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# Minimax control over unreliable communication channels☆

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ABSTRACT

In this paper, we consider a minimax control problem for linear time-invariant (LTI) systems over unreliable communication channels. This can be viewed as an extension of the $H^\infty$ optimal control problem, where the transmission from the plant output sensors to the controller, and from the controller to the plant are over sporadically failing channels. We consider two different scenarios for unreliable communication. The first one is where the communication channel provides perfect acknowledgments of successful transmissions of control packets through a clean reverse channel, that is the TCP (Transmission Control Protocol). Under this setting, we obtain a class of output feedback minimax controllers; we identify a set of explicit threshold-type existence conditions in terms of the $H^\infty$ disturbance attenuation parameter and the packet loss rates that guarantee stability and performance of the closed-loop system. The second scenario is one where there is no acknowledgment of successful transmissions of control packets, that is the UDP (User Datagram Protocol). We consider a special case of this problem where there is no measurement noise in the transmission from the sensors. For this problem, we obtain a class of corresponding minimax controllers by characterizing a set of (different) existence conditions. We also discuss stability and performance of the closed-loop system. We provide simulations to illustrate the results in all cases.


## 1. Introduction

Over the past decade, networked control systems (NCSs) have been introduced to capture the effect of communication limitations on control systems due to the widespread use of wireless communications, and the distributed or decentralized nature of modern control systems. The application areas of NCSs are very broad, with some examples being mobile sensor networks, remote surgery, unmanned aerial vehicles, and power and chemical plants (Hespanha, Naghshtabrizi, & Xu, 2007).

One major communication constraint in NCSs that has been considered in the literature is packet drops, also referred to as packet dropouts/losses or data erasure, between the controller and the plant, as well as between the sensors and the controller (Hespanha et al., 2007). Under this constraint, control and measurement packets that are generated respectively by controllers and sensors may not be transmitted/received reliably. Therefore, the objective should be to design effective control and estimation schemes that are able to tolerate a certain number of packet losses for stability and performance of the closed-loop system. Numerous types of models have been proposed to describe packet drops, but the Bernoulli-type model has been the most common one, because it captures generality and tractability (Hespanha et al., 2007; Imer, Yüksel, & Başar, 2006; Schenato, Sinopoli, Franceschetti, Poolla, & Sastry, 2007; Seiler & Sengupta, 2005; Sinopoli et al., 2004).

An initial study of networked control systems under Bernoulli-type packet drops for linear–quadratic–Gaussian (LQG) control systems was undertaken in Imer et al. (2006), where the link failure model was categorized based on whether the control packet reception is acknowledged (Transmission Control Protocol, TCP) or not (User Datagram Protocol, UDP). For the TCP-case, Imer et al. (2006) showed that the separation principle holds, the LQG controller is linear in the measurement, and the stability region is determined by the unstable modes of the system and control and measurement loss rates. For the UDP-case, it was shown in that paper that the optimal controller is linear under some conditions; however, there is *dual effect* between control and estimation (Bar-Shalom & Tse, 1974).

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The results in Imer et al. (2006) were extended by Schenato et al. (2007) to the noisy measurement case. Specifically, for the TCP-case, it was shown that the LQG controller in Imer et al. (2006) and the Kalman filter in Sinopoli et al. (2004) with the control input can be designed independently because there is no dual effect between filtering and control. The authors also characterized two independent critical values of control and measurement loss rates in terms of the unstable modes of the open-loop system. Furthermore, precise analytical bounds on these critical values were also provided. For the UDP-case, the authors showed that the optimal controller is generally nonlinear in the measurement, but is linear only for the perfect measurement case considered in Imer et al. (2006). The results for the TCP-case in Schenato et al. (2007) were extended to the multiple packet drop case in Garone, Sinopoli, Goldsmith, and Casavola (2012), to the generalized acknowledgment model in Garone, Sinopoli, and Casavola (2010), and to the limited transmission bandwidth case in Trivellato and Benvenuto (2010). The decentralized LQG control problem over TCP-networks was discussed in Chang and Lall (2011), and the LQG problem with Markovian packet losses was studied in Mo, Garone, and Sinopoli (2013). The stabilization issue and a characterization of the explicit critical values were also considered in Elia (2005), Elia and Eisenbeis (2011) and Silva and Pulgar (2011).

As for the $H^\infty$ control over unreliable communication channels, the framework of Markov jump linear systems (MJLSs) was mostly used in the literature with some related references being Costa, Fragoso, and Marques (2005), Geromel, Goncalves, and Fioravanti (2009), Ishii (2008), Lee and Dullerud (2006), Pan and Başar (1995), Sahebsara, Chen, and Shah (2008) and Seiler and Sengupta (2005). Related to the MJLS approach, Wang, Ho, Liu, and Liu (2009), Wang, Wang, and Wang (2013) and Wang, Yang, Ho, and Liu (2007) also studied $H^\infty$ control with random packet losses by identifying a set of (different) linear matrix inequalities. The controllers, however, were restricted to be time-invariant and are therefore suboptimal. We should mention that with the framework of MJLSs, the current mode of the Markov chain needs to be available to the controller, that is, the controller is mode dependent (Costa et al., 2005; Lee & Dullerud, 2006). Since the TCP-case does not have access to the current mode of the control packet loss information, as argued in Imer et al. (2006), the unreliable channel modeled by MJLSs is not equivalent to the TCP-case (or for that matter to the UDP-case). One of the most recent works of $H^\infty$ control over unreliable communication channels was presented in Shoukry, Araujo, Tabuada, Srivastava, and Johansson (2013); however, the communication degradation there was due to packet delays rather than packet drops.

In this paper, we study a minimax[2] control problem for LTI systems over unreliable communication channels. Unlike the previous work in Imer et al. (2006) and Schenato et al. (2007), this paper considers the case when the disturbance and the sensor noise are arbitrary and controlled by adversaries, instead of being stochastic with *a priori* specified statistics. We consider two different scenarios for unreliable communication channels: the TCP- and the UDP-case. Both channels are assumed to be temporally uncorrelated, which are modeled as two independent and identically distributed Bernoulli processes. These two different problems are formulated within the framework of stochastic zero-sum dynamic games, which enables us to develop worst-case ($H^\infty$) controllers under TCP- and UDP-like information structure.

We first consider the TCP-case. Due to its acknowledgment nature, we are able to apply the certainty equivalence principle developed in Başar and Bernhard (1995) and Didinsky (1994), where the deterministic $H^\infty$ optimal control was analyzed through three steps. By following these steps, we obtain a class of output feedback minimax controllers in terms of the $H^\infty$ disturbance attenuation parameter, say $\gamma$, and the control and measurement loss rates, where $\gamma$ is a parameter that measures robustness of the system against arbitrary disturbances as in standard $H^\infty$ control (Başar & Bernhard, 1995; Zhou, 1996). For the TCP-case, the minimax controller obtained is dependent on the acknowledged control packet loss information, which differs from the existing mode-dependent $H^\infty$ controller for MJLSs in the literature referenced above.

Specifically, the main results for the TCP-case can be summarized as follows:

(i) The existence of a minimax controller is dependent on $\gamma$, and the loss rates.
(ii) For given loss rates and $\gamma > 0$, if all the existence conditions are satisfied, then $\gamma$ is the attenuation level of the corresponding minimax controller.
(iii) The critical values of the control and measurement loss rates for closed-loop system stability and performance are functions of $\gamma$.
(iv) There is no separation between control and estimation.
(v) As $\gamma \to \infty$, the parametrized (in $\gamma$) minimax control system converges to the corresponding LQG control system in Imer et al. (2006) and Schenato et al. (2007).

Item (ii) implies that if $\gamma$ exists and is finite, then the corresponding admissible minimax controller achieves the disturbance attenuation level $\gamma$ for an arbitrary disturbance, that is, the $H^\infty$ norm of the closed-loop system is bounded above by $\gamma$ (Başar & Bernhard, 1995). As for item (v), the limiting behavior in terms of $\gamma$ implies that in view of item (ii), the disturbance does not play any role, since it is infinitely penalized; hence, the limiting behavior of the corresponding minimax controller collapses to the LQG controller as in the standard case discussed in Başar and Bernhard (1995) and Zhou (1996).

For the UDP-case, we consider the scenario when there is no measurement noise, which is a counterpart of the LQG problem discussed in Imer et al. (2006). We show that due to the absence of acknowledgments regarding control packet losses, there is dual effect between control and estimation, but the corresponding minimax controller parametrized by $\gamma$ is linear in the measurement. Such a dual effect problem did not arise in the $H^\infty$ control problem within the MJLS framework, since as already mentioned, the latter has access to the current mode of the Markov chain. We provide the (different) existence condition for the corresponding problem in terms of $\gamma$ and control and measurement loss rates. We also provide explicit expressions on the $H^\infty$ optimum disturbance attenuation parameter and the critical values for mean-square stability and performance of the closed-loop system. Moreover, we show that when $\gamma \to \infty$, the minimax control system collapses to the corresponding LQG system in Imer et al. (2006). Finally, from simulation results, we show that the stability and performance regions for the UDP-case is more severe than that of the TCP-case due to lack of acknowledgments.

We should mention that the TCP-case considered in this paper was discussed earlier in the conference versions in Moon and Başar (2013, 2014) in different forms. In these conference versions, some of the proofs were just outlined and/or were omitted due to space limitations. This paper provides complete results on the problem of minimax control with TCP-like unreliable communication channels, in addition to of course the UDP-case, as discussed above.

The organization of the paper is as follows. The problem formulation is stated in Section 2. Sections 3, 4, and 5 are for the TCP-case, which consider problems of state feedback minimax control, minimax estimation with intermittent observations (for

[2] In this paper, we will be using the qualifiers "$H^\infty$" and "minimax" interchangeably (Başar & Bernhard, 1995).

which complete development can be found in the companion paper Moon & Başar, submitted for publication), and the $H^\infty$ synthesis problem, respectively. A special case of the UDP problem is studied in Section 6. Section 7 provides numerical examples. We complete the paper with the concluding remarks of Section 8. In the Appendices A and B, a supporting lemma is provided, and the certainty equivalence principle of the $H^\infty$ control problem for the TCP-case is discussed.

*Notation*

$\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ denote, respectively, the spaces of $n$-dimensional and $m \times n$-dimensional real-valued vectors and matrices. We use $X > 0$ (resp. $X \geq 0$) when $X$ is a positive definite matrix (resp. positive semi-definite matrix). The transpose of a matrix $X$ (resp. a vector $x$) is denoted by $X^T$ (resp. $x^T$). For any $x \in \mathbb{R}^n$ and $S \geq 0$, we use $|x|_S^2 := x^T Sx$. $\{x_k\}$ (resp. $\{X_k\}$) denotes a sequence of vectors $x_k$ (resp. matrices $X_k$), $k = 0, 1, \ldots,$ with appropriate dimensions. $x_{0:k}$ is a shorthand for the vector $(x_0, x_1, \ldots, x_k)$. For any $X \in \mathbb{R}^{n\times n}$, $\rho(X)$ denotes the spectral radius of $X$. $\ell_2^n$ denotes the space of square-summable sequences, taking values in $\mathbb{R}^n$.

## 2. Problem formulation

We consider the following linear dynamical system:

$$x_{k+1} = Ax_k + \alpha_k Bu_k + Dw_k \tag{1a}$$

$$y_k = \beta_k Cx_k + Ew_k, \tag{1b}$$

where $x_k \in \mathbb{R}^n$ is the state; $u_k \in \mathbf{U} \subset \mathbb{R}^m$ is the control (actuator); $w_k \in \mathbf{W} \subset \mathbb{R}^p$ is the disturbance input as well as the measurement noise; $y_k \in \mathbf{Y} \subset \mathbb{R}^l$ is the sensor output; and $A, B, C, D, E$ are time-invariant matrices with appropriate dimensions. In (1), $\{w_k\}$ is a square-summable sequence, which is not necessarily stochastic. We further assume the following decompositions:

$$w_k = \begin{pmatrix}\bar{w}_k^T & v_k^T\end{pmatrix}^T, \qquad D = \begin{pmatrix}\bar{D} & 0\end{pmatrix}, \qquad E = \begin{pmatrix}0 & \bar{E}\end{pmatrix},$$

where $V = \bar{E}\bar{E}^T > 0$ and $\bar{E}$ is square and non-singular. Finally, we assume that the communication network is temporally uncorrelated, that is, $\{\alpha_k\}$ and $\{\beta_k\}$ in (1) are independent and identically distributed (i.i.d.) Bernoulli processes with $\mathbb{P}(\alpha_k = 1) = \alpha$ and $\mathbb{P}(\beta_k = 1) = \beta$, respectively. We denote the variance of $\alpha_k$ by $\bar{\alpha} := \alpha(1-\alpha)$.

The TCP-like information that is available to the controller is defined by

$$\begin{cases}\mathcal{I}_0 := \{y_0, \beta_0\} \\ \mathcal{I}_k := \{y_{0:k}, \alpha_{0:k-1}, \beta_{0:k}\}, \quad k \geq 1.\end{cases} \tag{2}$$

The UDP-like information is defined by

$$\begin{cases}\mathcal{G}_0 := \{y_0, \beta_0\} \\ \mathcal{G}_k := \{y_{0:k}, \beta_{0:k}\}, \quad k \geq 1.\end{cases} \tag{3}$$

Note that the major difference between (2) and (3) is that in (2), the acknowledgment signal, $\alpha_{0:k-1}$, is included, by which, as expected from the LQG case in Imer et al. (2006), the controller under (2) will provide better stability and performance.

A convention we adopt in this paper is one of zero-input strategy. That is, the actuator does not do anything when there are control packet losses. It was shown in Schenato (2009) that for the disturbance free case ($w_k \equiv 0$), using the one-step previous control packet in order to compensate for the current packet loss does not necessarily lead to better performance.

Let $\mathcal{U}$ and $\mathcal{W}$ be the appropriate spaces of control and disturbance policies, respectively. We define control and disturbance policies, $\mu \in \mathcal{U}$ and $\nu \in \mathcal{W}$, that consist of sequences of functions:

$$\mu = \{\mu_0, \mu_1, \ldots, \mu_{N-1}\}, \qquad \nu = \{\nu_0, \nu_1, \ldots, \nu_{N-1}\},$$

where $\mu_k$ and $\nu_k$ are Borel measurable functions which map the information set (2) or (3) into the control and disturbance spaces of $\mathbb{R}^m$ and $\mathbb{R}^p$, respectively. Note that in the spirit of the worst-case approach, the disturbance is assumed to know everything the controller does.

Now, our main objective in this paper is to obtain output feedback controllers over TCP- and UDP-networks, which minimize the following cost function:

$$\langle\langle \mathcal{T}_\mu^N \rangle\rangle := \sup_{(x_0, w_{0:N-1})} \frac{J^N(\mu,\nu)^{1/2}}{\mathbb{E}\left\{|x_0|_{Q_0}^2 + \sum_{k=0}^{N-1} |w_k|^2\right\}^{1/2}}, \tag{4}$$

where

$$J^N(\mu,\nu) = \mathbb{E}\left\{|x_N|_{Q_N}^2 + \sum_{k=0}^{N-1} |x_k|_Q^2 + \alpha_k |u_k|_R^2\right\},$$

where $Q, Q_N \geq 0$, $R, Q_0 > 0$, and $\mu$ and $\nu$ are the control and disturbance policies as introduced earlier. Note that the control $u_k$ incurs the additional cost only if it is applied to the plant. This can be viewed as an $H^\infty$ optimal control problem (Başar & Bernhard, 1995). It is worth noting that if $\alpha_k$ is included in (2), then the problem can be studied in the framework of Markov jump linear systems, and the optimal controller can then be obtained directly from Costa et al. (2005).

Associated with the system (1), we introduce the following zero-sum dynamic game that is parametrized by the disturbance attenuation parameter, $\gamma > 0$:

$$J_\gamma^N(\mu,\nu) = \mathbb{E}\Big\{|x_N|_{Q_N}^2 - \gamma^2 |x_0|_{Q_0}^2 + \sum_{k=0}^{N-1} |x_k|_Q^2 + \alpha_k |u_k|_R^2 - \gamma^2 |w_k|^2\Big\}, \tag{5}$$

subject to system (1a) and the measurement (1b).

Now, in view of (4) and (5), our main objective in this paper can be rephrased as one of obtaining a controller for (1) under the specified information structure such that it minimizes the cost function (5) while the disturbance maximizes the same cost function. In other words, we need to characterize a saddle point,[3] say $(\mu_\gamma, \nu_\gamma)$, for the zero-sum dynamic game (5) in terms of $\gamma$.

As in standard $H^\infty$ control (Başar & Bernhard, 1995), the existence of saddle-point solutions for (5) will be dependent on the value of $\gamma$. Therefore, we need to characterize the smallest value of $\gamma$, say $\gamma^*$, above which the saddle-point solutions exist. Then, by definition, for any $\gamma > \gamma^*$, the saddle point, $(\mu_\gamma, \nu_\gamma)$, exists, provided that $\gamma^*$ is finite. Moreover, for any finite $\gamma > \gamma^*$, $\mu_\gamma$ is a minimax controller that leads to finite upper value for the zero-sum dynamic game in (5), and achieves the performance level of $\gamma$ for (4), i.e., under $\mu_\gamma$, $\langle\langle \mathcal{T}_{\mu_\gamma}^N \rangle\rangle \leq \gamma$.

After characterizing a class of minimax controllers for the TCP and UDP-cases, the next goal is to examine such controllers with respect to the communication channel conditions. Specifically, given the controllers, we need to obtain the smallest values of $\alpha$ and $\beta$, say $\alpha_c$ and $\beta_c$, for the closed-loop system stability and performance. Obviously, $\alpha_c$ and $\beta_c$ are functions of $\gamma$, and $\gamma^*$ is a function of $\alpha$ and $\beta$.

In what follows, in Sections 3–5, we obtain a class of output feedback minimax controllers for the TCP-case. Toward that

[3] See Başar and Bernhard (1995) and Başar and Olsder (1999) for the definition of the saddle point of a zero-sum dynamic game. Normally, in going from (4) to (5), one would be looking for the minimax solution of (5), but as in Başar and Bernhard (1995), one could instead look for the saddle-point solution, without any loss of generality.

end, we apply the certainty equivalence principle discussed in Appendix B, in view of which the corresponding zero-sum dynamic game can be analyzed through three steps discussed in Appendix B. Note that the certainty equivalence principle was originally developed by Başar and Bernhard (1995) and Didinsky (1994) for the deterministic (no packet drops) $H^\infty$ control problem, and the results presented in Appendix B can be regarded as the *certainty equivalence principle* of the $H^\infty$ control problem for the TCP-case.

In Section 6, we obtain a (different) class of output feedback minimax controllers for the UDP-case. We consider a special case of this problem, where there is no measurement noise in (1b). As discussed in Section 6, the general minimax control problem for the UDP-case is hard, since there is no acknowledgment of control packet losses. In view of the certainty equivalence principle in Appendix B, this is because part (b) of the certainty equivalence principle cannot be applied to the UDP-case, which is shown in Section 4.

## 3. State feedback minimax control over the TCP-network

This section addresses part (a) of the certainty equivalence principle discussed in Appendix B. In particular, we obtain a state feedback minimax controller over the TCP-network.

### 3.1. Finite-horizon case

**Lemma 1.** *Consider the zero-sum dynamic game in (5) with a fixed $\gamma > 0$ and $\alpha \in [0, 1]$. Then:*

(i) *There exists a unique state feedback saddle-point solution if and only if*

$$\rho(D^T Z_{k+1} D) < \gamma^2, \quad \text{for all } k \in [0, N-1], \tag{6}$$

*where $Z_k$ is generated by the following generalized Riccati equation (GRE): $Z_N = Q_N$ and*

$$Z_k = Q + P_{u_k}^T(\alpha R + \bar{\alpha} B^T Z_{k+1} B) P_{u_k} - \gamma^2 P_{w_k}^T P_{w_k} + H_k^T Z_{k+1} H_k, \tag{7}$$

*where*

$$H_k = A - \alpha B P_{u_k} + D P_{w_k} \tag{8a}$$

$$P_{u_k} = (R + B^T(I + \alpha Z_{k+1} D M_k^{-1} D^T) Z_{k+1} B)^{-1} \times B^T (I + Z_{k+1} D M_k^{-1} D^T) Z_{k+1} A \tag{8b}$$

$$P_{w_k} = (\gamma^2 I - D^T(I - \alpha Z_{k+1} B L_k^{-1} B^T) Z_{k+1} D)^{-1} \times D^T (I - \alpha Z_{k+1} B L_k^{-1} B^T) Z_{k+1} A \tag{8c}$$

$$M_k = \gamma^2 I - D^T Z_{k+1} D \tag{8d}$$

$$L_k = R + B^T Z_{k+1} B. \tag{8e}$$

(ii) *The feedback saddle-point policies, $(\mu_\gamma^*, \nu_\gamma^*)$, can be written as*

$$u_k^* = \mu_k^*(\mathcal{I}_k) = -P_{u_k} x_k \tag{9}$$

$$w_k^* = \nu_k^*(\mathcal{I}_k) = P_{w_k} x_k, \quad k \in [0, N-1]. \tag{10}$$

(iii) *If $M_k$ has a negative eigenvalue for some $k$, then the zero-sum dynamic game does not admit a saddle point and the upper value of the game becomes unbounded.*

**Proof.** To prove parts (i) and (ii), we need to employ dynamic programming with the following value function (Başar & Olsder, 1999):

$$V_k(x_k) = \mathbb{E}\{x_k^T Z_k x_k | \mathcal{I}_k\},$$

where $Z_k \geq 0$ is given in (7) with $Z_N = Q_N$. Now, by induction, suppose the claim is true for $k+1$. That is, $V_{k+1}(x_{k+1})$ is the saddle-point value of the static zero-sum game at $k+1$ under (6). Then, since the information structure of the TCP-network is nested for all $k$, the cost-to-go at $k$ can be written as

$$V_k(x_k) = \min_{u_k} \max_{w_k} \mathbb{E}\Big\{h_k(x, u, w) + V_{k+1}(x_{k+1}) | \mathcal{I}_k\Big\} \tag{11}$$

$$= \max_{w_k} \min_{u_k} \mathbb{E}\Big\{h_k(x, u, w) + V_{k+1}(x_{k+1}) | \mathcal{I}_k\Big\}, \tag{12}$$

where $h_k(x, u, w) := |x_k|_Q^2 + \alpha_k |u_k|_R^2 - \gamma^2 |w_k|^2$. Under (6), the static zero-sum game above is strictly convex in $u_k$ and concave in $w_k$; hence there is a unique pair of minimizer and maximizer, which can be written as

$$u_k^* = -(R + B^T Z_{k+1} B)^{-1} B^T Z_{k+1}(A x_k + D w_k^*) =: \varphi_{1,k}(x_k, w_k^*)$$

$$w_k^* = (\gamma^2 I - D^T Z_{k+1} D)^{-1} D^T Z_{k+1}(A x_k + \alpha B u_k^*) =: \varphi_{2,k}(x_k, u_k^*).$$

The explicit expressions of $u_k^*$ and $w_k^*$ can be obtained by seeking fixed points of the above:

$$u_k^* = \varphi_{1,k}(x_k, \varphi_{2,k}(x_k, u_k^*)) = -P_{u_k} x_k$$

$$w_k^* = \varphi_{2,k}(x_k, \varphi_{1,k}(x_k, w_k^*)) = P_{w_k} x_k,$$

which is (9) and (10). Then the pair of (9) and (10) for each $k$ constitutes a saddle point for the static zero-sum game at $k$, and by substituting (9) and (10) into (11) (or (12)), we arrive at the GRE. Proceeding similarly, we can obtain the state feedback saddle-point strategies in (ii) with the GRE for all $k$, where the corresponding saddle-point value is $V_0(x_0)$.

To prove part (iii), suppose that it has a negative eigenvalue for some $\bar{k} \in [0, N-1]$. Then the corresponding static zero-sum game does not admit a saddle point. In fact, there exists a sequence of maximizer strategies by which the upper value of this static zero-sum game becomes unbounded at $\bar{k}$, which also proves the necessity of part (i) (Başar & Bernhard, 1995). □

### 3.2. Infinite-horizon case

We now discuss the infinite-horizon problem of state feedback minimax control over the TCP-network. Before presenting the result, we provide some preliminaries. In this section, we assume that $Q_N = 0$. We first state the infinite-horizon version of the solution in Lemma 1.

- The associated generalized algebraic Riccati equation (GARE) can be written as

$$\bar{Z} = Q + \bar{P}_u^T(\alpha R + \bar{\alpha} B^T \bar{Z} B)\bar{P}_u - \gamma^2 \bar{P}_w^T \bar{P}_w + \bar{H}^T \bar{Z} \bar{H}, \tag{13}$$

  where $\bar{H}$, $\bar{P}_u$, and $\bar{P}_w$ are infinite-horizon versions of (8) with respect to $\bar{Z}$.
- The stationary minimax controller and the worst-case disturbance are

$$\bar{u}_k^* = -\bar{P}_u x_k \tag{14}$$

$$\bar{w}_k^* = \bar{P}_w x_k. \tag{15}$$

- The infinite-horizon version of the existence condition in Lemma 1(i) is given by

$$\rho(D^T \bar{Z} D) < \gamma^2. \tag{16}$$

We also introduce the time-reverse notation, which is used in the next proposition that states the convergence of the GRE. Let $\tilde{Z}_k := Z_{N-k}$. Then the GRE in (7) can be rewritten as

$$\tilde{Z}_{k+1} = Q + \tilde{P}_{u_k}^T(\alpha R + \bar{\alpha} B^T \tilde{Z}_k B)\tilde{P}_{u_k} - \gamma^2 \tilde{P}_{w_k}^T \tilde{P}_{w_k} + \tilde{H}_k^T \tilde{Z}_k \tilde{H}_k, \tag{17}$$

where $\tilde{P}_{u_k}$, $\tilde{P}_{w_k}$, and $\tilde{H}_k$ are the time-reverse versions of (8) in Lemma 1 with respect to $\tilde{Z}_k$. The time-reverse version of the concavity condition can be written as

$$\rho(D^T\tilde{Z}_kD) < \gamma^2. \tag{18}$$

Then we have the following result.

**Proposition 1.** *Suppose $(A, B)$ is controllable and $(A, Q^{1/2})$ is observable. Define the sets*

$$\Gamma_1(\alpha) := \{\gamma > 0 : \bar{Z} \geq 0 \text{ solves (13) and holds (16)}\}$$
$$\Lambda_1(\gamma) := \{\alpha \in [0, 1) : \bar{Z} \geq 0 \text{ solves (13) and holds (16)}\}.$$

*Let $\gamma_1^*(\alpha) := \inf\{\gamma : \gamma \in \Gamma_1(\alpha)\}$ and $\alpha_c(\gamma) := \inf\{\alpha : \alpha \in \Lambda_1(\gamma)\}$. Then for any finite $\gamma > \gamma_1^*(\alpha)$ and $\alpha > \alpha_c(\gamma)$, as $k \to \infty$, $\{\tilde{Z}_k\} \to \bar{Z}^+$ where $\bar{Z}^+$ is a fixed point of (13) that satisfies (16).*

**Proof.** Let us first note some basic facts regarding the GARE in (13). In Başar and Bernhard (1995), it was proven that when $\alpha = 1$, (16) is a necessary and sufficient condition that guarantees convergence of the GRE in (17). In particular, for a fixed $\gamma > \gamma_1^*(1)$, given a fixed point of (13) that satisfies (16), $\{\tilde{Z}_k\}$ converges to $\bar{Z}^+$. Now, when $\alpha = 0$, (13) can be written as

$$\bar{Z} = A^T\bar{Z}A + Q + A^T\bar{Z}D(\gamma^2I - D^T\bar{Z}D)^{-1}D^T\bar{Z}A, \tag{19}$$

which is the algebraic Riccati equation (ARE) associated with the optimization problem of (A.1) in Appendix A.

If $A$ is stable, (19) has a solution that satisfies (16), which is also equivalent to saying that $\{\tilde{Z}_k\}$ converges to $\bar{Z}^+$ (Başar & Bernhard, 1995). When $A$ is unstable, since the maximum cost of (A.1) is not bounded, (19) does not admit any solution in the class of positive semi-definite matrices, which also shows that $\{\tilde{Z}_k\}$ does not converge for any $\gamma > 0$. Thus, $\Gamma(0)$ is empty when $A$ is unstable.

Due to definitions of $\gamma_1^*(\alpha)$ and $\alpha_c(\gamma)$, $\bar{Z}$ is a solution to (13) that satisfies (16). From Lemma A.1(ii), $\bar{Z}$ constitutes an upper bound on the GRE. Hence, we have (18), which guarantees monotonicity of the GRE from Lemma A.1(i). Then, we can conclude that the monotonic and bounded sequence $\{\tilde{Z}_k\}$ converges as $k \to \infty$. □

We also have the following result which shows the relationship between LQG and minimax control over the TCP-network.

**Proposition 2.** *Suppose that the assumptions in Proposition 1 hold. Then, as $\gamma \to \infty$, $\bar{Z}^+$ defined in Proposition 1 converges to the solution of the following ARE:*

$$\bar{Z}^+ = A^T\bar{Z}^+A - \alpha A^T\bar{Z}^+B(R + B^T\bar{Z}^+B)^{-1}B^T\bar{Z}^+A + Q.$$

**Proof.** The value of the soft-constrained zero-sum dynamic game decreases in $\gamma$ (Başar & Bernhard, 1995). Then the result follows immediately. □

We now state the main result of this section.

**Theorem 1.** *Suppose $(A, B)$ is controllable and $(A, Q^{1/2})$ is observable. Then for any finite $\gamma > \gamma_1^*(\alpha)$ and $\alpha > \alpha_c(\gamma)$, the following hold:*

(i) *The state feedback minimax controller is given by (14) with $\bar{Z}^+$.*
(ii) *Suppose $\alpha\bar{P}_u^TR\bar{P}_u - \gamma^2\bar{P}_w^T\bar{P}_w > 0$. Then the closed-loop system with the worst-case disturbance in (15), i.e., $x_{k+1} = (A - \alpha_kB\bar{P}_u + D\bar{P}_w)x_k$, is stable in the mean-square sense, that is, $\mathbb{E}\{|x_k|^2\} \to 0$ as $k \to \infty$ for all initial conditions.*
(iii) *The closed-loop system, i.e., $x_{k+1} = (A - \alpha_kB\bar{P}_u)x_k + Dw_k$, is bounded in the mean-square sense, that is, there exists $M \geq 0$ such that $\mathbb{E}\{|x_k|^2\} \leq M$ for all $k$ and initial conditions.*
(iv) *The state feedback minimax controller achieves the performance level of $\gamma$, that is, $\langle\langle \mathcal{T}^{\infty}_{\mu^*_\gamma} \rangle\rangle \leq \gamma$.*

**Proof.** Parts (i) follows from Proposition 1. To prove part (ii), by using (13), we have

$$\begin{aligned}
&\mathbb{E}\{|x_{k+1}|^2_{\bar{Z}^+}\} - \mathbb{E}\{|x_k|^2_{\bar{Z}^+}\} \\
&= \mathbb{E}\{x_k^T(\bar{H}^T\bar{Z}^+\bar{H} + \bar{\alpha}\bar{P}_u^TB^T\bar{Z}^+B\bar{P}_u - \bar{Z}^+)x_k\} \\
&= -\mathbb{E}\{x_k^TQx_k + \alpha x_k^T\bar{P}_u^TR\bar{P}_ux_k - \gamma^2x_k^T\bar{P}_w^T\bar{P}_wx_k\}.
\end{aligned}$$

Now, we have

$$\begin{aligned}
&\mathbb{E}\{x_{k+1}^T\bar{Z}^+x_{k+1}\} \\
&= x_0^T\bar{Z}^+x_0 - \sum_{i=0}^{k}\mathbb{E}\Big\{x_i^T(Q + \alpha\bar{P}_u^TR\bar{P}_u - \gamma^2\bar{P}_w^T\bar{P}_w)x_i\Big\}.
\end{aligned}$$

Since the left-hand side of the above equation is bounded below by zero, we have

$$\lim_{k\to\infty}\mathbb{E}\{x_k^T(Q + \alpha\bar{P}_u^TR\bar{P}_u - \gamma^2\bar{P}_w^T\bar{P}_w)x_k\} = 0.$$

Then in view of the observability assumption and $(\alpha\bar{P}_u^TR\bar{P}_u - \gamma^2\bar{P}_w^T\bar{P}_w) > 0$, $\mathbb{E}\{|x_k|^2\} \to 0$ as $k \to \infty$.

For part (iii), when $w_k \equiv 0$, we have

$$\begin{aligned}
Z &= \alpha\bar{P}_u^TR\bar{P}_u + (1 - \alpha)A^TZA \\
&\quad + \alpha(A - B\bar{P}_u)^TZ(A - B\bar{P}_u) + Q,
\end{aligned}$$

where $Z \geq 0$ exists due to Theorem 3 in Imer et al. (2006) and the relationship between the minimax control when $w_k \equiv 0$ and the LQG control. Then from (ii), we can show that $\mathbb{E}\{|x_k|^2\} \to 0$ as $k \to \infty$ for all initial conditions when $w_k \equiv 0$; hence, the result follows.

To prove part (iv), for any finite $\gamma > \gamma_1^*(\alpha)$, since the upper value of the game is bounded with $\bar{Z}^+$, we have the following inequality for all disturbances in $\ell_2^p$:

$$J^{\infty}(\mu_\gamma^*, w) \leq x_0^T\bar{Z}^+x_0 + \gamma^2\mathbb{E}\Big\{\sum_{k=0}^{\infty}|w_k|^2\Big\}.$$

By taking $x_0 = 0$, the result holds. □

## 4. Minimax estimation over the TCP-network

This section considers minimax estimation for the TCP-case, which corresponds to part (b) of the certainty equivalence principle in Appendix B. The detailed analysis on the results of this section can be found in Moon and Başar (submitted for publication). Here, we provide those results that are needed for the minimax control problem.

**Lemma 2.** *Consider the zero-sum dynamic game in (5) with $\alpha \in [0, 1]$, $\beta \in [0, 1]$, and a fixed $\gamma > 0$. Then:*

(i) *A stochastic minimax estimator (SME) exists if and only if*

$$\rho(\Sigma_kQ) < \gamma^2, \quad \forall k \in [0, N - 1], \tag{20}$$

*where $\Sigma_k$ is generated by the following generalized stochastic Riccati equation (GSRE): $\Sigma_0 = Q_0^{-1}$ and*

$$\Sigma_{k+1} = A(\Sigma_k^{-1} - \gamma^{-2}Q + \beta_kC^TV^{-1}C)^{-1}A^T + DD^T.$$

(ii) *The SME is*

$$\begin{aligned}
\bar{x}_{k+1} &= A\bar{x}_k + \alpha_kBu_k + A\Pi_k(\gamma^{-2}Q\bar{x}_k \\
&\quad + \beta_kC^TV^{-1}(y_k - C\bar{x}_k)),
\end{aligned} \tag{21}$$

*where the estimator gain $\Pi_k$ is given by $\Pi_k = (\Sigma_k^{-1} - \gamma^{-2}Q + \beta_kC^TV^{-1}C)^{-1}$.*

**Proof.** Since we seek a causal estimator, forward dynamic programming can be applied by introducing the quadratic *cost-to-come (worst past cost)* function $W_k(x_k) = \mathbb{E}\{-|x_k - \bar{x}_k|^2_{\Xi_k} + l_k|\mathcal{I}_{k-1}, u_{k-1}, \alpha_{k-1}\}$, where $\Xi_k > 0$, $\Xi_0 = \gamma^2 Q_0$, and $l_0 = 0$ (Başar & Bernhard, 1995; Didinsky, 1994; Moon & Başar, submitted for publication). Then the cost from the initial state to stage $k+1$ can be written as

$$\mathbb{E}\{|x_{k+1} - \bar{x}_{k+1}|^2_{\Xi_{k+1}} - l_{k+1}|\mathcal{I}_k, u_k, \alpha_k\}$$
$$= \min_{(\bar{w}_k, x_k)} \mathbb{E}\Big\{-|x_k|^2_Q - \alpha_k|u_k|^2_R + \gamma^2|\bar{w}_k|^2 + \gamma^2|y_k - \beta_k C x_k|^2_{V^{-1}} + |x_k - \bar{x}_k|^2_{\Xi_k} - l_k|\mathcal{I}_k, u_k, \alpha_k\Big\},$$

where the minimization is subject to (1a). Then we can obtain the SME and the corresponding GSRE by solving the above optimization problem under (20) (Moon & Başar, 2014, submitted for publication). This completes the proof. □

It is worth noting that the proof of Lemma 2 cannot be applied to the UDP-case due to the absence of acknowledgments. This shows that part (b) of the certainty equivalence principle discussed in Appendix B is not valid for the UDP-case.

We now construct the smallest values of $\gamma$ and $\beta$ for which the SME exists.

**Proposition 3.** *Suppose that $(A, C)$ is observable and $(A, D)$ is controllable. Define*

$$\Gamma_2(\beta) := \{\gamma > 0 : \rho(\Sigma_k Q) < \gamma^2, \forall k\}$$
$$\gamma_2^*(\beta) := \inf\{\gamma : \gamma \in \Gamma_2(\beta)\}$$
$$\Lambda_2(\gamma) := \{\beta \in [0, 1) : \rho(\Sigma_k Q) < \gamma^2, \forall k\}$$
$$\beta_c(\gamma) := \inf\{\beta : \beta \in \Lambda_2(\gamma)\}.$$

*Then, for any finite $\gamma > \gamma_2^*(\beta)$ and $\beta > \beta_c(\gamma)$, $\rho(\Sigma_k Q) < \gamma^2$ holds for all $k$; hence, the SME exists.*

**Remark 1.** (i) Due to the acknowledgment nature of the TCP-case, the SME is a function of control and measurement packet loss information, i.e. $\{\alpha_k\}$ and $\{\beta_k\}$.
(ii) The SME is time varying and random because the estimator gain depends on the GSRE that is a function of the measurement arrival process. Furthermore, when $\beta_k = 0$, while the Kalman filter in Schenato et al. (2007) is identical to the open-loop estimator, the SME performs the state estimation under the worst-case disturbance.
(iii) For any finite $\gamma > \gamma_2^*(\beta)$ and $\beta > \beta_c(\gamma)$, by induction, we can show that $P_k \leq \Sigma_k$ for all $k$, where $P_k$ with $P_0 = Q_0^{-1}$ is the error covariance matrix of the Kalman filter in Schenato et al. (2007). Moreover, as $\gamma \to \infty$, the SME and $\Sigma_k$ converge to the Kalman filter and $P_k$ in Schenato et al. (2007). □

## 5. Minimax control over the TCP-network

In this section, we consider part (c) of the certainty equivalence principle in Appendix B and therefore complete the design of the output feedback minimax control system over the TCP-network. Toward this end, we combine the results in Sections 3 and 4, and then introduce one additional existence condition for the worst-case state estimator.

**Theorem 2.** *For any $\gamma$, $\alpha$, and $\beta$, suppose (6) and (20) hold for all $k$, i.e. there exist the state feedback minimax controller and the SME over the TCP-network. Then:*

(i) *The worst-case state estimator, $\hat{x}_k$, exists if*

$$\rho(\Sigma_k Z_k) < \gamma^2, \quad \text{for all } k \in [0, N-1]. \tag{22}$$

(ii) *If the condition in* (i) *holds, then the worst-case state estimator can be written as*

$$\hat{x}_k = (I - \gamma^{-2}\Sigma_k Z_k)^{-1}\bar{x}_k, \tag{23}$$

*where $\bar{x}_k$ is generated by the SME in Lemma 2.*

(iii) *If the condition in* (i) *holds, then the output feedback minimax controller is given by (9) with (23). Furthermore, this controller achieves the disturbance attenuation performance level of $\gamma$, that is, we have $\langle\langle \mathcal{T}^N_{\mu^*_\gamma} \rangle\rangle \leq \gamma$.*

**Proof.** The proof can be found in Moon and Başar (2014). □

**Remark 2.** As expected from standard $H^\infty$ control theory, there are three conditions on $\gamma$; (20) is for the existence of the SME, (6) is related to the state feedback minimax controller, and (22) is the spectral radius condition that ensures the existence of the worst-case state estimator. Moreover, unlike the LQG case considered in Schenato et al. (2007), there is no separation between control and estimation due to (22). □

For the infinite-horizon case, we can use the theories developed in Sections 3 and 4 to obtain a corresponding output feedback minimax controller. This is done next; the proof is similar to that of Theorem 2.

**Theorem 3.** *Suppose that $(A, B)$ and $(A, D)$ are controllable, and $(A, Q^{1/2})$ and $(A, C)$ are observable. Define*

$$\Gamma_3(\alpha, \beta) := \{\gamma > 0 : \gamma > \gamma_1^*(\alpha), \gamma > \gamma_2^*(\beta), \rho(\Sigma_k \bar{Z}^+) < \gamma^2 \text{ holds for all } k\}$$
$$\gamma_3^*(\alpha, \beta) := \inf\{\gamma : \gamma \in \Gamma_3(\alpha, \beta)\},$$

*where $\bar{Z}^+$ is the solution of the GARE in (13) that satisfies (16). Then for any finite $\gamma > \gamma_3^*(\alpha, \beta)$, $\alpha > \alpha_c(\gamma)$, and $\beta > \beta_c(\gamma)$, the stationary output feedback minimax controller is given by (14) with the following worst-case state estimator:*

$$\hat{x}_k = (I - \gamma^{-2}\Sigma_k \bar{Z}^+)^{-1}\bar{x}_k, \tag{24}$$

*where $\bar{x}_k$ is generated by the SME in Lemma 2. Finally, we have $\langle\langle \mathcal{T}^\infty_{\mu^*_\gamma} \rangle\rangle \leq \gamma$.* □

**Remark 3.** (i) $\gamma_3^*(\alpha, \beta)$ is the smallest value of $\gamma$ that satisfies all the existence conditions, which is the optimum disturbance attenuation level of the original disturbance attenuation problem.
(ii) The optimum disturbance attenuation level is a function of $\alpha$ and $\beta$. In fact, $\gamma_3^*(1, 1)$ is related to the deterministic $H^\infty$ control problem, and $\gamma_3^*(0, 0)$ is analogous to the open-loop problem that is not finite when $A$ is unstable.
(iii) As can be seen from Theorem 3, the critical values, $\alpha_c(\gamma)$ and $\beta_c(\gamma)$, are coupled with each other through $\gamma$; hence, in general, their values are problem dependent and cannot be quantified analytically. This fact actually stems from standard $H^\infty$ control, in which the optimum disturbance attenuation level (the smallest value of $\gamma$ in the context of standard $H^\infty$ control) cannot be determined analytically, and a heuristic approach is generally used depending on the problem at hand (Başar & Bernhard, 1995; Zhou, 1996). □

We now discuss the limiting behavior of the output feedback minimax controller in Theorem 3 as $\gamma \to \infty$. Under this limit, from (24), we can easily see that $\hat{x}_k = \bar{x}_k$ for all $k$. Furthermore, the state feedback minimax controller as well as the SME collapse to the LQG system presented in Schenato et al. (2007) in view of Proposition 2 and Remark 1(iii).

For $\alpha_c(\gamma)$ and $\beta_c(\gamma)$, as can be seen from Proposition 2, $\alpha_c(\gamma)$ converges to that in Schenato et al. (2007) as $\gamma \to \infty$. As for $\beta_c(\gamma)$, first note that for any finite $\gamma > \gamma_2^*(\beta)$ and $\beta > \beta_c(\gamma)$, we have $\Sigma_k Q < \gamma^2 I$ for all $k$, which together with Remark 1(iii) implies that $\mathbb{E}\{P_k\}Q \leq \mathbb{E}\{\Sigma_k\}Q < \gamma^2 I$ for all $k$, where $P_k$ with $P_0 = \Sigma_0$ is the error covariance matrix for the LQG problem in Schenato et al. (2007) as defined in Remark 1(iii). Therefore, for any finite $\gamma > \gamma_2^*(\beta)$ and $\beta > \beta_c(\gamma)$, there exists $M := M(\gamma, \Sigma_0) > 0$ such that $\mathbb{E}\{P_k\}Q \leq \mathbb{E}\{\Sigma_k\}Q \leq MQ < \gamma^2 I$ for all $k$, where $M(\gamma, \Sigma_0)$ depends on $\gamma$ and the initial condition $\Sigma_0$. Now, since $\mathbb{E}\{\Sigma_k\}$ converges to $\mathbb{E}\{P_k\}$ as $\gamma \to \infty$ due to Remark 1(iii), in view of Theorem 5.5 in Schenato et al. (2007), as $\gamma \to \infty$, $\mathbb{E}\{P_k\} = \mathbb{E}\{\Sigma_k\} \leq M(\Sigma_0) < \infty$ for all $k$, where $M(\Sigma_0) > 0$ depends on $\Sigma_0$. Therefore, $\beta_c(\gamma)$ also converges to that in Schenato et al. (2007) as $\gamma \to \infty$.

The above discussion implies that the LQG critical conditions provided in Schenato et al. (2007) for the TCP-case are *necessary* for the minimax case in the sense that for the output feedback minimax controller that satisfies all the existence conditions in Theorem 3, $\alpha$ and $\beta$ will satisfy the conditions given in Schenato et al. (2007). This fact will be illustrated in detail with numerical examples in Section 7. Note that the necessity argument also follows from the fact that the value of the zero-sum dynamic game (5) decreases when $\gamma$ increases, and converges to that in Schenato et al. (2007) as $\gamma \to \infty$.

## 6. Minimax control over the UDP-network

In this section, we study the minimax control problem over the UDP-network.

### 6.1. Finite-horizon case

We first consider the finite-horizon problem. The UDP-like information structure and the associated cost function are given by (3) and (5), respectively. We assume that the linear dynamical system in (1) has no measurement noise ($E = 0$), and $C$ is the identity matrix, that is, in case of transmission the controller has perfect access to instantaneous value of the state. We then have the following linear dynamical system:

$$x_{k+1} = Ax_k + \alpha_k B u_k + D w_k, \qquad y_k = \beta_k x_k. \tag{25}$$

We let $\bar{\alpha} := \alpha - \alpha^2$, $\alpha' := 1 - \alpha$ and $\beta' := 1 - \beta$. We now obtain the output feedback minimax controller (and the worst-case disturbance) under the UDP-type information structure for (25).

**Lemma 3.** *Consider the zero-sum dynamic game in (5) with (25). For fixed $\gamma > 0$, $\alpha \in [0, 1]$ and $\beta \in [0, 1]$, we have the following result:*

(i) *There exists a unique output feedback saddle-point solution if and only if*

$$\rho(D^T Z_{k+1} D) < \gamma^2, \tag{26}$$

*where $Z_k$ is generated by the following coupled generalized Riccati equations (GREs): $Z_N = Q_N$, $U_N = 0$ and*

$$\begin{aligned} Z_k &= \check{H}_k^T Z_{k+1} \check{H}_k + Q - \gamma^2 \check{P}_{w_k}^T \check{P}_{w_k} \\ &\quad + \check{P}_{u_k}^T (\alpha R + \bar{\alpha} B^T Z_{k+1} B + \beta' \bar{\alpha} B^T U_{k+1} B) \check{P}_{u_k} \\ &= Q + A^T Z_{k+1} A - U_k + \beta' A^T U_{k+1} A \end{aligned} \tag{27}$$

$$\begin{aligned} U_k &= \beta' A^T U_{k+1} A + \check{P}_{w_k}^T (\gamma^2 I - D^T Z_{k+1} D) \check{P}_{w_k} \\ &\quad - \check{P}_{u_k}^T (\alpha R + \beta' \bar{\alpha} B^T U_{k+1} B) \check{P}_{u_k} \\ &\quad + 2\alpha \check{P}_{u_k}^T B^T Z_{k+1} A - 2 \check{P}_{w_k}^T B^T Z_{k+1} A \\ &\quad + 2\alpha \check{P}_{u_k}^T B^T Z_{k+1} D \check{P}_{w_k}, \end{aligned} \tag{28}$$

*where*

$$\begin{aligned} \check{H}_k &= A - \alpha B \check{P}_{u_k} + D \check{P}_{w_k} \\ \check{P}_{u_k} &= (S_k + \alpha B^T Z_{k+1} D M_k^{-1} D^T Z_{k+1} B)^{-1} \\ &\quad \times B^T (I + Z_{k+1} D M_k^{-1} D^T) Z_{k+1} A \\ \check{P}_{w_k} &= (M_k + \alpha D^T Z_{k+1} B S_k^{-1} B^T Z_{k+1} D)^{-1} \\ &\quad \times D^T (I - \alpha Z_{k+1} B S_k^{-1} B^T) Z_{k+1} A \\ S_k &= R + B^T (Z_{k+1} + \alpha' \beta' U_{k+1}) \\ M_k &= \gamma^2 I - D^T Z_{k+1} D. \end{aligned}$$

(ii) *The corresponding saddle-point solution can be written as follows, where $\hat{x}_k = \mathbb{E}\{x_k | \mathcal{G}_k\}$:*

$$u_k^* = -\check{P}_{u_k} \hat{x}_k \tag{29}$$

$$w_k^* = \check{P}_{w_k} \hat{x}_k, \quad k \in [0, N-1]. \tag{30}$$

**Proof.** To prove parts (i) and (ii), we need to employ dynamic programming or rather the Isaacs equation. At stage $N$, the value function is given by $V_N(x_N) = \mathbb{E}\{x_N^T Q_N x_N | \mathcal{G}_N\}$. It is easy to see that, from the dynamic programming equation, the cost-to-go from stage $N-1$ can be expressed as

$$V_{N-1}(x_{N-1}) = \min_{u_{N-1}} \max_{w_{N-1}} \mathbb{E}\Big\{h_{N-1}(x, u, w) + V_N(x_N) | \mathcal{G}_{N-1}\Big\} \tag{31}$$

$$= \max_{w_{N-1}} \min_{u_{N-1}} \mathbb{E}\Big\{h_{N-1}(x, u, w) + V_N(x_N) | \mathcal{G}_{N-1}\Big\} \tag{32}$$

$$= \mathbb{E}\{|x_{N-1}|^2_{Z_{N-1}} + |e_{N-1}|^2_{U_{N-1}} | \mathcal{G}_{N-1}\}, \tag{33}$$

where $h_{N-1}(x, u, w) := |x_{N-1}|_Q^2 + \alpha_{N-1} |u_{N-1}|_R^2 - \gamma^2 |w_{N-1}|^2$, $Z_{N-1}$ is the GRE in Lemma 1, $e_k := x_k - \hat{x}_k$ with $\hat{x}_k = \mathbb{E}\{x_k | \mathcal{G}_k\}$, and $U_{N-1}$ is given by

$$\begin{aligned} U_{N-1} &= -\alpha P_{u_{N-1}}^T (R + B^T Q_N B) P_{u_{N-1}} \\ &\quad + P_{w_{N-1}}^T (\gamma^2 I - D^T Q_N D) P_{w_{N-1}} + 2\alpha P_{u_{N-1}}^T B^T Q_N A \\ &\quad + 2\alpha P_{u_{N-1}}^T B^T Q_N D P_{w_{N-1}} - 2 P_{w_{N-1}} D^T Q_N A, \end{aligned}$$

where $P_{u_{N-1}}$ and $P_{w_{N-1}}$ are defined in Lemma 1. Note that $U_{N-1} \geq 0$. The equality in (33) is achieved by using the following saddle-point solution under (26):

$$u_{N-1}^* = -P_{u_{N-1}} \hat{x}_{N-1}, \qquad w_{N-1}^* = P_{w_{N-1}} \hat{x}_{N-1},$$

which can be achieved by solving the static zero-sum game in (31) (or (32)).

Note that as mentioned in Imer et al. (2006), the estimator error becomes a function of $u_{N-2}$ at stage $N-2$ so that there is dual effect. To see this, we first write the cost-to-go from stage $N-2$:

$$\begin{aligned} &V_{N-2}(x_{N-2}) \\ &= \min_{u_{N-2}} \max_{w_{N-2}} \mathbb{E}\big\{h_{N-2}(x, u, w) + V_{N-1}(x_{N-1}) | \mathcal{G}_{N-2}\big\} \end{aligned} \tag{34}$$

$$= \max_{w_{N-2}} \min_{u_{N-2}} \mathbb{E}\big\{h_{N-2}(x, u, w) + V_{N-1}(x_{N-1}) | \mathcal{G}_{N-2}\big\}. \tag{35}$$

Now, note that the estimation error at $k = N-2$ is zero when $\beta_{N-2} = 1$, and $e_{N-1} = A e_{N-2} + (\alpha_{N-2} - \alpha) u_{N-2}$, otherwise. Therefore, (34) (or (35)) yields a unique minimizer and maximizer under (26), which can be written as follows:

$$\begin{aligned} u_{N-2}^* &= -(R + B^T (Z_{N-1} + \alpha' \beta' U_{N-1}) B)^{-1} \\ &\quad \times B^T Z_{N-1} (A \hat{x}_{N-2} + D w_{N-2}^*) \\ &=: \psi_{1,N-2}(\hat{x}_{N-2}, w_{N-2}^*) \end{aligned}$$

$$w^*_{N-2} = (\gamma^2 I - D^T Z_{N-1} D)^{-1} D^T Z_{N-1}(A\hat{x}_{N-2} + \alpha B u^*_{N-2}) =: \psi_{2,N-2}(\hat{x}_{N-2}, u^*_{N-2}).$$

We obtain the saddle point, $(u^*_{N-2}, w^*_{N-2})$, for (34) (or (35)) by solving the above fixed-point equations:

$$u^*_{N-2} = \psi_{1,N-2}(\hat{x}_{N-2}, \psi_{2,N-2}(\hat{x}_{N-2}, u^*_{N-2})) = -\breve{P}_{u_{N-2}}\hat{x}_{N-2} \quad (36)$$

$$w^*_{N-2} = \psi_{2,N-2}(\hat{x}_{N-2}, \psi_{1,N-2}(\hat{x}_{N-2}, w^*_{N-2})) = \breve{P}_{w_{N-2}}\hat{x}_{N-2}. \quad (37)$$

Substituting (36) and (37) into (34) (or (35)), we obtain

$$V_{N-2}(x_{N-2}) = \mathbb{E}\{|x_{N-2}|^2_{Z_{N-2}} + |e_{N-2}|^2_{U_{N-2}}|\mathcal{G}_{N-2}\},$$

where $Z_{N-2}$ and $U_{N-2}$ are given in (27) and (28), respectively. Then proceeding similarly, the minimax controller and the worst-case disturbance that constitute a saddle point can be written as (29) and (30), respectively. This completes the proof. □

In summary, for the linear system given in (25), the corresponding minimax controller is (29), which is linear in the information $\mathcal{G}_k$ given by (3), and is a function of two coupled nonlinear GREs (27) and (28). If the above existence condition fails to hold, then the minimax controller does not exist. In fact, the value of the corresponding zero-sum dynamic game would then be infinite as we discussed in the TCP-case.

It should be mentioned that there is no known general solution to the problem of LQG control over the UDP-network for the noisy measurement model in (1b), since the associated optimization problem is then no longer convex, and the optimal LQG controller is generally nonlinear in the available information (Schenato et al., 2007). Also, there is no separation between control and estimation. We would naturally expect a similar difficulty to arise in the minimax control problem under the noisy measurement case.

We next proceed with the infinite-horizon case for again the additive noise free problem.

### 6.2. Infinite-horizon case

The infinite-horizon versions of the coupled GREs and the existence condition are provided below:

- The coupled GAREs are given by

$$Z = \breve{H}^T Z \breve{H} + Q - \gamma^2 \breve{P}_w^T \breve{P}_w + \breve{P}_u^T(\alpha R + \bar{\alpha} B^T Z B + \beta' \bar{\alpha} B^T U B)\breve{P}_u = Q + A^T Z A - U + \beta' A^T U A \quad (38)$$

$$U = \beta' A^T U A + \breve{P}_w^T(\gamma^2 I - D^T Z D)\breve{P}_w - \breve{P}_u^T(\alpha R + \beta' \bar{\alpha} B^T U B)\breve{P}_u + 2\alpha \breve{P}_u^T B^T Z A - 2\breve{P}_w^T B^T Z A + 2\alpha \breve{P}_u^T B^T Z D \breve{P}_w, \quad (39)$$

where $\breve{P}_u$, $\breve{P}_w$, and $\breve{H}$ are infinite-horizon versions of $\breve{P}_{u_k}$, $\breve{P}_{w_k}$, and $\breve{H}_k$, respectively.

- The corresponding minimax controller and the worst-case disturbance are given by

$$u^*_k = -\breve{P}_u \hat{x}_k \quad (40)$$

$$w^*_k = -\breve{P}_w \hat{x}_k. \quad (41)$$

- The existence condition can be written as

$$\rho(D^T Z D) < \gamma^2. \quad (42)$$

We need to obtain conditions on $\gamma$, $\alpha$ and $\beta$ that guarantee convergence of the coupled GREs in (27) and (28) under (26), which can be characterized by

$$\gamma^*_U(\alpha,\beta) = \inf\Big\{\gamma > 0 : \lim_{k\to\infty} \tilde{Z}_k = Z, \lim_{k\to\infty} \tilde{U}_k = U, Z \geq 0 \text{ and } U \geq 0 \text{ solve (38) and (39), and satisfy (42)}\Big\}$$

$$\alpha^U_c(\gamma,\beta) = \inf\Big\{\alpha \in [0,1) : \lim_{k\to\infty} \tilde{Z}_k = Z, \lim_{k\to\infty} \tilde{U}_k = U, Z \geq 0 \text{ and } U \geq 0 \text{ solve (38) and (39), and satisfy (42)}\Big\}$$

$$\beta^U_c(\gamma,\alpha) = \inf\Big\{\beta \in [0,1) : \lim_{k\to\infty} \tilde{Z}_k = Z, \lim_{k\to\infty} \tilde{U}_k = U, Z \geq 0 \text{ and } U \geq 0 \text{ solve (38) and (39), and satisfy (42)}\Big\},$$

where $\tilde{Z}_k$ and $\tilde{U}_k$ are the time-reverse equations of (27) and (28), respectively, as introduced in Section 3.2. Note that these parameters are coupled with each other. Also, if $\gamma > \gamma^*_U(\alpha,\beta)$, $\alpha > \alpha^U_c(\gamma,\beta)$, and $\beta > \beta^U_c(\gamma,\alpha)$, then the infinite-horizon minimax controller for the UDP-case is (40), provided that $\gamma$ is finite, which stabilizes the closed-loop system and achieves the disturbance attenuation level of $\gamma$. Moreover, as $\gamma \to \infty$, the critical values, $\alpha^U_c(\gamma,\beta)$ and $\beta^U_c(\gamma,\alpha)$, converge to the corresponding LQG values in Imer et al. (2006) and Schenato et al. (2007).

For the LQG problem, the explicit convergence conditions of the corresponding Riccati equations were obtained in Imer et al. (2006) and Schenato et al. (2007) when $B$ is invertible. Since the minimax controller is equivalent to the LQG controller when $\gamma$ asymptotically goes to infinity, those conditions are necessary for the minimax controller; that is, due to the existence condition, the conditions in Imer et al. (2006) and Schenato et al. (2007) are only necessary for the convergence of (27) and (28) even if $B$ is invertible. We should note that the general convergence conditions cannot be obtained analytically, because the critical values, $\alpha^U_c(\gamma,\beta)$ and $\beta^U_c(\gamma,\alpha)$, are coupled with each other.

We now state the main result of this section.

**Theorem 4.** *Suppose that* $(A,B)$ *and* $(A,D)$ *are controllable, and* $(A,Q^{1/2})$ *is observable. Suppose that* $\gamma > \gamma^*_U(\alpha,\beta)$ *is finite,* $\alpha > \alpha^U_c(\gamma,\beta)$ *and* $\beta > \beta^U_c(\gamma,\alpha)$*. Then:*

(i) *The minimax controller is given by (40).*

(ii) *Suppose* $\alpha \breve{P}_u^T R \breve{P}_u - \gamma^2 \breve{P}_w^T \breve{P}_w > 0$*. Then the closed-loop system with the worst-case disturbance in (41) and the estimation error are bounded in the mean-square sense, that is, there exist* $M, M' \geq 0$ *such that* $\mathbb{E}\{|x_k|^2\} \leq M$ *and* $\mathbb{E}\{|e_k|^2\} \leq M'$ *for all* $k$ *and initial conditions.*

(iii) *The closed-loop system with an arbitrary disturbance and the estimation error are bounded in the mean-square sense.*

(iv) *The minimax controller in* (i) *achieves the disturbance attenuation level of* $\gamma$.

**Proof.** Parts (i) and (iv) follow from the preceding discussion. To prove part (ii), by using (38) and (39), we have

$$\mathbb{E}\{|x_{k+1}|^2_Z - |x_k|^2_Z + |e_{k+1}|^2_U - |e_k|^2_U\} = -\mathbb{E}\{x_k^T(Q + \alpha \breve{P}_u^T R \breve{P}_u - \gamma^2 \breve{P}_w^T \breve{P}_w)x_k\} + \mathbb{E}\{e_k^T(\alpha \breve{P}_u^T R \breve{P}_u - \gamma^2 \breve{P}_w^T \breve{P}_w)e_k\}.$$

Summing up the above expression over $k$ yields

$$\mathbb{E}\{|x_{k+1}|_Z^2 + |e_{k+1}|_U^2\} = \mathbb{E}\{|x_0|_Z^2 + |e_0|_U^2\} + \sum_{i=0}^{k} \mathbb{E}\{e_i^T(\alpha \breve{P}_u^T R \breve{P}_u - \gamma^2 \breve{P}_w^T \breve{P}_w) e_i\} - \sum_{i=0}^{k} \mathbb{E}\{x_i^T(Q + \alpha \breve{P}_u^T R \breve{P}_u - \gamma^2 \breve{P}_w^T \breve{P}_w) x_i\}.$$

Note that for any $L \geq 0$, $\beta' \mathbb{E}\{x_k^T L x_k\} \geq \mathbb{E}\{e_k^T L e_k\}$, and we have $\alpha \breve{P}_u^T R \breve{P}_u - \gamma^2 \breve{P}_w^T \breve{P}_w > 0$. Therefore,

$$\mathbb{E}\{|x_{k+1}|_Z^2 + |e_{k+1}|_U^2\} \leq \mathbb{E}\{|x_0|_Z^2 + |e_0|_U^2\} - \sum_{i=0}^{k} \mathbb{E}\{x_i^T(Q + \beta(\alpha \breve{P}_u^T R \breve{P}_u - \gamma^2 \breve{P}_w^T \breve{P}_w)) x_i\}.$$

Since the left-hand side of the above inequality is bounded below by zero, we have

$$\lim_{k\to\infty} \mathbb{E}\{x_k^T(Q + \beta(\alpha \breve{P}_u^T R \breve{P}_u - \gamma^2 \breve{P}_w^T \breve{P}_w)) x_k\} = 0.$$

Since $\alpha \breve{P}_u^T R \breve{P}_u - \gamma^2 \breve{P}_w^T \breve{P}_w > 0$, in view of the observability assumption, we have the desired result.

For part (iii), when $w_k \equiv 0$, we have

$$Z = \alpha' A^T Z A + \alpha (A - B\breve{P}_u)^T Z (A - B\breve{P}_u) + \breve{P}_u^T(\alpha R + \bar{\alpha}\beta' B^T U B)\breve{P}_u + Q$$

$$U = \alpha A^T Z A - \alpha (A - B\breve{P}_u)^T Z (A - B\breve{P}_u) + \beta' A^T U A - \breve{P}_u^T(\alpha R + \bar{\alpha}\beta' B^T U B)\breve{P}_u,$$

where $Z \geq 0$ and $U \geq 0$ exist due to Theorem 9 in Imer et al. (2006) and the relationship between the minimax control when $w_k \equiv 0$ and the LQG control. Then from (ii), we can show that $\mathbb{E}\{|x_k|^2\}$ and $\mathbb{E}\{|e_k|^2\}$ are bounded when $w_k \equiv 0$; hence, the result follows. □

## 7. Numerical examples

We provide numerical examples to demonstrate the relationship between $\alpha$, $\beta$, and $\gamma$, and compare the disturbance attenuation performance for different values of $\gamma$.

### 7.1. Stability and performance region

Consider the following system:

$$x_{k+1} = A x_k + \alpha_k u_k + w_k, \tag{43}$$

where $A = 2$ and $A = 1.1$. We take $R = 1$ and $Q = 1$.

Fig. 1 shows the stability and performance region of (43) for the TCP-case. To obtain the region numerically, we use the following approach:

(S.1) Fix $\alpha = 1$ and take a sufficiently large value of $\gamma > 0$.
(S.2) Obtain the solution of the GARE in (13), and check the existence condition in (16).
(S.3) If the existence condition holds, decrease $\alpha$ and then go to (S.2). Otherwise, it is the critical value of $\alpha$ for that $\gamma$; go to the next step.
(S.4) Decrease $\gamma$ and fix $\alpha = 1$. Go to (S.2).

Fig. 1 shows (as goes with intuition) that the system needs a more reliable communication channel if the high level of disturbance attenuation is required. For both cases, $\alpha_c(\gamma) \to \alpha_R$ as $\gamma \to \infty$ where $\alpha_R = 1 - (1/2^2) = 0.75$ when $A = 2$ has been calculated in Imer et al. (2006). Moreover, as the plant

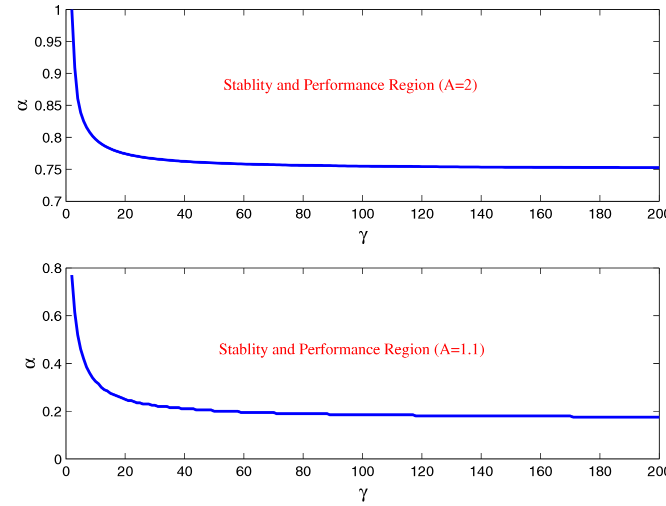



**Fig. 1.** Stability and performance region of (43) for the TCP-case.

becomes more open-loop unstable, the stability and performance region becomes smaller. This is an expected result, since the more open-loop unstable a plant is, the more frequently we need to measure its state and control it. Finally, it is easy to see that $\alpha > \alpha_R$ is a necessary condition for the existence of the state feedback minimax controller for the TCP-case.

The region of stability and performance for the UDP-case is shown in Fig. 2. We used the same approach above to obtain this plot. As expected, the UDP controller has a smaller stability and performance region in terms of $\alpha$, $\beta$, and $\gamma$ than the TCP-case. Moreover, as $\gamma \to \infty$, $\alpha_c^U(\gamma, \beta)$, and $\beta_c^U(\gamma, \alpha)$ converge to the corresponding critical values shown in Imer et al. (2006). This also shows that the condition given in Imer et al. (2006) is a necessary condition for the existence of the minimax controller for the UDP-case.

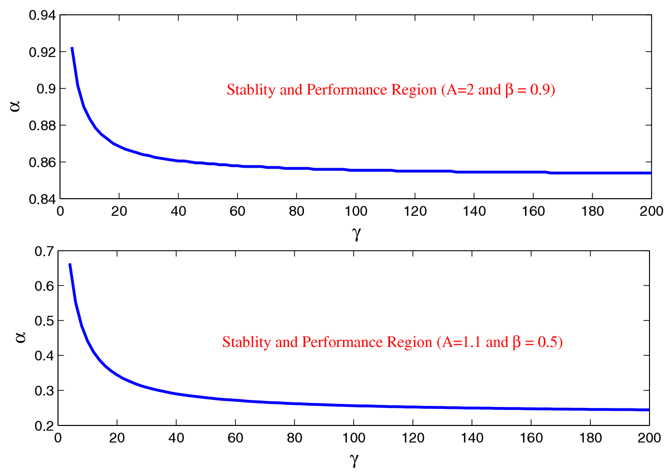
### 7.2. Disturbance attenuation performance (TCP-case)

We use the pendubot system as in Schenato et al. (2007), where the system and cost matrices can be found. Fig. 3 shows the existence regions of the state feedback minimax controller and the SME. This plot is also obtained by using the approach described in Section 7.1. The vertical axis is $\alpha$ for the state feedback controller, whereas it is $\beta$ for the SME. Note that the intersection of regions above the two lines guarantees the existence of the state feedback minimax controller as well as the SME. Moreover, as $\gamma \to \infty$, all these regions converge to the value of the LQG problem in Schenato et al. (2007).

Fig. 4 shows the disturbance attenuation performance of the minimax controller for different values of $\gamma$ when $w_k$ is Gaussian or a sinusoidal disturbance with amplitude of 0.01. We use $\alpha = 0.8$ and $\beta = 0.9$. As can be seen, when $\gamma = 20$, the minimax controller outperforms the LQG controller. Finally, as $\gamma \to \infty$, the performance of the minimax controller is identical to the corresponding LQG controller in Schenato et al. (2007).

## 8. Conclusions

In this paper, we have studied the minimax control problem for LTI systems over unreliable communication channels. We have considered two different scenarios for the communication channels: the TCP-case and the UDP-case. Unlike the previous work, we have considered the situation when the sensor noise and the disturbance are not necessarily stochastic processes, but are treated as adversarial inputs. The control problems are then



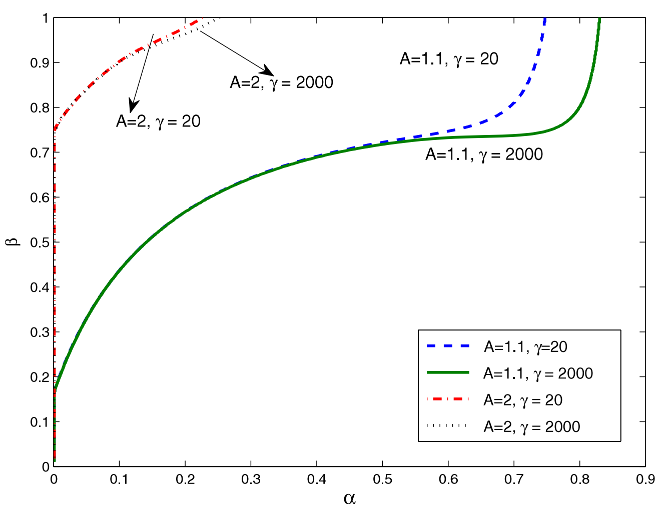




**Fig. 2.** Stability and performance region of (43) for the UDP-case.

naturally formulated within the framework of stochastic zero-sum dynamic games.

For both the TCP and UDP cases, we have obtained different classes of output feedback minimax controllers by characterizing the corresponding sets of existence conditions in terms of the $H^\infty$ disturbance attenuation parameter and the packet loss rates.

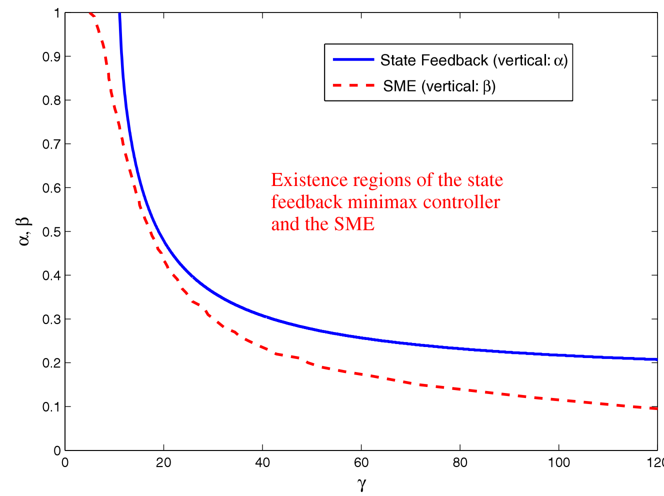
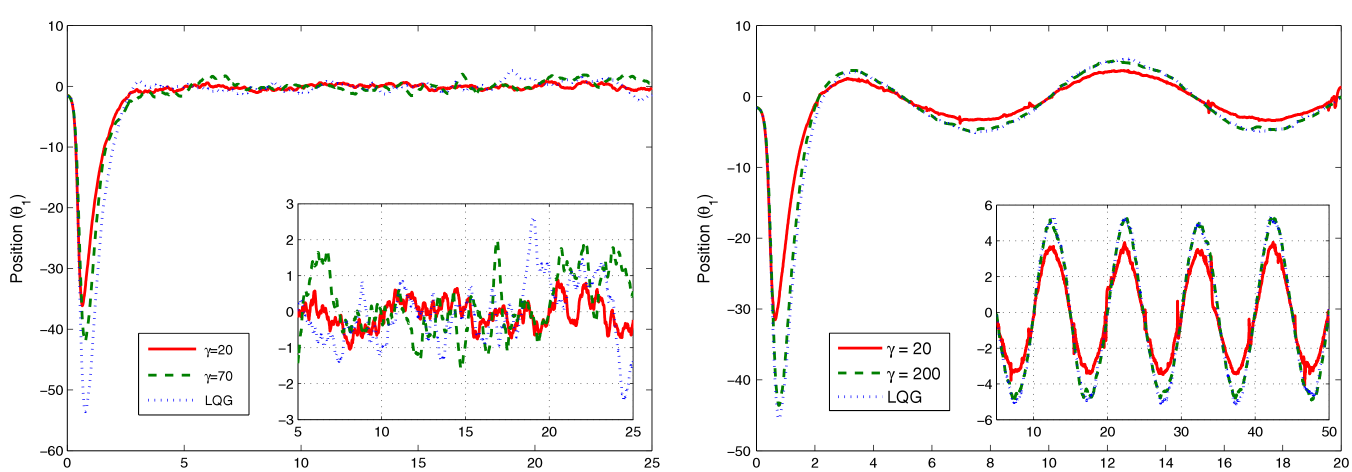



**Fig. 3.** Existence regions of the state feedback minimax controller and the SME for the pendubot system.

We have shown that stability and performance of the closed-loop system are determined by the disturbance attenuation parameter and the packet loss rates. Finally, as the disturbance attenuation parameter goes to infinity, the minimax controllers become equivalent to the corresponding LQG controllers (TCP or UDP controllers).

One possible extension of the results of this paper would be to the case when the communication channels are temporally correlated, such as $\{\alpha_k\}$ and $\{\beta_k\}$ being considered as two-state Markov processes with different transition probability matrices. Below we discuss the problem of state feedback minimax control over the TCP-network when $\{\alpha_k\}$ is temporally correlated.

The temporally correlated TCP-network can be modeled as a two-state irreducible and stationary Markov chain with the transition probability, which is also known as the Gilbert–Elliot channel model. In order to obtain the corresponding state feedback minimax controller, we need to use dynamic programming. This, however, requires more steps involved than those in Lemma 1, since depending on the acknowledged information, two different zero-sum games appear at each time $k$. This means that at $k$, the corresponding minimax controller is dependent on the available information of $\alpha_{k-1}$ due to the Markov property, provided that the associated existence conditions are satisfied. Moreover, in this case, the existence conditions will depend on $\gamma$ and the transition




(a) Gaussian disturbance.

(b) Sinusoidal disturbance with amplitude of 0.01.

**Fig. 4.** Disturbance attenuation performance with $\alpha = 0.8$ and $\beta = 0.9$.

probability rates, since the corresponding Riccati equation will do so. The detailed analysis on this problem is currently under study.

## Appendix A. Properties of the GARE (13)

**Lemma A.1.** *Suppose* $(A, B)$ *is controllable and* $(A, Q^{1/2})$ *is observable. Assume that given* $\gamma$ *and* $\alpha$, (18) *holds for all* $k$. *Assume further that the GARE has a solution* $Z := Z(\gamma, \alpha) \geq 0$ *which satisfies* (16). *Then* (i) $\tilde{Z}_k \leq \tilde{Z}_{k+1}$, *and* (ii) $Z \geq \tilde{Z}_k$ *for all* $k$.

**Proof.** (i) Note that $\tilde{Z}_k$ with $\alpha = 1$ is the GRE of the deterministic $H^\infty$ optimal control problem, and its monotonicity was proven in Başar and Bernhard (1995). When $\alpha = 0$, the GRE can be obtained by solving the following optimization problem:

$$\max_{w_{0:N-1}} |x_N|^2_{Q_N} + \sum_{k=0}^{N-1} |x_k|^2_Q - \gamma^2 |w_k|^2, \tag{A.1}$$

with the constraint of $x_{k+1} = Ax_k + Dw_k$. Then under the concavity condition, the monotonicity holds (Başar & Bernhard, 1995). To prove the general case, note that $Z_N = 0 \leq Z_{N-1}$. By induction, suppose $Z_{k+1} \geq Z_{k+2}$. Then we have

$$\begin{aligned}
V_k(x) &= |x|^2_{Z_k} \\
&= \min_u \max_w \Big[ \alpha' \mathbb{E}\Big\{ V_{k+1}(\bar{z}) + |x|^2_Q - \gamma^2 |w|^2 | \mathcal{I}_k \Big\} \\
&\quad + \alpha \mathbb{E}\Big\{ V_{k+1}(z) + |x|^2_Q + |u|^2_R - \gamma^2 |w|^2 | \mathcal{I}_k \Big\} \Big] \\
&\geq \min_u \max_w \Big[ \alpha' \mathbb{E}\Big\{ V_{k+2}(\bar{z}) + |x|^2_Q - \gamma^2 |w|^2 | \mathcal{I}_{k+1} \Big\} \\
&\quad + \alpha \mathbb{E}\Big\{ V_{k+2}(z) + |x|^2_Q + |u|^2_R - \gamma^2 |w|^2 | \mathcal{I}_{k+1} \Big\} \Big] \\
&= |x|^2_{Z_{k+1}} = V_{k+1}(x),
\end{aligned}$$

where $\alpha' = 1 - \alpha$, $z = Ax + Bu + Dw$ and $\bar{z} = Ax + Dw$. Here $V_k(x) = \mathbb{E}\{x^T Z_k x | \mathcal{I}_k\}$ is the saddle-point value of the dynamic game with only $N - k$ stages, (see Lemma 1). Hence, $Z_k \geq Z_{k+1}$ for all $k$. Then the result follows by reversing the time index.

(ii) When $\alpha = 1$ or $\alpha = 0$, $Z \geq \tilde{Z}_k$ for all $k$, which was shown in Başar and Bernhard (1995). To see the general case, when $k = 0$, $Z \geq \tilde{Z}_0 = 0$. Suppose $Z - \tilde{Z}_{k+1} \geq 0$. Under this assumption and the fact that $\alpha$ is positive, it can be checked that $Z - \tilde{Z}_k \geq 0$. Therefore, the result follows. □

## Appendix B. Certainty equivalence principle

The main idea of the certainty equivalence principle for $H^\infty$ control is as follows (Başar & Bernhard, 1995): *At time* $k$, *given the state feedback minimax controller, one should first look for the worst past disturbances that maximize the cost function under the specified information structure and then find the worst-case state estimator,* $\hat{x}_k$, *that corresponds to the worst past disturbances. If such an* $\hat{x}_k$ *exists, then the minimax controller can use it in place of the state* $x_k$ *to generate the control action.*

In this appendix, we show that the TCP problem formulated in Section 2 satisfies the three basic properties of the certainty equivalence principle in Başar and Bernhard (1995), but the UDP problem does not. Toward this end, we use the notation $s := \{s_k\} \in \mathbf{S}'$ and $s^\tau := \{s_k\}_{k=0}^{\tau} \in \mathbf{S}^\tau$.

Let the set of disturbances $\Omega$ be $(x_0, w) =: \omega \in \Omega := \mathbb{R}^n \times \mathbf{W}'$. For the LTI system, let the solutions of (1a) and (1b) be $x_t = \phi_t(u, w, x_0, \{\alpha_k\}_{k=0}^{t-1})$ and $y_t = \eta_t(u, w, x_0, \{\alpha_k\}_{k=0}^{t-1}, \beta_t)$. Then by using the inherent causality, $x_t = \phi_t(u^{t-1}, w^{t-1}, x_0, \{\alpha_k\}_{k=0}^{t-1})$ and $y_t = \eta_t(u^{t-1}, w^t, x_0, \{\alpha_k\}_{k=0}^{t-1}, \beta_t)$. Let $\Theta^\tau$ be the set of realizations of packet drops until $\tau \in [0, N-1]$. Now, for any given $\tau \in [0, N-1]$ and $(\bar{u}, \bar{y}, \bar{\kappa}) \in \mathbf{U}^\tau \times \mathbf{Y}^\tau \times \Theta^\tau$, we define the following subset $\Omega_\tau$ of $\Omega$:

$$\Omega_\tau(\bar{u}, \bar{y}, \bar{\kappa}) := \{\omega \in \Omega : \eta_k(\bar{u}, \omega, \bar{\kappa}) = \bar{y}_k,\ k = 0, 1, \ldots, \tau\},$$

where the set is compatible with all disturbance sequences in $\Omega$. We also introduce the following set which is the set of restrictions of the elements of $\Omega_\tau$ to $[0, \tau]$:

$$\Omega^\tau_\tau(\bar{u}, \bar{y}, \bar{\kappa}) := \{\omega^\tau \in \Omega^\tau : \omega \in \Omega_\tau(\bar{u}, \bar{y}, \bar{\kappa})\}.$$

Note that $\Omega_\tau$ and $\Omega^\tau_\tau$ are the sets that are related to the disturbances, which are compatible with observed sequences of control, measurement, and realizations of packet drops.

Now, it can be shown that the information process $(u, \omega, \kappa) \mapsto \{\Omega_\tau\}$ carries *consistent*, *perfect recall*, and *nonanticipative* properties introduced in Başar and Bernhard (1995, p. 249). Hence, we are now in a position to apply the certainty equivalence principle for the TCP problem. In particular, the zero-sum dynamic game for the TCP-case formulated in Section 2 can be studied through the following three steps:

(a) State feedback minimax control by assuming that the controller has the actual state information.
(b) Minimax estimation under the TCP-like information structure.
(c) Synthesis of the results in (a) and (b) by characterizing the worst-case state estimator, say $\hat{x}_k$, that will be used in the minimax controller obtained in part (a) by replacing the true state with $\hat{x}_k$.

For the UDP-case, on the other hand, due to the absence of acknowledgments, we cannot construct the above information process. Therefore, the zero-sum dynamic game of the UDP-case cannot be solved by the certainty equivalence principle.

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