Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

Minimax Control of MIMO Systems over Multiple TCP-like Lossy Networks

Jun Moon and Tamer Başar *

* *Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, Urbana, IL 61801; emails: {junmoon2, basar1}@illinois.edu.*

Abstract: This paper considers a minimax control problem over multiple packet dropping channels. The channel losses are assumed to be Bernoulli processes, and operate under the transmission control protocol (TCP); hence acknowledgments of control and measurement drops are available at each time. Under this setting, we obtain an output feedback minimax controller, which are implicitly dependent on rates of control and measurement losses. For the infinite-horizon case, we first characterize achievable H^{∞} disturbance attenuation levels, and then show that the underlying condition is a function of packet loss rates. We also address the converse part by showing that the condition of the minimum attainable loss rates for closed-loop system stability is a function of H^{∞} disturbance attenuation parameter. Hence, those conditions are coupled with each other. Finally, we show the limiting behavior of the minimax controller under the disturbance attenuation parameter.

Keywords: Minimax control, packet drops, LQG control, networked control systems

1. INTRODUCTION

Networked control captures scenarios where controllers, sensors, and the plant are connected over a network with communication constraints, where their communication links could be lossy and/or hampered by the limited capacity (Hespanha et al. (2007)). Our goal in this paper is to study one such class of systems where there are packet losses on links that carry sensor information to the controller and control signals to the plant along with adversarial inputs.

Specifically, we study a problem of H^{∞} control (minimax control) over multiple unreliable communication links where the links provide acknowledgments of control and measurement drops, and losses are modeled by Bernoulli processes. This problem is also known as control over TCP-like lossy networks (Imer et al. (2006)). The paper provides a complete generalization of the results obtained recently by Moon and Başar (2013a, 2014) where the single packet drop case was considered.

Linear-quadratic-Gaussian (LQG) control problems over single or multiple packet drop channels were already considered in numerous prior works in the literature. Imer et al. (2006) considered the single TCP-like packet dropping problem under the perfect measurement case. It was shown there that separation holds and the stability of the closed-loop system depends on the unstable modes of the system and the loss rate. Schenato et al. (2007) considered the noisy measurement LQG system by showing that the optimal controller in Imer et al. (2006) and the Kalman filter in Sinopoli et al. (2004) including the control input can be designed independently. They also provided more general upper and lower bounds on the stability margin. Garone et al. (2012) generalized the previous LQG results to multiple packet dropping channels.

In both the LQG case and its generalized H^{∞} control problem, packet drops can be captured within the framework of Markov jump linear systems (MJLSs), barring the specific information structure along with the acknowledgment scheme that pertains to packet dropping networks; see Pan and Başar (1995) and Costa et al. (2005) for a comprehensive treatment of H^{∞} control of continuous- and discrete-time MJLSs, respectively, with perfect and imperfect state information, but perfect Markov chain state information.

For the special structure of a single packet dropping network, and again for the H^{∞} control problem, most of the results in the literature have used the MJLS framework as the starting point, and have utilized LMIs and the bounded real lemma (Seiler and Sengupta (2005); Geromel et al. (2009); Ishii (2008)), with the downside being that, in contradistinction with the problem studied in this paper, they all work with full information on the Markov chain. This makes that approach not applicable to the problem in this paper where the TCP-type communication channel does not provide instantaneous packet loss information; further, the LMI-based MJLS approach is suboptimal (Schenato et al. (2007)) due to the assumption of stationarity.

In a series of papers by Moon and Başar (2013a,b, 2014), we have studied problems of minimax control and estimation over a single TCP-like packet dropping network. We have shown that i) the H^{∞} optimum disturbance attenuation level is a function of control and measurement

⋆ The first author was supported in part by the Fulbright Commission. This research was supported in part by the U.S. Air Force Office of Scientific Research (AFOSR) MURI grant FA9550-10-1-0573.

channel loss rates; ii) the critical packet dropping rate for closed-loop stability is a function of the H^∞ disturbance attenuation parameter; iii) separation does not hold; and iv) under a particular limit of the disturbance attenuation parameter, the minimax controller as well as the critical values of the loss rates converge to the corresponding ones in the LQG case. This paper extends these results to multiple packet dropping channels, and thus in a way also extends the LQG results of Garone et al. (2012) to the H^∞ control problem.

The structure of the paper is as follows. In Section 2, we formulate the minimax control problem over multiple TCP-like lossy networks. The finite-horizon case is considered in Section 3. The analysis of the infinite-horizon problem is in Section 4. A numerical example is included in Section 5. We end the paper with the concluding remarks of Section 6.

2. PROBLEM FORMULATION

Consider the following discrete-time linear system

$$x_{k+1} = Ax_k + B\Upsilon_k u_k + Dw_k,\ k = 0, 1, 2, ... \tag{1a}$$

$$y_k = \Pi_k C x_k + E v_k, \tag{1b}$$

where $x_k \in \mathbb{R}^n$ is the state; $u_k \in \mathbb{R}^m$ is the control; $w_k \in \mathbb{R}^p$ and $v_k \in \mathbb{R}^q$ are the disturbance input and the measurement noise, respectively, which are assumed to be arbitrary signals in ℓ_2; $y_k \in \mathbb{R}^q$ is the output; A, B, C, D, E are time invariant matrices with appropriate dimensions; and k is the time index. We also assume that E is nonsingular, and define $V := EE^T$.

The stochastic processes, $\Upsilon_k = \mathbf{diag}\{\alpha_k^1, ..., \alpha_k^m\}$ and $\Pi_k = \mathbf{diag}\{\beta_k^1, ..., \beta_k^q\}$, are sequences of matrices of i.i.d. stochastic processes where each component is a 0-1 Bernoulli processes with following mean values:

$$\bar{\Upsilon} = \mathbf{diag}\{\bar{\alpha}^1, ..., \bar{\alpha}^m\},\ \bar{\Pi} = \mathbf{diag}\{\bar{\beta}^1, ..., \bar{\beta}^q\},$$

where $\bar{\alpha}^i \in [0, 1]$ and $\bar{\beta}^j \in [0, 1]$ for all i and j, which of course completely describe them. Note that we have two packet dropping networks, with m and q channels, respectively. We assume that the channels are pairwise independent, but they need not be identically distributed.

We define the information that is available to the controller by

$$\begin{cases} \mathcal{I}_0 & := \{y_0,\ \beta_0\} \\ \mathcal{I}_k & := \{y_{0:k},\ u_{0:k-1},\ \Upsilon_{0:k-1},\ \Pi_{0:k}\},\ k \geq 1, \end{cases} \tag{2}$$

where $y_{0:k} := (y_0, ..., y_k)$ and the same notation applies to $u_{0:k-1}$, $\Upsilon_{0:k-1}$, and $\Pi_{0:k}$. Such an information structure is known as the *TCP-like* information structure due to full information on previous control link conditions. If (2) does not have $\Upsilon_{0:k-1}$, it is called *UDP-like* (Imer et al. (2006); Schenato et al. (2007)). On the other hand, MJLS formulations, as in Costa et al. (2005), consider information when the current value, Υ_k, is also included in (2).

Let $|\cdot|_S$ denote an appropriate weighted Euclidean norm or seminorm, weighted by the symmetric matrix S (with $S > 0$ or $S \geq 0$). Let $\mathcal{U}$ and $\mathcal{W}$ be sets of control and disturbance policies, respectively. We also let $\omega := (x_0, \nu, \{v_k\}) \in \Omega := \mathbb{R}^n \times \mathcal{W} \times \mathcal{V}$ where $\nu \in \mathcal{W}$ and $\mathcal{V}$ is the appropriate space for $\{v_k\}$. Our main objective in this paper is to find a controller that minimizes the following cost function:

$$\ll \mathcal{T}_\mu^N \gg := \sup_{\omega\in\Omega} \frac{J^N(\mu, \nu)^{1/2}}{(F^N)^{1/2}}, \tag{3}$$

where

$$F^N = \mathbb{E}\Big\{|x_0 - \tilde{x}_0|_{Q_0}^2 + \sum_{k=0}^{N-1} |w_k|^2 + |v_k|^2\Big\}$$

$$J^N(\mu, \nu) = \mathbb{E}\Big\{|x_N|_{Q_N}^2 + \sum_{k=0}^{N-1} |x_k|_Q^2 + |\Upsilon_k u_k|_R^2\Big\},$$

where $\tilde{x}_0$ is a known bias term which stands for the initial estimate of x_0; $Q, Q_N \geq 0$; and $R, Q_0 > 0$. Note that $\mu \in \mathcal{U}$ and $\nu \in \mathcal{W}$ consist of sequences of functions that map the information structure (2) into the controller and the disturbance spaces of $\mathbb{R}^m$ and $\mathbb{R}^p$, respectively, namely $u_k = \mu_k(\mathcal{I}_k)$ and $w_k = \nu_k(\mathcal{I}_k)$ [1] for all k. This optimization problem can be viewed as a modified version of the deterministic H^∞ control problem (Başar and Bernhard (1995)). In (3), the stochastic parameters of control and measurement drops are implicitly included to capture the lossy nature of the dynamical system (1).

By invoking the formulation of the corresponding soft-constrained game (Başar and Bernhard (1995)), the cost function of the zero-sum dynamic game that is parameterized by the disturbance attenuation parameter γ is given by

$$\begin{aligned} &J_\gamma^N(\mu, \nu) \\ &= \mathbb{E}\Big\{|x_N|_{Q_N}^2 - \gamma^2|x_0 - \tilde{x}_0|_{Q_0}^2 + \sum_{k=0}^{N-1} |x_k|_Q^2 \\ &\quad + |\Upsilon_k u_k|_R^2 - \gamma^2|w_k|^2 - \gamma^2|y_k - \Pi_k C x_k|_{V^{-1}}^2\Big\}, \end{aligned} \tag{4}$$

where the measurement equation (1b) is used with $v_k = E^{-1}(y_k - \Pi_k C x_k)$.

This completes the problem formulation of minimax control under multiple TCP-like packet dropping networks. Note that (4) is a zero-sum dynamic game in which the controller minimizes the cost function, while the unknown disturbance maximizes the same cost function. We need to characterize the saddle point of this, whenever it exists, where the existence will depend on the value of γ as well as the loss rates.

Since the perfect state information is not available, we need to establish an estimator policy that provides the worst-case estimated state that corresponds to the past worst-case disturbance. Moreover, we need to characterize the smallest value of γ which solves the original disturbance attenuation problem (3), and the range of values of $\bar{\Upsilon}$ and $\bar{\Pi}$ for the closed-loop system to be stable, in the sense to be clarified shortly.

3. FINITE-HORIZON MINIMAX CONTROL

3.1 Stochastic Minimax State Estimator Design

In this section, we establish a stochastic minimax estimator under the information structure (2).

[1] Note that an underlying assumption is that the worst-case disturbance has access to the same information as the controller.

Lemma 1. Consider the zero-sum dynamic game in (4) subject to (1) and (2) with $k \in [0,\ N-1]$, $\bar{\alpha}^i \in [0,1]$, $\bar{\beta}^j \in [0,1]$ for all i and j, and a fixed $\gamma > 0$. Then:

(i) A stochastic minimax state estimator (SMSE) exists if and only if

$$\rho(\Sigma_k Q) < \gamma^2 \text{ almost surely (a.s.) } \forall k \in [0, N-1], \quad (5)$$

where $\rho(\cdot)$ is spectral radius of $(\cdot)$ and Σ_k is generated by the following stochastic Riccati equation (SRE) (or stochastic error covariance matrix) with $\Sigma_0 = Q_0^{-1}$:

$$\Sigma_{k+1} \quad (6)$$
$$= A\left(\Sigma_k^{-1} - \gamma^{-2}Q + C^T\Pi_k V^{-1}\Pi_k C\right)^{-1} A^T + DD^T.$$

(ii) The SMSE is generated by

$$\bar{x}_0 = \tilde{x}_0$$
$$\bar{x}_{k+1} = A\bar{x}_k + B\Upsilon_k u_k \quad (7)$$
$$+ AT_k\left(\gamma^{-2}Q\bar{x}_k + C^T\Pi_k V^{-1}(y_k - \Pi_k C\bar{x}_k)\right),$$

where the estimator gain T_k can be written as

$$T_k = (\Sigma_k^{-1} - \gamma^{-2}Q + C^T\Pi_k V^{-1}\Pi_k C)^{-1}. \quad (8)$$

Proof. The proof is based on dynamic programming by introducing the quadratic *cost-to-come (worst past cost)* function under the information structure (2) (Başar and Bernhard (1995)). The detailed proof is similar to that of the single packet drop case, see Moon and Başar (2013b, 2014). □

Fact 2. Both (6) and (7) are forward-moving stochastic equations depending on past values of Π_k and Υ_k, and condition (5) is sample path dependent. □

3.2 Minimax Controller Design

In this section, we obtain a minimax controller for the dynamical system (1) under the information structure (2).

Lemma 3. Consider the zero-sum dynamic game in (4) subject to (1) and (2) with $k \in [0,\ N-1]$, $\bar{\alpha}^i \in [0,1]$, $\bar{\beta}^j \in [0,1]$ for all i and j, and a fixed $\gamma > 0$. Then:

(i) There exists a minimax controller if and only if (5) holds a.s. for all k, and

$$\phi_k(S_k) > 0,\ \forall k \in [0, N-1] \quad (9a)$$
$$\rho(D^T Z_{k+1} D) < \gamma^2,\ \forall k \in [0, N-1] \quad (9b)$$
$$\rho(\Sigma_k Z_k) < \gamma^2,\ \text{a.s. } \forall k \in [0, N-1], \quad (9c)$$

where Z_k and $\phi_k(S_k)$ are defined in (ii).

(ii) Z_k with $Z_N = Q_N$ is generated by the following generalized Riccati equation:

$$Z_k = A^T Z_{k+1} A + Q + P_{u_k}^T \phi_k(S_k) P_{u_k} \quad (10)$$
$$- P_{w_k}^T M_k P_{w_k} - 2P_{u_k}^T \bar{\Upsilon}^T B^T Z_{k+1} A$$
$$+ 2P_{w_k}^T D^T Z_{k+1} A - 2P_{u_k}^T \bar{\Upsilon}^T B^T Z_{k+1} D P_{w_k},$$

where

$$P_{u_k} = \left(\phi_k(S_k) + |D^T Z_{k+1} B\bar{\Upsilon}|^2_{M_k^{-1}}\right)^{-1} K_k \quad (11a)$$
$$P_{w_k} = \left(M_k + |\bar{\Upsilon}^T B^T Z_{k+1} D|^2_{\phi_k^{-1}(S_k)}\right)^{-1} L_k \quad (11b)$$
$$K_k = \bar{\Upsilon}^T B^T (I + Z_{k+1} D M_k^{-1} D^T) Z_{k+1} A \quad (11c)$$
$$L_k = D^T (I - Z_{k+1} B\bar{\Upsilon}\phi_k^{-1}(S_k)\bar{\Upsilon}^T B^T) Z_{k+1} A \quad (11d)$$
$$M_k = (\gamma^2 I - D^T Z_{k+1} D) \quad (11e)$$
$$S_k = (R + B^T Z_{k+1} B), \quad (11f)$$

where $\phi_k(X)$ can be obtained by solving

$$\phi_k(X) := \mathbb{E}\{\Upsilon_k^T X \Upsilon_k\}. \quad (12)$$

(iii) The minimax controller and the worst-case disturbance can be written as

$$u_k^* = \mu_k^*(\mathcal{I}_k) = -P_{u_k}\hat{x}_k \quad (13)$$
$$w_k^* = \nu_k^*(\mathcal{I}_k) = P_{w_k}\hat{x}_k, \quad (14)$$

where $\hat{x}_k$ is the worst-case estimated state that can be obtained by

$$\hat{x}_k = (I - \gamma^{-2}\Sigma_k Z_k)^{-1}\bar{x}_k, \quad (15)$$

where $\bar{x}_k$ is generated by the SMSE in Lemma 1.

Proof. Due to the certainty equivalence principle (Başar and Bernhard (1995)), we first need to obtain the state feedback minimax controller by assuming that the controller has access to full state information and the past control packet loss information, namely, $x_{0:k}$, $u_{0:k-1}$, and $\Upsilon_{0:k-1}$ for all k. Then, with the value function of $V_k(x_k) = \mathbb{E}\{x_k^T Z_k x_k | \mathcal{I}_k\}$ where $Z_N = Q_N$, the control law and the worst-case disturbance in parts (ii) and (iii) can be obtained by a value function iteration under the condition specified in (9a) and (9b) (Moon and Başar (2013b, 2014); Başar and Bernhard (1995)). Then under (9c), the worst-case estimated state in (15) can be obtained by

$$\hat{x}_k = \arg\max_{x_k} \mathbb{E}\{V_k(x_k) + W_k(x_k)|\mathcal{I}_k\}$$
$$= \arg\max_{x_k} \mathbb{E}\{|x_k|_{Z_k} - \gamma^2|x_k - \bar{x}_k|^2_{\Sigma_k^{-1}} + l_k|\mathcal{I}_k\},$$

where $W_k(x_k)$ is given in Lemma 1. This completes the proof of the lemma. □

Fact 4. The condition (9a) holds trivially for all k when we use the single packet drop model in Moon and Başar (2013a). □

We now state the main theorem of this section.

Theorem 5. Consider the stochastic dynamical system (1) with the cost function of (4) and the information structure (2). Suppose γ is fixed such that (5), (9a), (9b), and (9c) hold for all k. Then:

(i) The minimax controller (13) exists with the SMSE and the worst-case estimated state (15).
(ii) Under (13), the disturbance attenuation level of γ is achieved, that is, $\ll \mathcal{T}^N_{\mu^*_\gamma} \gg \leq \gamma$.
(iii) There is no separation of control and estimation due to the spectral radius condition (9c).
(iv) As $\gamma \rightarrow \infty$, the minimax controller with the SMSE converges to the solution of the corresponding LQG system. □

Note that γ has to satisfy the spectral radius condition (9c), which is related to (10) and (5). Therefore, we cannot design the minimax controller and the SMSE independently.

Fact 6. It was shown in Garone et al. (2012) that the LQG system features separation, i.e., the controller and the estimator can be designed independently. This fact and parts (iii) and (iv) in Theorem 5 imply that the separation holds in Theorem 5 when $\gamma \rightarrow \infty$. □

Fact 7. For the finite-horizon problem, the smallest value of γ that satisfies all conditions in Theorem 5 solves the original disturbance attenuation problem formulated in (3). □

4. INFINITE-HORIZON MINIMAX CONTROL

In this section, we analyze the limiting behavior of the minimax controller and the SMSE in Section 3 when $k, N \to \infty$ of the cost function (4) without the terminal constraint. We also show the relationship of γ, $\bar{\Upsilon}$, and $\bar{\Pi}$ to the existence and stabilizability of the minimax controller. We first provide the infinite-horizon version of the results of Section 3.

- The generalized algebraic Riccati equation (GARE) is
$$\begin{aligned}\bar{Z} &:= \bar{Z}(\gamma, \bar{\Upsilon}) \qquad (16)\\ &= A^T \bar{Z} A + Q + P_u^T \phi(R + B^T \bar{Z} B) P_u - P_w^T M P_w \\ &\quad - 2P_u^T \bar{\Upsilon}^T B^T \bar{Z} A + 2P_w^T D^T \bar{Z} A \\ &\quad - 2P_u^T \bar{\Upsilon}^T B^T \bar{Z} D P_w,\end{aligned}$$
where P_u, P_w, S, M, and $\phi(\cdot)$ are the infinite-horizon versions of (11) with respect to $\bar{Z}$, respectively.
- The infinite horizon version of the minimax controller and the worst-case disturbance can be written as
$$u_k = -P_u \hat{x}_k \qquad (17)$$
$$w_k = P_w \hat{x}_k. \qquad (18)$$
- The worst-case estimated state is
$$\hat{x}_k = (I - \gamma^{-2} \Sigma_k \bar{Z})^{-1} \bar{x}_k, \qquad (19)$$
where $\bar{x}_k$ is generated by the SMSE in (7).
- The infinite-horizon version of the set of existence conditions is
$$\phi(R + B^T \bar{Z} B) > 0 \qquad (20)$$
$$\rho(D^T \bar{Z} D) < \gamma^2 \qquad (21)$$
$$\rho(\Sigma_k \bar{Z}) < \gamma^2, \text{ a.s. } \forall k. \qquad (22)$$

Note that $\bar{\Upsilon}$ and $\bar{\Pi}$ are $m \times m$ and $q \times q$ matrices of control and measurement packet loss rates, respectively. Since the SMSE is time varying and random due to the SRE, the infinite-horizon version of the results of Lemma 1 is not generally available.

As we have seen, the existence conditions obtained in Section 3 need to be characterized in terms of the fixed point of the GARE in (16). Therefore, we first show the existence of a fixed point of (16).

Lemma 8. Suppose (A, B) is controllable, and $(A, Q^{1/2})$ is observable. Define
$$\begin{aligned}\Gamma_1(\bar{\Upsilon}) &:= \{\gamma > 0 \ : \ (20) \text{ and } (21) \text{ hold},\\ &\qquad \bar{Z} > 0 \text{ solves } (16).\}\\ \gamma_1^*(\bar{\Upsilon}) &:= \inf\{\gamma \in \Gamma_1(\bar{\Upsilon})\}.\end{aligned}$$
Assume that for any $\bar{\Upsilon}$, $\Gamma_1(\bar{\Upsilon})$ is nonempty and $\gamma > \gamma_1^*(\bar{\Upsilon})$. Then $\{Z_{k,N}\} \to \bar{Z}$ for each fixed k, as $N \to \infty$ where $\{Z_{k,N}\}$ is generated by the Riccati equation (10) and $\bar{Z}$ is a fixed point of (16) that satisfies (20) and (21).

Proof. First observe that the Riccati equation (16) generates a monotone sequence as $k = N, N-1, ..., 0$. From the definition, for a fixed γ, there is a fixed point of (16) that satisfies (20) and (21). This fixed point is in fact an upper bound on the Riccati equation (10), i.e., $\bar{Z} \geq Z_{k,N}$ for all k. To see this, consider (as in Başar and Bernhard (1995))
$$\begin{aligned}&\inf_{\mu \in \mathcal{U}} \sup_{\nu \in \mathcal{W}} J_\gamma^\infty(\mu, \nu)\\ &\geq \inf_{\mu \in \mathcal{U}} \mathbb{E}\Bigg\{ \sum_{k=N}^{\infty} |x_k|_Q^2 + |\Upsilon_k u_k|_R^2 \\ &\quad + \sum_{k=0}^{N-1} |x_k|_Q^2 + |\Upsilon_k u_k|_R^2 - \gamma^2 |\nu_k^*|^2 \Bigg\}\\ &\geq x_0^T Z_0 x_0,\end{aligned}$$
where ν_k^* is the worst-case disturbance of (14), and we used the state feedback cost function in Moon and Başar (2013a). Then it is a simple matter to show that a sequence which is nondecreasing and bounded from above converges to the fixed point of (16). Uniqueness follows from the observability assumption as in the deterministic minimax control problem in Başar and Bernhard (1995). □

For the SMSE, since the SRE is governed by $\{\Pi_k\}$, it does not admit any fixed points unless $\bar{\beta}^i = 1$ for all i. Therefore, the infinite-horizon version of the existence condition of the SMSE is analogous to its finite-horizon version:
$$\begin{aligned}\Gamma_2(\bar{\Pi}) &:= \{\gamma > 0 \ : \ (5) \text{ holds a.s. for all } k.\}\\ \gamma_2^*(\bar{\Pi}) &:= \inf\{\gamma \in \Gamma_2(\bar{\Pi})\}.\end{aligned}$$
It is easy to see that for a given $\bar{\Pi}$, if $\Gamma_2(\bar{\Pi})$ is nonempty and $\gamma > \gamma_2^*(\bar{\Pi})$, then (5) holds a.s. for all k.

In Garone et al. (2012), it was shown that the LQG controller requires the boundedness of $\mathbb{E}\{P_k\}$ where P_k is the error covariance matrix of the Kalman filter, since its upper bound constitutes an upper bound of the average cost of the LQG problem. Since this result coincides with our case when $\gamma \to \infty$, the boundedness of $\mathbb{E}\{\Sigma_k\}$ is necessarily required.

Let Θ^q be a collection of subsets of $\{1, 2, ..., q\}$ and let the matrix Δ_q^I be
$$\Delta_q^I := \mathbf{diag} \begin{cases} 1 & i \in I \in \Theta^q \\ 0 & \text{else.} \end{cases}$$
It is easy to check that Δ_q^I has the same dimension as of Π_k. The measurement loss rate that corresponds to Δ_q^I is defined by
$$\bar{\Delta}_q^I(\bar{\Pi}) := \left(\prod_{i \in I} \bar{\beta}^i \right) \left(\prod_{i \notin I} 1 - \bar{\beta}^i \right).$$
Note that this variable is determined by $\bar{\Pi}$. One such example is as follows: suppose $q = 3$ and $I' = \{1, 2\} \in \Theta^3$. Then $\Delta_3^{I'} = \mathbf{diag}\{1, 1, 0\}$ and $\bar{\Delta}_3^{I'}(\bar{\Pi}) = \bar{\beta}^1 \bar{\beta}^2 (1 - \bar{\beta}^3)$. Note also that by using the definitions of Δ and $\bar{\Delta}$, the existence condition (20) can be written as
$$\phi(R + B^T \bar{Z} B) = \sum_{I \in \Theta^m} \bar{\Delta}_m^I(\bar{\Upsilon}) \Delta_m^I (R + B^T Z B) \Delta_m^I.$$
The boundedness of $\mathbb{E}\{\Sigma_k\}$ can be stated as follows.

Lemma 9. Suppose $\gamma > \gamma_2^*(\bar{\Pi})$ is finite for a given $\bar{\Pi}$. Suppose (A, D) is controllable and (A, C) is observable. Define
$$\begin{aligned}\Gamma_3(\bar{\Pi}) &:= \{\gamma > 0 \ : \ \rho(\bar{\Sigma} Q) < \gamma^2\}\\ \gamma_3^*(\bar{\Pi}) &:= \inf\{\gamma \ : \ \gamma \in \Gamma_3(\bar{\Pi})\},\end{aligned}$$
where $\bar{\Sigma}$ is a fixed point of the following algebraic Riccati equation:

$$\bar{\Sigma} = \sum_{I \in \Theta^q} \bar{\Delta}_q^I(\bar{\Pi})\bar{\Sigma}^I \tag{23}$$

$$\bar{\Sigma}^I := A\left(\bar{\Sigma}^{-1} - \gamma^{-2}Q + C^T\Delta_q^I V^{-1}\Delta_q^I C\right)^{-1} A^T + DD^T.$$

Then:

(i) $\mathbb{E}\{\Sigma_k\} \leq \bar{\Sigma}_k = \sum_{I \in \Theta^q} \bar{\Delta}_q^I(\bar{\Pi})\bar{\Sigma}_k^I$, where
$$\bar{\Sigma}_{k+1}^I := A\left(\bar{\Sigma}_k^{-1} - \gamma^{-2}Q + C^T\Delta_q^I V^{-1}\Delta_q^I C\right)^{-1} A^T + DD^T.$$

(ii) If $\Gamma_3(\bar{\Pi})$ is nonempty and $\gamma > \max\{\gamma_2^*(\bar{\Pi}), \gamma_3^*(\bar{\Pi})\}$, then $\bar{\Sigma}_k \to \bar{\Sigma}$ as $k \to \infty$. Finally, under this condition, the SMSE exists and $\mathbb{E}\{\Sigma_k\} \leq \bar{\Sigma}$ for all k.

Proof. Part (i) can be shown by induction and applying Jensen's inequality to the expected value of the SRE (6), which is a concave function in Σ_k for all k (Moon and Başar (2013b)). For part (ii), note that from definition, $\bar{\Sigma}$ is a solution to (23) satisfying $\rho(\bar{\Sigma}Q) < \gamma^2$. Then the proof of the convergence of $\bar{\Sigma}_k$ to $\bar{\Sigma}$ is analogous to that of the single packet drop problem considered in Moon and Başar (2013b). This completes the proof. □

Fact 10. Note that under Lemma 9, we have $\mathbb{E}\{P_k\} \leq \mathbb{E}\{\Sigma_k\} \leq \bar{\Sigma}$ where P_k is the error covariance matrix in Garone et al. (2012), since $P_k \leq \Sigma_k$ a.s. for all k. Moreover as $\gamma \to \infty$, the upper bound coincides with the upper bound of $\mathbb{E}\{P_k\}$ in Garone et al. (2012). □

Finally, we introduce the infinite-horizon version of the spectral radius condition (9c):

$$\Gamma_4(\bar{\Upsilon}, \bar{\Pi}) := \{\gamma > 0 \ : \ \gamma > \gamma_1^*(\bar{\Upsilon}),\ \gamma > \gamma_2^*(\bar{\Pi}),\ \gamma > \gamma_3^*(\bar{\Pi}),\ (22) \text{ holds a.s. for all } k.\}$$
$$\gamma_4^*(\bar{\Upsilon}, \bar{\Pi}) := \inf\{\gamma \in \Gamma_4(\bar{\Upsilon}, \bar{\Pi})\}.$$

By convention, if $\Gamma_4(\bar{\Upsilon}, \bar{\Pi})$ is empty, then $\gamma_4^*(\bar{\Upsilon}, \bar{\Pi})$ is infinite; for example, if $\bar{\alpha}^i = \bar{\beta}^i = 0$ for all i and A is unstable, then $\Gamma_4(\bar{\Upsilon}, \bar{\Pi})$ is empty. Therefore, for fixed $\bar{\Upsilon}$ and $\bar{\Pi}$, if $\gamma > \gamma_4^*(\bar{\Upsilon}, \bar{\Pi})$ and is finite, then all existence conditions are satisfied. Note that from the above construction, we have characterized the minimum achievable disturbance attenuation level that leads to satisfaction of existence conditions. We now state the main theorem of this section.

Theorem 11. Consider the dynamical system (1) with the infinite-horizon version of the cost function (4). Suppose (A, B) and (A, D) are controllable, and $(A, Q^{1/2})$ and (A, C) are observable. For fixed $\bar{\Upsilon}$ and $\bar{\Pi}$, suppose $\gamma_4^*(\bar{\Upsilon}, \bar{\Pi})$ is finite and $\gamma > \gamma_4^*(\bar{\Upsilon}, \bar{\Pi})$. Then:

(i) The GARE (16) admits a unique fixed point that satisfies (20) and (21).
(ii) The SMSE and the worst-estimated state (19) exist.
(iii) The minimax controller can be obtained by (17) with the SMSE and (19).
(iv) The infinite-horizon version of the minimax controller (17) achieves the disturbance attenuation level of γ, that is, $\ll \mathcal{T}_{\mu_\gamma^*}^\infty \gg \leq \gamma$.
(v) Separation does not hold.
(vi) As $\gamma \to \infty$, the solution converges to that of the LQG problem. □

Now, assume that we have all the conditions of Theorem 11 satisfied. Consider the closed-loop system

$$x_{k+1} = Ax_k - B\Upsilon_k P_u(I - \gamma^{-2}\Sigma_k\bar{Z})^{-1}\bar{x}_k + DP_w(I - \gamma^{-2}\Sigma_k\bar{Z})^{-1}\bar{x}_k, \tag{24}$$

where $\bar{x}_k$ is generated by the SMSE in (7). Note that the closed-loop system (24) with the SMSE is time varying due to the estimator gain (8) and the SRE. In fact, the process of the SRE is not Markov, and its number of realizations grows by 2^{qk} depending on the sequence of measurement drops. Therefore, the closed-loop system cannot be seen as a finite or infinite Markov jump system as in Costa et al. (2005) and Costa and Fragoso (1995).

It is worth mentioning that if a given γ does not satisfy the condition in Theorem 11, then the minimax controller and the SMSE do not exist; therefore, it is not possible to stabilize the closed-loop system and achieve the performance level of γ.

The stability of the closed-loop system (24) can be achieved when $\bar{\beta}^i = 1$ for all i. In that case, there is a stationary estimator gain; hence the closed-loop system (24) with the minimax estimator is mean-square stable, that is, $\mathbb{E}\{|z_k|^2\} \to 0$ as $k \to \infty$ where $z_k = [x_k^T, \bar{x}_k^T]^T$. The proof is a direct modification of the single packet drop problem in Moon and Başar (2013a).

Fact 12. Note that the H^∞ optimum disturbance attenuation level of $\gamma_4^*(\bar{\Upsilon}, \bar{\Pi})$ is a function of control and measurement loss rates. Moreover, $\gamma_4^*(I_{m\times m}, I_{q\times q})$ where $I_{m\times m}$ (resp. $I_{q\times q}$) is the $m \times m$ (resp. $q \times q$) identity matrix is the optimum level, which is equivalent to the deterministic minimax control problem. The case when $\bar{\Upsilon}$ and $\bar{\Pi}$ are zero matrices is analogous to the open loop control problem; therefore, γ_4^* is not finite when A is unstable. □

We now discuss the converse part. To characterize the minimum attainable loss rates, define

$$\Lambda_1(\gamma) := \{\bar{\Upsilon} \ : \ (20) \text{ and } (21) \text{ hold, } \bar{Z} > 0 \text{ solves } (16).\}$$
$$\bar{\Upsilon}_c(\gamma) := \inf\{\bar{\Upsilon},\ \bar{\alpha}^i,\ \forall i \ : \ \bar{\Upsilon} \in \Lambda_1\}$$
$$\Lambda_2(\gamma) := \{\bar{\Pi} \ : \ (5) \text{ holds a.s. for all } k \text{ and } \rho(\bar{\Sigma}Q) < \gamma^2\}$$
$$\bar{\Pi}_c(\gamma) := \inf\{\bar{\Pi},\ \bar{\beta}^i,\ \forall i \ : \ \bar{\Upsilon} \in \Lambda_2\},$$

where $\gamma > \gamma_4^*(I_{m\times m}, I_{q\times q})$. Note that the infimization has to be taken over m control loss rates and q measurement loss rates. Moreover the above sets are empty when $\gamma < \gamma_4^*(I_{m\times m}, I_{q\times q})$. Now, for any $\bar{\Upsilon} \succcurlyeq \bar{\Upsilon}_c(\gamma)$ and $\bar{\Pi} \succcurlyeq \bar{\Pi}_c(\gamma)$ where $\succcurlyeq$ is componentwise inequality, we have all existence conditions holding.

However, unlike the single packet drop problem, $\bar{\Upsilon}_c(\gamma)$ and $\bar{\Pi}_c(\gamma)$ are hard to be characterized, since we have $m + q$ different rates that have to be considered while checking existence conditions. One alternative would be to fix all but one or two, and analyze the stability region with respect to those parameters. Another possibility is to have all $\bar{\alpha}^i$'s and $\bar{\beta}^j$'s the same, and along with γ, conduct the analysis in the 3-dimensional parameter space.

5. NUMERICAL EXAMPLE: EXISTENCE CONDITIONS

We examine the existence conditions of Section 4 for the batch reactor system considered in Garone et al. (2012). The plant is modeled as a 2 input-2-output system. The system matrices of A, B, and C are as in Garone et al. (2012), and the other parameters are taken as follows

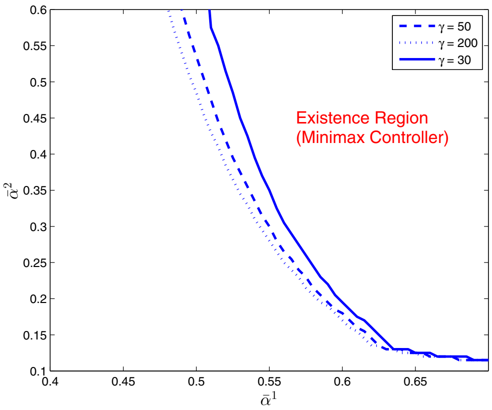

Fig. 1. Existence region of the minimax controller.

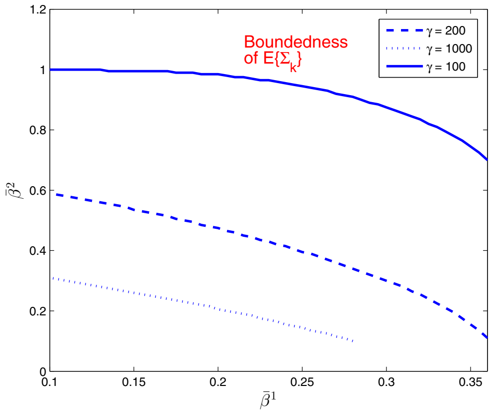

Fig. 2. Boundedness of $\mathbb{E}\{\Sigma_k\}$.

$$D = \sqrt{2}I_{4\times4},\ V = 0.001I_{2\times2},\ R = I_{2\times2},\ Q = I_{4\times4}.$$

Figure 1 shows the existence region of the minimax controller in Theorem 11. As can be seen, a smaller value of γ leads shrinkage of the existence region. This is because the system should need more reliable communication links if the disturbance attenuation is more important. A similar result is also achieved for the region of the boundedness of $\mathbb{E}\{\Sigma_k\}$ as shown in Fig 2. A detailed discussion on the relationship between the loss rates and γ can be found in Moon and Başar (2013a, 2014). It is worth mentioning that as $\gamma \to \infty$, the two regions converge to that of the LQG problem in Garone et al. (2012).

6. CONCLUDING REMARKS

In this paper, we have considered the minimax control problem over multiple TCP-like packet dropping networks. The communication channel is modeled as the Bernoulli-type packet losses. We have obtained the output feedback minimax controller. We have characterized the minimum disturbance attenuation level that can be achieved by the minimax controller. Moreover, we have obtained the minimum attainable control and measurement loss rates above which the minimax controller exists and is able to stabilize the system under some specific conditions. Finally, we have shown that as the disturbance attenuation parameter becomes unbounded, every result obtained in this paper specializes to corresponding results in the LQG case treated by Garone et al. (2012). Hence, there is a parallel with what is found in the deterministic H^∞ control case.

REFERENCES

Başar, T. and Bernhard, P. (1995). *H^∞ Optimal Control and Related Minimax Design Problems.* Birkhäuser, Boston, MA, 2nd edition.

Costa, O., Fragoso, M., and Marques, R. (2005). *Discrete-Time Markov Jump Linear Systems.* Springer-Verlag.

Costa, O.L.V. and Fragoso, M. (1995). Discrete-time LQ-optimal control problems for infinite Markov jump parameter systems. *IEEE Transactions on Automatic Control*, 40(12), 2076–2088.

Garone, E., Sinopoli, B., Goldsmith, A., and Casavola, A. (2012). LQG control for MIMO systems over multiple erasure channels with perfect acknowledgment. *IEEE Transactions on Automatic Control*, 57(2), 450–456.

Geromel, J., Goncalves, A., and Fioravanti, A. (2009). Dynamic output feedback control of discrete-time Markov jump linear systems through linear matrix inequalities. *SIAM Journal on Control and Optimization*, 48(2), 573–593.

Hespanha, J., Naghshtabrizi, P., and Xu, Y. (2007). A survey of recent results in networked control systems. *Proceedings of the IEEE*, 95(1), 138–162.

Imer, O.C., Yüksel, S., and Başar, T. (2006). Optimal control of LTI systems over unreliable communication links. *Automatica*, 42(9), 1429–1439.

Ishii, H. (2008). H^∞ control with limited communication and message losses. *Systems and Control Letters*, 57(4), 322–331.

Moon, J. and Başar, T. (2013a). Control over TCP-like lossy networks: A dynamic game approach. In *Proc. of American Control Conference*, 1581–1586. Washington, DC.

Moon, J. and Başar, T. (2013b). Estimation over lossy networks: A dynamic game approach. In *Proc. of the 52nd IEEE Conference on Decision and Control*, 2412–2417. Florence, Italy.

Moon, J. and Başar, T. (2014). Control over lossy networks: A dynamic game approach. In *Proc. of American Control Conference*. Portland, OR, USA.

Pan, Z. and Başar, T. (1995). H^∞ control of Markovian jump systems and solutions to associated piecewise-deterministic differential games. *In G.J. Olsder, editor, Annals of Dynamic Games*, 2, 61–94. Birkhäuser.

Schenato, L., Sinopoli, B., Franceschetti, M., Poolla, K., and Sastry, S. (2007). Foundations of control and estimation over lossy networks. *Proceedings of the IEEE*, 95(1), 163–187.

Seiler, P. and Sengupta, R. (2005). An H^∞ approach to networked control. *IEEE Transactions on Automatic Control*, 50(3), 356–364.

Sinopoli, B., Schenato, L., Franceschetti, M., Poolla, K., Jordan, M., and Sastry, S. (2004). Kalman filtering with intermittent observations. *IEEE Transactions on Automatic Control*, 49(9), 1453–1464.