2016 American Control Conference (ACC)
Boston Marriott Copley Place
July 6-8, 2016. Boston, MA, USA

A Unified Framework for l_p Analysis and Synthesis of Linear Switched Systems

Mohammad Naghnaeian, Petros G. Voulgaris, and Geir E. Dullerud

Abstract— **In this paper, we develop a new framework to analyze stability and stabilizability of Linear Switched Systems (LSS) as well as their gain computations. Our approach is based on a combination of state space operator descriptions and the Youla parametrization and provides a unified way for analysis and synthesis of LSS, and in fact of Linear Time Varying (LTV) systems, in any l_p induced norm sense. By specializing to the l_∞ case, we show how Linear Programming (LP) can be used to test stability, stabilizability and to synthesize stabilizing controllers that guarantee a near optimal closed-loop gain.**

I. INTRODUCTION

Linear Switched Systems (LSS) are a special class of hybrid systems and have been given a special attention due to their importance over the last fifteen years or so. One can refer to [1], [2], [3], and references therein for some of the works done in this area. Many aspects of such systems have been the subject of research. Stability analysis and stabilizability of such systems have been given a major attention, in the literature, see for example [4], [5], and [6].

Aside from stability, input-output properties of LSS are of practical importance. In particular, input-output gains of LSS and how they are possibly related to the gains of their LTI modes is a relevant question. For the l_2 induced gain, one can refer to papers such as [7], [8], [9], [10], and [11]. For slowly switching system, the worst-case $\mathscr{L}_2$ induced gain is studied in [9]. Interestingly enough, in this case, the gain of the switched system can be, in general, arbitrarily larger than that of its LTI modes. At the other extreme, the case of fast switching (when the rate of switching approaches infinity) was studied in [11] and shown that the $\mathscr{L}_2$ induced norm of a fast switching LSS is, in general, different than that of the average system. Moreover, recently in the context of l_2 induced gain, [12] proposes an LMI based control synthesis. The authors developed a sequence of LMIs of increasing size whose feasibility is equivalent to the closed-loop satisfying a certain performance level of quadratic type.

Most of the literature on the input-output properties of LSS addresses the quadratic types of performance. Regarding the other types of performance, in particular l_1 or l_∞ induced gains, very little has been done. This is precisely what this paper aims to address by integrating and expanding on some earlier and preliminary works of the current authors in [13], [14], [15] and [16].

Our perspective in this paper is greatly influenced by the fact that LSS reduce to Linear Time Varying (LTV) systems for a fixed switching sequence. Hence, we take the approach of first establishing the results for LTV systems and then tailoring them to LSS. In particular, we study the stability and stabilizability of LTV systems and reduce them to convex optimization problems. Then, we extend the results to LSS and argue how the stability/stabilizability problem can be converted to a partially nested sequence of Linear Programs (LP) in the l_∞ case. Furthermore, we study any induced norm of LSS and LTV systems. This, of course, includes the l_1, l_∞, and l_2 induced gains. We are, however, mainly interested in the l_∞ induced norm which can be computed via linear programming.

M. Naghnaeian is a PhD candidate with the Mechanical Science and Engineering Department, University of Illinois, Urbana, IL, USA naghnae2@illinois.edu

P. G. Voulgaris is with the Aerospace Engineering Department and the Coordinated Science Laboratory, University of Illinois, Urbana, IL, USA voulgari@illinois.edu

Geir E. Dullerud is with the Mechanical Science and Engineering Department and the Coordinated Science Laboratory, University of Illinois, Urbana, IL, USA dullerud@illinois.edu

This work was supported in part by the National Science Foundation under NSF Award NSF ECCS 10-27437 and AFOSR under award AF FA 9550-12-1-0193

The authors acknowledge AFOSR support under grant FA9550-15-1-0059

II. PRELIMINARIES

In this paper, $\mathbb{R}$ and $\mathbb{Z}_+$ denote the sets of real numbers and non-negative integers, respectively. The set of n-tuples $x = \{x(k)\}_{k=0}^{n-1}$ where $x(k)$s are real numbers is denoted by $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$, its l_∞ and l_1 norm are defined as $\|x\|_\infty = \max_{k\in\{0,1,\dots,n-1\}} |x(k)|$ and $\|x\|_1 = \sum_{k=0}^{n-1} |x(k)|$, respectively. Let $g = \{g(k)\}_{k=0}^{\infty}$ be a sequence where $g(k) \in \mathbb{R}^n$. Then, the l_∞ and l_1 norm of this sequence are defined as $\|g\|_\infty = \sup_{k\in\mathbb{Z}_+} \|g(k)\|_\infty$ and $\|g\|_1 = \sum_{k=0}^{\infty} \|g(k)\|_1$ whenever they are finite. The set of sequences whose l_∞ norm (l_1 norm) is finite is denoted by l_∞ (l_1). Given two normed spaces $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ and a linear operator $T : X \to Y$, its induced norm is defined as $\|T\| := \sup_{f\neq 0} \frac{\|Tf\|_Y}{\|f\|_X}$. Whenever both vector spaces are l_∞ , we use the notation $\|T\|_{l_\infty\text{-ind}}$.

Any linear causal map $T : x \in X \to y \in Y$ can be thought of as an infinite dimensional lower triangular matrix,

$$T = \begin{bmatrix} T_{0,0} & 0 & 0 & \cdots \\ T_{1,1} & T_{1,0} & 0 & \cdots \\ T_{2,2} & T_{2,1} & T_{2,0} & \\ \vdots & \vdots & & \ddots \end{bmatrix}. \quad (1)$$

By $\mathscr{R}[T]_n$ we mean the causal part of the n^{th} block row in the matrix representation of T, i.e. $\mathscr{R}[T]_n := \begin{bmatrix} T_{nn} & T_{n,n-1} & \cdots & T_{n0} \end{bmatrix}$. In terms of this representation $\|T\|_{l_\infty\text{-ind}} = \sup_n \|\mathscr{R}[T]_n\|_{l_\infty\text{-ind}}$. For a finite dimensional matrix $S = \begin{bmatrix} s_{11} & \cdots & s_{1n} \\ \vdots & & \vdots \\ s_{m1} & \cdots & s_{mn} \end{bmatrix} : \mathbb{R}^n \to \mathbb{R}^m$, $\|S\|_{l_\infty\text{-ind}} = \sup_i \sum_{j=1}^n |s_{ij}|$. We say a linear causal map $T : X \to Y$, with matrix representation (1), is Finite Impulse Response (FIR) of some order $M \in \mathbb{Z}_+$ if for any integer $n \geq M$, $\mathscr{R}[T]_n = \begin{bmatrix} 0 & \cdots & 0 & T_{n,M-1} & \cdots & T_{n0} \end{bmatrix}$. The standard delay operator is denoted by Λ. More precisely, for any $k \in \mathbb{Z}_+$ and any sequence $g = \{g(0), g(1), ...\}$,

$$\Lambda^k g = \left\{ \underbrace{0, ..., 0}_{k \text{ zeros}}, g(0), g(1), ... \right\},$$

and, with a slight abuse of notation, $\Lambda^{-k} g = \{g(k), g(k+1), ...\}$. A Linear Time-Varying (LTV) system

$$G : \begin{cases} x_{t+1} = A_t x_t + B_t u_t \\ y_t = C_t x_t + D_t u_t \end{cases},$$

where $u_t \in \mathbb{R}^m, x_t \in \mathbb{R}^n$, $y_t \in \mathbb{R}^p$, and $x_0 \in \mathbb{R}^n$ are input, state, output, and the initial condition of the system and A_t, B_t, C_t, and D_t are bounded matrices with appropriate dimensions for all $t \in \mathbb{Z}_+$, can be rewritten as

$$G : \begin{cases} \Lambda^{-1} x = \hat{A} x + \hat{B} u \\ y = \hat{C} x + \hat{D} u \end{cases}, \text{ with } x_0 \text{ given}, \quad (2)$$

where $x = \{x_t\}_{t=0}^\infty$, $y = \{y_t\}_{t=0}^\infty$, $u = \{u_t\}_{t=0}^\infty$, Λ is the delay operator,

$$\hat{A} = diag(A_0, A_1, ...) := \begin{bmatrix} A_0 & 0 & \cdots \\ 0 & A_1 & \\ \vdots & & \ddots \end{bmatrix},$$

and $\hat{B}$, $\hat{C}$, and $\hat{D}$ are defined analogously. We assume that $\hat{A}$, $\hat{B}$, $\hat{C}$, and $\hat{D}$ are bounded maps. It can be easily shown that (2) can also be written

$$G : \begin{cases} x = \left(I - \Lambda\hat{A}\right)^{-1} \Lambda\hat{B} u + \left(I - \Lambda\hat{A}\right)^{-1} \bar{x}_0 \\ y = \hat{C} x + \hat{D} u \end{cases}, \quad (3)$$

where $\bar{x}_0 = \{x_0, 0, 0, ...\}$. In (3), the effects of the initial condition on the state variables are made explicit through the mapping $\left(I - \Lambda\hat{A}\right)^{-1}$.

Definition 1: We say the LTV system G in (3) is stable if it is a bounded operator from $\begin{pmatrix} x_0 \\ u \end{pmatrix}$ to $\begin{pmatrix} x \\ y \end{pmatrix}$.

We note that stability in the sense of the above definition is equivalent to the boundedness of the mapping $\left(I - \Lambda\hat{A}\right)^{-1}$ in (3).

III. Linear Switched Systems

Throughout this paper, by a Linear Switched System (LSS), we mean a system P_σ that admits the following state-space representation:

$$P_\sigma : \begin{cases} x_{t+1} = A_{\sigma_t} x_t + B_{\sigma_t} u_t \\ y_t = C_{\sigma_t} x_t + D_{\sigma_t} u_t \end{cases}. \quad (4)$$

In (4), $\sigma = \{\sigma_k\}_{k=0}^\infty$ is called the switching sequence. For the mere simplicity of the presentation, we assume that σ assumes value in the set $\{1,2\}$, i.e., $\sigma \in \{1,2\}^\infty$. The extension to the case where σ assumes values in a more general but finite set is immediate. Sometimes, σ is restricted to be in the set of admissible switching sequences. We denote this set by Ξ which is a subset of all binary sequences. Notice that given $\sigma \in \Xi$, (4) defines a LTV system. Hence, one can define bounded linear operators $\hat{A}_\sigma$, $\hat{B}_\sigma$, $\hat{C}_\sigma$, and $\hat{D}_\sigma$ that depend on the switching sequence σ and rewrite (4) as

$$P_\sigma : \begin{cases} \Lambda^{-1} x = \hat{A}_\sigma x + \hat{B}_\sigma u \\ y = \hat{C}_\sigma x + \hat{D}_\sigma u \end{cases}, \quad (5)$$

where $\hat{A}_\sigma = diag\left(A_{\sigma_0}, A_{\sigma_1}, A_{\sigma_2}, ...\right)$ and $\hat{B}_\sigma$, $\hat{C}_\sigma$, and $\hat{D}_\sigma$ are defined analogously.

Equations (4) and (5) represent a LSS in its generic form. There are, also, important special classes of such systems that are of interest here; those are the systems whose state matrices, A-matrices, remain constant and are defined below:

Definition 2: Let M be a positive integer. We say a LSS P_σ is an input-output LSS of degree M if it is stable and admits the realization

$$P_\sigma : \begin{cases} x_{t+1} = A x_t + B_{\sigma_t} u_t \\ y_t = C_{\{\sigma_k\}_{k=t-M+1}^t} x_t + D_{\{\sigma_k\}_{k=t-M+1}^t} u_t \end{cases}. \quad (6)$$

The class of such systems is denoted by $\mathscr{S}_{IO}^M$ and $\mathscr{S}_{IO} = \bigcup_{M=1}^\infty \mathscr{S}_{IO}^M$.

We are also interested in a subclass of input-output LSS defined below:

Definition 3: Let M be a positive integer. We say a LSS P_σ is an output-only LSS of degree M if it is stable and admits the realization

$$P_\sigma : \begin{cases} x_{t+1} = A x_t + B u_t \\ y_t = C_{\{\sigma_k\}_{k=t-M+1}^t} x_t + D_{\{\sigma_k\}_{k=t-M+1}^t} u_t \end{cases}.$$

The class of such systems is denoted by $\mathscr{S}_O^M$ and $\mathscr{S}_O = \bigcup_{M=1}^\infty \mathscr{S}_O^M$.

The classes of input-output and output-only LSS are of particular interest for two reasons. First, any stable LSS can be approximated by elements of $\mathscr{S}_O$ and $\mathscr{S}_{IO}$ with arbitrary accuracy (see the next lemma). Second, we provide exact and tractable expressions to calculate the l_∞ induced norm of these systems.

Lemma 4: Let G_σ be a stable LSS and $\varepsilon > 0$. Then, there exist an integer M, $\bar{G}_\sigma \in \mathscr{S}_{IO}^M$, and $\tilde{G}_\sigma \in \mathscr{S}_O^M$ such that

$$\begin{aligned} \left\|G_\sigma - \bar{G}_\sigma\right\| &< \varepsilon, \\ \left\|G_\sigma - \tilde{G}_\sigma\right\| &< \varepsilon, \end{aligned}$$

for any switching sequence σ. Moreover, $\bar{G}_\sigma$ and $\tilde{G}_\sigma$ can be made FIR.

In the rest of this section, we show how to compute the l_∞ induced norm of a input-output LSS P_σ of degree M. It is obvious from (6) that the C and D-matrices of P_σ can assume 2^M values at each time instant t; each value associates with the segment $\{\sigma_t, \sigma_{t-1}, ..., \sigma_{t-M+1}\}$ of the switching sequence. Let $\mathscr{I}_M$ be the set of all binary sequences of size M, i.e.

$\mathscr{I}_M = \left\{ i = \{i_k\}_{k=0}^{M-1} : i_k \in \{1,2\} \right\}$. We notice that each $P_\sigma \in \mathscr{S}_{IO}^M$ can be associated with 2^{M+1} LTI systems denoted by $P_{i,j}$, where $i \in \mathscr{I}_M,\ j \in \{1,2\}$ and

$$P_{i,j} : \begin{cases} x_{t+1} = Ax_t + B_j u_t \\ y_t = C_i x_t + D_i u_t \end{cases}.$$

Then, the l_∞ gain of P_σ can be computed in terms of the impulse responses of the LTI systems $P_{i,j}$ denoted by $\left\{P_{i.j}(k)\right\}_{k=0}^{\infty}$, for $i \in \mathscr{I}_M$ and $j = 1,2$.

Lemma 5: Let P_σ be an input-output LSS of order M. Further, assume P_σ is multi-input and single-output (MISO). Then

$$\begin{aligned} \sup_\sigma \|P_\sigma\|_{l_\infty\text{-ind}} &= \sup_{i=\{i_k\}_{k=0}^{M-1} \in \mathscr{I}_M} \sum_{k=0}^{M-1} \left\|P_{i,i_k}(k)\right\|_{l_\infty\text{-ind}} \\ &+ \sum_{k=M}^{\infty} \max_j \left\|P_{i,j}(k)\right\|_{l_\infty\text{-ind}}. \end{aligned} \tag{7}$$

Remark 6: Although, Lemma 5 addresses MISO systems, it can be easily extended to MIMO systems. In fact, suppose $P_\sigma : u \to \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix}$ is a MIMO input-output LSS and let P_σ^k to be the MISO mapping from the input to the k^{th} output, i.e. $P_\sigma^k : u \to y_k$. Then, it can be verified that $\sup_\sigma \|P_\sigma\|_{l_\infty\text{-ind}} = \max_k \sup_\sigma \left\|P_\sigma^k\right\|_{l_\infty\text{-ind}}$ and Lemma 5 can be used to compute $\sup_\sigma \left\|P_\sigma^k\right\|_{l_\infty\text{-ind}}$ as P_σ^k is MISO.

Remark 7: The l_∞ gain computation (7) can be written as a LP. For simplicity, suppose P_σ is SISO and it is FIR of order $T > M$ for all switching sequences. Then,

$$\sup_\sigma \|P_\sigma\|_{l_\infty\text{-ind}} = \min \gamma,$$

such that for any $i = \{i_k\}_{k=0}^{M-1} \in \mathscr{I}_M,\ k \in \{0,1,...,M-1\}$, $k' \in \{M, M+1, ..., T-1\}$, and $j \in \{1,2\}$

$$\begin{aligned} \left|P_{i,i_k}(k)\right| &\le \gamma_i(k), \\ \left|P_{i,j}(k')\right| &\le \gamma_i(k'), \\ \sum_{s=0}^{T-1} \gamma_i(s) &\le \gamma. \end{aligned}$$

IV. Stability

In this section, we derive necessary and sufficient conditions for the stability of LTV systems as well as LSS. By stability, in this part, we mean stability in the sense of Definition 1. First, we present the results for LTV systems and then show how they can be extended to LSS. Notice that the LTV system G in (3) is stable if and only if the mapping $\left(I - \Lambda\hat{A}\right)^{-1}$ is stable. In other words, G is stable if and only if $\hat{A}$ stabilizes Λ. Invoking the Youla-Kucera parameterization, $\hat{A}$ stabilizes Λ if and only if there exists a stable LTV system Q_A such that

$$\hat{A} = Q_A (I + \Lambda Q_A)^{-1} = (I + Q_A \Lambda)^{-1} Q_A,$$

or equivalently,

$$\hat{A}(I + \Lambda Q_A) - Q_A = 0, \tag{8}$$

$$(I + Q_A \Lambda)\hat{A} - Q_A = 0. \tag{9}$$

Finding Q_A satisfying (8) or (9) and making them exact equalities is a computationally challenging task. However, as stability is a robust property, one can think of relaxing the above conditions while preserving the necessity and sufficiency of the results as follows:

Theorem 8: Consider the LTV system G in (3). Then G is a stable if and only if there exists an LTV system Q such that one of the following equivalent conditions hold

$$\left\|\hat{A}(I + \Lambda Q) - Q\right\| < 1, \tag{10}$$

$$\left\|(I + Q\Lambda)\hat{A} - Q\right\| < 1. \tag{11}$$

The above theorem holds as long as the norms in (10) and (11) are induced norms from any vector space to the same vector space. Furthermore, (10) and (11) are convex. And indeed, for the l_∞ induced norm, we will show how (10) can be cast as a linear program. To this end, suppose Q is FIR of order T. Then

$$\begin{aligned} &\mathscr{R}\left[\hat{A}(I + \Lambda Q) - Q\right]_T \\ &= A_T \begin{bmatrix} q_{T-1,0}(T-1) & \cdots & q_{T-1,T-1}(0) & I \end{bmatrix} \\ &- \begin{bmatrix} 0 & q_{T,1}(T-1) & \cdots & q_{T,T}(0) \end{bmatrix}, \end{aligned}$$

and (10) can be written as

$$\begin{aligned} &\left\|\hat{A}(I + \Lambda Q) - Q\right\|_{l_\infty\text{-ind}} \\ &= \sup_t \left\|\mathscr{R}\left[\hat{A}(I + \Lambda Q) - Q\right]_t\right\|_{l_\infty\text{-ind}} \\ &= \sup_{l,j} \Big\{ \left|e_l'\left[A_j - q_{j,T}(0)\right]\right| \\ &+ \sum_{s=1}^{T-1} \left|e_l'\left[A_j q_{j-1,T-s}(s-1) - q_{j,T-s}(s)\right]\right| \\ &+ \left|e_l^T\left[A_j q_{j-1,0}(T-1)\right]\right| \Big\} < 1, \end{aligned} \tag{12}$$

where the absolute value $|.|$ is taken component wise and e_l is a vector of all zeros and one for the l^{th} entry. It is obvious, from (12), that finding Q is a linear program.

A LSS reduces to a LTV system for a given switching sequence σ. Hence, its stability can be checked via Theorem 8 for that particular switching sequence. Clearly, if we want to check the stability of an LSS for every switching sequence, we have the check the stability of every induced LTV system as stated below:

Corollary 9: Consider the LSS P_σ as in (5). Let Ξ be a set of switching sequences containing the sequences of interest. Then, P_σ is stable for any $\sigma \in \Xi$ if and only if there exists a stable switching system Q_σ such that

$$\sup_{\sigma \in \Xi} \left\|\hat{A}_\sigma (I + \Lambda Q_\sigma) - Q_\sigma\right\| < 1, \tag{13}$$

$$\sup_{\sigma \in \Xi} \left\|(I + Q_\sigma \Lambda)\hat{A}_\sigma - Q_\sigma\right\| < 1. \tag{14}$$

We note that conditions (13) and (14) are in the so-called model matching form. In what follows, we discuss how (13) can be cast as a linear program if the norm is the l_∞ induced. Notice that given Q_σ satisfying (13), it can be approximated arbitrarily closely by an FIR input-output or output-only LSS. Therefore, the following holds true:

Theorem 10: Consider the LSS P_σ as in (5). Let Ξ be a set of switching sequences containing the sequences of interest.

Then, P_σ is stable for any $\sigma\in\Xi$ if and only if there exists an integer M such that one of the following holds:

$$\inf_{Q_\sigma\in\mathscr{S}_{IO}^M}\sup_{\sigma\in\Xi}\left\|\hat{A}_\sigma\left(I+\Lambda Q_\sigma\right)-Q_\sigma\right\| \quad < \quad 1, \tag{15}$$

$$\inf_{Q_\sigma\in\mathscr{S}_{O}^M}\sup_{\sigma\in\Xi}\left\|\hat{A}_\sigma\left(I+\Lambda Q_\sigma\right)-Q_\sigma\right\| \quad < \quad 1. \tag{16}$$

It is easy to see that for $Q_\sigma\in\mathscr{S}_{IO}^M$ or $Q_\sigma\in\mathscr{S}_{O}^M$, the mapping $\hat{A}_\sigma\left(I+\Lambda Q_\sigma\right)-Q_\sigma$ is indeed an input-output LSS of degree $M+1$. Therefore, Lemma 5 can be used to reduce (15) or (16) to LPs in the l_∞ case.

V. Gain Computation

In the previous section, we looked at the LTV systems (or LSS) as operators mapping $\begin{pmatrix} x_0 \\ u \end{pmatrix}$ to $\begin{pmatrix} x \\ y \end{pmatrix}$ and derived conditions for their boundedness (stability). For a bounded operator, we will proceed to quantifying its bound, a.k.a. its gain. Conventionally, in the context of finding the gain of linear system, the initial condition is set to zero and further, without loss of generality, only the effect of u on y is studied. We emphasize that our results hold true for any induced norm. In particular, the computations can be cast as a LP if the l_∞ induced norm is considered. First, we state the results for LTV systems:

Theorem 11: Consider the LTV system G in (3). Then G is stable and $\|G\|<1$ if and only if there exist a stable LTV Q and $\delta>0$ such that the following holds

$$\left\|\begin{bmatrix} [A(I+\Lambda Q)-Q] & \delta\left[A(I+\Lambda Q)-Q\right]\Lambda B \\ \frac{1}{\delta}\hat{C}(I+\Lambda Q) & \hat{C}(I+\Lambda Q)\Lambda B+\hat{D}\end{bmatrix}\right\|<1. \tag{17}$$

We note that (17) is not convex in both δ and Q. It is, however, convex given δ. Condition (17) can be further simplified for the l_∞ case as follows:

Corollary 12: The LTV system G is stable and $\|G\|_{l_\infty\text{-ind}}<\gamma$ for some $\gamma>0$ if and only if there exist a stable LTV Q and $\delta>0$ such that the following hold

$$\left\|\hat{A}(I+\Lambda Q)-Q\right\|_{l_\infty\text{-ind}} + \delta\left\|\hat{A}(I+\Lambda Q)\Lambda\hat{B}-Q\Lambda\hat{B}\right\|_{l_\infty\text{-ind}}<1, \tag{18}$$

$$\frac{1}{\delta}\left\|\hat{C}(I+\Lambda Q)\right\|_{l_\infty\text{-ind}} + \left\|\hat{C}(I+\Lambda Q)\Lambda\hat{B}+\hat{D}\right\|_{l_\infty\text{-ind}}<\gamma. \tag{19}$$

We note that if G is stable, (10) holds true and hence for sufficiently small value of δ, (18) and (19) admit a solution (Q,γ). Therefore, in principle, one can start from small values δ and gradually increase δ until either (18) becomes infeasible or the desired performance level γ is met. In fact, if $\|G\|_{l_\infty\text{-ind}}<\gamma$, then (18) and (19) admit a solution for large enough δ that is quantified in the next proposition.

Proposition 13: Suppose $\|G\|_{l_\infty\text{-ind}}<\gamma$. Then, the set of δ for which there exists a Q satisfying (18) and (19) contains the semi-infinite interval $(\delta_0,+\infty)$, where

$$\delta_0=\frac{\left\|\hat{C}\left(I-\Lambda\hat{A}\right)^{-1}\right\|_{l_\infty\text{-ind}}}{\gamma-\|G\|_{l_\infty\text{-ind}}}.$$

This proposition is particularly useful since it guarantees that if one keeps increasing δ and checking the feasibility of (18) and (19) for the given δ, the procedure eventually stops once δ is greater that δ_0. Theorem 11 can be extended for LSS as below:

Corollary 14: Let LSS P_σ be given as in (5). Then P_σ is stable and $\|P_\sigma\|_{l_\infty\text{-ind}}<\gamma$ for any $\sigma\in\Xi$ if and only if there exist $\delta,\gamma_i>0$, for $i\in\{1,2,3,4,\}$, a positive integer M, and $Q_\sigma\in\mathscr{S}_O^M$ such that

$$\begin{aligned} \gamma_1+\delta\gamma_2 &< 1, \\ \frac{1}{\delta}\gamma_2+\gamma_3 &< \gamma, \end{aligned}$$

and

$$\sup_{\sigma\in\Xi}\left\|\hat{A}_\sigma\left(I+\Lambda Q_\sigma\right)-Q_\sigma\right\|_{l_\infty\text{-ind}}<\gamma_1, \tag{20}$$

$$\sup_{\sigma\in\Xi}\left\|\hat{A}_\sigma\left(I+\Lambda Q_\sigma\right)\Lambda\hat{B}_\sigma-Q_\sigma\Lambda\hat{B}_\sigma\right\|_{l_\infty\text{-ind}}<\gamma_2, \tag{21}$$

$$\sup_{\sigma\in\Xi}\left\|\hat{C}_\sigma\left(I+\Lambda Q_\sigma\right)\right\|_{l_\infty\text{-ind}}<\gamma_3 \tag{22}$$

$$\sup_{\sigma\in\Xi}\left\|\hat{C}_\sigma\left(I+\Lambda Q_\sigma\right)\Lambda\hat{B}_\sigma+\hat{D}_\sigma\right\|_{l_\infty\text{-ind}}<\gamma_4. \tag{23}$$

Using Lemma 5 and Remark 7, one can cast (20)-(23) as LPs.

VI. Stabilizability

Consider a LTV system H with the exogenous input w, control input u, measured output y, and regulated output z

$$H:\begin{cases} \Lambda^{-1}x=\hat{A}x+\hat{B}^w w+\hat{B}^u u \\ z=\hat{C}^z x+\hat{D}^{zw}w+\hat{D}^{zu}u \\ y=\hat{C}^y x+\hat{D}^{yw}w \end{cases}. \tag{24}$$

It can be easily shown that a state-feedback controller $K:x\to u$ stabilizes the closed-loop if and only if $\hat{A}+\hat{B}K$ stabilizes Λ. According to Theorem 8, $\left[I-\Lambda\left(\hat{A}+\hat{B}K\right)\right]^{-1}$ is stable if and only if there exist two stable LTV systems Q and Z such that

$$\left\|\hat{A}(I+\Lambda Q)+\hat{B}^u Z-Q\right\|<1, \tag{25}$$

where $Z=K(I+\Lambda Q)$. Clearly, (25) is convex in Q and Z and can be seen as a state-feedback stabilizability check for (24). We further develop an output-feedback stabilizability test as follows:

Theorem 15: There exists a stabilizing output-feedback controller if and only if there exist stable LTV systems Q, Z^F, and Z^L such that

$$\begin{aligned} \left\|\hat{A}(I+\Lambda Q)+\hat{B}^u Z^F-Q\right\| &< 1, \\ \left\|(I+Q\Lambda)\hat{A}+Z^L\hat{C}^y-Q\right\| &< 1. \end{aligned} \tag{26}$$

In this case the controller is given by

$$K:\begin{cases} \chi=\Lambda\left(\hat{A}+\hat{B}^u F+L\hat{C}^y\right)\chi-\Lambda L y \\ u=F\chi \end{cases},$$

where $F=Z^F(I+\Lambda Q)^{-1}$ and $L=(I+Q\Lambda)^{-1}Z^L$.

It is obvious at this point that this theorem can be immediately extended to LSS by letting $Q\in\mathscr{S}_O^M$ and $Z^F,Z^L\in\mathscr{S}_O^{M+1}$. In this case, the mappings in (26) become input-output LSS of degree $M+1$ and (26) can be converted to a LP in the case of l_∞.

	Achievable l_∞ gain		
δ	$T=2$	$T=3$	$T=5$
1	17.71	15.02	11.72
5	4.71	4.42	3.97
10	3.11	2.80	2.63
50	2.01	1.72	1.49
100	2.01	1.68	1.41
1000	2.00	1.67	1.41
10000	2.00	1.67	1.41
	T : FIR order of Q_σ		

TABLE I

CLOSED-LOOP GAIN

VII. CONTROL SYNTHESIS

Based on our developments in the previous sections, one can synthesize controllers that guarantee certain performance level. In this paper, we discuss the state-feedback control synthesis; the output-feedback is the subject of our future research.

Consider a LSS plant given by

$$P_\sigma : \begin{cases} \Lambda^{-1}x = \hat{A}_\sigma x + \hat{B}^w_\sigma w + \hat{B}^u_\sigma u \\ z = \hat{C}^z_\sigma x + \hat{D}^{zw}_\sigma w + \hat{D}^{zu}_\sigma u \end{cases},$$

and a switching state-feedback controller $K_\sigma : x \to u$. The closed-loop, Φ_σ, is given by

$$\Phi_\sigma : \Lambda^{-1}x = \hat{A}^{cl}_\sigma x + \hat{B}^{cl}_\sigma w, z = \hat{C}^{cl}_\sigma x + \hat{D}^{cl}_\sigma w,$$

where

$$\begin{aligned} \hat{A}^{cl}_\sigma &= \hat{A}_\sigma + \hat{B}^u_\sigma K, \hat{B}^{cl}_\sigma = \hat{B}^w_\sigma, \\ \hat{C}^{cl}_\sigma &= \hat{C}^z + \hat{D}^{zu}K, \hat{D}^{cl}_\sigma = \hat{D}^{zw}_\sigma. \end{aligned}$$

From Corollary 14 and letting $Z_\sigma = K_\sigma(I + \Lambda Q_\sigma)$, we have that Φ_σ is stable and $\|\Phi_\sigma\|_{l_\infty\text{-ind}} < \gamma$ if and only if there exist $\delta, \gamma_i > 0$, for $i \in \{1,2,3,4,\}$, a positive integer M, $Q_\sigma \in \mathscr{S}^M_O$, and $Z_\sigma \in \mathscr{S}^{M+1}_O$ (or $Z_\sigma \in \mathscr{S}^{M+1}_{IO}$) such that

$$\begin{aligned} \gamma_1 + \delta\gamma_2 &< 1, \qquad (27) \\ \frac{1}{\delta}\gamma_2 + \gamma_3 &< \gamma, \end{aligned}$$

and

$$\sup_{\sigma\in\Xi} \left\| \hat{A}_\sigma (I + \Lambda Q_\sigma) + \hat{B}^u_\sigma Z_\sigma - Q_\sigma \right\|_{l_\infty\text{-ind}} < \gamma_1, \qquad (28)$$

$$\sup_{\sigma\in\Xi} \left\| \hat{A}_\sigma (I + \Lambda Q_\sigma) \Lambda \hat{B}^w_\sigma + \hat{B}^u_\sigma Z_\sigma \Lambda \hat{B}^w_\sigma - Q_\sigma \Lambda \hat{B}^w_\sigma \right\|_{l_\infty\text{-ind}} < \gamma_2, \qquad (29)$$

$$\sup_{\sigma\in\Xi} \left\| \hat{C}^z_\sigma (I + \Lambda Q_\sigma) + \hat{D}^{zu} Z_\sigma \right\|_{l_\infty\text{-ind}} < \gamma_3 \qquad (30)$$

$$\sup_{\sigma\in\Xi} \left\| \hat{C}^z_\sigma (I + \Lambda Q_\sigma) \Lambda \hat{B}^w_\sigma + \hat{D}^{zu} Z_\sigma \Lambda \hat{B}^w_\sigma + \hat{D}^{zw}_\sigma \right\|_{l_\infty\text{-ind}} < \gamma_4. \qquad (31)$$

We emphasize again that the mappings in (28)-(31) are input-output LSS and their l_∞ induced norm can be computed via LP.

Example 16: Consider the barbell of length l illustrated in Figure 1. There is mass of size $m = 1kg$ that jumps from one end of the barbell to the other end. The actuator torque

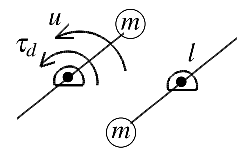

Fig. 1. a barbell with switching mass at the end

(control input) and the disturbance torque are labeled as u and τ, respectively. After letting l equal to the gravitational constant and discretizing the model at $2Hz$ we obtain $x_{t+1} = A_{\sigma_t}x_t + B^\tau_{\sigma_t}\tau_t + B^u_{\sigma_t}u_t$ where

$$\begin{aligned} A_1 &= \begin{bmatrix} 1.001 & 0.050 \\ 0.050 & 1.001 \end{bmatrix}, A_2 = \begin{bmatrix} 0.999 & 0.050 \\ -0.050 & 0.999 \end{bmatrix}, \\ B^\tau_1 &= B^\tau_2 = B^u_1 = B^u_2 = \begin{bmatrix} 0.001 \\ 0.050 \end{bmatrix}. \end{aligned}$$

We want to design a state-feedback controller, $K : x \to u$, that stabilizes the closed-loop and study the l_∞ gain of the closed-loop from the disturbance torque τ to the states and control input, i.e. $\begin{pmatrix} x \\ u \end{pmatrix}$. To this end, one can use (27) - (31). For this example, we let $Q_\sigma \in \mathscr{S}^1_O$ be FIR of some order T and $Z_\sigma \in \mathscr{S}^2_O$. Then we minimize (Q_σ, γ) subject to (27) - (31) for different values of δ as tabulated in Table I.

VIII. CONCLUSION

In this paper, we developed a new framework to analyze stability and stabilizability of LSS as well as their gain computations. We exploited the Youla parameterization in combination with the state-space operator descriptions to develop new stability and stabilizability tests. Furthermore, one can use this framework for control analysis and synthesis of LSS, as well as LTV systems, for any l_p induced norm. Also, we specialized our results to the l_∞ case and showed how LP can be used to test stability, stabilizability and to synthesize stabilizing controllers that guarantee a near optimal closed-loop gain. An appealing feature about our approach is that it relies on solving LPs which, in general, constitutes a reliable computational option to solve efficiently massive problems.

REFERENCES

[1] D. Liberzon, *Switching in systems and control*. Springer, 2003.
[2] H. Lin and P. Antsaklis, "Stability and stabilizability of switched linear systems: a survey of recent results," *Automatic Control, IEEE Transactions on*, vol. 54, no. 2, pp. 308–322, 2009.
[3] Z. Sun and S. S. Ge, *Switched linear systems: Control and design*. Springer, 2005.
[4] J.-W. Lee and G. E. Dullerud, "Uniform stabilization of discrete-time switched and markovian jump linear systems," *Automatica*, vol. 42, no. 2, pp. 205–218, 2006.
[5] F. Blanchini, S. Miani, and F. Mesquine, "A separation principle for linear switching systems and parametrization of all stabilizing controllers," *Automatic Control, IEEE Transactions on*, vol. 54, no. 2, pp. 279–292, 2009.
[6] A. Bhaya, F. Das, and C. Mota, "Equivalence of stability concepts for discrete time-varying systems," *International Journal of Robust and Nonlinear Control*, vol. 4, no. 6, pp. 725–740, 1994.

[7] P. Seiler and R. Sengupta, "An H_∞ approach to networked control," *Automatic Control, IEEE Transactions on*, vol. 50, no. 3, pp. 356–364, 2005.
[8] J. Lee and G. Dullerud, "Optimal disturbance attenuation for discrete-time switched and markovian jump linear systems," *SIAM Journal on Control and Optimization*, vol. 45, p. 1329, 2006.
[9] J. P. Hespanha, "Root-mean-square gains of switched linear systems," *Automatic Control, IEEE Transactions on*, vol. 48, no. 11, pp. 2040–2045, 2003.
[10] M. Margaliot and J. P. Hespanha, "Root-mean-square gains of switched linear systems: A variational approach," *Automatica*, vol. 44, no. 9, pp. 2398–2402, 2008.
[11] D. Roberson and D. Stilwell, "L_2 gain performance analysis of linear switched systems: fast switching behavior," in *American Control Conference, 2007. ACC'07.* IEEE, 2007, pp. 2084–2089.
[12] R. Essick, J.-W. Lee, and G. E. Dullerud, "Control of linear switched systems with receding horizon modal information," *Automatic Control, IEEE Transactions on*, vol. 59, no. 9, pp. 2340–2352, 2014.
[13] S. Jiang and P. Voulgaris, "Performance optimization of switched systems: a model matching approach," *Automatic Control, IEEE Transactions on*, vol. 54, no. 9, pp. 2058–2071, 2009.
[14] P. Voulgaris, "On the computation of l_∞ gains of switched systems," in *50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC) Orlando, FL, USA*, Dec 2011.
[15] M. Naghnaeian and P. Voulgaris, "On model matching problems of input-output switching systems," in *51th IEEE Conference on Decision and Control Conference, 2012, Maui, Hawaii*.
[16] ——, "Characterization and optimization of l_∞ gains of linear switched systems," *Automatic Control, IEEE Transactions on*, To appear.