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# A Three-Stage Colonel Blotto Game with Applications to Cyberphysical Security

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*Abstract*— **We consider a three-step three-player complete information Colonel Blotto game in this paper, in which the first two players fight against a common adversary. Each player is endowed with a certain amount of resources at the beginning of the game, and the number of battlefields on which a player and the adversary fights is specified. The first two players are allowed to form a coalition if it improves their payoffs. In the first stage, the first two players may add battlefields and incur costs. In the second stage, the first two players may transfer resources among each other. The adversary observes this transfer, and decides on the allocation of its resources to the two battles with the players. At the third step, the adversary and the other two players fight on the updated number of battlefields and receive payoffs. We characterize the subgame-perfect Nash equilibrium (SPNE) of the game in various parameter regions. In particular, we show that there are certain parameter regions in which if the players act according to the SPNE strategies, then (i) one of the first two players add battlefields and transfer resources to the other player (a coalition is formed), (ii) there is no addition of battlefields and no transfer of resources (no coalition is formed). We discuss the implications of the results on resource allocation for securing cyberphysical systems.**

## I. Introduction

The Colonel Blotto game models a scenario in which two players having certain resource levels fight over a finite number of battlefields. The players decide on the amount of resource they deploy on each battlefield in order to maximize their payoffs.

The case of players with symmetric resources and three battlefields was solved by Borel and Ville in [1] in 1938. Gross and Wagner [2] generalized the result of symmetric resources to an arbitrary number of battlefields; they also derived a Nash equilibrium of the game when there are two battlefields and asymmetric resources among the players. Until 2006, the two-player asymmetric-resource Colonel Blotto game with more than two battlefields remained unsolved. Roberson [3] completely characterized the equilibria for this case in 2006.

Following the fundamental work of [3] and [4], many interesting theoretical extensions followed, and numerous papers targeting specific domains of application have been published; see for example [5], [6], [7], [8], [9] among many others. There have also been experimental studies for various applications in [10], [11], [12] among many others. One of the first experimental studies that looked into network infrastructures is [13].

Two interesting variations of the Colonel Blotto game are considered in [6] and [7]. Kovenock et. al. in [6] add an extra stage to the game, during which both players could add extra battlefields to the initial number of battlefields by paying a cost. The authors investigate the parameters of the game for which additional battlefields are added in equilibrium. In [7], Kovenock and Roberson study a two-stage game in which a common adversary is engaged into two Colonel Blotto games with two separate, and seemingly unrelated players (possible allies). At the first stage of the game, unilateral transfers between the players (except the adversary) are allowed. The authors demonstrate that a positive transfer occurs (or in other words, a coalition is formed) for a range of parameter configurations. However, the authors do not compute the Nash equilibrium for the two-stage game.

In this paper, motivated by the models of [6] and [7], we consider a three-stage three-player game in which the first two players could do both: (i) add battlefields at the first stage and (ii) transfer resources among each other (if it improves the expected payoffs to both players) at the second stage. The third player is the adversary, who observes the updated number of battlefields and the amount of resource transferred among the players, and then allocates its resource to the battles with two players at the second stage. In the third stage, each of the first two players fights against the adversary on the updated number of battlefields. Our main contribution is that we compute the subgame-perfect Nash equilibrium (SPNE) of the three-stage game in certain regions of the parameter space.

Such settings allow us to gain insights into resource allocation decisions of networked parties, facing a common adversary with known resources. For example, consider two resource-constrained networks of servers and a resource-constrained hacker who wants to access the servers. The options available to the network operators are to invest in additional servers (equivalent to adding battlefields in our model) and share resources among each-other for securing the servers. The hacker observes the security level of each

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network and decides on the amount of resource it deploys to hack each of the servers of the two networks. The main questions we would like to answer in this setting are (i) When is it better for network operators to add additional servers? (ii) When should the network operators share their resources to make their network more secure and improve their expected payoffs? (iii) What is the worst-case (optimizing) behavior of the hacker in such a scenario?

### A. Outline of the paper

We formulate the three-stage Colonel Blotto game in Section II. In Section III, we recall the results about Nash equilibrium and expected payoffs to the players of the static Colonel Blotto game from [3]. Thereafter, we compute the SPNE of the three-stage game in Section IV in certain parameter regions of the game. We conclude our discussion in Section V.

### B. Notations

For a natural number $N$, we use $[N]$ to denote the set $\{1,\ldots,N\}$. $\mathbb{R}_+$ and $\mathbb{Z}_+$ respectively denote the set of all non-negative real numbers and integers. Let $\mathcal{X}_i, i\in[N]$ be non-empty spaces and consider $x_1\in\mathcal{X}_1,\ldots,x_N\in\mathcal{X}_N$. Then, $x_{1:N}$ denotes the set $\{x_1,\ldots,x_N\}$ and $\mathcal{X}_{1:N}$ denotes the product space $\mathcal{X}_1\times\cdots\times\mathcal{X}_N$.

## II. Problem Formulation

We now formulate the three-stage three-player Colonel Blotto game in this section. This is a complete information game, that is, at the second and third stages of the game, the actions taken by the players in the previous stages are common knowledge. The first two players fight against an adversary, call it $A$. Hereafter, we use Player 3 and $A$ interchangeably for the adversary. Players 1 and 2 are allowed to transfer resources among each other at the second stage of the game if it improves their payoff. This act of transferring a positive amount of resource can be thought of as a formation of coalition among the players.

The initial endowment of Player $i\in\{1,2\}$ and the adversary, respectively, are denoted by $\beta_i$ and $\alpha$. The initial number of battlefields on which the battle between Player $i\in\{1,2\}$ and the adversary will take place is denoted by $n_i$, and each battlefield carries a payoff denoted by $v_i>0$. We assume that $n_i\geq 3$ for $i\in\{1,2\}$.

### A. Information Structures and Strategies of the Players

The model of the game and parameters of the game are common knowledge among the players before the game begins. At the first stage, based only on the model and the parameters of the game, Player $i\in\{1,2\}$ decides on a non-negative integer $m_i\in\mathbb{Z}_+$, which denotes the number of battlefields Player $i$ wants to add to the existing battlefields, and pays a cost $cm_i^2$, where we assume $c>0$. The adversary does not take any action at the first stage.

The actions taken by the players at the first stage are common knowledge at the second stage. The second stage consists of two time steps. At the first time step, the first two players may choose to transfer resources among each other. We let $t_{i,j}:\mathbb{Z}_+^2\to[0,\beta_i]$ denote the control law of Player $i\in\{1,2\}$ at the second stage, which is the amount of resource Player $i$ transfers to Player $j\neq i$, $i,j\in\{1,2\}$ as a function of the number of battlefields added by both players at the first stage of the game. We assume that $t_{i,i}\equiv 0$ for $i\in\{1,2\}$. We use $r_i$ to denote the amount of resource available to Player $i\in\{1,2\}$ after the redistribution of resources. This is given by

$$r_i(t_{i,j},t_{j,i})=\beta_i+\sum_{j=1}^{2}(t_{j,i}-t_{i,j}). \tag{1}$$

The actions of the first two players (that is, the amount of resources transferred) are observed by the adversary at the second step at this stage of the game. The adversary then decides on the amount of resource it allocates to the battle with each player. In particular, the adversary decides on two functions $\alpha_i:\mathbb{Z}_+^2\times[0,\beta_1]\times[0,\beta_2]\to[0,\alpha]$ subject to the constraint that $\alpha_1(m_{1:2},t_{1,2},t_{2,1})+\alpha_2(m_{1:2},t_{1,2},t_{2,1})\leq\alpha$.

We now consider the third stage of the game in which the adversary engages in two battles against the first two players. At this stage, the players (including the adversary) know the numbers of battlefields added, the transfer among the first two players and the adversary's allocation of the resource to each battle. Given this information, each player needs to decide on the amount of resource it deploys on each battlefield. Thus, the final stage of the game consists of two static Colonel Blotto games.

For a given triple $\alpha_i,r_i\in\mathbb{R}_+$ and $m_i\in\mathbb{Z}_+$, let us define the sets

$$\mathcal{A}_i(\alpha_i,m_i):=\left\{\{\alpha_{i,k}\}_{k=1}^{n_i+m_i}\subset\mathbb{R}_+:\sum_{k=1}^{n_i+m_i}\alpha_{i,k}=\alpha_i\right\},$$
$$\mathcal{B}_i(r_i,m_i):=\left\{\{\beta_{i,k}\}_{k=1}^{n_i+m_i}\subset\mathbb{R}_+:\sum_{k=1}^{n_i+m_i}\beta_{i,k}=r_i\right\}.$$

At the final stage, the adversary fights against Player $i\in\{1,2\}$ on $n_i+m_i$ battlefields, where the resource levels of Player $i$ and the adversary, respectively, are $r_i$ and $\alpha_i$. It is well known that if the number of battlefields is greater than two, then Nash equilibrium of the players in static Colonel Blotto game exists only in mixed strategies [3]. Accordingly, the behavioral strategy of Player $i\in\{1,2\}$ at the third stage is denoted by $\mu_i$, such that $\mu_i(\alpha_i,r_i,m_i)\in\wp(\mathcal{B}_i(r_i,m_i))$, and the behavioral strategy of the adversary at the third stage is denoted by $(\nu_1,\nu_2)$, such that $\nu_i(\alpha_i,r_i,m_i)\in\wp(\mathcal{A}_i(\alpha_i,m_i)), i\in\{1,2\}$.

Henceforth, we use $\gamma^i$ to denote the strategy of Player $i$:

$$\begin{aligned}\gamma^i &:= \{m_i,t_{i,1},\ldots,t_{i,N},\mu_i\}, i\in\{1,2\}\\ \gamma^A &:= \{\alpha_1,\ldots,\alpha_N,\nu_1,\nu_2\}.\end{aligned}$$

Thus, each $\gamma^i$ is a collection of functions and the set of all such $\gamma^i$s is denoted by $\Gamma^i$. For ease of exposition, we drop the arguments of all functions $t_{i,j}$, $r_i$, $\alpha_i$, $\mu_i$, and $\nu_i$ $i,j\in\{1,2\}$ in subsequent discussions, and use the same notation to denote the actions taken by the player.

### B. Payoff Functions of the Players

At the third stage of the game, let us use $\beta_{i,k}$ and $\alpha_{i,k}$ to denote, respectively, the amount of resource Player $i$ and adversary deploy on battlefield $k \in [n_i + m_i]$. On every battlefield $k \in [n_i + m_i]$, the player who deploys maximum amount of resource wins and receives a payoff $v_i$. In case of a tie, the players share the payoff equally[1]. We let $p_{i,k}(\beta_{i,k}, \alpha_{i,k})$ denote the payoff that Player $i$ receives on the battlefield $k$, which is given by

$$p_{i,k}(\beta_{i,k}, \alpha_{i,k}) = \begin{cases} v_i & \beta_{i,k} > \alpha_{i,k}, \\ \frac{v_i}{2} & \beta_{i,k} = \alpha_{i,k}, \\ 0 & \text{otherwise}, \end{cases}$$

for $i \in \{1, 2\}$ and $k \in \{1, \ldots, n_i + m_i\}$. The adversary's payoff on a battlefield $k$ in the battle with Player $i$ is

$$p^A_{i,k}(\beta_{i,k}, \alpha_{i,k}) = v_i - p_{i,k}(\beta_{i,k}, \alpha_{i,k}).$$

We use $\pi_i$ to denote the expected cost functional of Player $i$ as a function of the strategies of all players. This is given by

$$\pi_i(\gamma^{1:3}) = \mathbb{E}\left[\sum_{k=1}^{n_i+m_i} p_{i,k}(\beta_{i,k}, \alpha_{i,k})\right] - cm_i^2, \quad i \in \{1, 2\},$$

$$\pi_3(\gamma^{1:3}) = \mathbb{E}\left[\sum_{i=1}^{N} \sum_{k=1}^{n_i+m_i} p^A_{i,k}(\beta_{i,k}, \alpha_{i,k})\right],$$

where the expectation is taken with respect to the probability induced on the random variables $\{\beta_{i,k}, \alpha_{i,k}\}_{i,k}$ by the choice of strategies of the players in the game. The model of the game and the payoff functions are common knowledge among the players. The Colonel Blotto game formulated above is referred to as $\mathbf{CB}(\underline{n}, \underline{\beta}, \alpha, \underline{v}, c)$.

A three-tuple of strategies $\{\gamma^{1\star}, \gamma^{2\star}, \gamma^{3\star}\}$ is said to form a Nash equilibrium if

$$\pi_i(\gamma^{1:3\star}) \quad \geq \quad \pi_i(\gamma^i, \gamma^{-i\star}),$$

for all possible $\gamma^i \in \Gamma^i$, $i \in [3]$. Since this is a game of perfect information with stagewise additive payoff functions, we can compute the subgame-perfect Nash equilibrium (SPNE) of the game. SPNE of a complete information game is a refinement of Nash equilibria of the game, which can be computed using a backward inductive algorithm. We refer the reader to [14, p. 72] and [15, Definition 5.14, p. 250] for the precise definition and properties of SPNE of complete information games.

### C. Research Questions and Solution Approach

We want to investigate the conditions under which in the game defined above, a coalition is formed in which the players transfer resources, or add additional battlefields. In particular, we want to know when

1) There is a positive transfer from one player to another. Note that in this scenario, the transfer should *increase or maintain* the payoffs to both, the transferring player as well as the player who accepts the transfer.
2) No transfer occurs at the second stage.
3) The adversary allocates all its resource to fight only one player.
4) The players have incentive to add new battlefields.

We first recall some preliminary results on the two-player static Colonel Blotto game from [3]. Solving the general problem formulated above is somewhat difficult due to discontinuity of expected payoff functions as a function of endowments of the players in the static game. Therefore, we restrict our attention to a subset of all possible parameter regions in order to keep the analysis tractable. We compute the parameter regions that feature the scenarios listed above in Section IV.

Our analysis of the subgame starting at the second stage is similar to the one considered in [7]. However, the authors in [7] do not compute the Nash equilibrium of the game; they restricted their attention to computing the best response strategies of the players. We derive here the SPNE of the game formulated above.

## III. Preliminary Results on Static Two-Player Colonel Blotto Game

Consider a two-players static Colonel Blotto game with $n$ battlefields. We let $r_i$ denote the resource of Player $i$. Define $\mathcal{R}_i := \{a \in \mathbb{R}^n_+ : \sum_{k=1}^n a_k \leq r_i\}$ and let $\partial\mathcal{R}_i$ be the boundary of the region $\mathcal{R}_i$. Then, the strategy of Player $i$ is a joint measure over the space $\mathcal{R}_i$. Let $\mu_i \in \wp(\mathcal{R}_i)$ be the strategy of Player $i$. Then, we let $\mathrm{Pr}^k_{\#}\mu_i$ denote the marginal of $\mu_i$ on the $k^{th}$ battlefield.

A player wins a battlefield if he deploys strictly larger amount of resource as compared with the other player on that battlefield. In case of a tie (both players deploying equal resources), the payoff is equally divided between the players. The payoff of winning a battlefield is given by $v$. Due to the cost functionals of the players, for a given strategy pair $(\mu_1, \mu_2)$ of the players, the expected cost to Player $i$ on battlefield $k \in [n]$ is dependent solely on the marginal distributions $(\mathrm{Pr}^k_{\#}\mu_1, \mathrm{Pr}^k_{\#}\mu_2)$.

The resources available to the players and the number of battlefields are common knowledge among the players. We call this Colonel Blotto game as $\mathbf{SCB}(\{1, r_1\}, \{2, r_2\}, n, v)$. Recall that [3] proved that Nash equilibrium exists in $\mathbf{SCB}(\{1, r_1\}, \{2, r_2\}, n, v)$ in mixed strategies of the players. We let $\mathtt{NE}(\mathbf{SCB}(\{1, r_1\}, \{2, r_2\}, n, v))$ denote the set of all Nash equilibria of the static Colonel Blotto game $\mathbf{SCB}(\{1, r_1\}, \{2, r_2\}, n, v)$. In the following lemma, we state the expected payoffs to each player if both players act according to Nash equilibrium strategies. For a proof of the lemma, see [3].

**Lemma 1:** For the static Colonel Blotto game $\mathbf{SCB}(\{1, r_1\}, \{2, r_2\}, n, v)$ with $n \geq 3$, suppose that $r_1$ and $r_2$ are such that $\frac{1}{n-1} \leq \frac{r_1}{r_2} \leq n - 1$. Then, the payoff

[1]It should be noted that if players play according to the Nash equilibrium strategies on the battlefields, then the case of both players having equal resource on a battlefield has a measure zero. Therefore, in equilibrium, the tie breaking rule does not affect the equilibrium expected payoffs.

functions of the players under Nash equilibrium $(\mu_1^\star, \mu_2^\star)$ are given by

$$P^1(\mathbf{SCB}(\{1,r_1\},\{2,r_2\},n,v)) = \begin{cases} nv\left(\frac{2}{n} - \frac{2r_2}{n^2 r_1}\right) & \text{if } \frac{1}{n-1} \le \frac{r_1}{r_2} < \frac{2}{n} \\ nv\left(\frac{r_1}{2r_2}\right) & \text{if } \frac{2}{n} \le \frac{r_1}{r_2} \le 1 \\ nv\left(1 - \frac{r_2}{2r_1}\right) & \text{if } 1 \le \frac{r_1}{r_2} \le \frac{n}{2} \\ nv\left(1 - \frac{2}{n} + \frac{2r_1}{n^2 r_2}\right) & \text{if } \frac{n}{2} < \frac{r_1}{r_2} < n-1 \end{cases},$$

$$P^2(\mathbf{SCB}(\{1,r_1\},\{2,r_2\},n,v)) = nv - P^1(\mathbf{SCB}(\{1,r_1\},\{2,r_2\},n,v)).$$

If $r_1 = 0$, then $P^1(\mathbf{SCB}(\{1,0\},\{2,r_2\},n,v)) = 0$.

*Remark 1:* Note that for a fixed $r_2, n$ and $v$, $r_1 \mapsto P^1(\mathbf{SCB}(\{1,r_1\},\{2,r_2\},n,v))$ is a concave monotonically increasing function in the parameter region $\frac{1}{n-1} \le \frac{r_1}{r_2} \le n-1$. Furthermore, $r_1 \mapsto P^1(\mathbf{SCB}(\{1,r_1\},\{2,r_2\},n,v))$ is a non-decreasing function on $\mathbb{R}_+$. □

## IV. SPNE of the Three-Stage Game

In this section, we compute the subgame-perfect Nash equilibrium for the game formulated in Section II. The SPNE of a complete information game is computed using a recursive algorithm. First, the Nash equilibrium for the game at the final stage is computed. Then, at any stage before the final stage, the Nash equilibrium for the subgame starting at that stage is considered and Nash equilibrium is computed for that game.

In what follows, we use $t := t_{1,2} - t_{2,1}$ to denote the total amount transferred from Player 1 to Player 2 at the second stage. The value of $t$ can take a negative value if Player 2 transfers its resource to Player 1. We use $r_1 := r_1(t) = \beta_1 - t$ and $r_2 := r_2(t) = \beta_2 + t$, respectively, to denote the resource levels of Player 1 and Player 2 after the transfer.

At the final stage, Player $i \in \{1,2\}$ and the adversary play a static Colonel Blotto game on $n_i + m_i$ battlefields with resource levels $r_i$ and $\alpha_i$, respectively. We have the following result.

**Lemma 2:** At the final stage, Player $i \in \{1,2\}$ and the adversary play a static Colonel Blotto game $\mathbf{SCB}(\{1,r_i\},\{2,\alpha_i\},n_i+m_i,v_i)$. Thus, the SPNE at the last stage is any pair of strategies $(\mu_i^\star,\nu_i^\star) \in \mathrm{NE}(\mathbf{SCB}(\{1,r_i\},\{2,\alpha_i\},n_i+m_i,v_i))$.

Consequently, we only need to compute the SPNE strategies of the players at the first two stages. We now restrict our analysis to a subset of all possible parameter regions in order to keep it tractable. In particular, we focus our attention to only those games in which if players act according to SPNE at the first two stages, then the ratio of $r_i$ and $\alpha_i$ lies in the interval $(\frac{2}{n_i}, \frac{n_i}{2})$, either for both $i \in \{1,2\}$ or $\alpha_i = 0$ for some $i \in \{1,2\}$. With this simplification, there are only four possible cases:

1) $2/n_1 < \alpha_1/r_1 < 1$ and $2/n2 < \alpha_2/r_2 < 1$
2) $2/n_1 < r_1/\alpha_1 < 1$ and $2/n_2 < r_2/\alpha_2 < 1$
3) $2/n_1 < \alpha_1/r_1 < 1$ and $2/n_2 < r_2/\alpha_2 < 1$
4) $2/n_1 < r_1/\alpha_1 < 1$ and $2/n_2 < \alpha_2/r_2 < 1$

However, Case 4 above is just Case 3 with index of the players interchanged. Therefore, we compute the SPNE of the game here only for the first three cases. Toward this end, we first compute the reaction curve of the adversary at the second stage in the next subsection.

*1) Preliminary Notation for Results:* We now define some notation that we use throughout the rest of the paper.

$$a_1(m_1,m_2,t) := \frac{\alpha}{1 + \sqrt{\frac{(n_2+m_2)v_2(\beta_2+t)}{(n_1+m_1)v_1(\beta_1-t)}}},$$

$$a_2(m_1,m_2,t) := \alpha - a_1(m_1,m_2,t),$$

$$\lambda_1(m_1,m_2,t) := \sqrt{\frac{(n_2+m_2)v_2(\beta_1-t)(\beta_2+t)}{(n_1+m_1)v_1}},$$

$$d(m_1,m_2,t) := \begin{cases} \alpha & \text{if } \frac{(n_1+m_1)v_1}{\beta_1-t} > \frac{(n_2+m_2)v_2}{\beta_2+t} \\ 0 & \text{if } \frac{(n_1+m_1)v_1}{\beta_1-t} < \frac{(n_2+m_2)v_2}{\beta_2+t} \\ \alpha \ w.p.\ p \in (0,1) & \text{if } \frac{(n_1+m_1)v_1}{\beta_1-t} = \frac{(n_2+m_2)v_2}{\beta_2+t} \end{cases}.$$

### A. Best Response of the Adversary

We first compute the best response strategies (also called reaction curves [15]) of the adversary in the game.

**Lemma 3:** Consider a game $\mathbf{CB}(\underline{n},\underline{\beta},\alpha,\underline{v},c)$. For a $t \in [-\beta_2,\beta_1]$, let $r_1 = \beta_1 - t$ and $r_2 = \beta_2 + t$. Fix $m_1, m_2 \in \mathbb{Z}_+$. The strategy of the adversary that maximizes its payoff is:

1) If $\frac{2}{n_1+m_1} < \frac{\alpha}{\beta_1-t} < 1$ and $\frac{2}{n_2} < \frac{\alpha}{\beta_2+t} < 1$, then $\alpha_1^*(m_1,m_2,t) = d(m_1,m_2,t)$.
2) If $\frac{2}{n_i+m_i} < \frac{r_i}{a_i(m_1,m_2,t)} < 1$, $i = 1,2$, then $\alpha_1^*(m_1,m_2,t) = a_1(m_1,m_2,t)$.
3) If $\frac{2}{n_1+m_1} < \frac{\alpha-\lambda_1(m_1,m_2,t)}{\beta_1-t} < 1$ and $\frac{2}{n_2+m_2} < \frac{\beta_2+t}{\lambda_1(m_1,m_2,t)} < 1$, then $\alpha_1^*(m_1,m_2,t) = \alpha - \lambda_1(m_1,m_2,t)$.

**Proof:** See [16] for a proof. ■

### B. Adversary with Least Resources

We now turn our attention to computing SPNE of the three-stage Colonel Blotto game formulated in Section II. First, we consider a case when the adversary has the least amount of resources among all players. We show that if the parameters of the game satisfy certain assumptions, then there exists a family of SPNEs in this game.

*1) Preliminary Notation for Theorem 4:* Let $\bar{m}_1 = \arg\max_{m_1 \in \mathbb{Z}_+} m_1 v_1 - c m_1^2$ and $\bar{m}_2 = \arg\max_{m_2 \in \mathbb{Z}_+} m_2 v_2 - c m_2^2$. Define

$$\bar{t}_{1,2}(m_1,m_2) = \frac{(n_2+m_2)v_2\beta_1 - (n_1+m_1)v_1\beta_2}{(n_1+m_1)v_1 + (n_2+m_2)v_2},$$

$$\bar{t}_{2,1}(m_1,m_2) = \frac{(n_1+m_1)v_1\beta_2 - (n_2+m_2)v_2\beta_1}{(n_1+m_1)v_1 + (n_2+m_2)v_2},$$

$$\zeta_1 = \bar{t}_{2,1}(0,\bar{m}_2) \qquad \zeta_2 = \bar{t}_{1,2}(\bar{m}_1,0).$$

**Theorem 4:** Consider a game $\mathbf{CB}(\underline{n},\underline{\beta},\alpha,\underline{v},c)$ with $\alpha < \min\{\beta_1,\beta_2\}$ and $\frac{2}{n_i} < \frac{\alpha}{\beta_i}$ for $i \in \{1,2\}$. If the parameters of the game satisfy either

$$\frac{(n_1+\bar{m}_1)v_1}{\beta_1} < \frac{n_2 v_2}{\beta_2}, \quad \left(1-\frac{\alpha}{2(\beta_2+\zeta_2)}\right)v_2 < c,$$
$$\frac{2}{n_1+\bar{m}_1} < \frac{\alpha}{\beta_1-\zeta_2} < 1, \quad \frac{2}{n_2} < \frac{\alpha}{\beta_2+\zeta_2} < 1, \quad (2)$$

or

$$\frac{n_1 v_1}{\beta_1} > \frac{(n_2+\bar{m}_2)v_2}{\beta_2}, \quad \left(1-\frac{\alpha}{2(\beta_1+\zeta_1)}\right)v_1 < c,$$
$$\frac{2}{n_2+\bar{m}_2} < \frac{\alpha}{\beta_2-\zeta_1} < 1, \quad \frac{2}{n_1} < \frac{\alpha}{\beta_1+\zeta_1} < 1, \quad (3)$$

then there is a family of SPNEs for this game given by

$$\alpha_1^\star(m_1,m_2) = d(m_1,m_2,t),$$
$$t_{1,2}^\star(m_1,m_2) = \begin{cases} t\in[0,\bar{t}_{1,2}(m_1,m_2)) & \text{if } \frac{(n_1+m_1)v_1}{\beta_1} < \frac{(n_2+m_2)v_2}{\beta_2} \\ 0 & \text{otherwise}\end{cases}$$
$$t_{2,1}^\star(m_1,m_2) = \begin{cases} t\in[0,\bar{t}_{2,1}(m_1,m_2)) & \text{if } \frac{(n_1+m_1)v_1}{\beta_1} > \frac{(n_2+m_2)v_2}{\beta_2} \\ 0 & \text{otherwise}\end{cases}$$
$$m_1^\star = \begin{cases} \bar{m}_1 & \text{if } \frac{(n_1+\bar{m}_1)v_1}{\beta_1} < \frac{n_2 v_2}{\beta_2} \\ 0 & \text{otherwise}\end{cases},$$
$$m_2^\star = \begin{cases} \bar{m}_2 & \text{if } \frac{n_1 v_1}{\beta_1} > \frac{(n_2+\bar{m}_2)v_2}{\beta_2} \\ 0 & \text{otherwise}\end{cases}.$$

Thus, along the equilibrium path, one player has an incentive to add battlefields and transfer some (or none) of its resources to the other player.

**Proof:** See [16] for a proof. ■

*Remark 2:* In the theorem above, if $c > v_1$, then $\bar{m}_1 = 0$. Similarly, if $c > v_2$, then $\bar{m}_2 = 0$. □

In the theorem above, there are three important points to note: The first point is that the adversary randomizes its action when $\frac{(n_1+m_1)v_1}{\beta_1-t} = \frac{(n_2+m_2)v_2}{\beta_2+t}$. Suppose $\frac{(n_1+m_1)v_1}{\beta_1} < \frac{(n_2+m_2)v_2}{\beta_2}$. Then, as $t$ increases, $\frac{(n_1+m_1)v_1}{\beta_1-t}$ increases while $\frac{(n_2+m_2)v_2}{\beta_2+t}$ decreases. The two quantities $\frac{(n_1+m_1)v_1}{\beta_1-t}$ and $\frac{(n_2+m_2)v_2}{\beta_2+t}$ become equal exactly when $t = \bar{t}_{1,2}(m_1,m_2)$. Thus, Player 1 will never transfer an amount equal to $\bar{t}_{1,2}(m_1,m_2)$ in this case since this action reduces its payoff. This is the reason why we see that Player 1 transfers any amount $t$ in the interval $[0,\bar{t}_{1,2}(m_1,m_2))$ when playing according to the SPNE.

The second point to note is that one player adds battlefields as well as transfer its resource to the other player if the value $c$ is small enough. The third point to note is that the player transferring its resource to the other player is the one with minimum $\frac{(n_i+m_i)v_i}{\beta_i}$, and not necessarily the player who has maximum resource level $\beta_i$. This is contrary to our intuition. A take-away from this result is that the rich player need not always be better off in a war as it may have more and/or highly valued battlefields to fight on.

This phenomena is also illustrated numerically in Figure 1 (a). In the red region, even though Player 1 has less resource than Player 2, Player 1 may choose to transfer some of its resource to Player 2. We now consider other scenarios in the next subsection.

### C. Other Cases

In this subsection, we consider the case in which the adversary has comparable or more resources than those of the other players. The SPNE of the game in such a case is as follows.

*1) Preliminary Notation for Theorem 5:*

$$\bar{t}_1(m_1,m_2) := \frac{(\beta_1-\beta_2)}{2} - \frac{(\beta_1+\beta_2)}{2} \times\sqrt{\frac{(n_1+m_1)v_1}{(n_1+m_1)v_1+(n_2+m_2)v_2}},$$
$$w_1(m_1,m_2) := (n_1+m_1)v_1 + \sqrt{(n_1+m_1)v_1((n_1+m_1)v_1+(n_2+m_2)v_2)},$$

$$\bar{m}_1 := \arg\max_{m_1\in\mathbb{Z}_+} m_1 v_1\left(1-\frac{\alpha}{2(\beta_1+\beta_2)}\right) - cm_1^2,$$
$$\zeta_1(m_1,m_2) := \frac{4(n_1+m_1)v_1\alpha^2}{(n_2+m_2)v_2(\beta_1+\beta_2)^2}.$$

**Theorem 5:** Consider a game $\mathbf{CB}(\underline{n},\underline{\beta},\alpha,\underline{v},c)$. The SPNE of the game is given as

1) Assume $c > \frac{\beta_1+\beta_2}{4\alpha}\max\{w_1(1,0)-w_1(0,0),v_2\}$ and let $\bar{t}_1 := \bar{t}_1(m_1,m_2)$. If $\frac{2}{n_i+m_i} < \frac{r_i(t)}{a_i(m_1,m_2,t)} < 1$, $i=1,2$, then
$$\alpha_1^\star(m_1,m_2,t) = a_1(m_1,m_2,t),$$
$$t_{1,2}^\star(m_1,m_2) = \begin{cases} \bar{t}_1 & \text{if } \frac{\beta_1-\beta_2}{2\beta_1\beta_2} > \sqrt{\frac{(n_1+m_1)v_1}{(n_2+m_2)v_2}} \\ 0 & \text{otherwise}\end{cases}$$
$$t_{2,1}^\star(m_1,m_2) = 0, \qquad m_1^\star = m_2^\star = 0.$$

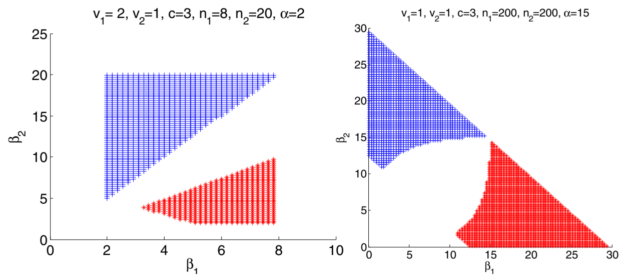
2) If $c > \frac{(\beta_1+\beta_2)v_2}{4\alpha}$, $\frac{2}{n_1+m_1} < \frac{\alpha-\lambda_1(m_1,m_2,t)}{(\beta_1-t)} < 1$, and $\frac{2}{n_2+m_2} < \frac{(\beta_2+t)}{\lambda_1(m_1,m_2,t)} < 1$, then
$$\alpha_1^\star(m_1,m_2,t) = \alpha-\lambda_1(m_1,m_2,t),$$
$$t_{1,2}^\star(m_1,m_2) = \begin{cases} \frac{\beta_1-\zeta_1(m_1,m_2)\beta_2}{\zeta_1(m_1,m_2)+1} & \text{if } \frac{\beta_1+\beta_2}{2\alpha} > \sqrt{\frac{(n_1+m_1)v_1\beta_2}{(n_2+m_2)v_2\beta_1}} \\ 0 & \text{otherwise.}\end{cases}$$
$$t_{2,1}^\star(m_1,m_2) = 0, \qquad m_1^\star = \bar{m}_1, \qquad m_2^\star = 0.$$

**Proof:** See [16] for a proof. ■

*Remark 3:* If $c > v_1\left(1-\frac{\alpha}{2(\beta_1+\beta_2)}\right)$, then $\bar{m}_1 = 0$, which implies $m_1^\star = 0$ in the Case 2 above. □

In the theorem above, there is a unique transfer among the players, and therefore, unique SPNE. The first case is that of adversary having a significantly larger amount of resources than the sum of the resources of the other two players. It is easy to note that Player 1 transfers to Player 2 in this case when $\frac{\beta_1-\beta_2}{2\beta_1\beta_2} > \sqrt{\frac{(n_1+m_1)v_1}{(n_2+m_2)v_2}} > 0$, which implies that $\beta_1 > \beta_2$. Thus, the rich player transfers resource to the poor player. Furthermore, the first two players do not add battlefields if the value of $c$ is high enough.

In the second case above, the adversary has a comparable resource level with respect to the resource levels of the other two players. In this case, like in the first case, the rich player transfers resource to the poor player. A graphical representation of when a transfer occurs and who transfers whom is given in Figure 1 (b).




(a) (b)

Fig. 1. (a) For fixed parameters $v_1 = 2, v_2 = 1, c = 3, n_1 = 8, n_2 = 20$, and $\alpha = 2$, Player 1 transfers to Player 2 in the red (bottom) region, whereas Player 2 transfers to Player 1 in the blue (upper) region. (b) For fixed parameters $v_1 = 1, v_2 = 1$, $c = 3$, $n_1 = 200, n_2 = 200$, and $\alpha = 15$, Player 1 transfers to Player 2 in the red (bottom) region, whereas Player 2 transfers to Player 1 in the blue (upper) region. In both figures, there is no addition of battlefield by any player in the colored region. In the white region, transfer may or may not occur.

In both cases above, it should be noted that even though both players fight with the adversary, there is a positive transfer of resource from the rich player to the poor player. This result was also reported in [7], but the authors did not compute the equilibrium behavior. Even though for every player, higher resource implies higher payoff (see Lemma 1), we see that a positive transfer takes place because the adversary observes the amount of resource transferred and changes its allocation appropriately in order to maximize its payoff.

In Case 2 above, Player 1 adds battlefields as well as transfer its resource to Player 2. This is similar to the behavior we saw in Theorem 4, where the adversary had least resources among all players. In Theorem 4, the transferring player may choose not to transfer any resource; on the contrary, in Theorem 5, we saw that the transferring player must transfer a unique positive amount of resource to the other player. It should also be noted that in all the cases above, if the value of $c$ was small, then adding battlefields is beneficial to Players 1 and 2. We consider the case of $c$ large enough here for ease of exposition.

We now have a qualitative picture of the behavior of resource-constrained players who could be attacked by a common resource-constrained adversary. Going back to the example of network operators and the hacker stated in Section I, we now know the parameter regions where it is beneficial for the network operators to form a coalition for sharing resources. If the cost of adding additional servers is small enough, it is in the best interest of network operators to add more servers. We also know the amount of resource the hacker will allocate to hack the servers in each network. In particular, if the hacker has very little resource as compared to the network operators, then the hacker attacks only one of the networks (see Theorem 4 for details). If the hacker has comparable or more resources than the network operators, the hacker divides its resource into two parts, where each part is used to attack each of the network operators (see Theorem 5 for details).

## V. Conclusion

In this paper, we formulated a three-stage Colonel Blotto game and computed the subgame-perfect Nash equilibrium of the game in various parameter regions. We showed that under some sufficient conditions, the players may have an incentive to add battlefields or form a coalition as it improves their expected payoffs. We discussed the implications of the results stated in this paper on securing cyberphysical-systems. In particular, it is beneficial for security agencies to increase the number of entities that can be under attack and/or form a coalition with other security agencies to share resources, which increases the overall security. The strategies discussed in this paper can be used to improve the security level of various cyberphysical systems that can potentially be attacked. In the future, we would like to extend the analysis to multi-player versions of this game.

## References

[1] E. Borel and J. Ville, *Applications de la théorie des probabilités aux jeux de hasard*. J. Gabay, 1938.
[2] O. Gross and R. Wagner, "A continuous Colonel Blotto game," *Rand*, 1950.
[3] B. Roberson, "The Colonel Blotto game," *Economic Theory*, vol. 29, no. 1, pp. 1–24, 2006.
[4] D. Kvasov, "Contests with limited resources," *Journal of Economic Theory*, vol. 136, no. 1, pp. 738–748, 2007.
[5] D. G. Arce, D. Kovenock, and B. Roberson, "Weakest-link attacker-defender games with multiple attack technologies," *Naval Research Logistics (NRL)*, vol. 59, no. 6, pp. 457–469, 2012.
[6] D. Kovenock, M. J. Mauboussin, and B. Roberson, "Asymmetric conflicts with endogenous dimensionality," *Korean Economic Review*, vol. 26, pp. 287–305, 2010.
[7] D. Kovenock and B. Roberson, "Coalitional Colonel Blotto games with application to the economics of alliances," *Journal of Public Economic Theory*, vol. 14, no. 4, pp. 653–676, 2012.
[8] P. H. Chia, "Colonel Blotto in web security," in *The Eleventh Workshop on Economics and Information Security, WEIS Rump Session*, 2012.
[9] P. Chia and J. Chuang, "Colonel Blotto in the phishing war," *Decision and Game Theory for Security*, pp. 201–218, 2011.
[10] S. Mago and R. Sheremeta, "Multi-battle contests: an experimental study," *Available at SSRN 2027172*, 2012.
[11] S. Chowdhury, D. Kovenock, and R. Sheremeta, "An experimental investigation of Colonel Blotto games," *Economic Theory*, vol. 52, no. 3, pp. 833–861, 2013.
[12] A. Arad and A. Rubinstein, "Multi-dimensional iterative reasoning in action: The case of the Colonel Blotto game," *Journal of Economic Behavior & Organization*, vol. 84, no. 2, p. 571585, 2012.
[13] M. McBride and D. Hewitt, "The enemy you can't see: An investigation of the disruption of dark networks," tech. rep., University of California-Irvine, Department of Economics, 2012.
[14] D. Fudenberg and J. Tirole, *Game Theory*. MIT Press, 1991.
[15] T. Başar and G. Olsder, *Dynamic Noncooperative Game Theory*. Society for Industrial Mathematics (SIAM) Series in Classics in Applied Mathematics, Philadelphia, 1999.
[16] A. Gupta, G. Schwartz, C. Langbort, S. S. Sastry, and T. Başar, "A three-stage Colonel Blotto game with applications to cyberphysical security," Tech. Rep. UCB/EECS-2014-19, EECS Department, University of California, Berkeley, March 2014. available online at http://www.eecs.berkeley.edu/Pubs/TechRpts/2014/EECS-2014-19.html.