially stable. Further, we show that the metastable state is in fact globally asymptotically stable. Finally, when the curing rates of a limited number of nodes can be controlled, we identify conditions under which the network can be stabilized to the origin. In particular, for path, star, and tree graphs, we characterize sufficient conditions for stabilization of the infection dynamics. Several simulations illustrate our results.

## Organization

Section II contains some mathematical preliminaries. In Section III, we introduce a game theoretic framework for modeling interactions over infected networks. We establish a connection between this framework and the  $n$ -intertwined model and study the stability of the latter with low curing rates in Section IV. Section V contains our results on the design of stabilizing controllers for infected networks. Numerical studies are provided in Section VI. Finally, Section VII gathers our conclusions and ideas for future work.

## II. MATHEMATICAL PRELIMINARIES

### Notation

We use the game theoretic notation  $x_{-i}$  to refer to the vector comprised of the optimization variables of all players except that of player  $i$ . We denote an edge between nodes  $i$  and  $j$  by  $\{i, j\}$ . For a real vector function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , the  $k$ -th derivative with respect to  $x_i$ , i.e.,  $\frac{\partial^k}{\partial x_i^k} f(x)$ , is denoted by  $\nabla_i^k f(x)$ . The Jacobian matrix of the dynamical system  $\dot{x} = \nabla f$  is denoted by  $J(x) = \nabla^2 f$ . In fact,  $J(x)$  is the Hessian of the function  $f$ . We denote the  $i$ -th eigenvalue of a matrix  $X$  by  $\lambda_i(X)$ . The identity matrix is denoted by  $I$ .

### Matrix Theory

We call square matrices  $X$  and  $Y$  similar if there exists a nonsingular matrix  $T$  such that  $Y = T^{-1}XT$ . An important property of similar matrices is that they share the same eigenvalues. A real square matrix  $X$  is called a Metzler matrix if its off-diagonal entries are nonnegative. It is well-known that Hurwitz Metzler matrices are diagonally stable

[11]. In particular, if  $X$  is a Hurwitz Metzler matrix, then there exists a diagonal matrix  $Q$  with positive entries such that

$$X^T Q + QX = -K, \quad (1)$$

where  $K$  is a positive definite matrix.

### III. GENERIC DYNAMICAL MODEL FOR INFECTED NETWORKS

Consider a network of  $n$  nodes that is described by a graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ , where  $\mathcal{V}$  is the set of vertices, and  $\mathcal{E}$  is the set of edges. Let  $A$  be the adjacency matrix of the graph with entries  $a_{ij} \in \mathbb{R}_{\geq 0}$ , where  $a_{ij} = 0$  if and only if  $\{i, j\} \notin \mathcal{E}$ . Let  $0 \leq x_i \leq 1$  be the rate with which node  $i$  sends messages. The objective function of each node  $i$ , denoted  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ , is comprised of a local utility function  $U_i : [0, 1] \rightarrow \mathbb{R}$ , and a component that is influenced by the neighboring agents of the node. The influence of node  $j$  on node  $i$  is described via the function  $\tilde{g}_{ij} : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ . We can then write the objective function of node  $i$  as

$$f_i(x_i, x_{-i}) = U_i(x_i) + \sum_{j \neq i} \tilde{g}_{ij}(x_i, x_j). \quad (2)$$

An interesting form of the influence function  $\tilde{g}_{ij}$  is the one used in the following:

$$f_i(x_i, x_{-i}) = U_i(x_i) + x_i \sum_{j \neq i} a_{ij} g_{ij}(x_j). \quad (3)$$

The benefit of working with the particular structure in (3) is twofold: (i) it highlights the fact that  $x_i$  is a rate as it multiplies the total influence of the neighboring nodes  $\sum_{j \neq i} a_{ij} g_{ij}(x_j)$ ; (ii) the second derivative of  $f_i$  with respect to  $x_i$  is independent of  $x_{-i}$ , which allows us to design concave games when  $U_i$  is selected to be concave in  $x_i$ .

Each node is interested in maximizing its own objective function  $f_i$ . Formally, we can write the problem of the  $i$ -th agent as

$$\max_{0 \leq x_i \leq 1} f_i(x_i, x_{-i}), \quad \text{for each fixed } x_{-i}. \quad (4)$$

When  $f_i$  is concave in  $x_i$ , and because the objective function of each player depends on the actions of other players, problem (4) describes a concave game [10]. The solution concept we are interested in studying here is the pure-strategy Nash equilibrium (PSNE).

**Definition 1 ([12]):** The vector  $x^*$  constitutes a PSNE if

$$f(x_i^*, x_{-i}^*) \geq f(x_i, x_{-i}^*),$$

for all  $i \in \{1, \dots, n\}$ .

According to this definition, no agent has an incentive to unilaterally deviate from the person-by-person optimal solution  $x^*$ . The next proposition establishes the existence and uniqueness of the PSNE for the game in (4), when the game is concave.

**Proposition 1 ([10]):** Under the following diagonal dominance condition:

$$2 |\nabla_i^2 U_i(x_i)| > \sum_{j \neq i} |\nabla_j \nabla_i (a_{ij} \tilde{g}_{ij}(x_i, x_j) + a_{ji} \tilde{g}_{ji}(x_j, x_i))|, \quad (5)$$

the concave game in (4) admits a unique PSNE.

### IV. STABILITY OF THE $n$ -INTERTWINED MARKOV MODEL

In the remaining parts of this paper, we take epidemic networks as a specific example of infected networks. In particular, we work with a recently proposed virus spread model called the the  $n$ -intertwined Markov model [1], which we will briefly review. This model captures the interdependence between the infection levels of a given node and its neighboring nodes. To this end, we assume that the graph  $\mathcal{G}$  is connected. In a disconnected graph, one should study the stability properties of each isolated connected component separately. The proposed model is based on viewing each node in the network as a Markov chain with two states: infected or cured. The curing and infection in the network are described by two independent Poisson processes with rates  $\delta > 0$  and  $\beta > 0$ , respectively. The transition rates between the two states depend on the infection probabilities of the neighboring nodes as well as the curing and infection rates. A mean-field approximation is made in [1] to capture the effect of neighbors on a given node via the total expected infection. This facilitates the derivation of an ODE that describes the evolution of the probability of infection of node  $i$ . Let  $p_i(t) \in [0, 1]$  be the infection probability of node  $i$  at time  $t \in \mathbb{R}_{\geq 0}$ . The  $n$ -intertwined Markov model is given by

$$\dot{p}_i(t) = -\delta p_i(t) + (1 - p_i(t)) \beta \sum_{j \neq i} a_{ij} p_j(t). \quad (6)$$

This model can be extended to a heterogeneous setting where each node is associated with a curing rate  $\delta_i$  and an infection rate  $\beta_i$  [13]–[15]. In particular, let  $p = [p_1, \dots, p_n]^T$ ,  $D = \text{diag}(\delta_1, \dots, \delta_n)$ ,  $P = \text{diag}(p_1, \dots, p_n)$ , and  $B = \text{diag}(\beta_1, \dots, \beta_n)$ . We can then re-write (6) for the general heterogeneous case in matrix form as follows:

$$\dot{p}(t) = (AB - D)p(t) - P(t)ABp(t). \quad (7)$$

#### A. The $n$ -intertwined Markov Model as a Concave Game

The following lemma establishes a relationship between virus spread in networks and concave games.

**Lemma 1:** The dynamics of the  $n$ -intertwined Markov model are best-response dynamics of a concave game whose players are the nodes, and their objective functions are

$$f_i(p_i, p_{-i}) = -\frac{\delta}{2} p_i^2 + p_i \left(1 - \frac{p_i}{2}\right) \beta \sum_{j \neq i} a_{ij} p_j. \quad (8)$$

*Proof:* Recall the objective functions defined in (2). Let  $U_i(p_i) = -\frac{\delta}{2} p_i^2$  and  $\tilde{g}_{ij}(p_i, p_j) = p_i \left(1 - \frac{p_i}{2}\right) a_{ij} \beta p_j$ . We then obtain

$$\nabla_i^2 f_i(x_i, x_{-i}) = -\delta - \beta \sum_{j \neq i} a_{ij} p_j < 0.$$

It is not hard to see that the dynamics in (6) corresponds to the gradient flow dynamics when the agents aim at maximizing their own objective functions (8). ■

In the homogeneous case, i.e., when the curing and infection rates do not vary per node, and when the graph is undirected, it was shown in [1] that, starting from an arbitrary initial

infection profile, the state converges to zero exponentially fast if

$$\lambda_1(A) < \frac{\delta}{\beta}, \quad (9)$$

where  $\lambda_1(A)$  is the largest eigenvalue of the adjacency matrix  $A$  of the graph. By applying the diagonal dominance condition in (5) to (8), we obtain

$$2\delta > \beta \sum_{j \neq i} a_{ij}(1 - p_i - p_j).$$

Define  $R_i := \sum_{j \neq i} a_{ij}$ . A sufficient condition for the above inequality to hold is

$$\frac{1}{2} \max_i R_i < \frac{\delta}{\beta}. \quad (10)$$

Note the similarities between (9) and (10). The two conditions are related by the Gershgorin Circle Theorem which states that every eigenvalue of  $A$  lies within at least one of the Gershgorin discs  $D(a_{ii}, R_i) = \{x \in \mathbb{R} : |x - a_{ii}| \leq R_i\}$ .

While (10) is more restrictive than (9), it is easier to compute and can be converted to a linear condition by requiring  $\frac{1}{2}R_i < \delta/\beta$  for all  $i$ . More importantly, when converted to the linear version, condition (10) can be checked in a distributed fashion.

In [13], the condition for exponential stability of the origin was extended to the heterogeneous setting. In principle, the following inequality provides a sufficient condition for stability of the origin

$$\lambda_1(AB - D) < 0. \quad (11)$$

This condition is transformed into a centralized eigenvalue equation in [13]. The extension of our condition to the heterogeneous case is straightforward:

$$\frac{1}{2} \sum_{j \neq i} a_{ij}\beta_j < \delta_i, \quad i = 1, \dots, n.$$

Note that this general condition still maintains the same attractive features of (10).

### B. Stability

Under the assumption that the steady-state exists, the equilibrium points of the dynamics (7) were derived in [1], [13]. Solving the equation

$$0 = (AB - D)p - PABp \quad (12)$$

for the steady-state leads to a quadratic equation in  $p_i$  which can have multiple solutions. However, it was shown in [13] that the origin  $p = 0$  is the only nonnegative vector that solves (12) when condition (11) is satisfied. Using the comparison lemma, it was shown in [15] that the origin is globally exponentially stable when (11) is satisfied. We provide an alternative proof to this fact using Lyapunov theory, which motivates some of the upcoming constructions.

**Proposition 2:** Under condition (11), the origin is globally exponentially stable.

*Proof:* Consider the quadratic Lyapunov function  $V(p) = \frac{1}{2}p^T p$ . We can then write

$$\begin{aligned} \dot{V}(p) &= p^T \dot{p} = p^T(AB - D)p - p^T PABp \\ &\leq p^T(AB - D)p \leq \lambda_1(AB - D)|p|^2 < 0, \quad p \neq 0. \end{aligned}$$

The first inequality follows because  $p^T PABp \geq 0$ . As for the second inequality, note that  $AB - D$  is similar to the symmetric matrix  $B^{1/2}AB^{1/2} - D$ ; hence,  $AB - D$  possesses real spectra, and the inequality follows. The last inequality holds because of (11). Since the Lyapunov function is quadratic, and its derivative has a quadratic upper bound, it follows that the state converges globally exponentially fast to the origin. This proves the proposition. ■

Note that the symmetry of  $B^{1/2}AB^{1/2} - D$  was crucial for obtaining a negative upper bound for the derivative of the Lyapunov function.

Interestingly, when condition (11) is violated, another valid probability vector arises in addition to the origin. This other equilibrium point is called the ‘‘metastable’’ state; we denote it by  $p^*$ . The metastable state has the property that  $p_i^* > 0$  for all  $i$ . This clearly shows that there will be a residual epidemic when the ratio  $\delta_i/\beta_i$  is below a certain threshold. Define  $\xi_i := \sum_{j \neq i} a_{ij}\beta_j p_j^*$  and  $\xi_i^* := \sum_{j \neq i} a_{ij}\beta_j p_j^*$ . We can then re-write (12) as

$$p_i^* = \frac{\xi_i^*}{\delta_i + \xi_i^*} = 1 - \frac{\delta_i}{\delta_i + \xi_i^*}. \quad (13)$$

Note that this expression of  $p_i^*$  is recursive, i.e., it depends on  $p_j^*$  which in turn depends on  $p_i^*$ . It was shown in [13] that  $p_i^*$  can be written as an infinite continued fraction that depends on the problem parameters only:  $A, \delta_i, \beta_i$ . Note that  $p_i^* < 1$  if and only if  $\delta_i > 0$ , which we assume throughout the paper. We also assume that  $\beta_i > 0$  for all  $i$ . The existence of this equilibrium is established in [1], [13]. As one of the main results of this paper, we study the stability properties of this equilibrium and compare our approach to earlier and parallel attempts in [8], [9]. Throughout the rest of this section, we assume that all the networks are undirected. By Lemma 1, we know that we can write the dynamics in (7) as  $\dot{p} = \nabla f$ , where  $\nabla f = [\nabla_1 f_1, \dots, \nabla_n f_n]^T$ , and  $f_i$  is defined in (8). We then recall from Section II that  $J(p) = \nabla^2 f$  is the Jacobian matrix of the dynamics (7).

**Lemma 2:** The origin is unstable when  $\lambda_1(AB - D) > 0$ .

*Proof:* Note that we have  $J(0) = AB - D$ . The linearized dynamics around the origin are then given by  $\dot{z} = (AB - D)z$ . Since  $\lambda_1(AB - D) > 0$ , using Lyapunov’s indirect method [16, Theorem 4.7, p. 139], it follows that the origin is unstable. ■

The following result is instrumental in proving our main result. Though it is a straightforward result, we will provide its proof for completeness.

**Proposition 3:** Let  $X$  be a Hurwitz Metzler matrix, and let  $Y$  be diagonal and positive definite. Then,  $YX$  is a Hurwitz Metzler matrix.

*Proof:* Since  $Y$  is a diagonal matrix with positive entries, multiplying it by  $X$  from the left (or right) does not change the sign of the entries of  $X$ . Hence,  $YX$  is a

Metzler matrix. To show that  $YX$  is Hurwitz, note that  $YX$  is similar to the matrix  $Y^{1/2}XY^{1/2}$ . Hence,  $YX$  is Hurwitz if and only if  $Y^{1/2}XY^{1/2}$  is Hurwitz. Since  $X$  is a Hurwitz Metzler matrix, there exists a positive diagonal matrix  $Q$  satisfying (1). Multiplying by  $Y^{1/2}$  from the right and left in (1) yields

$$Y^{1/2}X^TQY^{1/2} + Y^{1/2}QXY^{1/2} = -Y^{1/2}KY^{1/2}.$$

Since both  $Y$  and  $Q$  are diagonal, they commute. We can therefore write

$$(Y^{1/2}XY^{1/2})^TQ + Q(Y^{1/2}XY^{1/2}) = -Y^{1/2}KY^{1/2}.$$

Because  $Y^{1/2}KY^{1/2}$  is a positive definite matrix, the matrix  $Y^{1/2}XY^{1/2}$  is Hurwitz. This in turn implies that  $YX$  is also Hurwitz, as required. ■

**Theorem 1:** In a connected graph  $\mathcal{G}$ , the metastable state  $p^*$  is locally exponentially stable when  $\lambda_1(AB - D) > 0$ .

*Proof:* Similar to the previous lemma, we invoke Lyapunov's indirect method. Note that

$$J_{ii}(p^*) = -(\delta_i + \xi_i^*), \quad J_{ij}(p^*) = (1 - p_i^*)a_{ij}\beta_{ij}. \quad (14)$$

Using the definition of  $p^*$  in (13), we realize that  $J_{ii}(p^*) = -\delta_i/(1 - p_i^*)$ . More compactly, we can write

$$J(p^*) = (I - P^*)(-L + AB),$$

where  $P^* = \text{diag}(p_1^*, \dots, p_n^*)$  and  $L = (I - P^*)^{-2}D$ . It was shown in [13, Lemma 2] that the eigenvalues of the matrix  $-L + AB$  are real and negative, i.e., it is a Hurwitz matrix. Note that it is also a Metzler matrix. Recall that when  $\lambda_1(AB - D) > 0$ ,  $p^*$  is a valid probability vector with  $p_i^* < 1$  for all  $i$ . It then follows that  $I - P^*$  is a positive definite diagonal matrix. Using Lemma 3, we conclude that  $J(p^*)$  is Hurwitz, and  $p^*$  is locally exponentially stable. ■

The main theorem of this paper establishes the global stability of  $p^*$  using a quadratic Lyapunov function. An independent proof has appeared at the same time in [9]. However, the proof in [9] relies on the fact that, given  $p(0) \neq 0$ , there is a time by which  $p \in (0, 1]^n$ , but no rigorous proof was provided for this claim. Our proof is direct and does not rely on this fact. The proof in [8] makes use of LaSalle's Invariance Principle and other advanced tools from graph theory, which we completely avoid in our direct approach. Further comparison and discussion on these proofs, and other attempts in the literature, are left for a forthcoming publication.

**Theorem 2:** Assume that  $p(0) \neq 0$  and the graph  $\mathcal{G}$  is connected. Then, when  $\lambda_1(AB - D) > 0$ , the metastable state  $p^*$  is globally asymptotically stable.

*Proof:* Note that using Lemma 2, the set  $W = [0, 1]^n \setminus \{0\}$  is invariant under the evolutions of (7). Let  $p(0) \in W$  be any initial condition. Next, define the new state  $\tilde{p}_i = p_i - p_i^*$ . Also, let  $\tilde{\xi}_i = \xi_i - \xi_i^*$ . The dynamics of  $\tilde{p}_i$  are given by

$$\begin{aligned} \dot{\tilde{p}}_i &= \dot{p}_i = -\delta_i(\tilde{p}_i + p_i^*) + (1 - (\tilde{p}_i + p_i^*))(\tilde{\xi}_i + \xi_i^*) \\ &= -\delta_i\tilde{p}_i + (1 - p_i^*)\tilde{\xi}_i - \tilde{p}_i(\tilde{\xi}_i + \xi_i^*), \end{aligned}$$

which in terms of the Jacobian at  $p^*$  defined in (14) can be re-written as

$$\dot{\tilde{p}} = J(p^*)\tilde{p} - \tilde{P}AB\tilde{p},$$

where  $\tilde{P} = \text{diag}(\tilde{p}_1, \dots, \tilde{p}_n)$ .

Consider now the quadratic Lyapunov function  $V(\tilde{p}) = \frac{1}{2}\tilde{p}^T\tilde{p}$ . We can then write

$$\begin{aligned} \dot{V}(\tilde{p}) &= \tilde{p}^T J(p^*)\tilde{p} - \tilde{p}^T \tilde{P}AB\tilde{p} \\ &= \tilde{p}^T J(p^*)\tilde{p} + \tilde{p}^T \tilde{P}ABp^* - \tilde{p}^T \tilde{P}ABp. \end{aligned}$$

Evaluating (12) at  $p^*$  yields

$$ABp^* = (I - P^*)^{-1}Dp^*. \quad (15)$$

Note that  $\tilde{P}p^* = P^*\tilde{p}$ . We therefore have

$$\begin{aligned} \tilde{p}^T \tilde{P}ABp^* &= \tilde{p}^T \tilde{P}(I - P^*)^{-1}Dp^* \\ &= \tilde{p}^T (I - P^*)^{-1}DP^*\tilde{p}, \end{aligned}$$

where the first equality follows from (15), and the second equality hold because all the matrices involved are diagonal.

Define  $\Sigma := (I - P^*)^{-1}DP^*$ . Consider now the following matrix:

$$\begin{aligned} J(p^*) + \Sigma &= (I - P^*)(-(I - P^*)^{-2}D + AB) \\ &\quad + (I - P^*)^{-1}DP^* \\ &= -D + (I - P^*)AB. \end{aligned}$$

From (15), it follows that  $(J(p^*) + \Sigma)p^* = 0$ . Since the graph  $\mathcal{G}$  is connected, it follows by Theorem 1 in [13] that the symmetric matrix  $(I - P^*)^{-1}(J(p^*) + \Sigma)B^{-1}$  has a unique zero eigenvalue, and the rest of its eigenvalues are negative. This implies that the eigenvalues of  $J(p^*) + \Sigma$  are real and nonpositive. To see this, define the matrix  $M := (I - P^*)^{1/2}B^{1/2}$ , and note that the matrix  $J(p^*) + \Sigma$  is similar to the symmetric matrix  $M(I - P^*)^{-1}(J(p^*) + \Sigma)B^{-1}M$ . This implies that the eigenvalues of  $J(p^*) + \Sigma$  are real. The quadratic form  $x^T M(I - P^*)^{-1}(J(p^*) + \Sigma)B^{-1}Mx$  is nonpositive for every  $x \in \mathbb{R}^n$ , because  $M$  is a nonsingular matrix, and the matrix  $(I - P^*)^{-1}(J(p^*) + \Sigma)B^{-1}$  is negative semidefinite. It follows that  $M(I - P^*)^{-1}(J(p^*) + \Sigma)B^{-1}M$  is negative semidefinite, which implies that the eigenvalues of  $J(p^*) + \Sigma$  are nonpositive.

Noting that the vector  $\tilde{P}\tilde{p} = [\tilde{p}_1^2, \dots, \tilde{p}_n^2]^T$  is nonnegative, we can then write

$$\begin{aligned} \dot{V}(\tilde{p}) &= \tilde{p}^T (J(p^*) + \Sigma)\tilde{p} - \tilde{p}^T \tilde{P}ABp \\ &\leq -\tilde{p}^T \tilde{P}ABp \leq 0. \end{aligned}$$

Given that the set  $W$  is invariant under (7), we have that  $\dot{V}(\tilde{p}) = 0$  if and only if  $p = p^*$ , which completes the proof. ■

## V. CONTROL DESIGN

In this section, we will investigate the possibility of reducing the infection by altering the curing rates at the limited number of nodes belonging to a set  $S_{\text{control}} \subset \mathcal{V}$ . For this reason, throughout this section, we replace  $\delta_i$  with  $u_i(t)$ , where  $i \in S_{\text{control}}$ . Given the necessary conditions presented in the recent paper [17], we will use the assumption that there

exists a small curing rate of  $\alpha_i$  at any node in  $\mathcal{F} = \mathcal{V} \setminus S_{\text{control}}$ . This amount of self healing may, however, not be enough to stabilize the system to the origin. We are interested in answering the following question: *When condition (11) is initially violated, can we stabilize the system to the origin by controlling the nodes in  $S_{\text{control}}$  only?*

By construction, we have  $S_{\text{control}} \cap \mathcal{F} = \emptyset$  and  $S_{\text{control}} \cup \mathcal{F} = \{1, \dots, n\}$ . Let  $U(t)$  be a diagonal matrix such that  $U_{ii}(t) = u_i(t)$  if and only if  $i \in S_{\text{control}}$  and zero otherwise. Similarly, let  $\Gamma$  be a diagonal matrix such that  $\Gamma_{ii} = \alpha_i$  if and only if  $i \in \mathcal{F}$  and zero otherwise. The dynamics can then be written as:

$$\dot{p}(t) = (AB - \Gamma - U(t))p(t) - P(t)ABp(t),$$

Note that this system is affine in controls. To see this, define  $h(p) = (AB - \Gamma)p - PABp$  and  $g_i(p) := -p_i e_i$ , where  $\{e_1, \dots, e_n\}$  is the fundamental basis. We can then write

$$\dot{p} = h(p) + \sum_{i \in S_{\text{control}}} g_i(p)u_i.$$

When zero is unstable for the drift vector field  $\dot{p} = h(p)$ , the only feasible design problem, when the controllers must be bounded, is to find a control  $u$  that would drive  $p^*$  as close as possible to zero. We are currently investigating this problem.

In what follows we consider two special cases for which a limited number of controllers can stabilize the system.

**Lemma 3:** The star graph can be stabilized by placing an appropriate controller at the root node and arbitrarily small  $\alpha$ -self-loops everywhere else.

*Proof:* We will proceed by showing that the function  $V(p) = \frac{1}{2}p^T p$  is a control Lyapunov function (CLF). Without loss of generality, let node 1 be the root. The dynamics of all other nodes is then given by  $\dot{p}_i = -\alpha_i p_i + (1 - p_i)\alpha_{i1}\beta_1 p_1$ . A necessary and sufficient condition for  $V(p)$  to be a CLF is

$$\frac{\partial}{\partial p} V(p)^T g_1(p) = -p_1^2 = 0 \implies \frac{\partial}{\partial p} V(p)^T h(p) < 0, \quad p \neq 0.$$

But when  $p_1 = 0$ , we have

$$\frac{\partial}{\partial p} V(p)^T h(p) = p^T (AB - \Gamma)p - p^T PABp = -p^T \Gamma p,$$

which is negative. Hence,  $V(p)$  is indeed a CLF, and we can stabilize the system using Sontag's universal controller [18]. ■

Note that Sontag's controller requires the controlling node to have knowledge of the entire state. In the above, the root node is connected to all the nodes, and hence it has access to the state vector  $p$ .

**Lemma 4:** In an odd (or even) length path graph, a maximum of  $(n - 1)/2$  (or  $n/2$ ) controllers are required to stabilize the network, provided that all other nodes implement arbitrarily small  $\alpha$ -self-loops.

*Proof:* The proof is similar to the star graph case. We will show that  $V(p) = \frac{1}{2}p^T p$  is a CLF. Let us place the

controllers at nodes  $\{2, 4, \dots\}$ . Then, from the structure of  $A$ , it follows that  $p^T ABp = 0$  when

$$\frac{\partial}{\partial p} V(p)^T (g_2(p), g_4(p), \dots) = - \sum_{i \in S_{\text{control}}} p_i^2 = 0.$$

This implies that  $\frac{\partial}{\partial p} V(p)^T h(p) = -p^T \Gamma p$ , and  $V(p)$  is a CLF. The size of  $S_{\text{control}}$  follows from the way we have placed the controllers. This concludes the proof. ■

Similar results can be obtained for other classes of graphs. The key idea behind the above results is to place the controllers in such a way that no path can be drawn between two nodes in  $\mathcal{F}$  without passing through a node in  $S_{\text{control}}$ . For example, in a tree with an even number of levels, stabilization can be achieved by controlling the nodes in every other level, and placing arbitrarily small  $\alpha$ -self-loops everywhere else.

## VI. NUMERICAL STUDIES

We will demonstrate the global stability of  $p^*$ , which we proved in Theorem 2. The infection rates, the edge weights, and the initial infection profile were generated randomly. The curing rates were selected in a way that violates (11).

Fig. 1 shows the state of a ring graph with 20 nodes. The figure also plots the Lyapunov function  $V(\tilde{p}) = \frac{1}{2}\tilde{p}^T \tilde{p}$ . As claimed, the system converges to the strictly positive state  $p^*$ , and the Lyapunov function decays monotonically to zero.

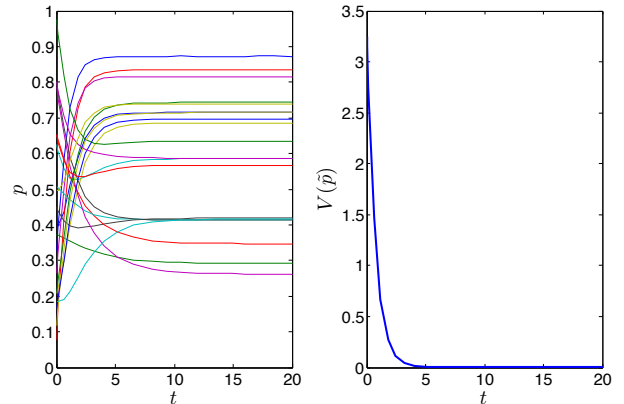


Fig. 1: Stabilization of a ring graph with 20 nodes.

Fig. 2 shows the same simulation for a random graph with 100 nodes. The probability that an edge occurs in the graph was selected to be  $3/10$ . The specific graph realization used in this experiment contained 1704 edges. Again, we observe that the state converges to  $p^*$ . It is interesting to note that convergence here is faster than the case of the ring graph.

Next, we will compare the performance of Sontag's universal controller to a constant controller based on the cost of control as given by  $\int_0^T u_i(t)dt$ . The horizon of the simulation,  $T$ , is chosen to be 100. Consider a star graph with 10 nodes. By Lemma 3, we know that it suffices to control the root node to stabilize the network. Let node 1 be at the root. We assume that the rest of the nodes implement a self loop  $\alpha = 0.1$ . Fig. 3 illustrates the performance of a constant controller  $u_1 = 8$ , while the performance of Sontag's universal controller is shown in Fig. 4. We observe

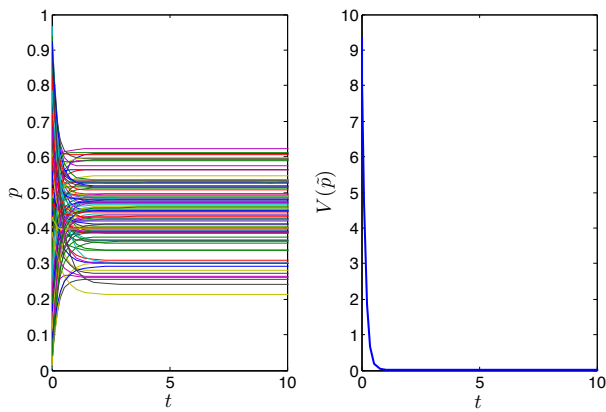


Fig. 2: Stabilization of a random graph with 100 nodes and 1704 edges.

that the stabilization properties of both controllers are similar. However, Sontag’s universal controller incurs a lower cost compared to the constant controller; the total cost incurred by the constant controller is 800, while that incurred by Sontag’s controller is 738.6.

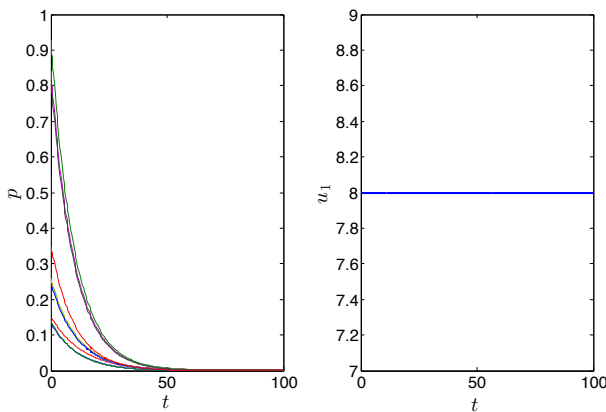


Fig. 3: A star graph with a constant controller implemented at the root.  $n = 10$ .

## VII. CONCLUSION

We have proposed a dynamical model that describes the interaction among nodes as a concave game and demonstrated that the  $n$ -intertwined model is a special case of it. This alternative description provides a new condition, which can be checked collectively by agents, for the stability of the origin. When the curing rates in the network are low, we show that the metastable state  $p^*$  is locally exponentially stable. Moreover, we provided a Lyapunov-based proof to show that  $p^*$  is globally asymptotically stable. Finally, we proposed a method that allows for stabilizing the state to the origin using a limited number of controllers.

Future work will focus on the formulation of various optimal control problems in terms of curing rates, understanding the stability properties of directed graphs, further investigation of the fundamental limitations to stabilizability of infected networks, and decentralized control designs.

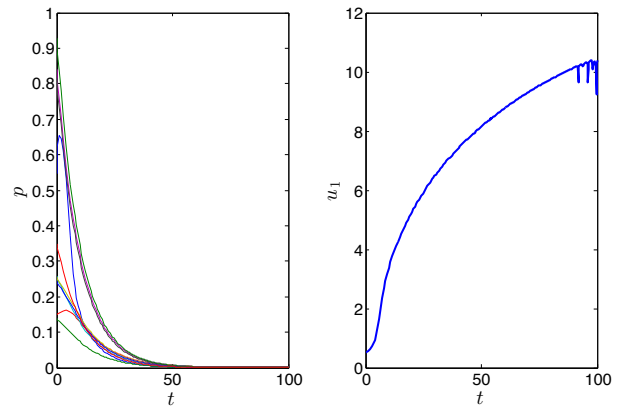


Fig. 4: A star graph with Sontag’s universal controller implemented at the root.  $n = 10$ .

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