Control over Lossy Networks: A Dynamic Game Approach

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Abstract— This paper considers a minimax control (H^∞ control) problem for linear time-invariant (LTI) systems where the communication loop is subject to a TCP-like packet drop network. The problem is formulated within the zero-sum dynamic game framework. The packet drop network is governed by two independent Bernoulli processes that model control and measurement packet losses. Under this constraint, we obtain a dynamic output feedback minimax controller. For the infinite-horizon case, we provide necessary and sufficient conditions in terms of the packet loss rates and the H^∞ disturbance attenuation parameter under which the minimax controller exists and is able to stabilize the closed-loop system in the mean-square sense. In particular, we show that unlike the corresponding LQG case, these conditions are coupled and therefore cannot be determined independently.

I. Introduction

The problem of control over unreliable communication channels has been an active research area in networked control systems since connections between sensors and controllers, and controllers and actuators are more commonly made over wireless media. This convenience of remote operation is offset by possible loss of performance due to link failures and packet drops [1].

Since packet losses occur in a random manner, especially in remote and distributed control systems, it is natural to study conditions on the control and/or measurement loss rates that a control system can tolerate and still maintain a reliable operation. Although numerous models have been proposed to model packet drops in control systems, the Bernoulli-type binary switch packet drop model has been studied exclusively in recent years because it captures generality and tractability [2]–[17].

In the linear-quadratic-Gaussian (LQG) framework, reference [2] is regarded as one of the first attempts to find a solution to this problem. This paper categorized types of lossy links based on whether the packet reception is acknowledged (*TCP-like* protocol) or not (*UDP-like* protocol), and showed that with unreliable perfect state measurements and the TCP-like protocol, the optimal controller is linear in the available measurement, and the separation principle holds. The paper also obtained the solution for the UDP-case, in which case there is no separation of estimation and control. References [3]–[7] extended these results to the general LQG controller and showed that the separation principle still holds for the TCP-case. These papers also provided the minimum

levels of probability of losses for control and measurement packets, above which the LQG controller which accounts for failures is able to stabilize the system, and showed that these conditions are not coupled.

In the H^∞ control case, the authors in [12] and [13] captured packet drops via the Markov jump linear system (MJLS) theory. The controllers obtained there are restricted to be time-invariant and dependent on the current Markov state; hence they are suboptimal and cannot accommodate the TCP-like information structure [2]. The performance of the H^∞ controller for continuous time systems with packet drops was studied in [14]. However, an upper bound was imposed on the number of packet drops, which makes the network not a general lossy one. In [15] and [16], a stationary H^∞ controller was obtained in terms of a set of linear matrix inequalities (LMIs). In [18], Markov state dependent controller was obtained for periodic jump systems by making use of the results in [12]. One of the most recent works on H^∞ control over a packet drop-like network is [17], but there communication constraint is more on packet delays rather than packet drops. It seems like, to date, the problem of discrete-time minimax control under a TCP-like information structure has not yet been addressed.

Accordingly, in this paper, we study a minimax control problem for linear time-invariant (LTI) systems with TCP-like information on packet drops. In [19] and [20], we obtained respectively the state feedback minimax controller and the stochastic minimax estimator for the TCP-case, which can also be seen as generalizations of [2] and [21], respectively. We build on those results in this paper to obtain an output feedback minimax controller under TCP-like lossy connections, which constitutes an extension to the results of [3]. By applying the certainty equivalence principle developed in [22] and [23], for worst-case design problems, we show again that the separation principle does not hold in general due to the coupled spectral radius condition which is a function of the H^∞ disturbance attenuation parameter as well as the failure probabilities. For the infinite-horizon case, we provide necessary and sufficient conditions in terms of the packet drop rates and the H^∞ disturbance attenuation parameter under which the minimax controller exists and is able to stabilize the closed-loop system. In particular, we show that unlike the LQG case, these conditions are coupled and therefore cannot be determined independently.

The paper is organized as follows. In Section II, we formulate the problem. In Section III, we obtain the minimax estimator and the controller, and synthesize them. In Section IV, we study the infinite-horizon problem. Simulation results are provided in Section V. Conclusions are drawn in Section

II. PROBLEM FORMULATION

Consider the discrete-time linear dynamical system

$$x_{k+1} = Ax_k + \alpha_k Bu_k + Dw_k, \ k = 0, 1, 2, ... \quad (1a)$$

$$y_k = \beta_k Cx_k + Ev_k, \quad (1b)$$

where $x_k \in \mathbb{R}^n$ is the state; $u_k \in \mathbb{R}^m$ is the control (actuator); $w_k \in \mathbb{R}^p$ and $v_k \in \mathbb{R}^l$ are the disturbance input and the measurement noise, respectively, and are assumed to be arbitrary signals in ℓ_2; $y_k \in \mathbb{R}^l$ is the output; A, B, C, D, and E are time invariant matrices with appropriate dimensions; and k is the time index. We also assume that E is nonsingular, and define $V := EE^T$.

The stochastic processes of $\{\alpha_k\}$ and $\{\beta_k\}$ in (1) model a packet drop network from the controller to the actuator and from the sensor to the controller, respectively. We assume that they are independent binary switches and therefore are i.i.d. processes with distributions of $\mathbb{P}(\alpha_k = 1) = \alpha$ and $\mathbb{P}(\beta_k = 1) = \beta$, respectively. We let $\bar{\alpha} := \alpha(1 - \alpha)$.

We define the information that is available to the controller:

$$\begin{cases} \mathcal{I}_0 &:= \{y_0, \ \beta_0\} \\ \mathcal{I}_k &:= \{y_{0:k}, \ u_{0:k-1}, \ \alpha_{0:k-1}, \ \beta_{0:k}\}, \ k \geq 1, \end{cases} \quad (2)$$

where $y_{0:k} := (y_0, ..., y_k)$ and the same notation applies to $u_{0:k-1}$, $\alpha_{0:k-1}$, and $\beta_{0:k}$. Such an information structure is known as the *TCP-like* information structure due to full information on previous control link conditions. If the information structure (2) does not have $\alpha_{0:k-1}$, it is called *UDP-like* [2], [3]. Furthermore, if α_k is included in (2), we can design a controller by using the MJLS approach in [24].

Let $\mathcal{U}$ and $\mathcal{W}$ be the appropriate spaces of control and disturbance policies, respectively. The control and disturbance policies, $\mu \in \mathcal{U}$ and $\nu \in \mathcal{W}$, that consist of a sequence of functions, are defined by $\mu = \{\mu_0, ..., \mu_{N-1}\}$ and $\nu = \{\nu_0, ..., \nu_{N-1}\}$, respectively, where μ_k and ν_k are functions which map the information set $\mathcal{I}_k$ in (2) into the control and disturbance spaces of $\mathbb{R}^m$ and $\mathbb{R}^p$, respectively, namely, $u_k = \mu_k(\mathcal{I}_k)$ and $w_k = \nu_k(\mathcal{I}_k)$ for all k. Note that in the spirit of the worst-case approach, the disturbance is assumed to know everything the controller does. We also define $\omega := (x_0, \nu, \{v_k\}) \in \Omega := \mathbb{R}^n \times \mathcal{W} \times \mathcal{V}$ where $\mathcal{V}$ is the appropriate space for $\{v_k\}$.

Let $|\cdot|_S$ denote an appropriate weighted Euclidean norm or seminorm, weighted by the symmetric matrix S (with $S > 0$ or $S \geq 0$). Our main objective in this paper is to find a controller that minimizes the following cost function:

$$\ll \mathcal{T}_\mu^N \gg := \sup_{\omega \in \Omega} \frac{(J^N(\mu, \nu))^{1/2}}{(\Psi^N)^{1/2}}, \quad (3)$$

where

$$\Psi^N = \mathbb{E}\left\{|x_0 - \tilde{x}_0|_{Q_0}^2 + \sum_{k=0}^{N-1} |w_k|^2 + |v_k|^2\right\}$$

$$J^N(\mu, \nu) = \mathbb{E}\left\{|x_N|_{Q_N}^2 + \sum_{k=0}^{N-1} |x_k|_Q^2 + \alpha_k |u_k|_R^2\right\},$$

where $\tilde{x}_0$ is a known bias term which stands for the initial estimate of x_0; $Q, Q_N \geq 0$; and $R, Q_0 > 0$. Without the stochastic parameters, such a problem formulation is known as the H^∞ control problem [22].

By invoking the formulation of the soft-constrained game in [22], the cost function of the zero-sum dynamic game that is parameterized by the disturbance attenuation parmeter γ is given by

$$J_\gamma^N(\mu, \nu) \quad (4)$$

$$= \mathbb{E}\left\{|x_N|_{Q_N}^2 - \gamma^2 |x_0 - \tilde{x}_0|_{Q_0}^2 + \sum_{k=0}^{N-1} |x_k|_Q^2 \right.$$

$$\left. + \alpha_k |u_k|_R^2 - \gamma^2 |w_k|^2 - \gamma^2 |y_k - \beta_k Cx_k|_{V^{-1}}^2\right\},$$

where the measurement equation (1b) is used with $v_k = E^{-1}(y_k - \beta_k Cx_k)$. Note that the control and intermittent observations are included in the cost function (4) with stochastic variables $\{\alpha_k\}$ and $\{\beta_k\}$ as multiplying factors.

This completes the formulation of the output feedback minimax control problem under packet drops. Our goal is to obtain a controller for the system (1) under the information structure (2) such that the cost function is minimized while $(w_{0:N-1}, x_0)$ maximizes the same cost function. In other words, we need to find a saddle-point $(\mu_\gamma^*, (\nu_\gamma^*, x_0^*)) \in \mathcal{U} \times (\mathcal{W} \times \mathbb{R}^n)$ for the zero-sum game (4) parameterized by γ. If this exists, as shown in [22], the one corresponding to the smallest value of γ is an H^∞ controller that minimizes the original cost function of (3).

Along the way, since the controller does not have access to the actual state information, we also need to establish a minimax estimator policy under the information structure (2) that provides the worst-case information on the state. Finally, we have to characterize conditions on the smallest value of γ that determines the minimum value of the original disturbance attenuation problem of (3), and the smallest values of α and β that guarantee stability of the closed-loop system.

Remark 1: The zero-sum game formulated in this section is considered to have an imperfect information structure since it does not have access to the actual state variables [22]. $\square$

Remark 2: In this paper, the zero-input strategy is used, that is, the actuator does not do anything when there are control packet losses. It has been shown in [25] that for the disturbance free case, holding the one-step previous control packet in order to compensate for the current packet loss does not necessarily lead to a better performance. $\square$

III. FINITE-HORIZON MINIMAX CONTROL

We solve the optimization problem (4) by applying the certainty equivalence principle developed for worst-case designs in [22] and [23].

A. Minimax Estimator Design

Lemma 1: Consider the zero-sum dynamic game in (4) subject to (1) and (2) with $k \in [0, \ N-1]$, $\alpha \in [0, 1]$, $\beta \in [0, 1]$, and a fixed $\gamma > 0$.

(i) A stochastic minimax state estimator (SMSE) exists if

$$\rho(\Sigma_k Q) < \gamma^2 \text{ almost surely (a.s.) } \forall k \in [0, N-1], \quad (5)$$

where $\rho(\cdot)$ is spectral radius, and Σ_k is generated by the following stochastic Riccati equation (SRE) with $\Sigma_0 = Q_0^{-1}$:

$$\Sigma_{k+1} = A \left(\Sigma_k^{-1} - \gamma^{-2}Q + \beta_k C^T V^{-1} C \right)^{-1} A^T + DD^T; \tag{6}$$

(ii) The SMSE with $\bar{x}_0 = \tilde{x}_0$ is

$$\bar{x}_{k+1} = A\bar{x}_k + \alpha_k B u_k \tag{7}$$
$$+ A\Pi_k \left(\gamma^{-2}Q\bar{x}_k + \beta_k C^T V^{-1}(y_k - C\bar{x}_k) \right),$$

where the estimator gain Π_k can be written as

$$\Pi_k = (\Sigma_k^{-1} - \gamma^{-2}Q + \beta_k C^T V^{-1} C)^{-1}. \tag{8}$$

Proof. Since we seek a causal linear estimator, forward dynamic programming can be applied by introducing the quadratic *cost-to-come (worst past cost)* function $W_k(x_k) = \mathbb{E}\left\{-|x_k - \bar{x}_k|^2_{\Xi_k} + l_k|\mathcal{I}_k\right\}$ where $\Xi_k > 0$, $\Xi_0 = \gamma^2 Q_0$, and $l_0 = 0$ [20], [22], [23]. Then the cost from the initial state to stage $k+1$ can be written as

$$\mathbb{E}\left\{ -|x_{k+1} - \bar{x}_{k+1}|^2_{\Xi_{k+1}} + l_{k+1}|\mathcal{I}_{k+1} \right\}$$
$$= \max_{(w_k, x_k)} \mathbb{E}\Big\{ |x_k|^2_Q + \alpha_k|u_k|^2_R - \gamma^2|w_k|^2 \tag{9}$$
$$- \gamma^2|y_k - \beta_k C x_k|^2_{V^{-1}} - |x_k - \bar{x}_k|^2_{\Xi_k} + l_k|\mathcal{I}_{k+1} \Big\}.$$

For convenience, we consider the negative of the LHS of (9) and therefore a corresponding minimization problem on the RHS. For the existence of a unique minimizer, by Lemma 6.1 in [22], it is necessary to have $\Xi_k - Q > 0$ for all k. Then the minimum cost at stage $k+1$ is

$$\mathbb{E}\left\{ |x_{k+1} - \bar{x}_{k+1}|^2_{\Xi_{k+1}} - l_{k+1}|\mathcal{I}_{k+1} \right\}$$
$$= \mathbb{E}\Big\{ |x_{k+1} - \alpha_k B u_k - A(\Xi_k + \beta_k \gamma^2 C^T V^{-1} C - Q)^{-1}$$
$$\times (\Xi_k \bar{x}_k + \beta_k \gamma^2 C^T V^{-1} y_k)|^2_{\bar{\Xi}_k^{-1}}$$
$$+ |-\Xi_k \bar{x}_k - \beta_k \gamma^2 C^T V^{-1} y_k|^2_{(\Xi_k + \beta_k \gamma^2 C^T V^{-1} C - Q)^{-1}}$$
$$+ \alpha_k|u_k|^2_R + |\bar{x}_k|^2_{\Xi_k} + \gamma^2|y_k|^2_{V^{-1}} - l_k|\mathcal{I}_{k+1} \Big\},$$

where $\bar{\Xi}_k := A(\Xi_k + \beta_k \gamma^2 C^T V^{-1} C - Q)^{-1} A^T + \gamma^{-2} DD^T$ and we made use of the fact that $\beta_k^2 = \beta_k$. Since this is true for all k, the dynamic equation for $\bar{x}_k$ can be written as

$$\bar{x}_{k+1} = A(\Xi_k + \beta_k \gamma^2 C^T V^{-1} C - Q)^{-1} \tag{10a}$$
$$\times (\Xi_k \bar{x}_k + \beta_k \gamma^2 C^T V^{-1} y_k) + \alpha_k B u_k$$

where

$$\Xi_{k+1} \tag{10b}$$
$$= (A(\Xi_k + \beta_k \gamma^2 C^T V^{-1} C - Q)^{-1} A^T + \gamma^{-2} DD^T)^{-1}.$$

Now, with $\Xi_0 = \gamma^2 Q_0$, let $\Sigma_k := \gamma^2 \Xi_k^{-1}$. Then the necessary and sufficient condition becomes (5). Also (10a) and (10b) can be rewritten as (7) and (6), respectively. This completes the proof. □

Remark 3: The SMSE in Lemma 1 is time varying and random, since it is a function of β_k and α_k. □

Remark 4: Note that if (5) does not hold, then the solution of the maximization problem (9) does not have a unique solution. In fact, there exists a sequence of maximizer strategies which makes the maximization value infinite. Therefore, the condition (5) is crucial. □

To characterize the existence condition in Lemma 1(i), it is useful to find a value $\eta \in [0,1]$ such that with $\tilde{\Sigma}_k$ generated as below,

$$\tilde{\Sigma}_0 = Q_0^{-1}$$
$$\tilde{\Sigma}_{k+1} = A(\tilde{\Sigma}_k^{-1} - \gamma^{-2}Q + \eta C^T V^{-1} C)A^T + DD^T,$$

and $\Sigma_k \leq \tilde{\Sigma}_k$ a.s. for all k. Clearly, if $\rho(\tilde{\Sigma}_k Q) < \gamma^2$ for a given γ, then condition (5) holds. It should be noted that the value of η depends on β since the SRE (6) is a function of $\{\beta_k\}$. In general, $\eta \leq \beta$. Furthermore, such an upper bound always exists since $\Sigma_k \leq \tilde{\Sigma}_k$ almost surely for all k with $\eta = 0$.

B. Minimax Controller Design

Lemma 2: Consider the zero-sum dynamic game in (4) subject to (1) and (2) with $k \in [0, N-1]$, $\alpha \in [0,1]$, $\beta \in [0,1]$, and a fixed $\gamma > 0$.

(i) There exists a minimax controller if and only if (5) holds a.s. for all k, and

$$\rho(D^T Z_{k+1} D) < \gamma^2, \quad \forall k \in [0, N-1] \tag{11a}$$
$$\rho(\Sigma_k Z_k) < \gamma^2, \quad \text{a.s. } \forall k \in [0, N-1], \tag{11b}$$

where Z_k is generated by the Riccati equation defined in (ii);

(ii) The Riccati equation is $Z_N = Q_N$ and

$$Z_k = Q + P_{u_k}^T \left(\alpha R + \bar{\alpha} B^T Z_{k+1} B \right) P_{u_k} - \gamma^2 P_{w_k}^T P_{w_k}$$
$$+ H_k^T Z_{k+1} H_k, \tag{12}$$

where

$$H_k = A - \alpha B P_{u_k} + D P_{w_k} \tag{13a}$$
$$P_{u_k} = \left(R + K_{u_k}^1 Z_{k+1} B \right)^{-1} K_{u_k}^2 Z_{k+1} A \tag{13b}$$
$$K_{u_k}^1 = B^T \left(I + \alpha Z_{k+1} D M_k^{-1} D^T \right) \tag{13c}$$
$$K_{u_k}^2 = B^T \left(I + Z_{k+1} D M_k^{-1} D^T \right) \tag{13d}$$
$$P_{w_k} = \left(\gamma^2 I - K_{w_k} Z_{k+1} D \right)^{-1} K_{w_k} Z_{k+1} A \tag{13e}$$
$$K_{w_k} = D^T \left(I - \alpha Z_{k+1} B L_k^{-1} B^T \right) \tag{13f}$$
$$L_k = \left(R + B^T Z_{k+1} B \right) \tag{13g}$$
$$M_k = \left(\gamma^2 I - D^T Z_{k+1} D \right); \tag{13h}$$

(iii) The minimax controller and the worst-case disturbance can be written as

$$u_k = \mu_k^*(\mathcal{I}_k) = -P_{u_k} \hat{x}_k \tag{14}$$
$$w_k = \nu_k^*(\mathcal{I}_k) = P_{w_k} \hat{x}_k, \tag{15}$$

where $\hat{x}_k$ is the worst-case estimated state that can be obtained by

$$\hat{x}_k = (I - \gamma^{-2} \Sigma_k Z_k)^{-1} \bar{x}_k, \tag{16}$$

where $\bar{x}_k$ is generated by the SMSE in Lemma 1.

Proof. For the state feedback case, we have shown in [19] that (11a) guarantees uniqueness of the minimax controller under control packet drops, which can be obtained by using backward induction with the cost-to-go $V_k(x_k) = \mathbb{E}\{x_k^T Z_k x_k | \mathcal{I}_k\}$. Now, by applying the certainty equivalence principle introduced in [22] and [23], under (11b) and the information structure (2), the worst-case estimated state (16) can be obtained as

$$\hat{x}_k = \arg \max_{x_k} \mathbb{E}\{V_k(x_k) + W_k(x_k) | \mathcal{I}_k\}$$
$$= \arg \max_{x_k} \mathbb{E}\{|x_k|_{Z_k} - \gamma^2 |x_k - \bar{x}_k|_{\Sigma_k^{-1}}^2 + l_k | \mathcal{I}_k\},$$

where $W_k(x_k)$ is from Lemma 1. This completes the proof.$\square$

Remark 5: There are three conditions on γ for the existence of the minimax controller in Lemma 2: (5) is for the existence of the SMSE that performs state estimation under measurement losses; (11a) is related to the state feedback minimax controller under control packet drops; and (11b) is the spectral radius condition that ensures the existence of the worst-case estimated state. $\square$

From Lemmas 1 and 2, the minimax controller and the SMSE cannot be designed independently even for the TCP-case since the spectral radius condition (11b) is coupled with two Riccati equations (6) and (12). Note that this is a direct consequence of the worst-case design approach in H^∞ control discussed in [22]. For the LQG case in [2] and [3], on the other hand, the separation principle holds between control and estimation.

The closed-loop system with the minimax controller in Lemma 2 and the SMSE in Lemma 1 cannot be seen as a Markov jump linear system (MJLS) discussed in [24] since the information structure (2) does not contain the current mode of the control loss information, α_k, and the estimator gain (8) is time varying due to the SRE. Therefore, we cannot apply the results in [24] to this closed-loop system.

Remark 6: Due to the SMSE, the worst-case estimated state $\hat{x}_k$ is also random and is a function of $\{\beta_k\}$. $\square$

We now state the main theorem for this section.

Theorem 1: Given the link conditions $\alpha \in [0,1]$ and $\beta \in [0,1]$, suppose there exists a finite value of $\gamma^F > 0$, and the smallest such value satisfying (5), (11a), and (11b) for all $k \in [0, N-1]$. Let $\gamma > \gamma^F$. Then a minimax controller achieving the disturbance attenuation level γ, μ_γ^*, is given by (14), that is $\ll \mathcal{T}_{\mu_\gamma^*}^N \gg \leq \gamma$.

Proof. Let $\mu_\gamma^* = (\mu_0^*, ..., \mu_{N-1}^*)$, $\nu_\gamma^* = (\nu_0^*, ..., \nu_{N-1}^*)$. Substitution of the minimax strategy μ_γ^* into the system (1a) and the estimator (7) leads to the following closed-loop system, in terms of $z_k = [x_k^T, \bar{x}_k^T]^T$:

$$z_{k+1} = \bar{A}_k z_k + \bar{D}_k w_k,$$

where $\bar{A}_k$ and $\bar{D}_k$ are matrices that are associated with the closed-loop state z_k. Also the cost function (4) can be written

as

$$J_\gamma^N(\mu^*, \nu)$$
$$= \mathbb{E}\{z_N^T \begin{pmatrix} Q_N & 0 \\ 0 & 0 \end{pmatrix} z_N - \gamma^2 z_0^T \begin{pmatrix} Q_0 & 0 \\ 0 & 0 \end{pmatrix} z_0$$
$$+ \sum_{k=0}^{N-1} z_k^T \begin{pmatrix} Q & 0 \\ 0 & \alpha_k P_{u_k}^T R P_{u_k} \end{pmatrix} z_k - \gamma^2 |w_k|^2\}.$$

Now, assume that $x_0 = \tilde{x}_0 = 0$. Since $\gamma > \gamma^F$, we have $J_\gamma^N(\mu_\gamma^*, \nu) \leq J_\gamma^N(\mu_\gamma^*, \nu_\gamma^*) \leq J_\gamma^N(\mu, \nu_\gamma^*)$ for all $(\mu, \nu) \in \mathcal{U} \times \mathcal{W}$ [22]. Therefore $J_\gamma^N(\mu_\gamma^*, \nu_\gamma^*) = 0$, and the result follows.$\square$

IV. INFINITE-HORIZON MINIMAX CONTROL

In this section, we discuss the infinite-horizon minimax control problem over the packet drop network when $k, N \to \infty$ in the cost function (4) without the terminal constraint and under the additional assumptions that (A, B) and (A, D) are controllable, and (A, C) and $(A, Q^{1/2})$ are observable. We define $\bar{Z} := \bar{Z}(\gamma, \alpha)$ where the value of the matrix is dependent on γ and α.

- The algebraic Riccati equation of (12) can be written as

$$\bar{Z} = \bar{H}^T \bar{Z} \bar{H} + \bar{P}_u^T \left(\alpha R + \bar{\alpha} B^T \bar{Z} B\right) \bar{P}_u \quad (17)$$
$$- \gamma^2 \bar{P}_w^T \bar{P}_w + Q,$$

  where $\bar{P}_u$, $\bar{P}_w$, and $\bar{H}$ are obtained by replacing Z_k in (13) with $\bar{Z}$, respectively.
- The infinite horizon minimax controller is

$$u_k = -\bar{P}_u \hat{x}_k. \quad (18)$$

- The infinite-horizon version of the worst-case estimated state is

$$\hat{x}_k = (I - \gamma^{-2} \Sigma_k \bar{Z})^{-1} \bar{x}_k, \quad (19)$$

  where $\bar{x}_k$ is generated by the SMSE in (7).
- The infinite-horizon version of the existence condition (11a) can be written as

$$\rho(D^T \bar{Z} D) < \gamma^2. \quad (20)$$

Proposition 1.
1) Assume that (11a) holds for all k. Define the sets

$$\Gamma_1(\alpha) := \{\gamma > 0 \ : \ \bar{Z}^+ > 0 \text{ solves (17)}$$
$$\text{and satisfies (20).}\}$$
$$\Lambda_1(\gamma) := \{\alpha \in [0,1] \ : \ \bar{Z}^+ > 0 \text{ solves (17)}$$
$$\text{and satisfies (20).}\}$$

Define $\gamma_1^*(\alpha) \triangleq \inf\{\gamma : \gamma \in \Gamma_1(\alpha)\}$ and $\alpha_c(\gamma) \triangleq \inf\{\alpha : \alpha \in \Lambda_1(\gamma)\}$. Suppose that one of the following two conditions holds:
(a) for any α, $\Gamma_1(\alpha)$ is nonempty and $\gamma > \gamma_1^*(\alpha)$
(b) $\gamma > \gamma_1^*(1)$ and $\alpha > \alpha_c(\gamma)$.
Then $\{\tilde{Z}_k\} \to \bar{Z}^+$ as $k \to \infty$ where $\bar{Z}^+$ is the unique fixed point of (17) that satisfies (20).

2) Suppose $\Gamma_1(\alpha)$ is not empty with α_1 and α_2. If $\alpha_1 > \alpha_2$, then $\gamma_1^*(\alpha_1) < \gamma_1^*(\alpha_2)$. Also $\gamma_1^*(\alpha) \geq \gamma_1^*(1)$ for all α.

3) If $\gamma_1 > \gamma_2 > \gamma_1^*(1)$, then $\alpha_c(\gamma_1) < \alpha_c(\gamma_2)$. Also as $\gamma \to \infty$, $\alpha_c(\gamma) \to \alpha_R$ where α_R is the critical value of α in the LQG problem in [2], [3].

Proof. The proof is in [19]. $\qquad\square$

As can be seen, $\gamma_1^*(\alpha)$ and $\alpha_c(\gamma)$ are coupled with each other; therefore, they cannot be determined independently. This is the main difference between the linear-quadratic-regulator (LQR) in [2] where the critical value of the failure rate is only a function of the eigenvalues of A and rank of B. Now, for the estimation part, the SRE is stochastic as a function of $\{\beta_k\}$, which does not allow for any fixed points. The detailed properties of the SRE are discussed in [20].

Proposition 2: We define the following sets for (5):

$$\Gamma_2(\beta) := \{\gamma > 0 \ : \ \rho(\Sigma_k Q) < \gamma^2, \ \text{a.s.} \ \forall k\}$$
$$\Lambda_2(\gamma) := \{0 \leq \beta \leq 1 \ : \ \rho(\Sigma_k Q) < \gamma^2, \ \text{a.s.} \ \forall k\}.$$

Let $\gamma_s^*(\beta) := \inf\{\gamma > 0 : \ \gamma \in \Gamma_2(\beta)\}$ and $\beta_s(\gamma) := \inf\{0 \leq \beta \leq 1 : \ \beta \in \Lambda_2(\gamma)\}$. Now, suppose that one of the following two conditions holds:

(a) for any β, $\Gamma_2(\beta)$ is nonempty and $\gamma > \gamma_s^*(\beta)$

(b) $\gamma > \gamma_s^*(1)$ and $\beta > \beta_s(\gamma)$.

Then the condition in Lemma 1 holds almost surely for all k [20]. $\qquad\square$

Remark 7: It is easy to show that $\gamma_s^*(\beta)$ and $\beta_c(\gamma)$ have the same properties as in Proposition 1 2) and 3). Also as $\gamma \to \infty$, $\beta_s(\gamma) \to 0$. $\qquad\square$

Now, to obtain bounds on $\mathbb{E}\{\Sigma_k\}$, consider the following modified Riccati equations [20]:

$$\breve{\Sigma}_{k+1} = (1-\beta)A\breve{\Sigma}_k A^T + DD^T \qquad (21)$$
$$\bar{\Sigma}_{k+1} = (1-\beta)\Big(A\big(\bar{\Sigma}_k^{-1} - \gamma^{-2}Q\big)^{-1}A^T + DD^T \Big) \qquad (22)$$
$$+ \beta\Big(A\big(\bar{\Sigma}_k^{-1} - \gamma^{-2}Q + C^T V^{-1} C\big)^{-1}A^T + DD^T \Big),$$

where $\breve{\Sigma}_0 = \bar{\Sigma}_0 = Q_0^{-1}$. Let $\bar{\gamma}^*(\beta)$ and $\bar{\beta}_c(\gamma)$ be the smallest values of γ and β for which (22) converges. The detailed convergence conditions are in [20]. Let $\gamma > \gamma_2^*(\beta) := \max\{\gamma_s^*(\beta), \bar{\gamma}^*(\beta)\}$. and $\beta > \beta_c(\gamma) := \max\{\beta_s(\gamma), \bar{\beta}_c(\gamma)\}$. Then $\mathbb{E}\{\Sigma_k\}$ is bounded where (21) and (22) generate lower and upper bounds, respectively [20].

Finally, to express the infinite-horizon version of (11a), define

$$\Gamma_3(\alpha, \beta) := \{\gamma > 0 \ : \ \gamma > \gamma_1^*(\alpha), \ \gamma > \gamma_2^*(\beta),$$
$$\rho(\Sigma_k \bar{Z}^+) < \gamma^2, \ \text{a.s.} \ \forall k\}$$
$$\gamma_3^*(\alpha, \beta) := \inf\{\gamma > 0 : \ \gamma \in \Gamma_3(\alpha, \beta)\},$$

Note that for any $\gamma > \gamma_3^*(\alpha, \beta)$, $\alpha > \alpha_c(\gamma)$, and $\beta > \beta_c(\gamma)$, γ satisfies all existence conditions. This implies that under this condition, the worst-case estimated state (19) exists with $\bar{Z}^+$. Let $\gamma^* := \gamma_3^*(\alpha, \beta)$. We now state the main theorem for this section.

Theorem 2: Suppose that γ^* is finite, $\gamma > \gamma^*$, $\alpha > \alpha_c(\gamma)$, and $\beta > \beta_c(\gamma)$. The minimax controller achieving the disturbance attenuation level γ, μ_γ^*, can be written as (18) with the solution of (17) that satisfies (20), and the worst-case estimated state (19). In other words, $\ll \mathcal{T}_{\mu_\gamma^*}^\infty \gg \leq \gamma$. $\qquad\square$

Corollary 1: Suppose that we have the same conditions as in Theorem 2. If $\beta = 1$, then the closed-loop system with the SMSE and the worst-case disturbance is stable in the mean-square stable, that is, $\mathbb{E}\{|z_k|^2\} \to 0$ as $k \to \infty$, where $z_k = [x_k^T, \bar{x}_k^T]^T$.

Proof. Note that for the state feedback case, it was already shown that the minimax controller stabilizes the closed-loop system in the mean-square sense, i.e., $\mathbb{E}\{|x_k|^2\} \to 0$ as $k \to \infty$ [20]. Since we have the perfect measurement link, there is a stationary minimax estimator gain with $\gamma > \gamma^*$ and the estimation error, $x_k - \bar{x}_k$, is independent of u_k and α_k for all k, which implies its asymptotic stability by Theorem 6.4 in [22]. Then the result follows immediately. $\qquad\square$

Remark 8: In the deterministic case, as $\gamma \to \infty$, the closed-loop system converges to the LQG system [22]. We can see a similar result for the problem of this paper. That is, as $\gamma \to \infty$, the controller and estimator converge to the LQG system with link failures in [2] and [3]. Therefore, the separation principle holds when γ is sufficiently large. This also implies that $\alpha_c(\gamma)$ and $\beta_c(\gamma)$ also converge to the critical values in [2], [3], and therefore become independent of each other. $\square$

Remark 9: If one of the conditions in Theorem 2 fails, then the value of the zero-sum game is infinite; therefore, the minimax controller does not exist. $\qquad\square$

Remark 10: Due to the spectral radius condition, the optimum disturbance attenuation level is a function of α and β. In fact, $\gamma_3^*(1,1)$ is the best achievable level which is that of the system with a perfect communication link. Note that $\gamma_3^*(0,0)$ is for the open-loop case and if A is unstable, $\gamma_3^*(0,0)$ is not finite. $\qquad\square$

Remark 11: The boundedness of $\mathbb{E}\{\Sigma_k\}$ is necessarily required for the infinite-horizon minimax control case since it is a sufficient condition to have bounded average cost for the infinite-horizon LQG control in [3]. $\qquad\square$

V. SIMULATION RESULTS

In this section, simulation results are provided to illustrate the disturbance rejection performance. We use the pendubot system where the system matrices are given in [3]. Figure 1 shows the convergence region of (12). Note that the controller needs more reliable control communication link if a high disturbance attenuation performance is required. Figure 2 shows the disturbance attenuation performance of the minimax controller with sinusoidal disturbances with amplitude of 0.01. Notice that the minimax controller outperforms the LQG controller. Note also that as $\gamma \to \infty$, the performance of the H^∞ controller becomes identical to that of the LQG system with link failures.
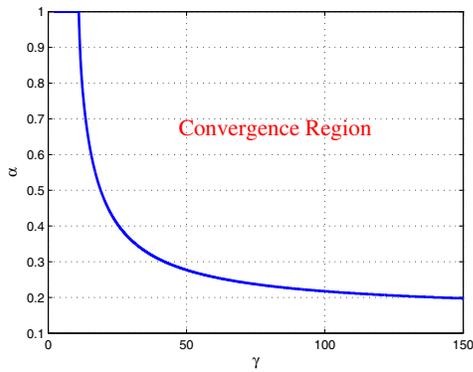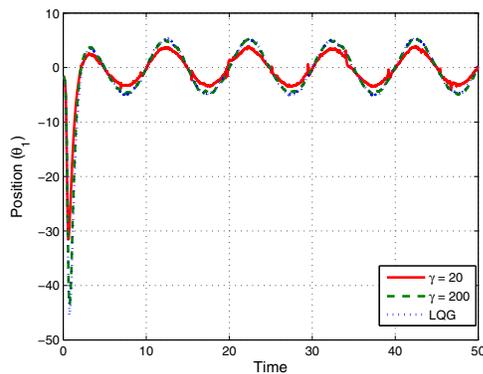
Fig. 1. Convergence region of (12).

Fig. 2. Disturbance rejection performance when $\alpha = 0.8$ and $\beta = 0.9$ under sinusoidal disturbances.

VI. Conclusions

In this paper, we have presented expressions for the minimax controller for LTI systems for a TCP-like packet drop network. We have formulated the problem within the zero-sum dynamic game framework. The packet drop network is assumed to be characterized by two independent Bernoulli processes that model control and measurement packet losses. For the infinite-horizon case, we have provided sufficient conditions under which the minimax controller exists and is able to stabilize the closed-loop system. In particular, we have proven that unlike the LQG case, these conditions are coupled with each other; hence they cannot be determined independently. Finally, we have shown that if, in the controller parameterized by the disturbance attenuation parameter γ, γ becomes arbitrarily large, then in the limit, each of the results obtained in this paper specialize to the corresponding results in the LQG case with link failures, treated in [2], [3].

References

[1] J. Hespanha, P. Naghshtabrizi, and Y. Xu, "A survey of recent results in networked control systems," *Proc. of the IEEE*, vol. 95, no. 1, pp. 138–162, Jan. 2007.

[2] O. C. Imer, S. Yüksel, and T. Başar, "Optimal control of LTI systems over unreliable communication links," *Automatica*, vol. 42, no. 9, pp. 1429–1439, 2006.

[3] L. Schenato, B. Sinopoli, M. Franceschetti, K. Poolla, and S. Sastry, "Foundations of control and estimation over lossy networks," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 163–187, Jan. 2007.

[4] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, and S. Sastry, "Optimal linear LQG control over lossy networks without packet acknowledgment," in *Proc. of the Decision and Control (CDC)*, Dec. 2006, pp. 392–397.

[5] E. Garone, B. Sinopoli, A. Goldsmith, and A. Casavola, "LQG control for MIMO systems over multiple erasure channels with perfect acknowledgment," *IEEE Transactions on Automatic Control*, vol. 57, no. 2, pp. 450–456, Feb. 2012.

[6] E. Garone, B. Sinopoli, and A. Casavola, "LQG control over lossy TCP-like networks with probabilistic packet acknowledgement," *International Journal of Systems, Control, and Communications*, vol. 2, no. 1/2/3, pp. 55–81, 2010.

[7] M. Trivellato and N. Benvenuto, "State control in networked control systems under packet drops and limited transmission bandwidth," *IEEE Transactions on Communications*, vol. 58, no. 2, pp. 611–622, 2010.

[8] N. Elia, "Remote stabilization over fading channels," *Systems and Control Letters*, vol. 54, no. 3, pp. 237–249, 2005.

[9] N. Elia and J. Eisenbeis, "Limitations of linear control over packet drop networks," *IEEE Transactions on Automatic Control*, vol. 56, no. 4, pp. 826–841, April 2011.

[10] E. I. Silva and S. A. Pulgar, "Control of LTI plants over erasure channels," *Automatica*, vol. 47, no. 8, pp. 1729–1736, 2011.

[11] V. Gupta, N. Martins, and J. Baras, "Optimal output feedback control using two remote sensors over erasure channels," *IEEE Transactions on Automatic Control*, vol. 54, no. 7, pp. 1463–1476, 2009.

[12] P. Seiler and R. Sengupta, "An H^∞ approach to networked control," *IEEE Transactions on Automatic Control*, vol. 50, no. 3, pp. 356–364, March 2005.

[13] J. Geromel, A. Goncalves, and A. Fioravanti, "Dynamic output feedback control of discrete-time markov jump linear systems through linera matrix inequalities," *SIAM Journal on Control and Optimization*, vol. 48, no. 2, pp. 573–593, 2009.

[14] H. Gao, "Network-based H^∞ output tracking control," *IEEE Transactions on Automatic Control*, vol. 53, pp. 655–667, 2008.

[15] Z. Wang, F. Yang, D. W. C. Ho, and X. Liu, "Robust H^∞ control for networked systems with random packet losses," *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, vol. 37, no. 4, pp. 916–924, 2007.

[16] M. Sahebsara, T. Chen, and S. L. Shah, "Optimal H_∞ filtering in networked control systems with multiple packet dropouts," *Systems and Control Letters*, vol. 57, no. 9, pp. 696–702, 2008.

[17] Y. Shoukry, J. Araujo, P. Tabuada, M. Srivastava, and K. H. Johansson, "Minimax control for cyber-physical systems under network packet scheduling attacks," in *Proceedings of the 2nd ACM international conference on High confidence networked systems*, New York, NY, USA, 2013, pp. 93–100.

[18] H. Ishii, "H^∞ control with limited communication and message losses," *Systems and Control Letters*, vol. 57, no. 4, pp. 322–331, 2008.

[19] J. Moon and T. Başar, "Control over TCP-like lossy networks: A dynamic game approach," in *Proc. of American Control Conference*, Washington, DC, June 2013, pp. 1581–1586.

[20] ——, "Estimation over lossy networks: A dynamic game approach," in *Proc. of the 52nd IEEE Conference on Decision and Control*, Florence, Italy, 2013, pp. 2412–2417.

[21] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, M. Jordan, and S. Sastry, "Kalman filtering with intermittent observations," *IEEE Trans. on Autom. Control*, vol. 49, no. 9, pp. 1453–1464, Sept. 2004.

[22] T. Başar and P. Bernhard, *H^∞ Optimal Control and Related Minimax Design Problems*, 2nd ed. Boston, MA: Birkhäuser, 1995.

[23] G. Didinsky, "Design of minimax controllers for nonlinear systems using cost-to-come methods," Ph.D. dissertation, University of Illinois, Urbana-Champaign, IL, 1994.

[24] O. Costa, M. Fragoso, and R. Marques, *Discrete-Time Markov Jump Linear Systems*. Springer-Verlag, 2005.

[25] L. Schenato, "To zero or to hold control inputs with lossy links?" *IEEE Transactions on Automatic Control*, vol. 54, no. 5, pp. 1093–1099, May 2009.