Detection of Transient Instabilities in Multi-Agent Systems and Swarms

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***Abstract*— In this paper, we introduce the novel concept of *transient instabilities* in multi-agent systems and swarms, i.e., a small disturbance leads to increasing-amplitude oscillations throughout the swarm, which results in a large number of inter-agent collisions. This instability is dominant in the transient phase of the system and it does not appear in the steady-state behavior of the system, as each agent uses Lyapunov-stable feedback control laws.**

We present a rigorous definition of transient instability in swarms, and discuss its key properties. We also present a sufficient condition to check if a swarm will be transient stable. Using both theoretical techniques and numerical results, we show that a simple Proportional-Integral-Derivative (PID) feedback-tracking control law is not suitable for these applications. We also present a robust control law for Euler-Lagrangian systems that maintains transient stability across static swarms or time-varying reconfiguration in swarms. Both theoretical results and numerical simulations are presented to demonstrate the effectiveness of our proposed approach.

I. Introduction

Multi-agent systems and swarms, consisting of formations or constellations of small satellites or teams of aerial and ground robots, can be used for a variety of applications in space, in air, and on ground. It is commonly assumed in the control theory literature that proving the Lyapunov-stability of a swarm's dynamics is sufficient to ascertain the stability of the swarm. In this paper we show that this assumption is incorrect, i.e., there exist formation geometries/shapes and Lyapunov-stable feedback control laws that can lead to multiple inter-agent collisions within the swarm.

We use the following toy problem to describe the concepts in this paper. The evolution of a 2-dimensional (2D) swarm of agents is shown in Fig. 1, where each agent tries to maintain a constant distance with its preceding agents along both dimensions. A small disturbance in either dimension leads to increasing-amplitude oscillations throughout the swarm, which results in a large number of inter-agent collisions. These oscillations die out with time and the steady-state behavior is asymptotically stable. We define this phenomenon of increasing-amplitude oscillations within the swarm that leads to inter-agent collisions as *transient instability*. This instability is dominant in the transient phase of the system and it does not appear in the steady-state behavior of the system.

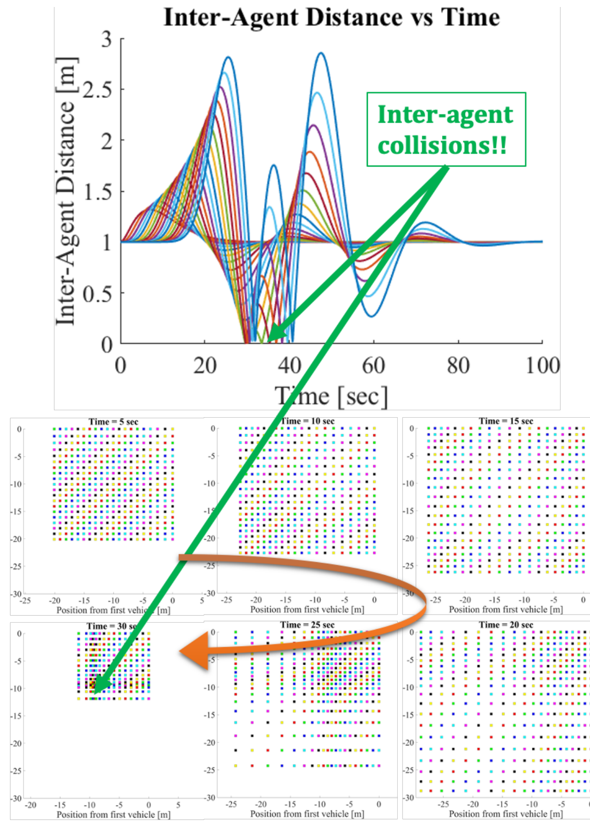

Fig. 1: Motion of a 2D swarm of agents, where each agent tries to maintain a constant distance with its preceding agents along both dimensions. A small disturbance in either dimension leads to increasing-amplitude oscillations throughout the swarm, which results in a large number of inter-agent collisions. We define this phenomenon as *transient instability*.

Since each agent's motion is asymptotically stable, all stability techniques in traditional control theory (like gain/phase margin and root locus for linear systems and Lyapunov theory or contraction theory for nonlinear systems) cannot detect these transient instabilities because each agent's motion is technically *stable* according to the standard stability definition used in traditional control theory.

A. Literature Survey

Transient instabilities were first discovered in one-dimensional (1D) vehicle convoys in the 1970s [1], where it is often denoted as *string instability*. Many mitigation strategies have been proposed for 1D vehicle convoys in [2],

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[3], [4], [5], [6]. A recent review paper [7] lists the state-of-the-art results in string stability of vehicle platoons.

To our knowledge, this is the first paper to show that such transient instabilities can also arise in 2D or 3D swarm formations. The main difficulties in applying the vehicle convoys solutions to swarms are as follows:

- Swarms are usually deployed in higher dimensions (2D and 3D), and the coupling between the dimensions introduces new complexities not discussed in the previous literature on 1D vehicle convoys.
- The dynamics of vehicle convoys is usually represented by linear time-varying equations, hence tools from linear systems control theory are usually used in the solution approach. The dynamics of swarm agents are often nonlinear (e.g. spacecraft, quadrotor) hence solutions techniques from linear systems control theory cannot be directly applied.

B. Main Contributions

The main contributions and organization of this paper are as follows:

- We first rigorously define transient instabilities in swarms, and discuss its various features in Section II.
- We present a mathematical framework or "theoretical tool" to detect transient instabilities in swarms in Section III. For a special case, we also prove that this tool is a *sufficient condition* for avoiding transient instabilities in swarms.
- In Section IV, we show that the standard Proportional-Integral-Derivative (PID) control law does not satisfy this condition for linear systems.
- In Section V, a robust control law [8], previously introduced for non-swarm applications, satisfies this condition and always avoids transient instabilities for linear systems.
- Moreover, this robust control law also preserves transient stability in the swarm, even when the agents reconfigure and track time-varying inter-agent distances as shown in Section VI.

Finally, Section VII concludes the paper.

II. Definition of Transient Instability in Swarms

In order to understand transient instability in swarms, let us first discuss the example in Fig. 1. The indices used in this example are described in Fig. 2. Here is a 2D swarm of $N \times M$ agents, where $N, M \in \mathbb{N}$. Each $(i,j)^{\text{th}}$ follower agent (shown in blue) is trying to maintain a constant separation distance $D \in \mathbb{R}$ from the right $(i-1,j)^{\text{th}}$ agent and up $(i,j-1)^{\text{th}}$ agent. The leader $(1,1)^{\text{th}}$ agent (shown in red) decides the desired trajectory of the swarm, that all the follower agents will follow. The leader+follower agents at the start of each axis (shown in green) also ensure that the swarm maintains the desired trajectory selected by the leader agent.

Let $\mathbf{x}_{i,j}(t) = [p_{i,j,x}(t)\, p_{i,j,y}(t)\, v_{i,j,x}(t)\, v_{i,j,y}(t)]$ denote the state of the $(i,j)^{\text{th}}$ agent in the swarm at time t, where $p_{i,j,x}$, $p_{i,j,y}$ and $v_{i,j,x}$, $v_{i,j,y}$ are the positions and velocities of the agent in the X-axis and Y-axis respectively. The dynamics of the $(i,j)^{\text{th}}$ follower agent $(\forall i \in \{1,\dots,N\}, \forall j \in \{1,\dots,M\})$ is given by:

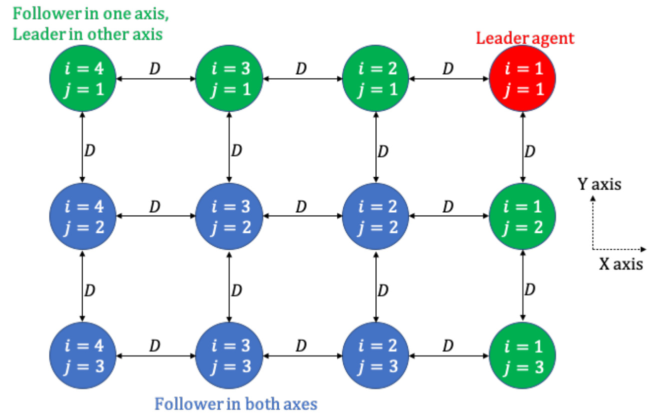


Fig. 2: Indices used in the 2D swarm of agents example.

$$\dot{p}_{i,j,x}(t) = v_{i,j,x}(t), \tag{1}$$
$$\dot{p}_{i,j,y}(t) = v_{i,j,y}(t), \tag{2}$$
$$\dot{v}_{i,j,x}(t) = -\mu\, v_{i,j,x}(t) + u_{i,j,x}(t), \tag{3}$$
$$\dot{v}_{i,j,y}(t) = -\mu\, v_{i,j,y}(t) + u_{i,j,y}(t). \tag{4}$$

Here $u_{i,j,x}(t)$, $u_{i,j,y}(t)$ are the control forces per unit mass along X-axis and Y-axis, and μ is the linearized drag force (damping) coefficient per unit mass. Note that there are no disturbance terms in the system dynamics equations.

Let the desired state of the $(i,j)^{\text{th}}$ agent in the swarm be denoted by $\bar{\mathbf{x}}_{i,j}(t) = [\bar{p}_{i,j,x}(t)\, \bar{p}_{i,j,y}(t)\, \bar{v}_{i,j,x}(t)\, \bar{v}_{i,j,y}(t)]$, where $\bar{p}_{i,j,x}$, $\bar{p}_{i,j,y}$ and $\bar{v}_{i,j,x}$, $\bar{v}_{i,j,y}$ are the positions and velocities in the X-axis and Y-axis respectively. Since the $(i,j)^{\text{th}}$ follower agent is aiming to maintain the distance D from the right $(i-1,j)^{\text{th}}$ agent and the up $(i,j-1)^{\text{th}}$ agent, the desired state $\bar{\mathbf{x}}_{i,j}(t)$ is given by:

$$\bar{p}_{i,j,x}(t) = p_{i-1,j,x}(t) - D\,, \tag{5}$$
$$\bar{p}_{i,j,y}(t) = p_{i,j-1,y}(t) - D\,, \tag{6}$$
$$\bar{v}_{i,j,x}(t) = v_{i-1,j,x}(t)\,, \tag{7}$$
$$\bar{v}_{i,j,y}(t) = v_{i,j-1,y}(t)\,. \tag{8}$$

Note that the desired states of agents are represented in multiple local reference frames, and not pegged to a single reference frame. It might be possible to remove transient instabilities by stating all desired trajectories in a single reference frame only [9], but this is not possible in all situations.

The Proportional-Integral-Derivative (PID) control scheme of the $(i,j)^{\text{th}}$ follower agent is given by:

$$u_{i,j,x}(t) = -h_1\left(p_{i,j,x}(t) - \bar{p}_{i,j,x}(t)\right) - h_2\left(v_{i,j,x}(t) - \bar{v}_{i,j,x}(t)\right) - h_3 \int_0^t \left(p_{i,j,x}(\tau) - \bar{p}_{i,j,x}(\tau)\right) d\tau \tag{9}$$

$$u_{i,j,y}(t) = -h_1\left(p_{i,j,y}(t) - \bar{p}_{i,j,y}(t)\right) - h_2\left(v_{i,j,y}(t) - \bar{v}_{i,j,y}(t)\right) - h_3 \int_0^t \left(p_{i,j,y}(\tau) - \bar{p}_{i,j,y}(\tau)\right) d\tau \tag{10}$$

The gains of the PID controller (namely h_1, h_2, h_3 in Eq. (9,10)) can be tuned using pole placement. In Fig. 1, we use the gains from [1], i.e., $h_1 = 0.25$, $h_2 = 0.8$, $h_3 = 0.025$, $\mu = 0.1$.

The key features of these transient instabilities in Fig. 1 are as follows:

- A small disturbance in the leader's motion, leads to increasing amplitude oscillations down the chain of agents. Once the amplitude of oscillations is greater than the desired inter-agent distance, then inter-agent collisions will happen. This is extremely undesirable and we want to avoid such situations in swarms.
- Since each agent's motion is asymptotically stable, all stability techniques in traditional control theory (like gain/phase margin and root locus for linear systems and Lyapunov theory or contraction theory for nonlinear systems [10]) cannot detect these transient instabilities because each agent's motion is technically *stable* according to the standard stability definition used in traditional control theory. In other words, if each agent's motion is individually analyzed, then it passes the stability checks in traditional control theory and we cannot even detect this transient instability within the swarm.

Obviously, the problem of transient instabilities has to be addressed before swarms can be deployed in the real world. In order to detect these transient instabilities, we need to analyze the swarm as a whole and pay special attention to the interaction between agents and the emergent behaviors of the swarm.

We now present a new definition of transient instabilities in swarms. We know $\mathbf{x}_{i,j}(t) = [p_{i,j,x}(t)\, p_{i,j,y}(t)\, v_{i,j,x}(t)\, v_{i,j,y}(t)]$ represents the state vector of the $(i,j)^{\text{th}}$ follower agent in the swarm at time t. Similarly, let $\mathbf{x}_{i-1,j}(t)$, $\mathbf{x}_{i+1,j}(t)$, $\mathbf{x}_{i,j-1}(t)$, $\mathbf{x}_{i,j+1}(t)$ be the states of the $(i-1,j)^{\text{th}}$, $(i+1,j)^{\text{th}}$, $(i,j-1)^{\text{th}}$, $(i,j+1)^{\text{th}}$ neighboring agents in the swarm respectively.

Definition 1 (Transient Instability in Swarm): Transient instability is defined as the phenomenon of increasing-amplitude oscillations of inter-agent distances within the swarm that leads to inter-agent collisions, such that either or both of the following inequalities hold:

$$\begin{aligned}&\sup_{\tau}\left(\|p_{i,j,x}(t)\|_{\mathscr{L}_2(\tau)} - \|p_{i-1,j,x}(t)\|_{\mathscr{L}_2(\tau)}\right)\\ &\qquad < \sup_{\tau}\left(\|p_{i+1,j,x}(t)\|_{\mathscr{L}_2(\tau)} - \|p_{i,j,x}(t)\|_{\mathscr{L}_2(\tau)}\right),\\ &\qquad\qquad \forall i \in \{2,\ldots,N-1\}, \forall j \in \{1,\ldots,M\}.\end{aligned} \tag{11}$$

$$\begin{aligned}&\sup_{\tau}\left(\|p_{i,j,y}(t)\|_{\mathscr{L}_2(\tau)} - \|p_{i,j-1,y}(t)\|_{\mathscr{L}_2(\tau)}\right)\\ &\qquad < \sup_{\tau}\left(\|p_{i,j+1,y}(t)\|_{\mathscr{L}_2(\tau)} - \|p_{i,j,y}(t)\|_{\mathscr{L}_2(\tau)}\right),\\ &\qquad\qquad \forall i \in \{1,\ldots,N\}, \forall j \in \{2,\ldots,M-1\}.\end{aligned} \tag{12}$$

Here $\|\cdot\|_{\mathscr{L}_2}$ is the norm defined by $\|\cdot\|_{\mathscr{L}_2(\tau)} := \sqrt{\int_0^\tau \|\cdot\|^2 dt}$ for a large but finite τ [10], on the space

$$\mathscr{L}^2(\tau) = \left\{ f : f \text{ is measurable and } \left(\int_\tau \|f\|^2 d\mu\right)^{1/2} < \infty \right\}. \tag{13}$$

□

Def. 1 succinctly captures the effect of increasing-amplitude oscillations within the swarm, by comparing the maximum distance between the positions of two neighboring agents over time.

III. Mathematical Tool to Check Transient Stability in Swarms

We need a mathematical framework or "theoretical tool" to check if a swarm will exhibit transient stability, i.e., it will avoid the instability defined in Def. 1. Therefore, we present the following tool for *transient stability* in swarms, which is a *sufficient condition* to avoid transient instabilities.

Assumption 1: The desired states of neighboring agents $\bar{\mathbf{x}}_{i-1,j}(t)$, $\bar{\mathbf{x}}_{i+1,j}(t)$, $\bar{\mathbf{x}}_{i,j-1}(t)$, $\bar{\mathbf{x}}_{i,j+1}(t)$ do not force any inter-agent collisions with $\bar{\mathbf{x}}_{i,j}(t)$ and always maintain sufficient distance between neighbors.

In other words, the desired states in Eq.(5)–(8) are well designed and don't force the agent to collide with each other.

Definition 2 (Transient Stability in Swarm): The mathematical framework or "theoretical tool" to check if a swarm will exhibit transient stability is given by:

$$\sup_{\tau} \frac{\|p_{i,j,x}(t)\|_{\mathscr{L}_2(\tau)}}{\|\bar{p}_{i,j,x}(t)\|_{\mathscr{L}_2(\tau)}} \le 1, \forall i \in \{1,\ldots,N\}, \forall j \in \{1,\ldots,M\}, \tag{14}$$

$$\sup_{\tau} \frac{\|p_{i,j,y}(t)\|_{\mathscr{L}_2(\tau)}}{\|\bar{p}_{i,j,y}(t)\|_{\mathscr{L}_2(\tau)}} \le 1, \forall i \in \{1,\ldots,N\}, \forall j \in \{1,\ldots,M\}, \tag{15}$$

where $\mathscr{L}_2$-norm is defined in Eq. (13).

In the linear case, the left-hand side of Eq. (14) is equivalent to [10], [11]:

$$\begin{aligned}\sup_{\tau} \frac{\|p_{i,j,x}(t)\|_{\mathscr{L}_2(\tau)}}{\|\bar{p}_{i,j,x}(t)\|_{\mathscr{L}_2(\tau)}} = \left\| \frac{\mathbf{P}_{i,j,x}(s)}{\bar{\mathbf{P}}_{i,j,x}(s)} \right\|_{\mathscr{H}_\infty} = \max_{\omega} \left| \frac{\mathbf{P}_{i,j,x}(\imath\omega)}{\bar{\mathbf{P}}_{i,j,x}(\imath\omega)} \right|,\\ \forall i \in \{1,\ldots,N\}, \forall j \in \{1,\ldots,M\},\end{aligned} \tag{16}$$

where $\imath$ represents the imaginary unit vector and $\mathbf{P}_{i,j,x}(s)$ is the Laplace transform of $p_{i,j,x}(t)$. Therefore, alternatively, in the linear case, transient stability in swarms can be checked using:

$$\max_{\omega} \left| \frac{\mathbf{P}_{i,j,x}(\imath\omega)}{\bar{\mathbf{P}}_{i,j,x}(\imath\omega)} \right| \le 1, \forall i \in \{1,\ldots,N\}, \forall j \in \{1,\ldots,M\}, \tag{17}$$

$$\max_{\omega} \left| \frac{\mathbf{P}_{i,j,y}(\imath\omega)}{\bar{\mathbf{P}}_{i,j,y}(\imath\omega)} \right| \le 1, \forall i \in \{1,\ldots,N\}, \forall j \in \{1,\ldots,M\}, \tag{18}$$

□

We now show that the above mathematical framework to detect transient stability in swarms Def. 2 ensures that the

	h_1	h_2	h_3	μ	A_p	ω_p [Hz]	ω_{co} [Hz]	Inter-agent collision
Trial 1	0.25	0.8	0.025	0.1	1.2010	0.2840	0.5870	Yes
Trial 2	0.31	2.1	0.01	0.1	1.0180	0.1000	0.4472	No
Trial 3	0.7	1.0	0	0.1	1.2444	0.6204	1.0909	Yes
Trial 4	0.3	2.9	0	0.1	1	0	0.1000	No
Trial 5	0.3	3.0	0.01	0.1	1.0110	0.0577	0.2000	No
Trial 6	0.3	3.0	0.03	0.1	1.0339	0.0999	0.2695	No
Trial 7	0.2	1.7	0.02	0.1	1.0608	0.1088	0.3050	No
Trial 8	0.2	1.7	0.02	0.1	1.1850	1.2079	2.3306	No
Trial 9	100	11	1.0	0.1	1.3930	8.2048	14.0638	No
Trial 10	0	3.0	0.9	0.1	1.0200	0.3495	0.4664	Yes
Trial 11	2.0	23.0	0.9	0.1	1.0199	0.1978	0.4664	No
Trial 12	500	20	0.9	0.1	1.5471	19.3427	31.5593	No

TABLE I: Table with A_p (38), ω_p (39), and ω_{co} (40) for different trial values.

swarm will avoid the instability defined in Def. 1 for a special case.

Theorem 1: In a leader-follower swarm (as shown in Fig. 2), where each agent tries to maintain a constant distance D from its neighboring agents and also starts in this configuration, if the swarm satisfies the condition in Eq. (14,15), then the swarm won't have increasing-amplitude oscillations in inter-agent distances described in Eq. (11,12). Then the swarm will be transient stable.

Proof: We only show the steps in the proof for the X-axis, as the proof along Y-axis is exactly similar.

Triangle inequality on Eq. (5) and $\|D\|_2 = \|-D\|_2$ gives:

$$\|\bar{p}_{i,j,x}(t)\|_{\mathcal{L}_2(\tau)} \leq \|p_{i-1,j,x}(t)\|_{\mathcal{L}_2(\tau)} + \|D\|_2\,, \quad \forall i \in \{2,\ldots,N\}, \forall j \in \{1,\ldots,M\}\,, \tag{19}$$

Substituting Eq. (19) into Eq. (14) gives:

$$\sup_\tau \frac{\|p_{i,j,x}(t)\|_{\mathcal{L}_2(\tau)}}{\|p_{i-1,j,x}(t)\|_{\mathcal{L}_2(\tau)} + \|D\|_2} \leq \sup_\tau \frac{\|p_{i,j,x}(t)\|_{\mathcal{L}_2(\tau)}}{\|\bar{p}_{i,j,x}(t)\|_{\mathcal{L}_2(\tau)}} \leq 1\,, \quad \forall i \in \{2,\ldots,N\}, \forall j \in \{1,\ldots,M\}\,, \tag{20}$$

Eq. (20) implies that:

$$\frac{\|p_{i,j,x}(t)\|_{\mathcal{L}_2(\tau)}}{\|p_{i-1,j,x}(t)\|_{\mathcal{L}_2(\tau)} + \|D\|_2} \leq 1\,, \forall \tau \in \mathbb{R}^+ \quad \forall i \in \{2,\ldots,N\}, \forall j \in \{1,\ldots,M\}\,, \tag{21}$$

Since $\left(\|p_{i-1,j,x}(t)\|_{\mathcal{L}_2(\tau)} + \|D\|_2\right)$ is positive, we get:

$$\|p_{i,j,x}(t)\|_{\mathcal{L}_2(\tau)} \leq \|p_{i-1,j,x}(t)\|_{\mathcal{L}_2(\tau)} + \|D\|_2\,, \forall \tau \in \mathbb{R}^+ \quad \forall i \in \{2,\ldots,N\}, \forall j \in \{1,\ldots,M\}\,, \tag{22}$$

$$\|p_{i,j,x}(t)\|_{\mathcal{L}_2(\tau)} - \|p_{i-1,j,x}(t)\|_{\mathcal{L}_2(\tau)} \leq \|D\|_2\,, \forall \tau \in \mathbb{R}^+ \quad \forall i \in \{2,\ldots,N\}, \forall j \in \{1,\ldots,M\}\,, \tag{23}$$

Taking the $\sup_\tau$ over Eq. (23) gives:

$$\sup_\tau \left(\|p_{i,j,x}(t)\|_{\mathcal{L}_2(\tau)} - \|p_{i-1,j,x}(t)\|_{\mathcal{L}_2(\tau)}\right) \leq \|D\|_2\,, \quad \forall i \in \{2,\ldots,N\}, \forall j \in \{1,\ldots,M\}\,, \tag{24}$$

Since each agent starts at time $t = 0$ with a distance D from its preceding agent, we get that the distance between first and second agent at time $t = 0$ is:

$$\left(\|p_{2,jx,}(0)\|_{\mathcal{L}_2(0)} - \|p_{1,j,x}(0)\|_{\mathcal{L}_2(0)}\right) = \|D\|_2\,, \quad \forall j \in \{1,\ldots,M\}\,, \tag{25}$$

Taking the $\sup_\tau$ over all time gives:

$$\begin{aligned}\|D\|_2 &= \left(\|p_{2,jx,}(0)\|_{\mathcal{L}_2(0)} - \|p_{1,j,x}(0)\|_{\mathcal{L}_2(0)}\right) \\ &\leq \sup_\tau \left(\|p_{2,j,x}(t)\|_{\mathcal{L}_2(\tau)} - \|p_{1,j,x}(t)\|_{\mathcal{L}_2(\tau)}\right)\,, \\ &\quad \forall j \in \{1,\ldots,M\}\,,\end{aligned} \tag{26}$$

If the swarm were to exhibit increasing-amplitude oscillations, then from Eq. (11) and Eq. (26), we get:

$$\begin{aligned}\|D\|_2 &\leq \sup_\tau \left(\|p_{2,j,x}(t)\|_{\mathcal{L}_2(\tau)} - \|p_{1,j,x}(t)\|_{\mathcal{L}_2(\tau)}\right) < \ldots \\ &< \sup_\tau \left(\|p_{i,jx}(t)\|_{\mathcal{L}_2(\tau)} - \|p_{i-1,j,x}(t)\|_{\mathcal{L}_2(\tau)}\right) \\ &< \sup_\tau \left(\|p_{i+1,j,x}(t)\|_{\mathcal{L}_2(\tau)} - \|p_{i,j,x}(t)\|_{\mathcal{L}_2(\tau)}\right) < \ldots\,, \\ &\quad \forall j \in \{1,\ldots,M\}\,,\end{aligned} \tag{27}$$

Eq. (27), where $\|D\|_2$ is the strict lower-bound, contradicts with Eq. (24), where $\|D\|_2$ is the upper bound. Hence increasing-amplitude oscillations in inter-agent distance is not possible. ■

In the next section, we study the effect of different kinds of control laws on swarm transient stability.

IV. BEHAVIOR OF PID CONTROL LAWS

We first focus on the behavior of the PID control law introduced in Eq. (9,10). The closed loop equations of motion for the follower agents ($\forall i \in \{2,\ldots,N\}, \forall j \in \{2,\ldots,M\}$) are:

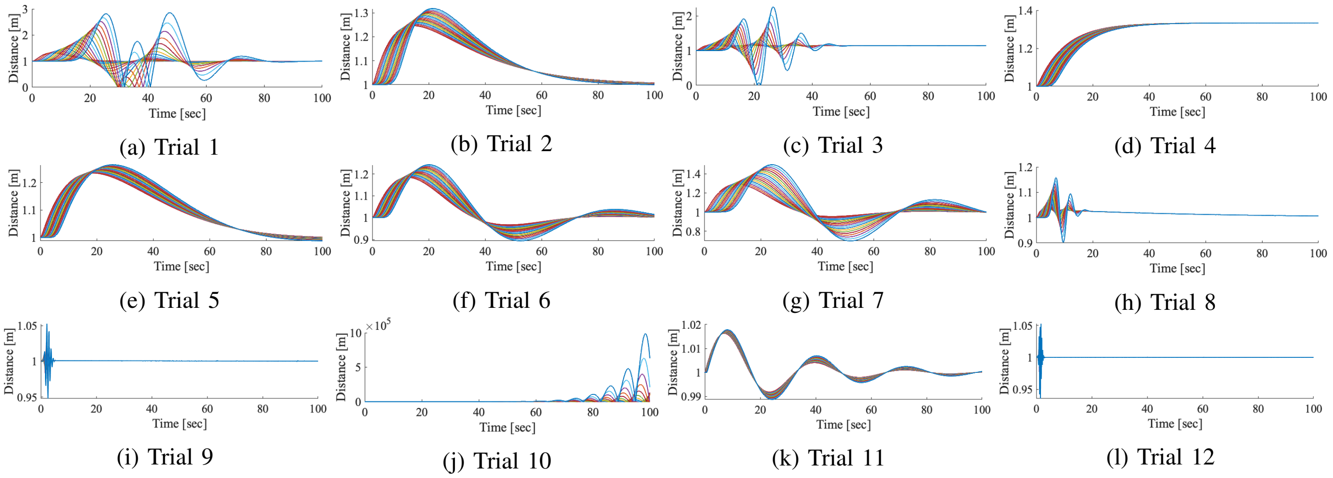
$$\dot{p}_{i,j,x}(t) = v_{i,j,x}(t)\,, \tag{28}$$

$$\dot{p}_{i,j,y}(t) = v_{i,j,y}(t)\,, \tag{29}$$

$$\dot{v}_{i,j,x}(t) = -\mu v_{i,j,x}(t) + h_1\left(\bar{p}_{i,j,x}(t) - p_{i,j,x}(t)\right) + h_2\left(\dot{\bar{p}}_{i,j,x}(t) - v_{i,j,x}(t)\right) + h_3 \int_0^t \left(\bar{p}_{i,j,x}(t) - p_{i,j,x}(\tau)\right) d\tau\,, \tag{30}$$

$$\dot{v}_{i,j,y}(t) = -\mu v_{i,j,y}(t) + h_1\left(\bar{p}_{i,j,y}(t) - p_{i,j,y}(t)\right) + h_2\left(\dot{\bar{p}}_{i,j,y}(t) - v_{i,j,y}(t)\right) + h_3 \int_0^t \left(\bar{p}_{i,j,y}(t) - p_{i,j,y}(\tau)\right) d\tau\,. \tag{31}$$

Since the closed loop system is linear, we can directly use the transient stability tool shown in Eq. (17,18). Taking



(a) Trial 1 (b) Trial 2 (c) Trial 3 (d) Trial 4 (e) Trial 5 (f) Trial 6 (g) Trial 7 (h) Trial 8 (i) Trial 9 (j) Trial 10 (k) Trial 11 (l) Trial 12

Fig. 3: Evolution of inter-agent distance with time for all trials in Table I using the PID control law. Note the inter-agent collisions in Trials 1, 3, and instability in Trial 10.

Laplace transform of Eq. (28)–(31):

$$s\mathbf{P}_{i,j,x}(s) = \mathbf{V}_{i,j,x}(s)\,, \tag{32}$$

$$s\mathbf{P}_{i,j,y}(s) = \mathbf{V}_{i,j,y}(s)\,, \tag{33}$$

$$s\mathbf{V}_{i,j,x}(s) = -\mu\mathbf{V}_{i,j,x}(s) + h_1\left(\bar{\mathbf{P}}_{i,j,x}(s) - \mathbf{P}_{i,j,x}(s)\right) + h_2\left(s\bar{\mathbf{P}}_{i,j,x}(s) - \mathbf{V}_{i,j,x}(s)\right) + \frac{h_3}{s}\left(\bar{\mathbf{P}}_{i,j,x}(s) - \mathbf{P}_{i,j,x}(s)\right)\,, \tag{34}$$

$$s\mathbf{V}_{i,j,y}(s) = -\mu\mathbf{V}_{i,j,y}(s) + h_1\left(\bar{\mathbf{P}}_{i,j,y}(s) - \mathbf{P}_{i,j,y}(s)\right) + h_2\left(s\bar{\mathbf{P}}_{i,j,y}(s) - \mathbf{V}_{i,j,y}(s)\right) + \frac{h_3}{s}\left(\bar{\mathbf{P}}_{i,j,y}(s) - \mathbf{P}_{i,j,y}(s)\right)\,. \tag{35}$$

In order to apply Eq. (17,18), we evaluate the ratios:

$$\frac{\mathbf{P}_{i,j,x}(s)}{\bar{\mathbf{P}}_{i,j,x}(s)} = \frac{h_2 s^2 + h_1 s + h_3}{s^3 + (\mu + h_2)s^2 + h_1 s + h_3}\,, \tag{36}$$

$$\frac{\mathbf{P}_{i,j,y}(s)}{\bar{\mathbf{P}}_{i,j,y}(s)} = \frac{h_2 s^2 + h_1 s + h_3}{s^3 + (\mu + h_2)s^2 + h_1 s + h_3}\,. \tag{37}$$

In order to understand the transient behavior of the swarm, we evaluate a number of trial cases, shown in Table I and the evolution of their inter-agent distances, shown in Fig. (3). Let us define *peak magnitude* A_p (38), *frequency at peak magnitude* ω_p (39), and *maximum crossover frequency* ω_{co} (40), after which the magnitude of the ratio $\left|\frac{\mathbf{P}_{i,j,x}(\imath\omega)}{\bar{\mathbf{P}}_{i,j,x}(\imath\omega)}\right|$ is always smaller than 1.

$$A_p = \max_{\omega}\left|\frac{\mathbf{P}_{i,j,x}(\imath\omega)}{\bar{\mathbf{P}}_{i,j,x}(\imath\omega)}\right|\,, \qquad \text{(Same as Eq. (17))} \tag{38}$$

$$\omega_p = \arg\max_{\omega}\left|\frac{\mathbf{P}_{i,j,x}(\imath\omega)}{\bar{\mathbf{P}}_{i,j,x}(\imath\omega)}\right|\,, \tag{39}$$

$$\omega_{co} = \max_{\omega}\ \text{such that}\ \left(\left|\frac{\mathbf{P}_{i,j,x}(\imath\omega)}{\bar{\mathbf{P}}_{i,j,x}(\imath\omega)}\right| = 1\right)\,. \tag{40}$$

In Fig. 4, we plot the A_p and ω_{co} for many different positive PID gains (i.e., $h_1 > 0$, $h_2 > 0$, and $h_3 > 0$ in Eq. (9,10)). We see a clear band of good gains, where the minimum inter-agent distance is lower-bounded and does not lead to collisions (see red, yellow, green, blue, and magenta dots) and bad gains where the agents collide (see black dots).

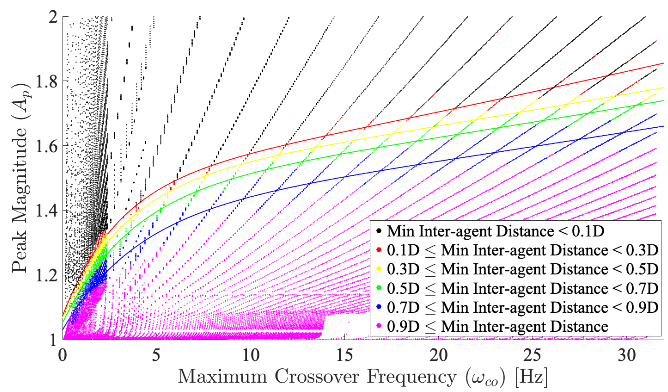


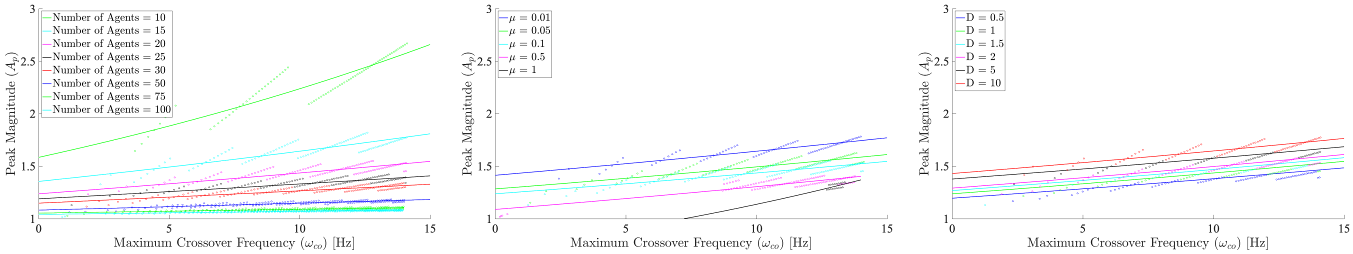
Fig. 4: Minimum inter-agent distance for different PID gains, for a swarm with 20×20 agents and $\mu = 0.1$, $D = 1$. The lines indicate an approximate fit of those colored points. All examples shown here have steady-state inter-agent distance $\in [0.99D, 1.01D]$, i.e., the swarm is asymptotically stable.

This shows that $A_p \leq 1$ in Eq. (17) is a *sufficient condition* for transient stability in swarms, but not a *necessary condition*.

Now we want to understand the effect of different parameters in these simulations. We vary only the following parameters (number of agents in the $N \times N$ swarm, damping μ, inter-agent distance D) in the following simulations:

- Fig. 5a shows that increasing the number of agents in the swarm pushes the lines downwards towards $A_p = 1$ line, i.e. the blue line in Fig. 4 converges to the $A_p = 1$ line.
- Fig. 5b shows that increasing the damping pushes the lines downwards towards the $A_p = 1$ line.
- Fig. 5c shows that reducing the desired distance between agents D pushes the lines downwards towards the $A_p = 1$ line.

This shows that the transient stability tool $A_p \leq 1$ in Eq. (17,18) is also a *necessary condition* when the number of agents in the swarm N, M is very large, or the damping μ is



(a) Number of agents in an $N \times N$ swarm. (b) Damping μ in agent's dynamics (c) Desired inter-agent distance D

Fig. 5: These plots show the effect of changing different parameters on the transient stability of the swarm. The lines indicate an approximate fit of those color points, with minimum inter-agent distance $\in [0.7D, 0.9D]$ and steady-state inter-agent distance $\in [0.99D, 1.01D]$. That is, all of these lines represent the blue line in Fig. 4.

large, or the inter-agent distance D is quite small. Moreover, since it is almost impossible to get peak magnitude $A_p \leq 1$ for any positive PID gains in Eq. (9,10), we can claim that it is impossible to have transient stability in a swarm with a simple PID control law.

V. Behavior of Robust Control Laws

We now use the robust control scheme derived for the attitude control of spacecraft with disturbances [8], which is a modified version of the control law for Euler-Lagrangian systems [12]. The control law for the $(i,j)^{\text{th}}$ follower agent, using separation measurements and aiming to maintain distance D from right $(i-1,j)^{\text{th}}$ agent and up $(i,j-1)^{\text{th}}$ agent, is given by:

$$u_{i,j,x}(t) = \dot{\Omega}_{i,j,x}(t) + \mu\Omega_{i,j,x}(t) + h_2\left(\Omega_{i,j,x}(t) - v_{i,j,x}(t)\right) \tag{41}$$

where

$$\Omega_{i,j,x}(t) = \dot{\bar{p}}_{i,j,x}(t) + h_1\left(\bar{p}_{i,j,x}(t) - p_{i,j,x}(t)\right), \tag{42}$$

$$\bar{p}_{i,j,x}(t) = p_{i-1,j,x}(t) - D, \tag{43}$$

$$\dot{\bar{p}}_{i,j,x}(t) = \dot{p}_{i-1,j,x}(t) = v_{i-1,j,x}(t), \tag{44}$$

$$\ddot{\bar{p}}_{i,j,x}(t) = \ddot{p}_{i-1,j,x}(t) = \dot{v}_{i-1,j,x}(t), \tag{45}$$

Only equations for X-axis are shown here, as the Y-axis is exactly similar. The closed loop equations of motion are:

$$\dot{p}_{i,j,x}(t) = v_{i,j,x}(t), \tag{46}$$

$$\begin{aligned}\dot{v}_{i,j,x}(t) = & -\mu v_{i,j,x}(t) + \ddot{\bar{p}}_{i,j,x}(t) + h_1\left(\dot{\bar{p}}_{i,j,x}(t) - \dot{p}_{i,j,x}(t)\right) \\ & + \mu\dot{\bar{p}}_{i,j,x}(t) + \mu h_1\left(\bar{p}_{i,j,x}(t) - p_{i,j,x}(t)\right) \\ & + h_2\left(\dot{\bar{p}}_{i,j,x}(t) + h_1\left(\bar{p}_{i,j,x}(t) - p_{i,j,x}(t)\right) - v_{i,j,x}(t)\right)\end{aligned} \tag{47}$$

Simplifying Eq. (47) gives:

$$\begin{aligned}\dot{v}_{i,j,x}(t) = & \ddot{\bar{p}}_{i,j,x}(t) + (h_2 - \mu + h_1)\left(\dot{\bar{p}}_{i,j,x}(t) - v_{i,j,x}(t)\right) \\ & + (\mu h_1 + h_2 h_1)\left(\bar{p}_{i,j,x}(t) - p_{i,j,x}(t)\right)\end{aligned} \tag{48}$$

Taking the Laplace transform:

$$\begin{aligned}s^2\mathbf{P}_{i,j,x}(s) = & s^2\bar{\mathbf{P}}_{i,j,x}(s) + (h_2 - \mu + h_1)(s\bar{\mathbf{P}}_{i,j,x}(s) - s\mathbf{P}_{i,j,x}(s)) \\ & + (\mu h_1 + h_2 h_1)(\bar{\mathbf{P}}_{i,j,x}(s) - \mathbf{P}_{i,j,x}(s)),\end{aligned} \tag{49}$$

$$\frac{\mathbf{P}_{i,j,x}(s)}{\bar{\mathbf{P}}_{i,j,x}(s)} = \frac{s^2 + (h_2 - \mu + h_1)s + (\mu h_1 + h_2 h_1)}{s^2 + (h_2 - \mu + h_1)s + (\mu h_1 + h_2 h_1)} = 1. \tag{50}$$

Eq. (50) shows that robust control law always satisfies $A_p \leq 1$ in Eq. (17). This means that the swarm will be transient stable for all values of h_1 and h_2. Using the gain values in Table I, we see in Fig. 6 that the swarm agents avoid inter-agent collisions for all trials.

Note that damping (μ) is not necessary for ensuring swarm transient stability using this robust control law, as shown in Fig. 7 where damping $\mu = 0$.

VI. Behavior of Above Control Laws under Swarm Reconfiguration

We now wish to reconfigure the swarm such that the $(i,j)^{\text{th}}$ follower agent maintains a time-varying distance $(D_0 + D_1 t)$ from the right $(i-1,j)^{\text{th}}$ agent and up $(i,j-1)^{\text{th}}$ agent for the first 50 sec, and then holds the final inter-agent distance.

We first use the robust control law, introduced in Section V. Only equations for X-axis are shown here:

$$u_{i,j,x}(t) = \dot{\Omega}_{i,j,x}(t) + \mu\Omega_{i,j,x}(t) + h_2\left(\Omega_{i,j,x}(t) - v_{i,j,x}(t)\right) \tag{51}$$

where

$$\Omega_{i,j,x}(t) = \dot{\bar{p}}_{i,j,x}(t) + h_1\left(\bar{p}_{i,j,x}(t) - p_{i,j,x}(t)\right), \tag{52}$$

$$\bar{p}_{i,j,x}(t) = p_{i-1,j,x}(t) - D_0 - D_1 t, \tag{53}$$

$$\dot{\bar{p}}_{i,j,x}(t) = \dot{p}_{i-1,j,x}(t) - D_1 = v_{i-1,j,x}(t) - D_1, \tag{54}$$

$$\ddot{\bar{p}}_{i,j,x}(t) = \ddot{p}_{i-1,j,x}(t) = \dot{v}_{i-1,j,x}(t), \tag{55}$$

Note that the desired separation distance keeps increasing with time t. The closed loop equations of motion (only for the X-axis) are:

$$\dot{p}_{i,j,x}(t) = v_{i,j,x}(t), \tag{56}$$

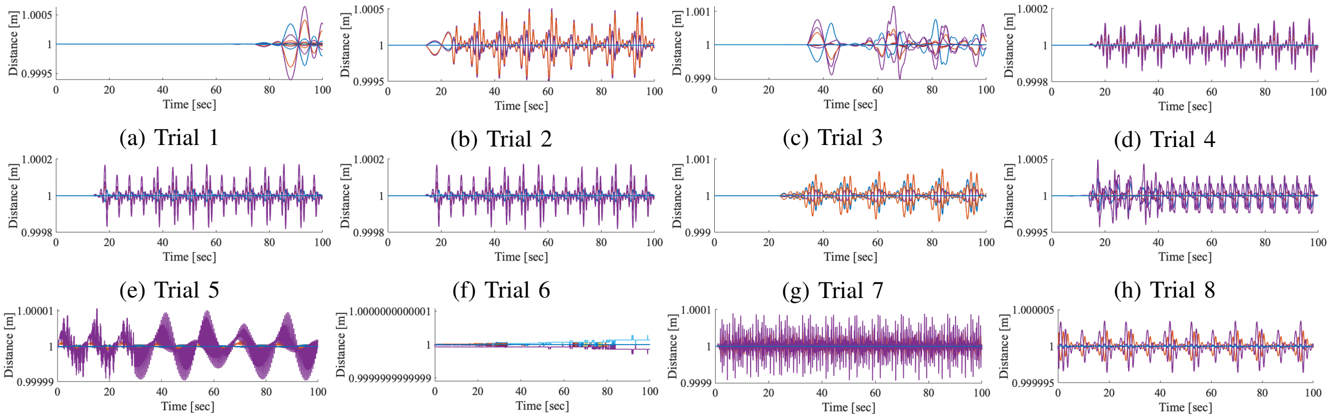
$$\begin{aligned}\dot{v}_{i,j,x}(t) = & \ddot{\bar{p}}_{i,j,x}(t) + (h_2 - \mu + h_1)\left(\dot{\bar{p}}_{i,j,x}(t) - v_{i,j,x}(t)\right) \\ & + (\mu h_1 + h_2 h_1)\left(\bar{p}_{i,j,x}(t) - p_{i,j,x}(t)\right)\end{aligned} \tag{57}$$

Taking the Laplace transform:

$$\begin{aligned}s^2\mathbf{P}_{i,j,x}(s) = & s^2\bar{\mathbf{P}}_{i,j,x}(s) + (h_2 - \mu + h_1)(s\bar{\mathbf{P}}_{i,j,x}(s) - s\mathbf{P}_{i,j,x}(s)) \\ & + (\mu h_1 + h_2 h_1)(\bar{\mathbf{P}}_{i,j,x}(s) - \mathbf{P}_{i,j,x}(s)),\end{aligned} \tag{58}$$

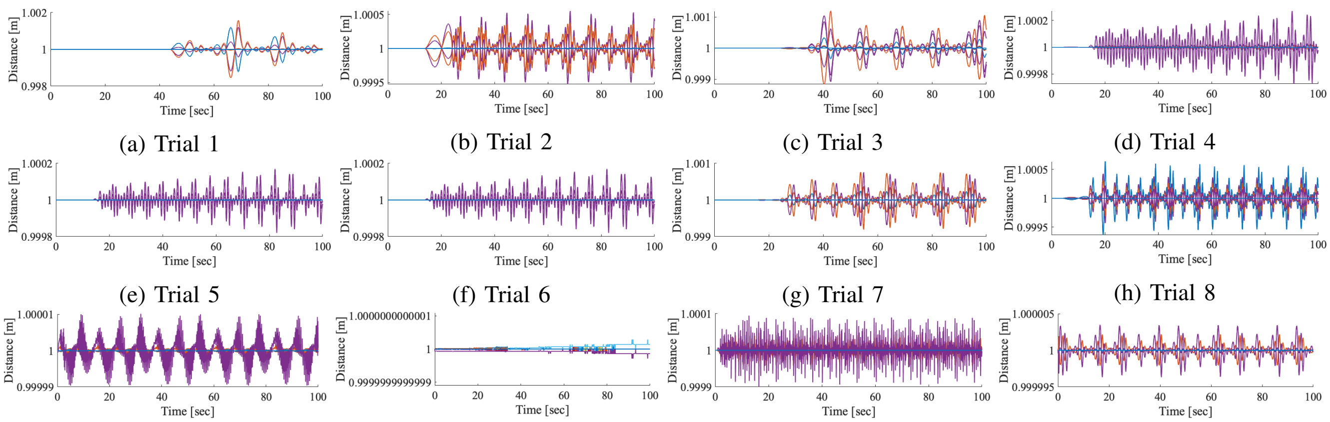
$$\frac{\mathbf{P}_{i,j,x}(s)}{\bar{\mathbf{P}}_{i,j,x}(s)} = \frac{s^2 + (h_2 - \mu + h_1)s + (\mu h_1 + h_2 h_1)}{s^2 + (h_2 - \mu + h_1)s + (\mu h_1 + h_2 h_1)} = 1. \tag{59}$$

Eq. (59) shows that robust control law always satisfies $A_p \leq 1$ in Eq. (17). This means that the swarm will be



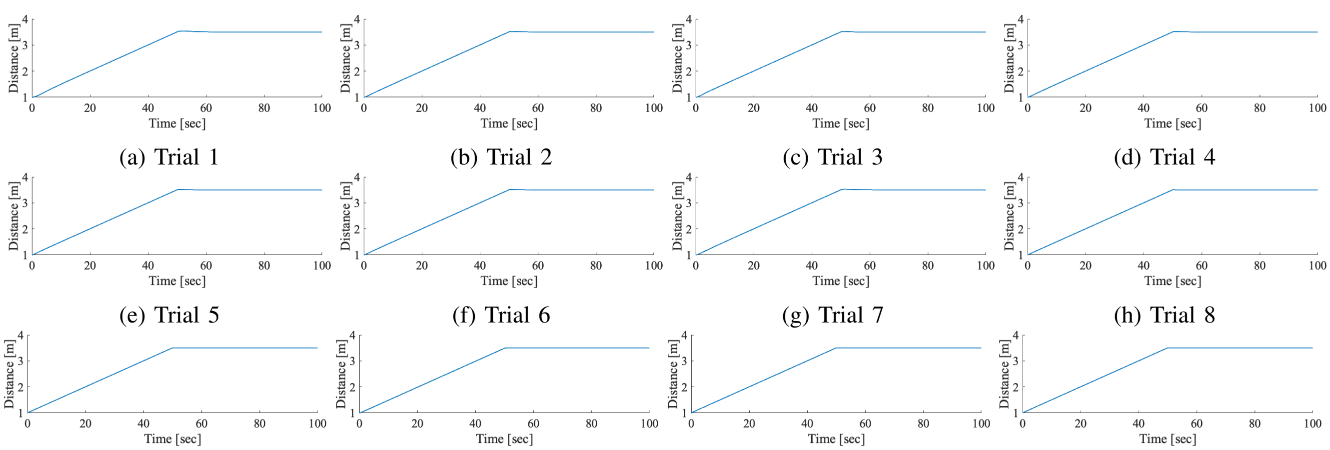
(a) Trial 1 (b) Trial 2 (c) Trial 3 (d) Trial 4
(e) Trial 5 (f) Trial 6 (g) Trial 7 (h) Trial 8
(i) Trial 9 (j) Trial 10 (k) Trial 11 (l) Trial 12

Fig. 6: Evolution of inter-agent distance with time for all trials in Table I using the robust control law. Note that all trajectories are transient stable. The variation of inter-agent distance in the Y-axis is very small.



(a) Trial 1 (b) Trial 2 (c) Trial 3 (d) Trial 4
(e) Trial 5 (f) Trial 6 (g) Trial 7 (h) Trial 8
(i) Trial 9 (j) Trial 10 (k) Trial 11 (l) Trial 12

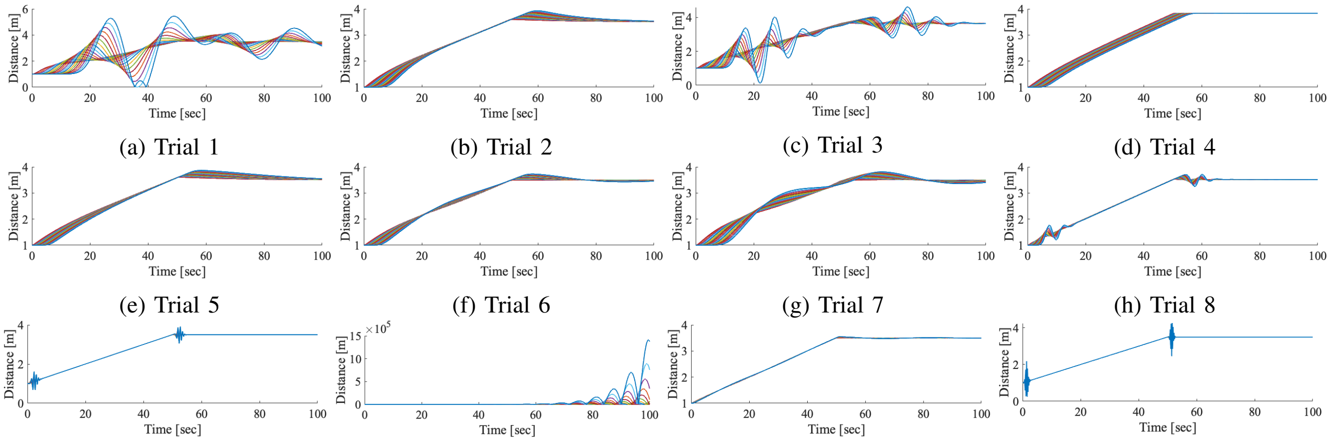
Fig. 7: Evolution of inter-agent distance with time for all trials in Table I using the robust control law, with damping $\mu = 0$. Note that all trajectories are transient stable. The variation of inter-agent distance in the Y-axis is very small.



(a) Trial 1 (b) Trial 2 (c) Trial 3 (d) Trial 4
(e) Trial 5 (f) Trial 6 (g) Trial 7 (h) Trial 8
(i) Trial 9 (j) Trial 10 (k) Trial 11 (l) Trial 12

Fig. 8: Evolution of inter-agent distance with time for all trials in Table I using the robust control law for swarm reconfiguration. Note that all trajectories are transient stable.

transient stable for all values of h_1 and h_2. Using the gain values in Table I and $D_0 = 1$ m and $D_1 = 0.05$ m/sec; we see in Fig. 8 that the swarm agents avoid inter-agent collisions for all trials. As expected, this reconfiguration could cause



(a) Trial 1 (b) Trial 2 (c) Trial 3 (d) Trial 4
(e) Trial 5 (f) Trial 6 (g) Trial 7 (h) Trial 8
(i) Trial 9 (j) Trial 10 (k) Trial 11 (l) Trial 12

Fig. 9: Evolution of inter-agent distance with time for all trials in Table I using the PID control law for swarm reconfiguration. Note the inter-agent collisions in Trial 1 and instability in Trial 10.

inter-agent collisions under a bad controller. For example, if the PID controller in Section IV and the the gain values in Table I are used, where $D_0 = 1$ m and $D_1 = 0.05$ m/sec; we see in Fig. 9 that the swarm agents do have inter-agent collisions and instability even though the agents are moving away from each other.

VII. Conclusions

We first introduced the concept of transient instabilities in multi-agent systems and swarms, which is new concept in the swarm and control theory literature. We presented a mathematical framework or "theoretical tool" to detect if a swarm is going to be transient stable. We proved that tool is a *sufficient condition* for transient stability in swarms, but it tends to become a *necessary condition* when the swarm size is very large or other parameters are changed. We showed how a simple PID control law, irrespective of choice of gains, is not appropriate for transient stability in swarms. On the other hand, a robust control law for Euler-Lagrangian system guarantees transient stability in swarms, while the agents are tracking static or time-varying inter-agent distances.

The theoretical results shown in this paper are for a very special case, but we show that the tools also work for more general cases. Future work will focus on developing theoretical arguments for the general cases too.

Future work will also focus on applying these techniques to nonlinear swarm systems and more complex geometries. We also plan to focus on the development of general mitigation strategies that can guarantee swarm stability under all conditions. We envisage that these techniques will be widely used to check the stability of swarms, before they are deployed in the field.

Acknowledgment

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