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Distributed optimization by myopic strategic interactions and the price of heterogeneity

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***Abstract*— This paper is concerned with the tradeoffs between low-cost heterogenous designs and optimality. We study a class of constrained myopic strategic games on networks which *approximate* the solutions to a constrained quadratic optimization problem; the Nash equilibria of these games can be found using best-response dynamical systems, which only use local information. The notion of price of heterogeneity captures the quality of our approximations. This notion relies on the structure and the strength of the interconnections between agents. We study the stability properties of these dynamical systems and demonstrate their complex characteristics, including abundance of equilibria on graphs with high sparsity and heterogeneity. We also introduce the novel notions of social equivalence and social dominance, and show some of their interesting implications, including their correspondence to consensus. Finally, using a classical result of Hirsch [1], we fully characterize the stability of these dynamical systems for the case of star graphs with asymmetric interactions. Various examples illustrate our results.**

I. INTRODUCTION

In the past decade, distribution and decentralization of tasks have allowed for achieving global objectives, costly to achieve in a centralized manner, using groups of individuals; see [2], [3] and references therein. Heterogeneity, nevertheless, can result in a more complex set of solutions or deviation from optimality, e.g. [4]. Our current work investigates the tradeoffs between heterogeneity and performance. In particular, we revisit a class of distributed optimization problems where the optimization variable is the state of a network, and we introduce a class of constrained myopic strategic networked games to *approximate* their solutions.

Literature review

This work has connections with the literature on games on networks and distributed optimization. The design of distributed dynamical systems for optimization of a sum of convex functions has been studied intensively in recent years; see e.g. [5], [6], [7], [8]. These works rely on communicating the estimates of agents and running a consensus-based gradient flow dynamics.

Regarding the literature on games on networks, this class of games has also been studied intensively recently; see [9] and references within. The most related work to ours is [10], where the piecewise linear best-response dynamical systems of strategic interactions have been introduced and studied. The very recent work [11] studies these dynamical systems for the case of asymmetric interactions. Finally, we use techniques for stability analysis of piecewise linear systems [12], [13], [14], cooperative and noncooperative dynamical systems [15], [1], the notions of best-response dynamics [16], [17] and the price of anarchy from game theory [18].

Statement of contributions

Our first contribution is the introduction of a class of myopic strategic games with piecewise best-response dynamics for *approximating* the solutions of constrained quadratic optimization problems. The novel notion of price of heterogeneity captures the quality of this approximation. This notion relies on the structure of the underlying network and the dependencies on the actions of other agents, termed as the coefficient of heterogeneity. In contrast to the existing distributed optimization dynamics that rely on communicating the estimates of agents about the state, the best-response dynamical systems corresponding to this class of games can be solved efficiently and only using local information. Moreover, unlike the continuous-time distributed optimization flow, this dynamical system is of first order.

As our second contribution, we study the stability properties of the best-response dynamical systems of myopic games on networks. Using Lyapunov techniques for piecewise linear systems, we provide an alternative proof for the convergence of these dynamical systems for scenarios with low coefficient of heterogeneity. For the cases with high coefficients of heterogeneity, we show the complex characteristics of these dynamical systems, including abundance of equilibria for networks with high sparsity. We also introduce the novel notions of social equivalence and social dominance, and show some interesting implications, including their correspondence to consensus. Finally, using a classical result of Hirsch [1], we fully characterize the stability of these dynamical systems for the case of star graphs with asymmetric interactions. We provide some simulations and suggest a class of open problems related to the notion of the price of heterogeneity. Due to space limitation, most of the proofs are omitted and will appear elsewhere.

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II. MATHEMATICAL PRELIMINARIES

We start with some notational conventions. Let $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}_{\geq 0}$, $\mathbb{Z}$, and $\mathbb{Z}_{\geq 1}$ denote the sets of real, complex, nonnegative real, integer, and positive integer numbers, respectively. The set of eigenvalues of a matrix $A \in \mathbb{R}^{n\times n}$, $n \in \mathbb{Z}_{\geq 1}$, is denoted by $\operatorname{spec}(A) \subset \mathbb{C}$.

A. Graph theory preliminaries

A *directed graph*, or simply *digraph*, is a pair $\mathcal{G} = (V, E)$, where V is a finite set called the vertex set and $E \subseteq V \times V$ is the edge set. In this paper, we mostly work with the class of *undirected graphs*, or simply *graphs*, where the edge set E consists of unordered pairs of vertices. We assume that these graphs have no self-loop. The *degree* of a vertex $v \in V$, denoted $d(v)$, is the number of in-neighbors and out-neighbors of v. A *weighted graph* is a triplet $\mathcal{G} = (V, E, A)$, where (V, E) is a graph and $A \in \mathbb{R}^{n\times n}_{\geq 0}$ is the *adjacency matrix*. We denote the entries of A by a_{ij}, $i, j \in \{1, \ldots, n\}$, entry $a_{ij} > 0$ if and only if $(v_i, v_j) \in E$ and $a_{ij} = 0$, otherwise. Unless mentioned otherwise, we further assume that $a_{ij} = 1$ when $(v_i, v_j) \in E$. If a matrix A satisfies this property, we say that A is a *weight assignment* of the graph $\mathcal{G} = (V, E)$. For a weighted graph, the weighted degree of v_i, $i \in \{1, \ldots, n\}$, is $d^{\mathrm{w}}(v_i) = \sum_{j=1}^{n} a_{ij}$.

III. PROBLEM STATEMENT

Let $\mathcal{G} = (V, E)$, where $V = \{1, \ldots, n\}$, $n \in \mathbb{Z}_{\geq 1}$, be an undirected graph, and consider the optimization problem

$$
\begin{aligned}
\text{minimize} \quad & f(x) = \frac{1}{2}\sum_{i=1}^{n} x_i^2, \\
& \delta \sum_{j\in\mathcal{N}_i} a_{ij}x_j + x_i \geq 1, \\
& 0 \leq x_i \leq 1 \quad \text{for all } i \in V, \qquad (1)
\end{aligned}
$$

where $A = [a_{ij}]$ is an adjacency matrix associated with $\mathcal{G}$ and $\delta \in [0, 1]$. Some remarks are now in order.

First, more general forms of the optimization problem above can be considered; we have chosen this form as it simplifies clarifying our ideas without much loss of generality. Second, many practical problems fall into the framework described above. For example, consider a coverage/resource allocation problem on the region $U = \cup_{i=1}^{n} U_i \subset \mathbb{R}^2$ using a group of agents in V; see Figure 1. Each agent $i \in V$ has limited resources and is responsible for providing coverage in the region U_i by allocating $x_i \in [0, 1]$ to it. In order to cover U_i, agent i needs to allocate one unit of resources to it (i.e., this agent needs to operate fully for covering this region alone). When $U_i \cap U_j \neq \emptyset$, agent $i \in V$ can reduce the resources it is allocating to U_i by counting on the fact that $j \in V$ is also allocating some resources to this region, with the discounted factor of δa_{ij}. In other words, parameter δ identifies the strength of the interconnections between neighboring agents and is assumed to be the same

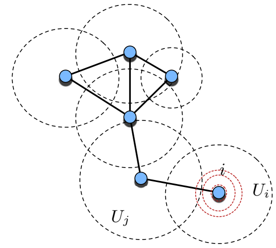


Fig. 1. A region $U = \sum_{i=1}^{n} U_i$ is shown. Each U_i is equipped with a sensor v_i. If $U_i \cap U_j \neq \emptyset$, then v_j also can provide some coverage for U_i, the quality of this is captured by δa_{ij}.

for all agents, for simplicity. For this reason, from now on, we refer to δ as the *coefficient of heterogeneity*.

It is easy to see that (1) is feasible and its solutions can be found in a centralized manner, for example, by the primal-dual interior-point method [19]. Henceforth, let $x^*_{\text{opt}} \in [0, 1]^n$ be a solution of (1) and let us define $f^* := f(x^*_{\text{opt}})$.

Next, note that it is possible to cast this problem in the framework of consensus-based distributed optimization [5], if each agent $i \in V$ can compute an estimate $x^*_i \in \mathbb{R}^n$ of the solution of (1) and can/is willing to share these estimates with its neighbors. Nevertheless, for large values of $n \in \mathbb{Z}_{\geq 1}$, such a distributed optimization scheme can easily become cumbersome. One thus wonders how suboptimal it would be for the agents to attempt to solve (1) *myopically*, i.e., agent $i \in V$ solves

$$
\begin{aligned}
\text{minimize} \quad & f_i(x) = \frac{1}{2}x_i^2, \\
& \delta \sum_{j\in\mathcal{N}_i} a_{ij}x_j + x_i \geq 1, \\
& 0 \leq x_i \leq 1. \qquad (2)
\end{aligned}
$$

The key point about (2) is that agent $i \in V$ only requires the values $x_j \in \mathcal{N}_i$ to choose x_i, as characterized in the following result.

Lemma 3.1: **(The solutions of** (2) **for each agent):** Given $\{x_j\}_{i=1}^{|\mathcal{N}_i|}$, the solution to (2) is given by

$$
x^*_i = \begin{cases} 0 & \delta \sum_{j\in\mathcal{N}_i} a_{ij}x_j \geq 1 \\ 1 - \delta \sum_{j\in\mathcal{N}_i} a_{ij}x_j & \text{otherwise.} \end{cases} \qquad (3)
$$

The dependency of the solutions for agent $i \in V$ turns the collection of the minimization problems given by (2) to a *game* over $\mathcal{G}$. Indeed, by Lemma 3.1, the set of reaction curves of the game G with the set of players V, the strategy set of each player being $[0, 1]$, and the payoff of the ith player being u_i, where

$$
u_i(x_i, x_{-i}) = \frac{1}{2}x_i^2 + \delta x_i \sum_{j\in\mathcal{N}_i} a_{ij}x_j - x_i,
$$

exactly corresponds to (3); note that this construction is certainly not unique. We term the class of games whose set of Nash equilibria coincides with the solutions of (2) as the class of *myopic strategic games with piecewise linear reaction curves*. Interestingly, this class is precisely the class of games on networks studied in [10]. In the rest of this paper, we often drop the term *with piecewise linear reaction curves* and refer to these as myopic strategic games. We have the following definition.

Definition 3.2: **(Price of Heterogeneity):** Let $x^*_{\text{lsg}} \in [0,1]^n$ be a pure strategy Nash equilibrium of the myopic strategic game associated with the optimization problem (1). Then we term

$$\text{PoH}_{f^*}(x^*_{\text{lsg}}) = \frac{f^*}{\sum_{i=1}^n f_i((x^*_{\text{lsg}})_i)} \tag{4}$$

as the *Price of Heterogeneity* (PoH) associated with x^*_{lsg} for (1).

Note that PoH is a quality measure associated with a given *Nash equilibrium* and by its definition, it is always less than one. This notion can also be casted as the notion of price of anarchy [18]. One reason as to why we have called this notion the price of heterogeneity is related to its correlation with the values of δ in our setting; when there is no interconnection between agents and δ is zero, the solution of the two problems match. As we will see later, increasing δ makes the Nash equilibria less efficient. The other reason is that here we start with an optimization problem, instead of a game, and the notion of price of heterogeneity captures the quality of the equilibria of myopic strategic games that, as a designer, we have associated to it.

IV. Piecewise linear best-response dynamics on networks

In this section, we study the best-response dynamical systems for the class of myopic strategic games with piecewise linear reaction curves given by

$$\dot{x}_i = \max\{1 - \delta \sum_{j=1}^n a_{ij} x_j, 0\} - x_i, \quad x_i(0) \in [0,1], \tag{5}$$

for all $i \in \{1, \ldots, n\}$. These dynamical systems have been recently introduced in [10]. As a step toward characterizing the price of heterogeneity, we revisit the stability properties of (5). In particular, we provide independent proofs for some of the cases studied in [10] and provide some key results in Section V.

First, note that it is easy to verify that the trajectories of (5) are bounded.

Lemma 4.1: **(Convergence for critical δ):** Let $\mathcal{G}$ be any weighted graph with the set of vertices V. If

$$\delta < \delta^* = \frac{1}{\max_{v_i \in V}(d^{\text{w}}(v_i))},$$

the Nash equilibrium of the strategic network game is unique and (5) is convergent to it.

We are next interested in characterizing scenarios where, unlike Lemma 4.1, some agents *switch* their strategies over time, according to (3). We start with the following definition.

Definition 4.2: **(Active and passive sets):** Let $\boldsymbol{x} = (x_1, \ldots, x_n) \in [0,1]^n$. We say that agent v_i is *active* if $1 - \delta \sum_{j=1}^n a_{ij} x_j > 0$ and *passive* otherwise. We denote by $\mathsf{S}_{\text{active}}, \mathsf{S}_{\text{passive}} : [0,1]^n \rightrightarrows V$, where $\mathsf{S}_{\text{active}}(\boldsymbol{x}) \subset V$ and $\mathsf{S}_{\text{passive}}(\boldsymbol{x}) \subset V$ are the subsets of active and passive agents, respectively. Given a set of active agents $V_{\text{a}} \subset V$, we call the set

$$X_{\text{active}}(V_{\text{a}}) = \{\boldsymbol{x} \in [0,1]^n \mid \mathsf{S}_{\text{active}}(\boldsymbol{x}) = V_{\text{a}}\}$$

the V_{a}-*set*. The V_{p}-*set* is defined similarly.

An immediate corollary of Definition 4.2 is that $X_{\text{active}}(V_{\text{a}}) \cup X_{\text{passive}}(V \backslash V_{\text{a}}) = [0,1]^n$. Given $V_{\text{a}} \subset V$, one can relabel the rows and the columns of the adjacency matrix A and partition it into the active and passive blocks,

$$A = \begin{pmatrix} A_a & A_{ap} \\ A_{pa} & A_p \end{pmatrix}.$$

The following key result characterizes $\delta^{**} > \delta^*$ for which (5) is asymptotically stable; although a proof of this result is given in [10], we give here an alternate proof.

Theorem 4.3: **(Sufficient conditions for asymptotic stability of** (5)**):** Suppose $\mathcal{G}$ is undirected. If

$$\lambda_{\text{min}}(\mathsf{I} + \delta A) > 0,$$

where $\lambda_{\text{min}}(\mathsf{I} + \delta A)$ is the minimum eigenvalue of $(\mathsf{I} + \delta A)$, then (5) is asymptotically stable.

Henceforth, we denote the maximum value of δ for which $\lambda_{\text{min}}(\mathsf{I} + \delta A) > 0$ by δ^{**}. The following examples show that studying the stability properties of (5) for $\delta > \delta^{**}$ is in general difficult. This is because these dynamical systems can become unstable on some active set $X_{\text{active}}(V_{\text{a}})$, $V_{\text{a}} \subset V$; nevertheless, the trajectories may eventually leave this region and be convergent. We demonstrate these difficulties with some examples.

Example 4.4: **(Stability for $\delta > \delta^{**}$):** Let us consider the network of Figure 2(b).

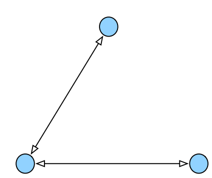


Fig. 2. A path graph with 3 agents.

The best-response dynamical system is

$$\begin{aligned}
\dot{x}_1 &= \begin{cases} 1 - x_1 - \delta x_2 & 1 - \delta x_2 > 0, \\ -x_1 & \text{otherwise;} \end{cases} \\
\dot{x}_2 &= \begin{cases} 1 - x_2 - \delta(x_1 + x_3) & 1 - \delta(x_1 + x_3) > 0, \\ -x_2 & \text{otherwise;} \end{cases} \\
\dot{x}_3 &= \begin{cases} 1 - x_3 - \delta x_2 & 1 - \delta x_2 > 0, \\ -x_3 & \text{otherwise;} \end{cases}
\end{aligned} \tag{6}$$

By Lemma 4.1, and since $\max_{i \in \{1,2,3\}} d^{\mathrm{w}}(i) = 2$, for all $\delta < \frac{1}{2}$, the best-response dynamical system is asymptotically convergent to $\boldsymbol{x}^* = (\delta A + \mathsf{I}_n)^{-1}\mathbf{1}$. Let us now study the implications of Theorem 4.3 for the case where $\delta \geq \frac{1}{2}$. By symmetry, we need to consider the only nontrivial case when all agents are active. In this case, we compute $\lambda_{\min}(\mathsf{I} + \delta A) = 1 - \sqrt{2}\delta$, which is positive by choosing $\delta < \frac{1}{\sqrt{2}}$. This result thus implies the asymptotic stability of (6) for the nontrivial case where $\frac{1}{2} \leq \delta < \frac{1}{\sqrt{2}}$. We next show that more is true and (6) is indeed asymptotically stable for any $\frac{1}{\sqrt{2}} \leq \delta < 1$. We start by proving that the subspace

$$W = \{\boldsymbol{x} = (x_1, x_2, x_3) \in [0,1]^3 \mid x_1 + x_3 \geq \frac{1}{\delta} \text{ and } x_2 \leq \frac{1}{2\delta}\},$$

is invariant under the flow. Note that since $x_1 + x_3 \geq \frac{1}{\delta}$, agent 2 is passive and thus when $\boldsymbol{x}(0) \in W$, for all $t \in \mathbb{R}_{>0}$, $x_2(t) \leq x_2(0)$. Moreover, since $x_2 \leq \frac{1}{2\delta}$ and in W

$$\dot{x}_1 + \dot{x}_3 = 2 - (x_1 + x_3) - 2\delta x_2,$$

we conclude that $x_1(t) + x_3(t) \leq x_1(0) + x_3(0)$, for all $t \in \mathbb{R}_{>0}$, thus proving our claim. Next, we show that any trajectory starting from any initial condition $\boldsymbol{x} \notin W$, will eventually enter W. Note that (6) has no equilibrium in the region $U = \{\boldsymbol{x} = (x_1, x_2, x_3) \in [0,1]^3 \mid x_1 + x_3 < \frac{1}{\delta}\}$; moreover, $\mathsf{I}_3 + \delta A$ has one negative eigenvalue and thus any trajectory starting from an initial condition $\boldsymbol{x}(0) \in U$ will eventually leave this region, entering W. Also, since agent 2 is passive in $[0,1]^3 \backslash (U \cup W)$, any trajectory starting from this region will eventually leave this region, yielding the result. Next, since the flow is invariant in W, using a Lyapunov argument on this region, we conclude that any flow entering W will asymptotically converge. •

In Section VI we show that the property demonstrated in this example indeed holds true for any star graph.

Example 4.5: **(Abundance of equilibria):** Consider the graph shown in Figure 3. First, note that for any path graph

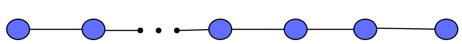


Fig. 3. An undirected path graph is shown.

with $n \in \mathbb{Z}_{\geq 1}$ vertices and for any choice of δ, the matrix $\mathsf{I} + \delta A$ has at most one zero eigenvalue. This is because the eigenvalues of A can be written in terms of Chebyshev polynomials [20], which are orthogonal polynomials and thus have distinct eigenvalues. Let us now consider the case where $n = 4$, and choose $\delta = -1/\lambda_{\min}(A) \cong 0.6180$. Then any point in the set $\{x \in [0,1]^4 \mid (\mathsf{I} + \delta A)\boldsymbol{x} = \mathbf{1}\}$ is an equilibrium of (5). •

Example 4.6: **(Emergence of unstable equilibria in stable submanifolds):** Consider the path graph shown in Figure 3 with $n = 4$ and let $\delta = 0.9$, which by Example 4.5 is larger than δ^{**}. Using the Stable Manifold Theorem [21], the stable submanifold S_{stable} is of dimension three. Let us take the initial condition to be $x_0 = (0.3717, 0.6015, 0.6015, 0.3717)^T \in S_{\text{stable}}$. Figure 4(a) shows the trajectories of (5) starting from x_0. These trajectories are asymptotically convergent to $x^* = (0.9174, 0.0917, 0.0917, 0.9174)^T \in S_{\text{stable}}$. Nevertheless, with a small perturbation, in the direction orthogonal to S_{stable}, the trajectories leave S_{stable}; see Figure 4(b). •

The phenomena observed in the examples above demonstrate another complex facet of decentralizing tasks: even if one gives up on optimality, high coefficient of heterogeneity results in the dependency to initial conditions and thus lack of global asymptotic properties.

V. Social Relation and the Behavior of the Dynamics

In this section, we study some general properties of the dynamics (5). But before doing that, let us define the two concepts of social equivalence and social dominance.

Definition 5.1: **(Social equivalence and social dominance):** We say that i and j are socially equivalent if $\mathcal{N}_i = \mathcal{N}_j$. We say that i is socially dominated by j if $\mathcal{N}_i \subseteq \mathcal{N}_j \setminus \{i\}$.

Social equivalency and social dominance have interesting implications. The first observation is that social equivalence implies consensus.

Lemma 5.2: **(Social equivalence implies consensus):** Consider the best response dynamics on an arbitrary graph $\mathcal{G} = (V, E)$ and an arbitrary $\delta \in [0,1]$. If i, j are socially equivalent, then $\lim_{t \to \infty}(x_i(t) - x_j(t)) = 0$.

The second general result is related to the implications of social dominance in the behavior of the dynamical systems (5). The first implication is that if one agent is socially dominated by another agent, and at some point of the time the dominated agent invests more, then she should invest more energy forever. Subsequently, we show that if this does not happen, i.e. a socially dominated i (by j) keeps working less than j forever, then either the neighbors of i are very hard-working or the neighbors of j other than i are asymptotically lazy.

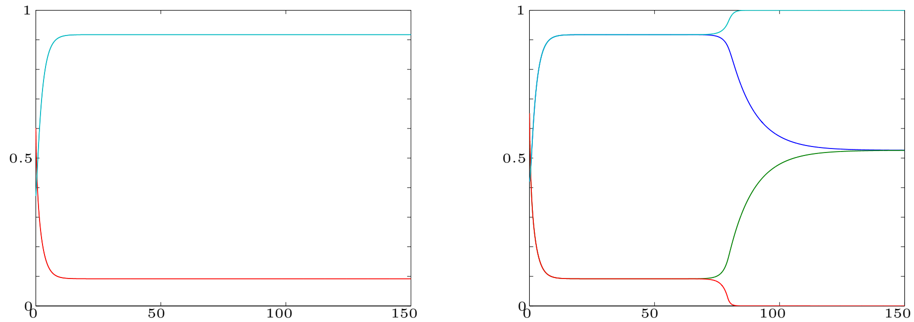
Lemma 5.3: **(Implications of social dominance):** Consider an arbitrary graph $\mathcal{G}$ and the associated dynamics (5), and suppose that $i \in \{1, \ldots, n\}$ is dominated by j.

a. If $x_i(T) \geq x_j(T)$ for some time $T \geq 0$, then $x_i(t) \geq x_j(t)$ for all $t \geq T$.
b. If for all $t \geq 0$, we have $x_i(t) < x_j(t)$, then we have

$$\liminf_{t \to \infty} \Big(\sum_{\ell \in \mathcal{N}_j \setminus \mathcal{N}_i} x_\ell(t) \Big) \Big(1 - \delta \sum_{\ell \in \mathcal{N}_i} x_\ell(t)\Big) \leq 0.$$

VI. The Stability of Star Configuration

In this section, we show the stability of the star configuration. But before doing that let us consider a generalization of the dynamics in the two-dimensional case. For this, consider



(a) (b)

Fig. 4. (a) and (b) correspond to the trajectories of (5) for a path graph with 4 agents with the initial conditions of x_0 and $x_0 + 0.05\mathbf{1}_4$.

the following dynamics:

$$\begin{aligned}\dot{x}_1(t) &= \max\{1 - \delta_1 x_2(t), 0\} - x_1(t)\\ \dot{x}_2(t) &= \max\{1 - \delta_2 x_1(t), 0\} - x_2(t)\end{aligned} \tag{7}$$

where $\delta_1, \delta_2 \geq 0$ are constants. Note that in this case, unlike the case of the dynamics (5) in the two-dimensional case, the δ parameter is agent dependent (similar to [11], we term such an interconnection *asymmetric*). Also, note that here δ_is, can be arbitrarily large. Fortunately, dynamics (7) is a stable dynamics for any choice of parameters $\delta_1, \delta_2 \geq 0$.

Lemma 6.1: **(Convergence for asymmetric interactions of two agents):** The dynamics (7) is convergent for any initial condition $x(0) \in \mathbb{R}^2$.

Now consider the star configuration and the dynamics (5) on this configuration, i.e. the graph $\mathcal{G} = (\{1, \ldots, n\} \cup \{*\}, E)$ where $E = \{\{*, i\} \mid i \in \{1, \ldots, n\}\}$.

Theorem 6.2: **(Convergence for star graphs with asymmetric interactions):** The dynamics (5) is stable for an arbitrary star.

Proof: Let $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$. Then, the dynamics governing the $*$ vertex would be:

$$\dot{x}_* = \max\{1 - \delta \sum_{i=1}^{n} x_i, 0\} - x_* = \max\{1 - n\delta\bar{x}, 0\} - x_*. \tag{8}$$

Also, note that for any leaf $i \in \{1, \ldots, n\}$, we have $\dot{x}_i = 1 - \delta x_* - x_i$, and hence the dynamics governing $\bar{x}$ would be:

$$\dot{\bar{x}} = 1 - \delta x_* - \bar{x}. \tag{9}$$

Thus (8) and (9) define a flow in $\mathbb{R}^2$, which is a special case of the dynamics (7), and hence by Lemma 6.1 $(x_*, \bar{x})$ is convergent. On the other hand, note that by Lemma 5.2 $\lim_{t\to\infty}(x_i(t) - x_j(t)) = 0$ for all $i, j \in \{1, \ldots, n\}$ and since $\lim_{t\to\infty} \bar{x}(t)$ exists, the result follows. ■

VII. Simulations and Discussions

In this section, we demonstrate how the value of PoH changes with the coefficient of heterogeneity, and the size and sparsity of the network. In all cases, we have used the best-response dynamical system, randomly initialized, to compute a random Nash equilibrium of the corresponding myopic strategic games. As was demonstrated before, for large values of δ the Nash equilibrium of these games is not unique and thus the selection of Nash equilibrium is dependent on the initial conditions.

In Figures 5(a) (b), (d), and (f) we have depicted PoH versus δ, which we refer to as the PoH *diagram*. (a) and (b) demonstrate that for path graphs with relatively large coefficients of heterogeneity, the solution of the myopic strategic game still matches the optimal one. This is in particular interesting when one compares the values of the solutions to the optimization problem to the case with no interconnections. For example, consider the path graph with 20 agents (Figure 5(b)), for $\delta = 0.2$, where PoH $= 1$. For this case $f^* = 5.3015$, which comparing to the case with no interconnections, where this value is 20, is a large improvement.

Another interesting feature (see Figures 5(c-f)), is that when the network is sparse, the approximation with myopic strategic games performs better; this is intuitive, as for low sparsity a centralized designer has more flexibility in finding a better solution. Investigating these properties analytically is a current direction of our research.

VIII. Conclusions and Future Work

We have studied the tradeoffs between optimality and heterogeneity for a class of optimization problems over networks, using a class of myopic strategic games which approximate their solutions. We have studied the stability properties of the corresponding best-response dynamical systems and provided new insights in them using classical stability results for piecewise linear systems and the novel notions of social equivalence and social dominance. We have fully characterized the stability properties of these systems for the case of star graphs with asymmetric interactions.

Many avenues for future research appear to be open, including the design of other classes of strategic games, not

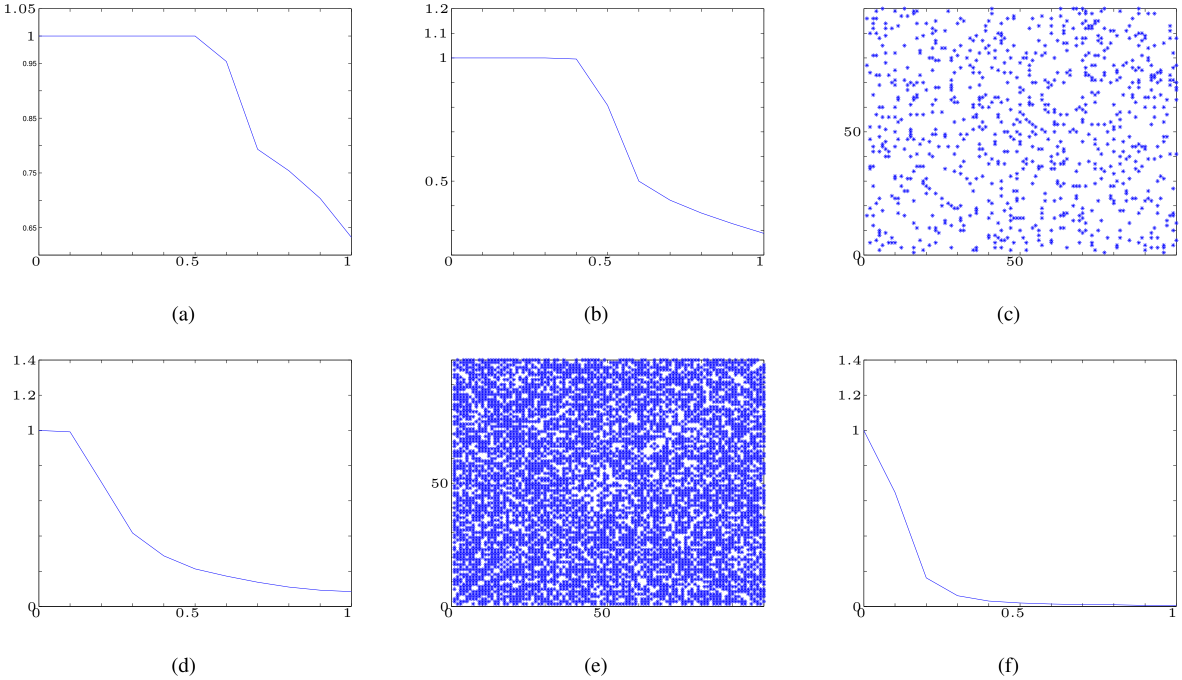


Fig. 5. (a) and (b) show PoH diagrams for path graphs with 4 and 20 agents, respectively. (c) and (e), respectively, show sparse (8.3 neighbors on the average) and dense (56 neighbors on the average) adjacency matrices with 100 agents; (d) and (f), respectively, show the PoH diagrams for (c) and (d).

completely myopic, for better approximations of distributed optimization problems, the study of asymmetric interactions for larger classes of games, characterizing analytic bounds on the price of heterogeneity for different classes of networks, studying the scenarios with time-varying networks, and investigating other applications of myopic approximation.

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