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A lifting approach to $\mathcal{L}_2$-gain analysis of periodic event-triggered and switching sampled-data control systems

W.P.M.H. Heemels G.E. Dullerud A.R. Teel

Abstract— **In this work we are interested in the stability and $\mathcal{L}_2$-gain of hybrid systems with linear flow dynamics, periodic time-triggered jumps and nonlinear possibly set-valued jump maps. This class of hybrid systems includes various interesting applications such as periodic event-triggered control. In this paper we also show that sampled-data systems with arbitrarily switching controllers can be captured in this framework by requiring the jump map to be set-valued. We provide novel conditions for the internal stability and $\mathcal{L}_2$-gain analysis of these systems adopting a lifting-based approach. In particular, we establish that the internal stability and contractivity in terms of an $\mathcal{L}_2$-gain smaller than 1 are equivalent to the internal stability and contractivity of a particular discrete-time set-valued nonlinear system. Despite earlier works in this direction, these novel characterisations are the first necessary and sufficient conditions for the stability and the contractivity of this class of hybrid systems. The results are illustrated through multiple new examples.**

I. Introduction

In this paper we will study hybrid systems [1] of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\xi\\ \tau\end{bmatrix} = \begin{bmatrix}A\xi + Bw\\ 1\end{bmatrix}, \text{ when } \tau \in [0,h] \tag{1a}$$

$$\begin{bmatrix}\xi^+\\ \tau^+\end{bmatrix} \in \phi(\xi)\times\{0\}, \text{ when } \tau = h \tag{1b}$$

$$z = C\xi + Dw. \tag{1c}$$

The states of this hybrid system consist of $\xi \in \mathbb{R}^{n_\xi}$ and a timer variable $\tau \in \mathbb{R}_{\geq 0}$. The variable $w \in \mathbb{R}^{n_w}$ denotes the disturbance input and z the performance output. Moreover, A, B, C, D are constant real matrices of appropriate dimensions, $h \in \mathbb{R}_{>0}$ is a positive timer threshold, and $\phi : \mathbb{R}^{n_\xi} \rightrightarrows \mathbb{R}^{n_\xi}$ denotes an arbitrary nonlinear possibly set-valued map with $\phi(0) = \{0\}$. Note that $\phi(0) = \{0\}$ guarantees that the set $\{\begin{bmatrix}\xi\\ \tau\end{bmatrix} \mid \xi = 0 \text{ and } \tau \in [0,h]\}$ is an equilibrium set of (1) in absence of disturbances ($w = 0$).

Maurice Heemels is with the Control System Technology Group, Department of Mechanical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands, e-mail: m.heemels@tue.nl. Geir E. Dullerud is with the Mechanical Science and Engineering Department, and Coordinated Science Laboratory, University of Illinois, Urbana, IL 61801, e-mail: dullerud@illinois.edu. Andrew R. Teel is with the Department of Electrical and Computer Engineering, University of California, Santa Barbara, CA, USA, e-mail: teel@ece.ucsb.edu.

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Interpreting the dynamics of (1) indicates that (1) has *periodic* time-triggered jump conditions, i.e., jumps take place at times kh, $k \in \mathbb{N}$ (when $\tau(0) = h$), according to a nonlinear jump map as given by (1b). In between the jumps the system flows according to the differential equation in (1a). This class of systems is of relevance in various applications and includes the closed-loop systems arising from periodic event-triggered control (PETC) for linear systems [2], networked control with constant transmission intervals and a shared networked requiring network protocols [3], [4], reset control systems [5], [6], [7], [8], [9] with periodically verified reset conditions, and sampled-data saturated controls [10], see [11], [12]. Furthermore, in this paper we will show that also sampled-data systems with arbitrarily switching controllers can be captured in this framework by allowing the jump map to be set-valued. In fact, the previously mentioned applications all had single-valued and piecewise affine (PWA) jump maps ϕ. Given this broad collection of relevant applications fitting the description (1), it is of interest to study the stability and $\mathcal{L}_2$-gain analysis from disturbances w to output z for this class of systems. This paper recounts the main results of [11] and illustrates their application through multiple, novel examples, including the mentioned arbitrarily switching sampled-data control systems. Moreover, extensions are presented with respect to [11] to accommodate the study of set-valued dynamics. The study of set-valued dynamics is not only motivated by new applications such as the arbitrarily switching sampled-data controllers that directly require models as in (1) with set-valued ϕ. Set-valued functions ϕ become also important when robust stability and robust $\mathcal{L}_2$-gain statements are desired, which often require regularization of discontinuous jump maps, as, for instance, appear in the applications of PETC, networked control systems and reset control systems. This regularisation is then needed to obtain outer semi-continuity properties of the jump maps, which are needed for providing robust stability and performance properties, see [13] and Remark III.1 below for more details.

Earlier works, see, e.g., [2], [10], [1], [12] already studied how to provide conditions that lead to upper bounds on the $\mathcal{L}_2$-gain for hybrid systems of the form (1) with ϕ a piecewise linear (PWL) or piecewise affine (PWA) map. Using Riccati differential equations to construct suitable Lyapunov functions, LMI-based conditions providing upper

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bounds on the $\mathcal{L}_2$-gain were derived. However, the existing results only provided *sufficient* conditions and it is not clear how far these conditions are from necessity. Inspired by this question, we present in this paper a new perspective on the problem using ideas from lifting as adopted in sampled-data control of linear systems [14], [15], [16], [17], [18], [19]. The classical literature on sampled-data control, see, e.g., [14], [15], [16], [17], [18], [19] all required the *linearity* of both plant and controller. Obviously, for systems of the form (1) the linearity property does not hold when ϕ is nonlinear. In this paper we review and illustrate the main results of [11] with novel examples and new applications, in additional to providing extensions towards set-valued maps ϕ. We will show that the internal stability and contractivity of the hybrid system (1) (in the sense of an $\mathcal{L}_2$ gain smaller than 1) are equivalent to the internal stability and contractivity of a specific discrete-time set-valued nonlinear system. As such, to the best of our knowledge, the present work and [11] are the first to show that lifting-like techniques are also useable without linearity properties and still lead to computationally friendly checks (in this case in terms of semi-definite programming problems). This new method provides much better bounds for the contractivity of (1) than the earlier results in [2], [10], [1] and [12] due to the fact that a *necessary and sufficient* condition is obtained in terms of the ℓ_2-gain of a specific discrete-time system, instead of only sufficient conditions. Through various new examples and applications we will show how the results can be applied in a numerically tractable manner.

II. PRELIMINARIES

For X, Y Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_X$ and $\langle \cdot, \cdot \rangle_Y$, respectively, a linear operator $U : X \to Y$ is called isometric if $\langle Ux_1, Ux_2 \rangle_Y = \langle x_1, x_2 \rangle_X$ for all $x_1, x_2 \in X$. We denote by $U^* : Y \to X$ the (Hilbert) adjoint operator that satisfies $\langle Ux, y \rangle_Y = \langle x, U^* y \rangle_X$ for all $x \in X$ and all $y \in Y$. Note that U being isometric is equivalent to $U^* U = I$. The operator U is called an isomorphism if it is an invertible mapping. The induced norm of U (provided it is finite) is denoted by $\|U\|_{X,Y} = \sup_{x \in X \setminus \{0\}} \frac{\|Ux\|_Y}{\|x\|_X}$. If the induced norm is finite we say that U is a bounded linear operator. If $X = Y$ we write $\|U\|_X$ and if X, Y are clear from the context we use the notation $\|U\|$.

To a Hilbert space X with inner product $\langle \cdot, \cdot \rangle_X$, we can associate the Hilbert space $\ell_2(X)$ consisting of infinite sequences $\tilde{x} = (\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \ldots)$ with $\tilde{x}_i \in X$, $i \in \mathbb{N}$, satisfying $\sum_{i=0}^{\infty} \|\tilde{x}_i\|_X^2 < \infty$, and the inner product $\langle \tilde{x}, \tilde{y} \rangle_{\ell_2(X)} = \sum_{i=0}^{\infty} \langle \tilde{x}_i, \tilde{y}_i \rangle_X$. We denote $\ell_2(\mathbb{R}^n)$ by ℓ_2 when $n \in \mathbb{N}_{\geq 1}$ is clear from the context. We also use the notation $\ell(X)$ to denote the set of all infinite sequences $\tilde{x} = (\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \ldots)$ with $\tilde{x}_i \in X$, $i \in \mathbb{N}$. As usual, we denote by $\mathbb{R}^n$ the standard n-dimensional Euclidean space with inner product $\langle x, y \rangle = x^\top y$ and norm $|x| = \sqrt{x^\top x}$ for $x, y \in \mathbb{R}^n$. $\mathcal{L}_2^n([0, \infty))$ denotes the set of square-integrable functions defined on $\mathbb{R}_{\geq 0} := [0, \infty)$ and taking values in $\mathbb{R}^n$ with $\mathcal{L}_2$-norm $\|x\|_{\mathcal{L}_2} = \sqrt{\int_0^\infty |x(t)|^2 dt}$ and inner product $\langle x, y \rangle_{\mathcal{L}_2} = \int_0^\infty x^\top(t) y(t) dt$ for $x, y \in \mathcal{L}_2^n([0, \infty))$. If n is clear from the context we also write $\mathcal{L}_2$. We also use square-integrable functions on subsets $[a, b]$ of $\mathbb{R}_{\geq 0}$ and then we write $\mathcal{L}_2^n([a, b])$ (or $\mathcal{L}_2([a, b])$ if n is clear from context) with the inner product and norm defined analogously. The set $\mathcal{L}_{2,e}^n([0, \infty))$ consists of all locally square-integrable functions, i.e., all functions x defined on $\mathbb{R}_{\geq 0}$, such that for each bounded domain $[a, b] \subset \mathbb{R}_{\geq 0}$ the restriction $x \mid_{[a,b]}$ is contained in $\mathcal{L}_2^n([a, b])$. We also will use the set of essentially bounded functions defined on $\mathbb{R}_{\geq 0}$ or $[a, b] \subset \mathbb{R}_{\geq 0}$, which are denoted by $\mathcal{L}_\infty^n([0, \infty))$ or $\mathcal{L}_\infty^n([a, b])$ with the norm given by the essential supremum denoted by $\|x\|_{\mathcal{L}_\infty}$ for an essentially bounded function x. A function $\beta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is called a $\mathcal{K}$-function if it is continuous, strictly increasing and $\beta(0) = 0$.

Consider the discrete-time system of the form

$$\begin{pmatrix} \xi_{k+1} \\ r_k \end{pmatrix} \in \psi(\xi_k, v_k) \tag{2}$$

with disturbance input $v_k \in V$, performance output $r_k \in R$, and state $\xi_k \in \mathbb{R}^{n_\xi}$ at discrete time $k \in \mathbb{N}$, where V and R are Hilbert spaces, and $\psi : \mathbb{R}^{n_\xi} \times V \rightrightarrows \mathbb{R}^{n_\xi} \times R$ a set-valued mapping.

Definition II.1 The discrete-time system (2) is said to have an ℓ_2-gain from v to r smaller than γ if there exist a $\gamma_0 \in [0, \gamma)$ and a $\mathcal{K}$-function β such that, for any $v \in \ell_2(V)$ and any initial state $\xi_0 \in \mathbb{R}^{n_\xi}$, the corresponding solutions to (2) satisfy

$$\|r\|_{\ell_2(R)} \leq \beta(|\xi_0|) + \gamma_0 \|v\|_{\ell_2(V)}. \tag{3}$$

Sometimes we also use the terminology γ-contractivity if this property holds. Moreover, 1-contractivity is also called contractivity.

Definition II.2 The discrete-time system (2) is said to be internally stable if there is a $\mathcal{K}$-function β such that, for any $v \in \ell_2(V)$ and any initial state $\xi_0 \in \mathbb{R}^{n_\xi}$, the corresponding solutions ξ to (2) satisfy

$$\|\xi\|_{\ell_2} \leq \beta(\max(|\xi_0|, \|v\|_{\ell_2(V)})). \tag{4}$$

Note that since $\|\xi\|_{\ell_\infty} \leq \|\xi\|_{\ell_2}$ and $\|\xi\|_{\ell_2} < \infty$ implies $\lim_{k \to \infty} \xi_k = 0$, we also have global attractivity and Lyapunov stability properties of the origin when the discrete-time system is internally stable.

III. APPLICATIONS OF MODELLING FRAMEWORK

Several applications fit the hybrid system models (1), see [11], [12]. We will demonstrate here shortly how PETC applications fit the models (1) and show also how sampled-data systems with arbitrarily switching controllers can be modelled as in (1).

A. Periodic Event-Triggered Control Systems

ETC is a control strategy that is designed to reduce the amount of computation and communication in a control system by updating and communicating sensor and actuator data only as needed to guarantee certain stability or performance properties, see, e.g., [20] for a recent overview. The ETC strategy that we consider in this paper combines ideas from periodic sampled-data control and ETC, leading to so-called periodic event-triggered control (PETC) systems [2]. In PETC, the event-triggering condition is verified periodically in time instead of continuously as in standard ETC, see, e.g., [21], [22], [23]. As such, at every sampling time it is decided whether or not new measurements and control signals need to be transmitted and updated, respectively.

We assume the plant to be given by

$$\frac{\mathrm{d}}{\mathrm{d}t}x_p = A_p x_p + B_{pu}u + B_{pw}w, \tag{5}$$

which will be controlled using a PETC strategy specified by

$$u(t) = K\hat{x}_p(t), \quad \text{for} \quad t \in \mathbb{R}_{\geq 0}, \tag{6}$$

where $\hat{x}_p \in \mathbb{R}^{n_p}$ is a left-continuous signal given for $t \in (t_k, t_{k+1}]$, $k \in \mathbb{N}$, by

$$\hat{x}_p(t) = \begin{cases} x_p(t_k), & \text{when } \xi(t_k)^\top Q \xi(t_k) > 0, \\ \hat{x}_p(t_k), & \text{when } \xi(t_k)^\top Q \xi(t_k) \leq 0, \end{cases} \tag{7}$$

where $\xi := [x_p^\top \ \ \hat{x}_p^\top]^\top$ and $t_k = kh$, $k \in \mathbb{N}$, are the sampling times with $h > 0$ the sampling period, see Fig. 1. Note that $\hat{x}_p$ can be interpreted as the most recently received measurement of the state x_p available at the controller. Whether or not $\hat{x}_p(t_k)$ is transmitted at time t_k is determined as follows: If at time t_k it holds that $\xi^\top(t_k)Q\xi(t_k) > 0$, the state information $x_p(t_k)$ is sent at time t_k to the controller and $\hat{x}_p$ and u are updated accordingly. However, if $\xi^\top(t_k)Q\xi(t_k) \leq 0$, the current state information is not transmitted and $\hat{x}_p$ and u keep the same value for (at least) another sampling interval. In [2] it was shown that such quadratic event-triggering conditions form a relevant class of event generators, as many popular versions can be written in this form. For instance, the event-triggering condition proposed in [21] is given by $\|\hat{x}_p(t_k) - x_p(t_k)\| > \rho\|x_p(t_k)\|$ with $\rho > 0$ a positive constant, which can obviously be written in the quadratic form in (7). The complete closed-loop PETC system can be written as the hybrid system (1), by combining (5), (6) and (7), leading to $A = \begin{bmatrix} A_p & B_{pu}K \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} B_{pw} \\ 0 \end{bmatrix}$, and ϕ a PWL map as in

$$\phi(\xi) = \begin{cases} \{J_1\xi\}, & \text{when } \xi^\top Q\xi > 0 \\ \{J_2\xi\}, & \text{when } \xi^\top Q\xi \leq 0 \end{cases} \tag{8}$$

with $J_1 = \begin{bmatrix} I_{n_p} & 0 \\ I_{n_p} & 0 \end{bmatrix}$ and $J_2 = I_{n_\xi}$. Clearly, in addition to the static state-feedback controllers (6) discussed here, also dynamic output-feedback PETC and output-based event-triggering conditions can be easily modeled in the framework of (1), see [2].

Remark III.1 For the PETC applications mentioned above, but also for networked and reset control applications, see [11], the function ϕ is given by (8), which is a discontinuous map not satisfying outer semi-continuity properties in the sense that the graph of the map is closed. For studying general *robust* stability and *robust* $\mathcal{L}_2$-gain properties of (1), it is shown in [13], [24] that outer semi-continuity of the jump map plays a crucial role. By using Krasovskii regularisations of the jump map outer semi-continuity is obtained, which modifies the map (8) to the outer semi-continuous map

$$\phi(\xi) = \begin{cases} \{J_1\xi\}, & \text{when } \xi^\top Q\xi > 0 \\ \{J_1\xi, J_2\xi\}, & \text{when } \xi^\top Q\xi = 0 \\ \{J_2\xi\}, & \text{when } \xi^\top Q\xi < 0. \end{cases} \tag{9}$$

Note that now $\phi(\xi)$ contains multiple values for certain ξ. Hence, in order to establish robust versions of internal stability and contractivity, it is useful to work with the map (9) instead of the one in (8), see [13], [24] for further details. Hence, as already indicated in the introduction, this provides a further motivation for extending the results of [11] towards set-values mappings ϕ. Note that for the PETC application the use of the regularised map in (9) indicates that when $\xi^\top Q\xi = 0$ two situations can happen as either the state transmission and control update take place or not.

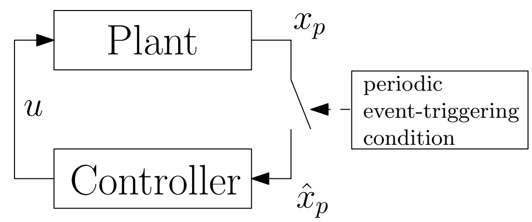

Fig. 1. Schematic representation of an event-triggered control system.

B. Sampled-data systems with switching controllers

The study of switching (linear) systems under arbitrary switching has received ample attention both in continuous-time and discrete-time settings, see, e.g., [25], [26], [27]. Also $\mathcal{L}_2$- and ℓ_2-gains of continuous-time and discrete-time switched linear systems, respectively, under arbitrary switching were studied, see, e.g., [28]. However, the $\mathcal{L}_2$-gain analysis of a continuous-time linear system under switching *sampled-data* linear controllers was, to the best of the authors' knowledge, not addressed before. To study and formalise this problem we will focus here for simplicity on sampled-data static state-feedback control, although the extension to discrete-time output-based dynamic controllers is straightforward.

Therefore, we consider the plant as in (5) and assume that this plant for a given performance output z when the system is controlled by the feedback law

$$u(t) = K_{\sigma_k} x_p(t_k), \text{ when } t \in (t_k, t_{k+1}] \tag{10}$$

for some $\sigma_k \in \mathcal{M} := \{1, 2, \ldots, M\}$ with $t_k = kh$, $k \in \mathbb{N}$. Hence, σ_k indicates which controller gain is activated at time t_k, $k \in \mathbb{N}$, and this "control mode" can switch in an arbitrary fashion. Hence, by introducing the state variable $\xi := [x_p^\top \ \ u^\top]^\top$ we can capture this setup in the framework of (1) by taking $A = \begin{bmatrix} A_p & B_{pu} \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} B_{pw} \\ 0 \end{bmatrix}$, and ϕ a set-valued map as in

$$\phi(\xi) = \{J_1\xi, J_2\xi, \ldots, J_M\xi\} \tag{11}$$

with $J_i := \begin{bmatrix} I & 0 \\ K_i & 0 \end{bmatrix}$, $i \in \mathcal{M}$. Hence, an $\mathcal{L}_2$-gain analysis of a continuous-linear system under switching *sampled-data* linear controllers boils down to an $\mathcal{L}_2$-gain analysis of a hybrid system in the form (1).

IV. Stability and contractivity notions

The objective of the paper is to study the $\mathcal{L}_2$-gain and internal stability of the system (1). We focus here on $\mathcal{L}_2$-gain smaller than 1, called contractivity, but note that by proper scaling of C and D matrices in (1), one can determine if the $\mathcal{L}_2$-gain is smaller than any value $\gamma \in \mathbb{R}_{>0}$.

Definition IV.1 The hybrid system (1) is said to have an $\mathcal{L}_2$-gain from w to z smaller than γ (or the system is γ-*contractive*) if there exist a $\gamma_0 \in [0, \gamma)$ and a $\mathcal{K}$-function β such that, for any $w \in \mathcal{L}_2$ and any initial condition given by $\xi(0) = \xi_0$ and $\tau(0) = h$, the corresponding solutions to (1) satisfy $\|z\|_{\mathcal{L}_2} \leq \beta(|\xi_0|) + \gamma_0\|w\|_{\mathcal{L}_2}$. The system is called *contractive*, if it is 1-contractive.

Definition IV.2 The hybrid system (1) is said to be *internally stable* if there exists a $\mathcal{K}$-function β such that, for any $w \in \mathcal{L}_2$ and any initial condition given by $\xi(0) = \xi_0$ and $\tau(0) = h$, the corresponding solutions to (1) satisfy $\|\xi\|_{\mathcal{L}_2} \leq \beta(\max(|\xi_0|, \|w\|_{\mathcal{L}_2}))$.

Note that the requirement $\|\xi\|_{\mathcal{L}_2} \leq \beta(\max(|\xi_0|, \|w\|_{\mathcal{L}_2}))$ in the definition of internal stability is rather natural here as we assume $\mathcal{L}_2$-disturbances and investigate $\mathcal{L}_2$-gains. Indeed, analogous to Definition IV.1 in which a bound is imposed on the $\mathcal{L}_2$-norm of the output z (expressed in terms of a bound on $|\xi_0|$ and $\|w\|_{\mathcal{L}_2}$), we require in Definition IV.2 that the norm of the state trajectory ξ is confined by more flexible (nonlinear) bounds in terms of on $|\xi_0|$ and $\|w\|_{\mathcal{L}_2}$.

Below we will establish that this definition of internal stability implies also global attractivity of the origin (i.e., $\lim_{t\to\infty} \xi(t) = 0$ for all $w \in \mathcal{L}_2$, $\xi(0) = \xi_0$ and $\tau(0) = h$) and also Lyapunov stability of the origin as we will have $\|\xi\|_{\mathcal{L}_\infty} \leq \beta'(\max(|\xi_0|, \|w\|_{\mathcal{L}_2}))$ for some $\mathcal{K}$-function β', see Proposition V.1 below.

V. Internal stability and $\mathcal{L}_2$-gain analysis

In this section we will analyze the $\mathcal{L}_2$-gain and the internal stability of (1) using ideas from lifting [14], [15], [16], [17], [18], [19].

A. Lifting

To study contractivity, we introduce the lifting operator $W : \mathcal{L}_{2,e}[0,\infty) \to \ell(\mathcal{K})$ with $\mathcal{K} = \mathcal{L}_2[0,h]$ given for $w \in \mathcal{L}_{2,e}[0,\infty)$ by $W(w) = \tilde{w} = (\tilde{w}_0, \tilde{w}_1, \tilde{w}_2, \ldots)$ with

$$\tilde{w}_k(s) = w(kh + s) \text{ for } s \in [0,h] \tag{12}$$

for $k \in \mathbb{N}$. Obviously, W is a linear isomorphism mapping $\mathcal{L}_{2,e}[0,\infty)$ into $\ell(\mathcal{K})$ and, moreover, W is isometric as a mapping from $\mathcal{L}_2[0,\infty)$ to $\ell_2(\mathcal{K})$. Using this lifting operator, we can rewrite the model in (1) as

$$\xi_{k+1} = \hat{A}\xi_k^+ + \hat{B}\tilde{w}_k \tag{13a}$$
$$\xi_k^+ \in \phi(\xi_k) \tag{13b}$$
$$\tilde{z}_k = \hat{C}\xi_k^+ + \hat{D}\tilde{w}_k \tag{13c}$$

in which ξ_0 is given and $\xi_k = \xi(kh^-) = \lim_{s\uparrow kh} \xi(s)$, $k \in \mathbb{N}_{\geq 1}$, and $\xi_k^+ = \xi(kh^+) = \lim_{s\downarrow kh} \xi(s) = \xi(kh)$ (assuming that ξ is continuous from the right) for $k \in \mathbb{N}$, and $\tilde{w} = (\tilde{w}_0, \tilde{w}_1, \tilde{w}_2, \ldots) = W(w) \in \ell_2(\mathcal{K})$ and $\tilde{z} = (\tilde{z}_0, \tilde{z}_1, \tilde{z}_2, \ldots) = W(z) \in \ell(\mathcal{K})$. Here we assume in line with Definition IV.1 that $\tau(0) = h$ in (1). Moreover,

$$\hat{A} : \mathbb{R}^{n_\xi} \to \mathbb{R}^{n_\xi} \qquad \hat{B} : \mathcal{K} \to \mathbb{R}^{n_\xi}$$
$$\hat{C} : \mathbb{R}^{n_\xi} \to \mathcal{K} \qquad \hat{D} : \mathcal{K} \to \mathcal{K}$$

are given for $x \in \mathbb{R}^{n_\xi}$ and $\omega \in \mathcal{K}$ by

$$\hat{A}x = e^{Ah}x \tag{14a}$$
$$\hat{B}\omega = \int_0^h e^{A(h-s)}B\omega(s)ds \tag{14b}$$
$$(\hat{C}x)(\theta) = Ce^{A\theta}\xi \tag{14c}$$
$$(\hat{D}\omega)(\theta) = \int_0^\theta Ce^{A(\theta-s)}B\omega(s)ds + D\omega(\theta), \tag{14d}$$

where $\theta \in [0,h]$.

By determining the trajectories of (1) explicitly, comparing them to the expressions (14) and exploiting that W is an isometric isomorphism, it is not hard to see that (13) is contractive if and only if (1) is contractive. Moreover, by extending a result in [11], we have the following proposition.

Proposition V.1 *The following statements hold:*

- *The hybrid system* (1) *is internally stable if and only if the discrete-time system* (13) *is internally stable.*
- *The hybrid system* (1) *is contractive if and only if the discrete-time system* (13) *is contractive.*
- *Furthermore, in case* (1) *is internally stable, it also holds that* $\lim_{t\to\infty} \xi(t) = 0$ *and there exists a* $\mathcal{K}$-*function* β' *such that* $\|\xi\|_{\mathcal{L}_\infty} \leq \beta'(\max(|\xi_0|, \|w\|_{\mathcal{L}_2}))$ *for all* $w \in \mathcal{L}_2$, $\xi(0) = \xi_0$ *and* $\tau(0) = h$.

B. Main result

The following result is an extension of the main result of [11]. The proof can be obtained based on the proofs in [11]. Note that a necessary condition for (1) and its lifted version (13) to be contractive is that the induced gain $\|\hat{D}\|_{\mathcal{K}} < 1$.

Theorem V.2 *Consider system* (1) *and its lifted version* (13) *with* $\|\hat{D}\|_{\mathcal{K}} < 1$. *Define the discrete-time nonlinear system*

$$\begin{pmatrix} \bar{\xi}_{k+1} \\ r_k \end{pmatrix} \in \begin{pmatrix} A_d \\ C_d \end{pmatrix} \phi(\bar{\xi}_k) + \begin{pmatrix} B_d \\ 0 \end{pmatrix} v_k \tag{15}$$

with A_d, B_d *and* C_d *real matrices of appropriate dimensions satisfying*

$$A_d = \hat{A} + \hat{B}\hat{D}^*(I - \hat{D}\hat{D}^*)^{-1}\hat{C} \tag{16a}$$

$$B_d B_d^\top = \bar{B}\bar{B}^* = \hat{B}(I-\hat{D}^*\hat{D})^{-1}\hat{B}^* \text{ and}$$
$$C_d^\top C_d = \bar{C}^*\bar{C} = \hat{C}^*(I-\hat{D}\hat{D}^*)^{-1}\hat{C}. \qquad (16b)$$

The system (1) *is internally stable and contractive if and only if the system* (15) *is internally stable and contractive.*

Remark V.3 To explicitly compute the discrete-time system (15) provided in Theorem V.2 we need to determine the operators $\hat{B}\hat{D}^*(I-\hat{D}\hat{D}^*)^{-1}\hat{C}$, $\hat{B}(I-\hat{D}^*\hat{D})^{-1}\hat{B}^*$, and $\hat{C}^*(I-\hat{D}\hat{D}^*)^{-1}\hat{C}$ to obtain the triple (A_d, B_d, C_d) in (15). These matrices can be computed explicitly based on the procedures in [29] under the assumption that $\|\hat{D}\|_{\mathcal{K}} < 1$. The condition $\|\hat{D}\|_{\mathcal{K}} < 1$ can be tested using Lemma 3.2 in [29] or Theorem 13.5.1 in [14]. See [11], where also an alternative test is given, and the complete procedure is described.

C. Main results translated for applications

In this subsection we will show how the main result as formulated in Theorem V.2 translates for the applications mentioned explicitly in Section III.

1) PETC applications: As shown in Section III, for the PETC applications the nonlinear mapping ϕ in the hybrid system (1) is piecewise linear as in (8) or, in case additional robustness properties are required (see Remark III.1), as in (9). As a consequence, the system (15) translates into a PWL system (using (9) and, for convenience of notation, writing ξ instead of $\bar{\xi}$) given by

$$\begin{pmatrix}\xi_{k+1}\\ r_k\end{pmatrix} \in \begin{cases} \left\{\begin{pmatrix}A_1\xi_k + B_d v_k\\ C_1\xi_k\end{pmatrix}\right\} & \text{if } \xi_k^\top Q\xi_k > 0\\ \left\{\begin{pmatrix}A_1\xi_k + B_d v_k\\ C_1\xi_k\end{pmatrix}, \begin{pmatrix}A_2\xi_k + B_d v_k\\ C_2\xi_k\end{pmatrix}\right\} & \text{if } \xi_k^\top Q\xi_k = 0\\ \left\{\begin{pmatrix}A_2\xi_k + B_d v_k\\ C_2\xi_k\end{pmatrix}\right\} & \text{if } \xi_k^\top Q\xi_k \le 0 \end{cases} \qquad (17)$$

for $k \in \mathbb{N}$, with $A_i = A_d J_i$, and $C_i = C_d J_i$, $i = 1, 2$. Hence, to study the internal stability and contractivity of (1) we have to determine the internal stability and contractivity of the (set-valued) *discrete-time PWL linear system* (17). Due to the PWL structure this analysis can be done by combining ideas from dissipativity theory [30], [31] and piecewise quadratic Lyapunov/storage functions [32], [33], which leads to (sufficient) LMI-based conditions for testing stability and contractivity.

2) Sampled-data system with arbitrarily switching controllers: For the control of the continuous-time linear system under arbitrarily switching sampled-data controllers as discussed in Section III, the map ϕ in the hybrid system (1) is set-valued as in (11). With this map ϕ, the system (15) translates into a discrete-time switching linear system (for convenience of notation we write ξ instead of $\bar{\xi}$) given by

$$\xi_{k+1} = A_{\sigma_k}\xi_k + B_d v_k \qquad (18a)$$
$$r_k = C_{\sigma_k}\xi_k, \qquad (18b)$$

where $\sigma_k \in \mathcal{M} = \{1, 2, \ldots, M\}$ denotes the mode at discrete time $k \in \mathbb{N}$, and $A_i = A_d J_i$, and $C_i = C_d J_i$, $i \in \mathcal{M}$.

Due to Theorem V.2 the internal stability and the contractivity of a continuous-time linear system under arbitrarily switching sampled-data controllers are equivalent to $\|\hat{D}\|_{\mathcal{K}} < 1$ and the internal stability and contractivity of (18) under arbitrary switching. The latter can be guaranteed by finding a mode-dependent quadratic Lyapunov function $V(\xi, \sigma) = \xi^\top P_\sigma \xi$ [27] that satisfies a dissipation inequality [30], [31] of the type

$$V(\xi_{k+1}, \sigma_{k+1}) - V(\xi_k, \sigma_k) \le - \varepsilon\xi_k^\top\xi_k - r_k^\top r_k + \gamma_0 v_k^\top v_k, \; k \in \mathbb{N} \qquad (19)$$

for some $\gamma_0 \in [0, 1)$ and $\varepsilon > 0$. This dissipation inequality is guaranteed to hold for some $\gamma_0 \in [0, 1)$ and some $\varepsilon > 0$, if the LMI conditions (note the strictness in the inequalities)

$$\begin{bmatrix} A_i^\top P_j A_i - P_i + C_i^\top C_i & A_i^\top P_j B_d\\ B_d^\top P_j A_i & B_d^\top P_j B_d - I\end{bmatrix} \prec 0, \; i, j \in \mathcal{M}$$

and

$$P_i \succ 0, \; i \in \mathcal{M}$$

are feasible. Hence, if these LMIs are satisfied it is guaranteed that (18) is internally stable and contractive, which together with $\|\hat{D}\|_{\mathcal{K}} < 1$, implies then the internal stability and the contractivity of (1) with ϕ as in (11).

VI. Numerical Examples

In this section, we illustrate the presented theory for the PETC and switching sampled-data control applications discussed in Section III. In both examples, the plant (5) is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}x_p = \begin{bmatrix}1 & 2\\ -2 & 1\end{bmatrix}x_p + \begin{bmatrix}0\\ 1\end{bmatrix}u + \begin{bmatrix}1\\ 1\end{bmatrix}w \qquad (20)$$

which is open-loop unstable.

A. PETC application

In this example, the plant (20) will be controlled using a PETC strategy specified by (6), (7), in which $K = [-0.45 \;\; -3.25]$. At sampling times $t_k = kh$, $k \in \mathbb{N}$, we will transmit the state $x_p(t_k)$ to the controller and update the control action when $\|K\hat{x}_p(t_k) - Kx_p(t_k)\| > \rho\|K\hat{x}_p(t_k)\|$. When $\|K\hat{x}_p(t_k) - Kx_p(t_k)\| = \rho\|K\hat{x}_p(t_k)\|$ it is arbitrary if a transmission occurs or not, see Remark III.1. This PETC setup corresponds to

$$Q = \begin{bmatrix}(1-\rho^2)K^\top K & -K^\top K\\ -K^\top K & K^\top K\end{bmatrix} \qquad (21)$$

in the set-valued function ϕ given in (9) for (1).

To study the internal stability and the $\mathcal{L}_2$-gain of (1) we have to determine the contractivity of the *discrete-time PWL linear system* (17) for various scaled values of C and D (next to checking $\|\hat{D}\|_{\mathcal{K}} < 1$). We will perform such an analysis based on the method discussed in Subsection V-C.1 using dissipation inequalities similar to (19) (including S-procedure relaxations where possible) based on the PWQ Lyapunov/storage functions of the form

$$V(\xi_k, \sigma_k) = \xi_k^\top P_{\sigma_k}\xi_k \qquad (22)$$

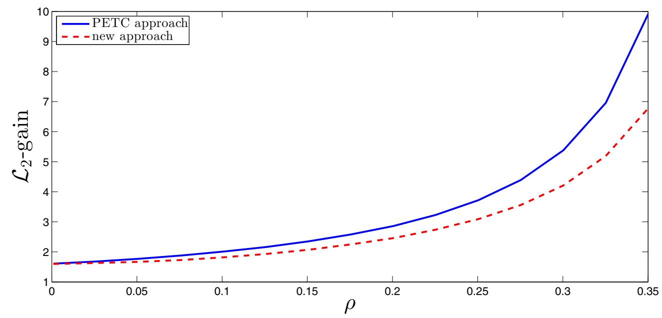

Fig. 2. Upper bound of the $\mathcal{L}_2$-gain as a function of the triggering parameter ρ with $h = 0.19$ fixed. The solid (blue) line is based on [2], while the dashed (red) line uses the new results presented in this paper.

using the partition of $\mathbb{R}^{n_\xi}$ in the regions

$$\Omega_i := \left\{ \xi \in \mathbb{R}^{n_\xi} \,\middle|\, \xi^\top X_i \xi \geq 0 \right\},\ i \in \{1, \ldots, N\} \qquad (23)$$

with X_i, $i \in \{1, \ldots, N\}$, symmetric matrices such that $\cup_{i=1}^{N} \Omega_i = \mathbb{R}^{n_\xi}$ and $\Omega_i \cap \Omega_j$ is of zero measure for all $i, j \in \{1, \ldots, N\}$, $i \neq j$. In addition, $\{\xi \in \mathbb{R}^{n_\xi} \mid \xi^\top Q \xi \leq 0\} = \bigcup_{i=1}^{N_1} \Omega_i$ and $\{\xi \in \mathbb{R}^{n_\xi} \mid \xi^\top Q \xi \geq 0\} = \bigcup_{i=N_1+1}^{N} \Omega_i$ for some $N_1 < N$ should hold. The value of σ_k in (22) to be used at time k in state ξ_k is given by

$$\sigma_k = \begin{cases} \min\{j \mid \xi_k \in \Omega_j\} & \text{if } \xi_k^\top Q \xi_k \neq 0, \\ N_1 & \text{if } \xi_k^\top Q \xi_k = 0, \xi_{k+1} = A_1 \xi_k + B_d v_k, r_k = C_1 \xi_k, \\ N_1 + 1 & \text{if } \xi_k^\top Q \xi_k = 0, \xi_{k+1} = A_2 \xi_k + B_d v_k, r_k = C_2 \xi_k, \end{cases}$$

where we assume that $\{\xi \in \mathbb{R}^{n_\xi} \mid \xi^\top Q \xi = 0\} = \Omega_{N_1} \cap \Omega_{N_1+1}$. The value of σ_k when $\xi_k^\top Q \xi_k = 0$ complies with the dynamics chosen at time k in (17). This method leads to LMIs using various S-procedure relaxations. Due to space limitations the detailed expressions for the LMIs are omitted.

For the example here, we take $N_1 = 1$ and $N = 4$ and use a partition inspired by [7], [34]. This results in Fig. 2 and Fig. 3. In fact, also the upper bounds on the $\mathcal{L}_2$-gain of (1) corresponding to the *sufficient* conditions obtained in the earlier works [2], [10] are provided. In Fig. 2 we observe that the new conditions lead to significantly better bounds than the existing ones as we built upon *necessary and sufficient conditions* using the discrete-time PWL system (17) obtained via lifting. In fact, it can be shown that the methodology of [2], [10] is equivalent to a conservative test for contractivity of the discrete-time PWL system (17) using *quadratic* Lyapunov/storage functions. Based on our new lifting-based techniques more insights are obtained due to the fact that we have necessary and sufficient conditions, which, in addition, allow the use of more versatile Lyapunov functions such as the PWQ ones as in (22) and additional S-procedure relaxations in the formulation of the LMIs that could not be used in [2], [10].

B. Sampled-data system with arbitrarily switching controllers

In this example, we will consider the control of the system (20) in a sampled-data fashion using two arbitrarily switched state-feedback controllers as in (10), in which $K_1 =$

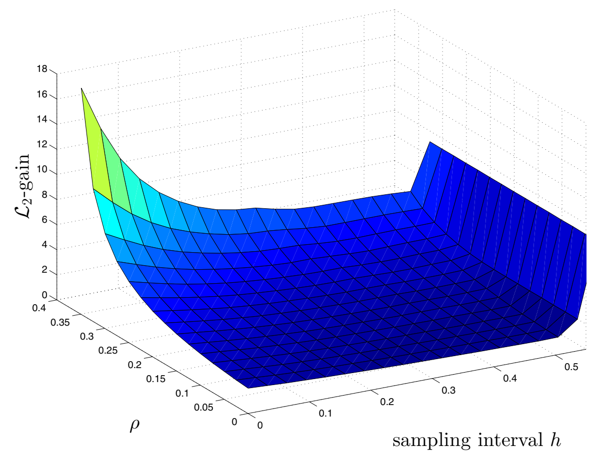

Fig. 3. Upper bound of the $\mathcal{L}_2$-gain as a function of the sampling interval h and triggering parameter ρ.

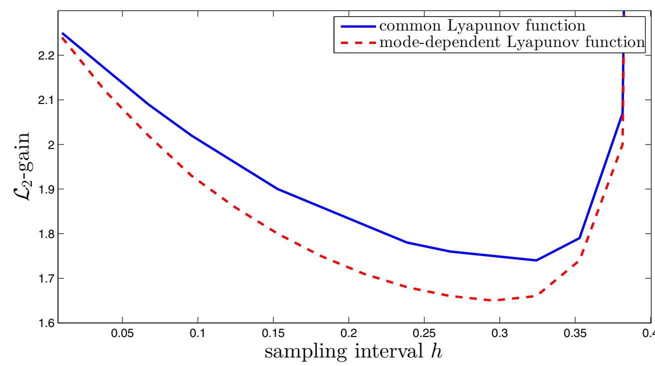

Fig. 4. Upper bound of the $\mathcal{L}_2$-gain as a function of the sampling interval h using both a common and a mode-dependent Lyapunov function.

$[-0.45 \quad -3.25]$ and $K_2 = [-2.28 \quad -4.74]$. Following the procedure and LMIs given in Section V-C.2 the results as in Fig. 4 are obtained for various values of the sampling period h based on a mode-dependent quadratic Lyapunov function. For comparison we also plotted the results that would be obtained by the direct application/generalisation of the results in [2], [10], [1] based on τ-dependent *common* quadratic Lyapunov/storage functions of the form $V(\xi) = \xi^\top P(\tau) \xi$ for (1). Again, we observe a clear improvement provided by the new results compared to these existing ones.

VII. Conclusions

In this work we studied the internal stability and the $\mathcal{L}_2$-gain of hybrid systems that have linear flow dynamics, periodic time-triggered jumps and arbitrary nonlinear jump maps. This class of hybrid systems is relevant for various applications including PETC, control over communication networks with quadratic protocols such as the TOD protocol, continuous-time linear systems controlled by arbitrarily switching sampled-data control laws and many more. We have established novel necessary and sufficient conditions for both the internal stability and the contractivity (in the sense of $\mathcal{L}_2$-gains) for these dynamical systems in terms of the internal stability and the contractivity (now in the sense of ℓ_2-gains) of an appropriate discrete-time nonlinear (possibly set-valued) system. These conditions are the first that are both necessary and sufficient. These new insights were obtained by adopting a lifting-based perspective for the problem, like in [11], and lead to numerically tractable methods to study the internal stability and the $\mathcal{L}_2$-gain of the studied hybrid systems class, in spite of the fact that linearity, which is an assumption usually needed in the lifting literature, does not hold.

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