ECE 484: Principles of Safe Autonomy (Fall 2025)
Lecture 9
Control (part 2)

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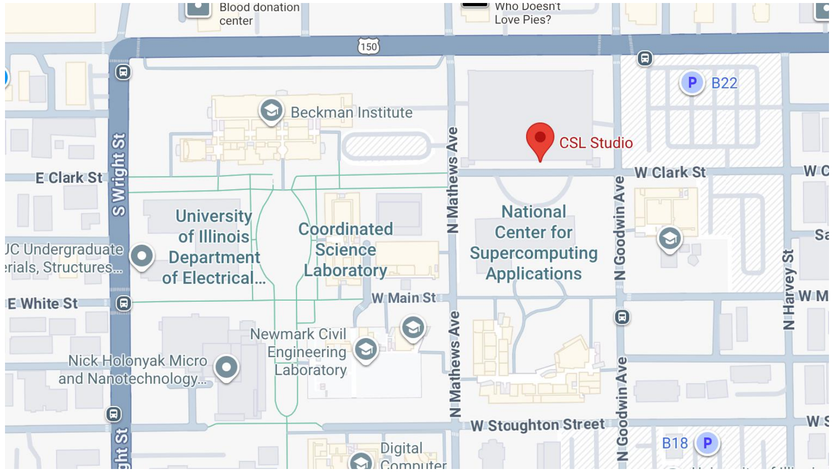
Announcements

- Field Trip to CSL Studio for F1 Tenth, GRAIC, and Drone projects (10/2, 11 am)
  - No regular class in this class room on 10/2
  - 1206 W Clark St, Urbana, IL 61801



- Project group sign up will open the day after (10/3).
  - Group limit: 4, with an exception of groups of 5 for GEM.

- Groups will be finalized the Tuesday after (10/7).
  - Students who do not sign up will be randomly assigned.

- Pay close attention to any announcements on Campuswire in case anything changes

Outline

- Modeling the control problem
  - Differential Equations; solutions and their properties
  - Bang-bang control
- Control design ←
  - PID
  - State feedback
  - MPC (brief)
- Requirements
  - Stability
  - Lyapunov theory and its relation to invariance

On-off control of a room heater with a thermostat

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), u(t))$$
$$\boldsymbol{u}(t) = g(x(t))$$

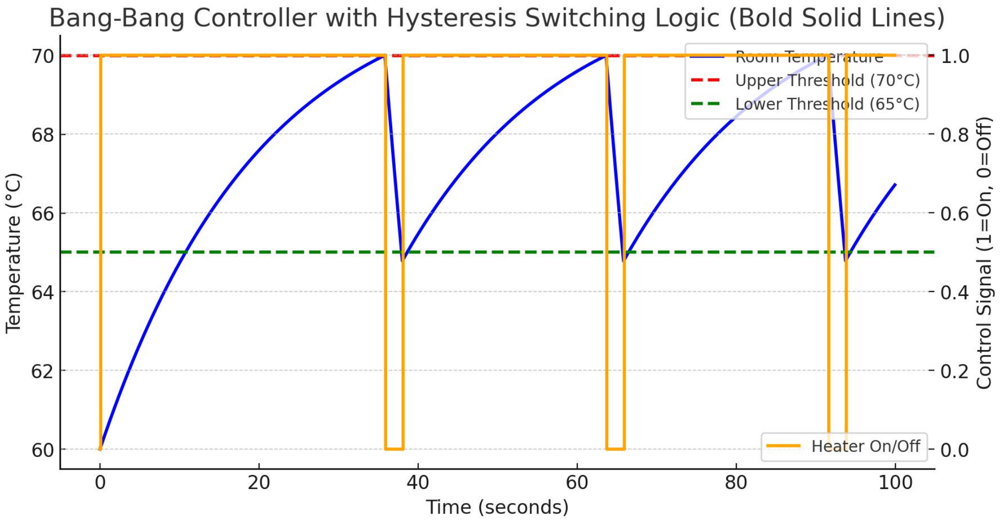
A simple thermostat controller

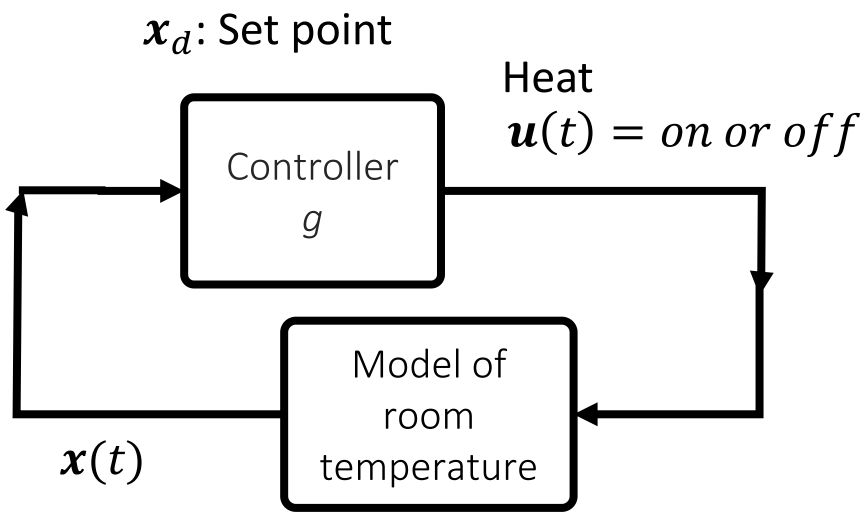
$g(x(t))$:

if $x(t) \geq x_d$ then $u(t)$ = off

else if $x(t) \leq x_d - \varepsilon$ then $u(t)$ = on

This is called bang-bang control



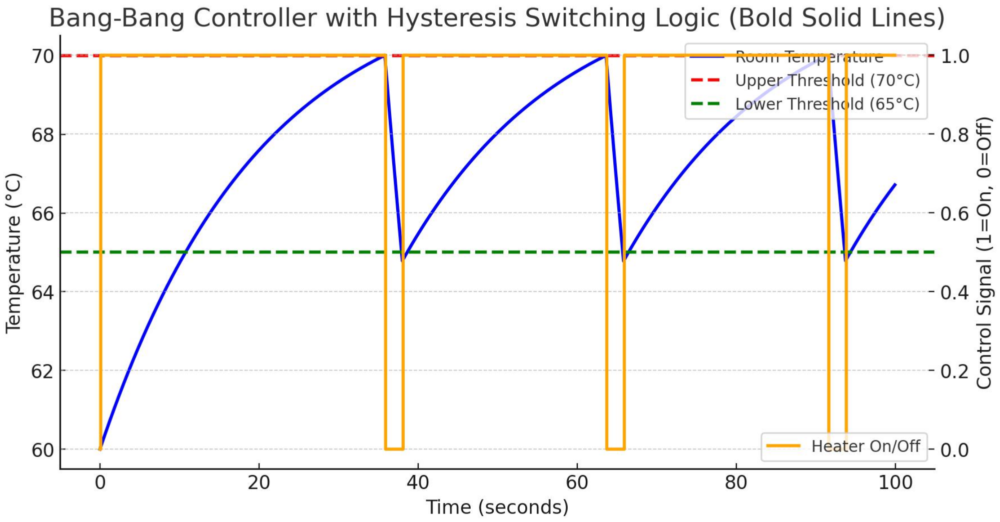


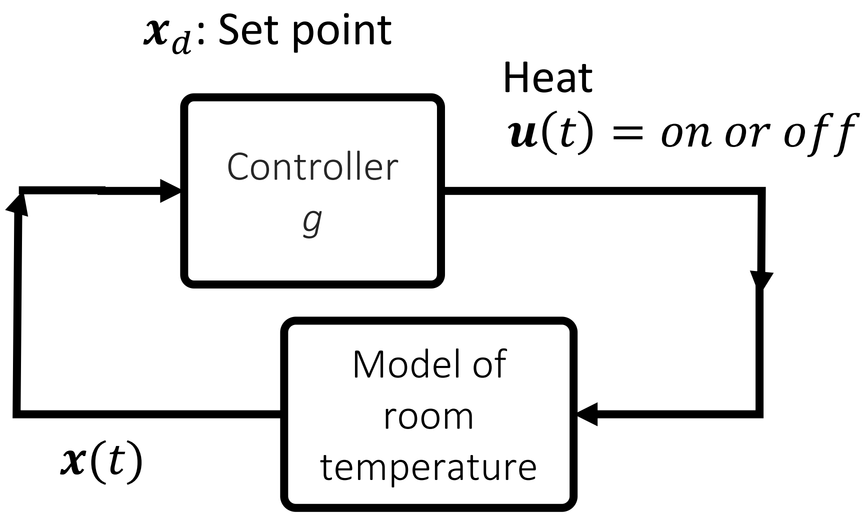
On-off control of a room heater with a thermostat

Bang-bang control is a feasible strategy when the controlled variable is observable

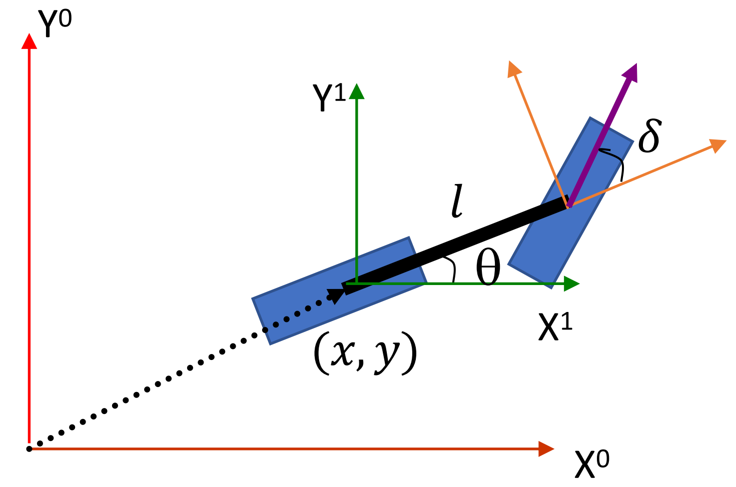
Disadvantages

- **Usually not energy efficient**
- Overshoots and undershoots because of inertia and delays
- Causes excess stress on the actuators
- Can cause the system to become unstable (to be defined later)





Review: Rear Wheel Model (Bicycle model)



Plant state: real wheel pose) = $\boldsymbol{x_B}: \mathbb{R}^3 = \begin{bmatrix} x_B \\ y_B \\ \theta_B \end{bmatrix}$

Control input: front wheel steering angle $u: \mathbb{R}$ = δ_B

Model parameters: car length (l) speed (v_B)

$f: \mathbb{R}^4 \rightarrow \mathbb{R}^3$

$\dot{\boldsymbol{x}}_B = f(\boldsymbol{x_B}, u)$

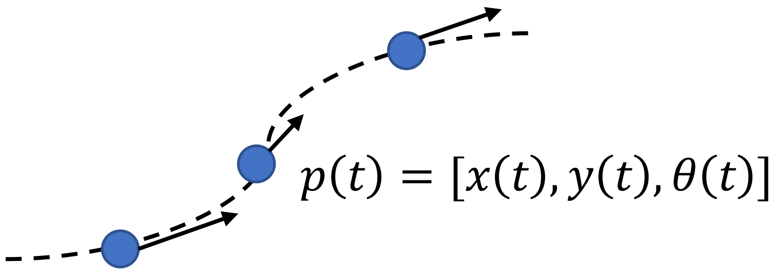
$$\begin{bmatrix} \dot{x}_B \\ \dot{y}_B \\ \dot{\theta}_B \end{bmatrix} = \begin{bmatrix} v_B \cos\theta_B \\ v_B \sin\theta_B \\ \frac{v_B}{l} \tan\delta_B \end{bmatrix}$$

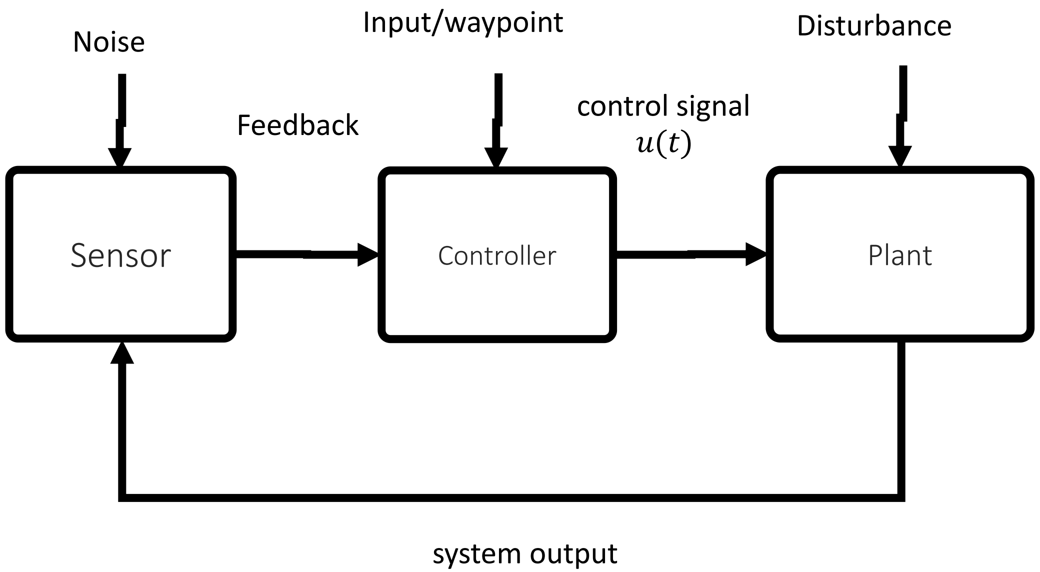
Path following control

The path **to be followed** by a robot is typically represented by a parameterized curve (e.g., parameterized by time)

This path is computed by a higher-level planner (e.g., using hybrid A*, RRT)

Each point in the path defines the desired instantaneous pose $p(t)$ of the vehicle

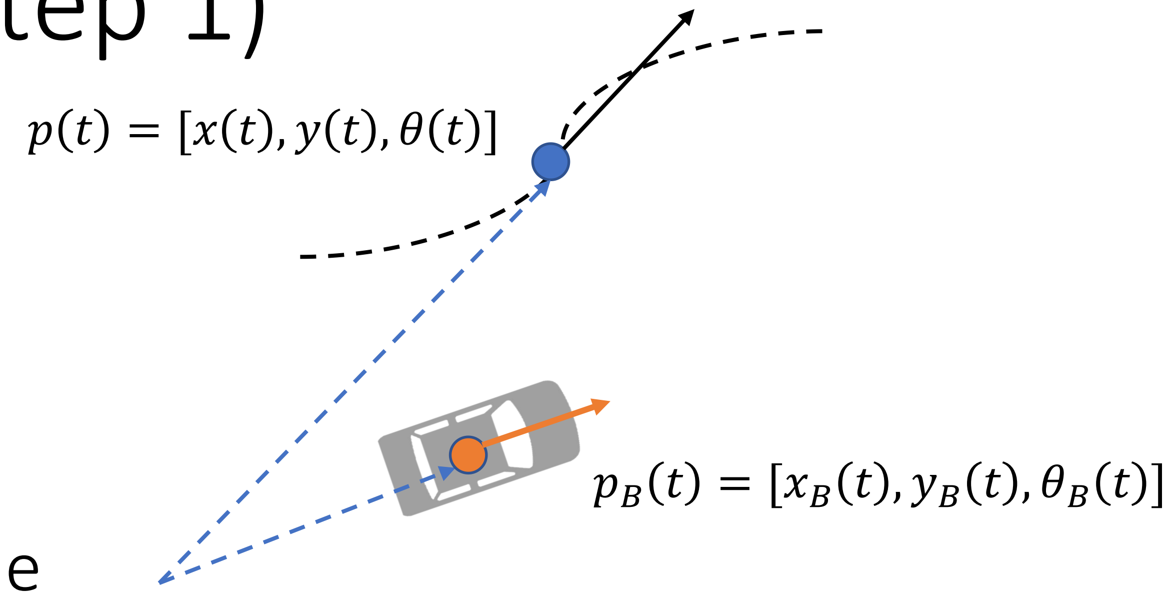




Path following control (Step 1)

Desired instantaneous pose $p(t)$

How to define error between actual pose $p_B(t)$ and desired pose $p(t)$ in the form of $x_d(t) - x(t)$ *so that then we can develop a control law*



Bang bang controller for bicycle model (Step 2)

Dynamics $\begin{bmatrix} \dot{x}_B \\ \dot{y}_B \\ \dot{\theta}_B \end{bmatrix} = \begin{bmatrix} v\, cos\theta_B \\ v\, sin\theta_B \\ \frac{v}{l} tan\delta_B \end{bmatrix}$

Heading error: $e_h = \theta_B - \theta$

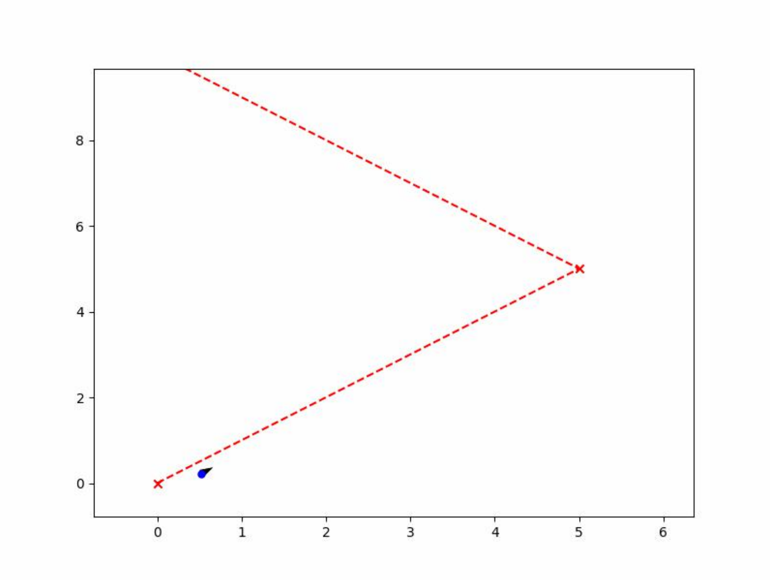
Cross track error: $e_d = \pm \, ||(x_B, y_B) - (x, y)||$

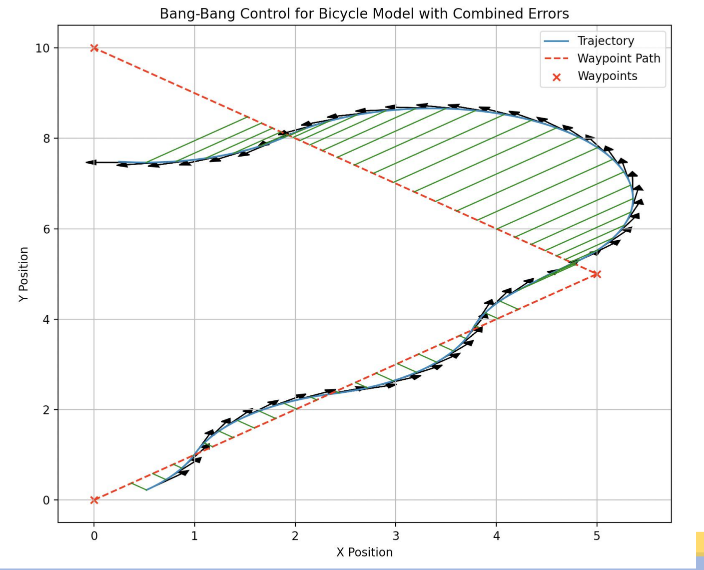
(**signed distance,** depending on which side the bicyle is)

Combined error: $e = e_h + \alpha e_d$

Bang-bang controller:

if $e > 0$ then $\delta = \delta_{max}$ else $\delta = -\delta_{max}$





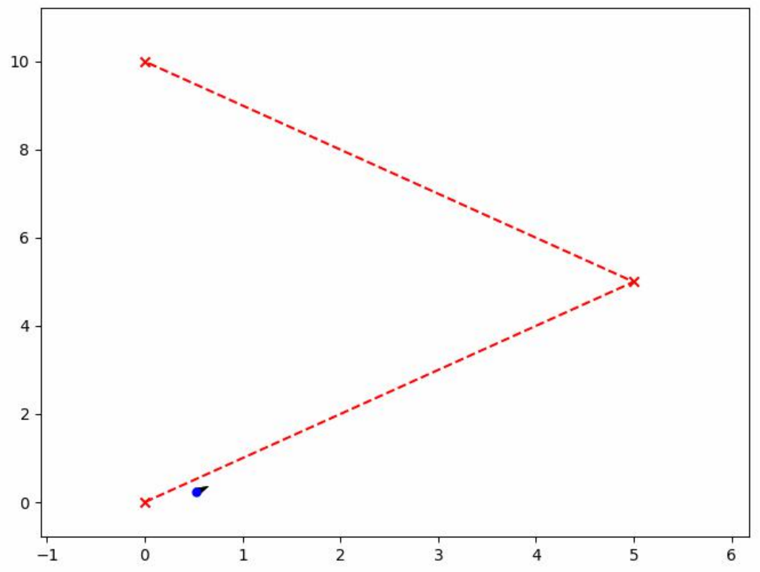
Proportional control

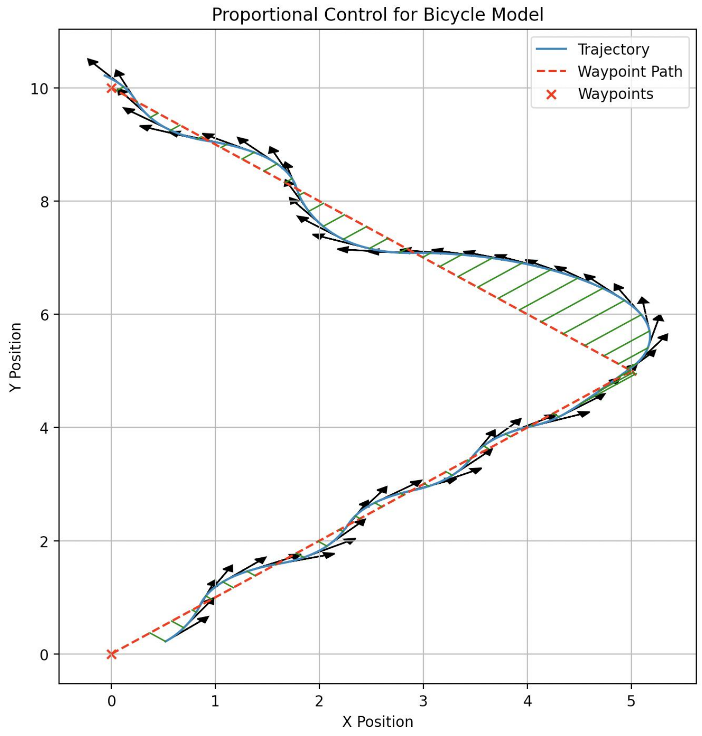
Dynamics $\begin{bmatrix} \dot{x}_B \\ \dot{y}_B \\ \dot{\theta}_B \end{bmatrix} = \begin{bmatrix} v\, cos\theta_B \\ v\, sin\theta_B \\ \frac{v}{l} tan\delta_B \end{bmatrix}$

Heading error: $e_h = \theta_B - \theta$

Cross track error: $e_d = \pm \, ||(x_B, y_B) - (x, y)||$

Proportional controller $\delta = -K_h e_h + -K_d e_d$





More complete path following control

Desired instantaneous pose $p(t)$

The error vector measured vehicle coordinates
$\mathrm{e}(t) = [\delta_s(t), \delta_n(t), \delta_\theta(t), \delta_v(t)]$

$[\delta_s, \delta_n]$ define the coordinate errors in the vehicle's reference frame: along track error and cross track error

- Along track error: distance ahead or behind the target in the instantaneous direction of motion.

$$\delta_s = \cos\left(\theta_B(t)\right)\left(x(t) - x_B(t)\right) + sin\left(\theta_B(t)\right)\left(y(t) - y_B(t)\right)$$

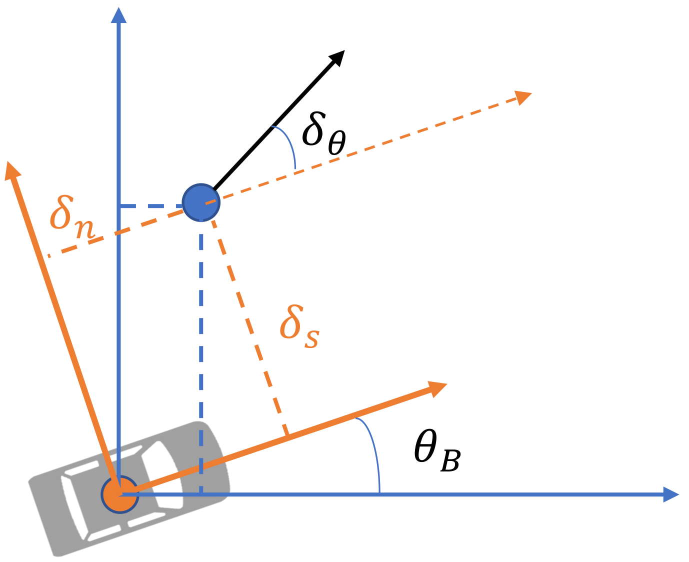
- Cross track error: portion of the position error orthogonal to the intended direction of motion

$$\delta_n = -\sin\left(\theta_B(t)\right)\left(x(t) - x_B(t)\right) + cos\left(\theta_B(t)\right)\left(y(t) - y_B(t)\right)$$

- Heading error

$$\delta_\theta = \theta(t) - \theta_B(t)$$
$$\delta_v = v(t) - v_B(t)$$

$p(t) = [x(t), y(t), \theta(t), v(t)]$



$p_B(t) = [x_B(t), y_B(t), \theta_B(t), v_B(t)]$

A Proportional controller

Plant $\dot{x}(t) = u(t) + d(t)$, where $d(t)$ is a small disturbance signal

The goal is the drive the plant state to a target steady state value, say $x_d = 70°$

Idea: Make the control input negatively proportional to the error: **Negative feedback**

Error: $e(t) = x(t) - x_d$

Proportional controller: $u(t) =- K_p e(t)$, the constant K_p is called **controller gain**

Using proportional (P) **negative feedback**

$$u(t) =- K_P e(t) =- K_P(x(t) - x_d)$$

$$\dot{x}(t) =- K_P x(t) + K_P x_d + d(t)$$

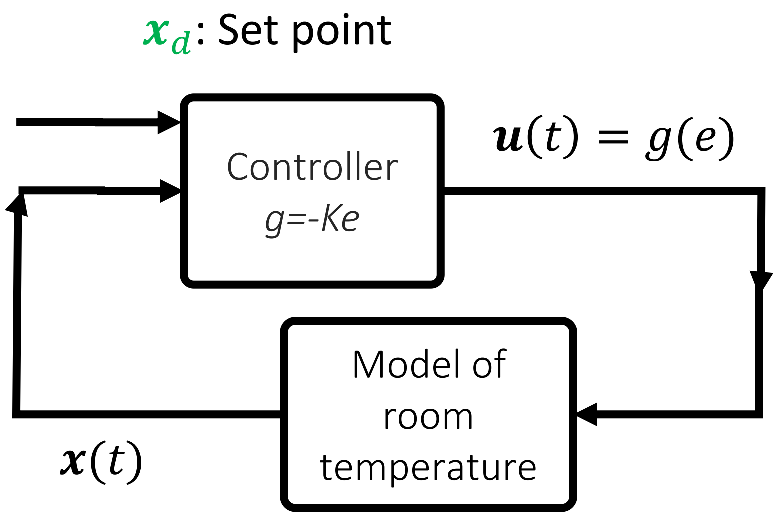
Consider a **constant disturbance** d_{ss} (e.g., room energey loss)

$$\dot{x}(t) =- K_P x(t) + K_P x_d + d_{ss}$$

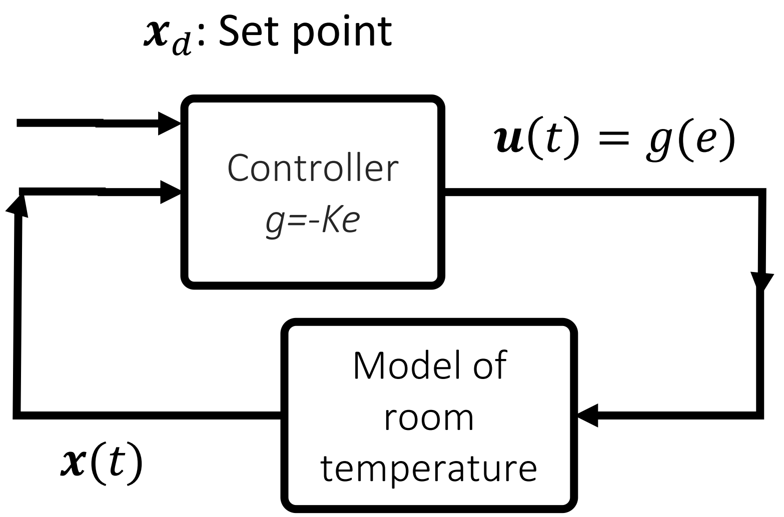
What is the steady state value? Trick: set RHS = 0

Set $-K_P x(t) + K_P x_d + d_{ss} = 0$

$$x(t) = x_{ss} := \frac{d_{ss}}{K_P} + x_d$$



Proportional controller example



With constant disturbance d_{ss} we rewrite the ODE

$$\dot{x}(t) = -K_P x(t) + K_P x_d + d_{ss} \text{ with } x_{ss} = \frac{d_{ss}}{K_P} + x_d$$

$$\dot{x}(t) = K_P(x_{ss} - x(t))$$

The solution of this ODE (Transient behavior) is:

$$x(t) = x_{ss} + (x(0) - x_{ss})e^{-tK_p}$$

Rewrite:

$$x(t) = x(0)e^{-tK_p} + x_{ss}(1 - e^{-tK_p})$$

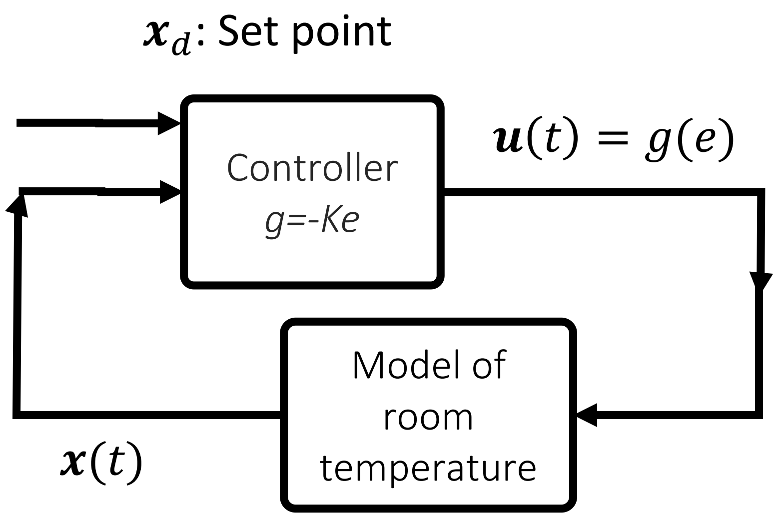
General solution of first-order linear DE

$x(t) = x_{ss} + Ce^{-K_p t}$

Setting t=0

$x(0) = x_{ss} + C$

Proportional Controller



Transient behavior of the control system

$$x(t) = x(0)e^{-tK_p} + x_{ss}(1 - e^{-tK_p}); x_{ss} = \frac{d_{ss}}{K_P} + x_d$$

The proportional controller uses negative feedback to track the desired setpoint smoothly

Steady state error may not be 0

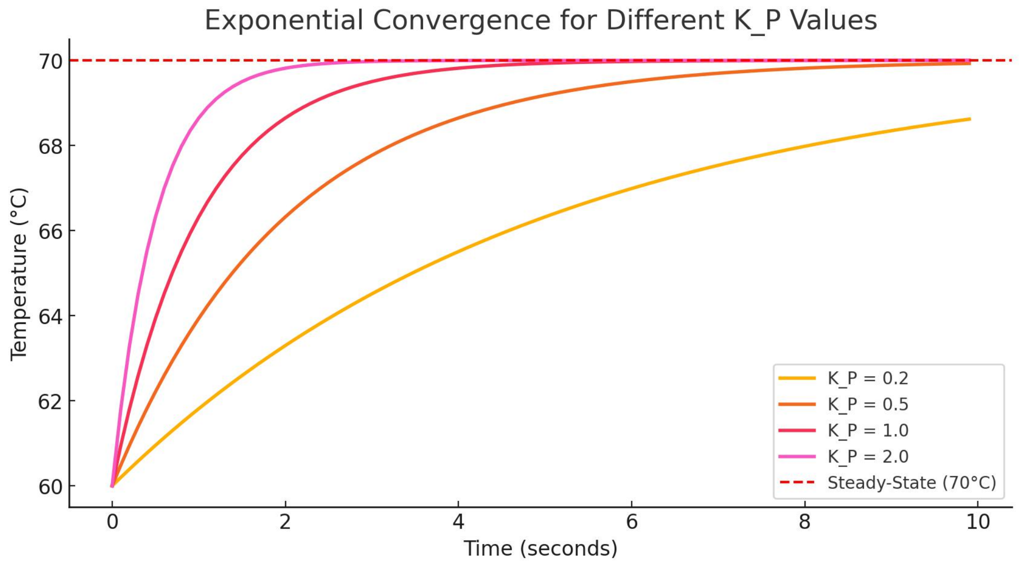
Larger proportional gain K_P more reactive the controller and faster the system converges to the target state K_P

Larger K_P implies smaller steady state tracking error

For systems with delays and inertia high proportional gain can cause oscillations or overshoots

There may be actuator limits that prevent $u(t) =- K_P e(t) =- K_P(x(t) - x_d)$ to be a feasible control input

(e.g., the room loses energy very fast and heater power insufficient)



The PID controller

Error: difference of desired and measured $e(t) = x_d - x(t)$

$$u(t) = K_P e(t) + K_I \int_0^t e(\tau) d\tau + K_D \frac{de(t)}{dt}$$

Tune K_P, K_I, K_d for the required performance

P (Proportional): Corrects based on the current error

Reacts to errors quickly (may lead to oscillations), may not be sufficient to remove steady state error

I (Integral): Corrects based on the accumulated past error (“memory”).

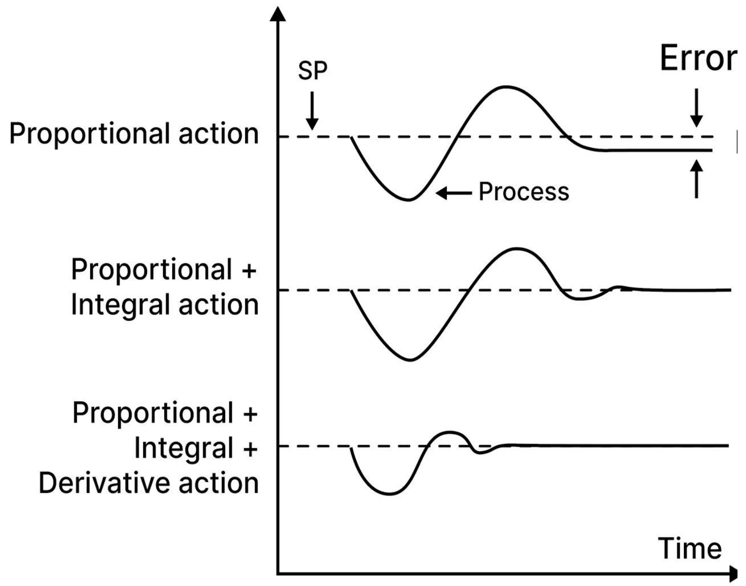
Removes steady state error

D (Derivative): Predicts future error based on the rate of change.

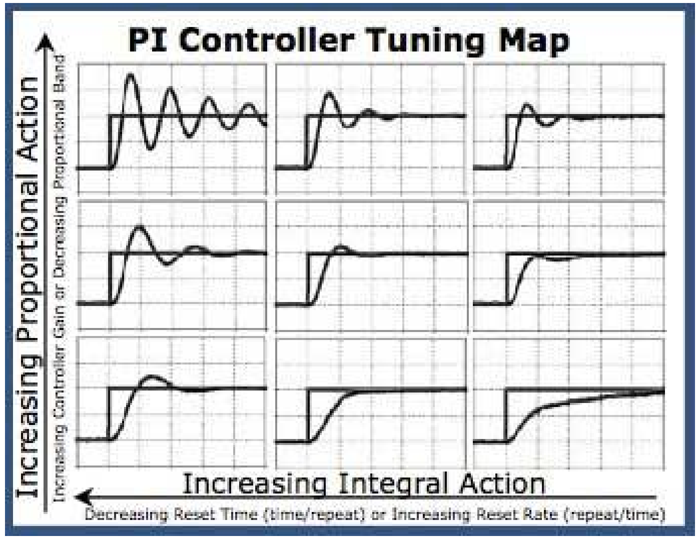
Dampens oscillations, reduce overshoots, but derivatives can be sensitive to noise

PD control: $K_I = 0$

PI control: $K_D = 0$

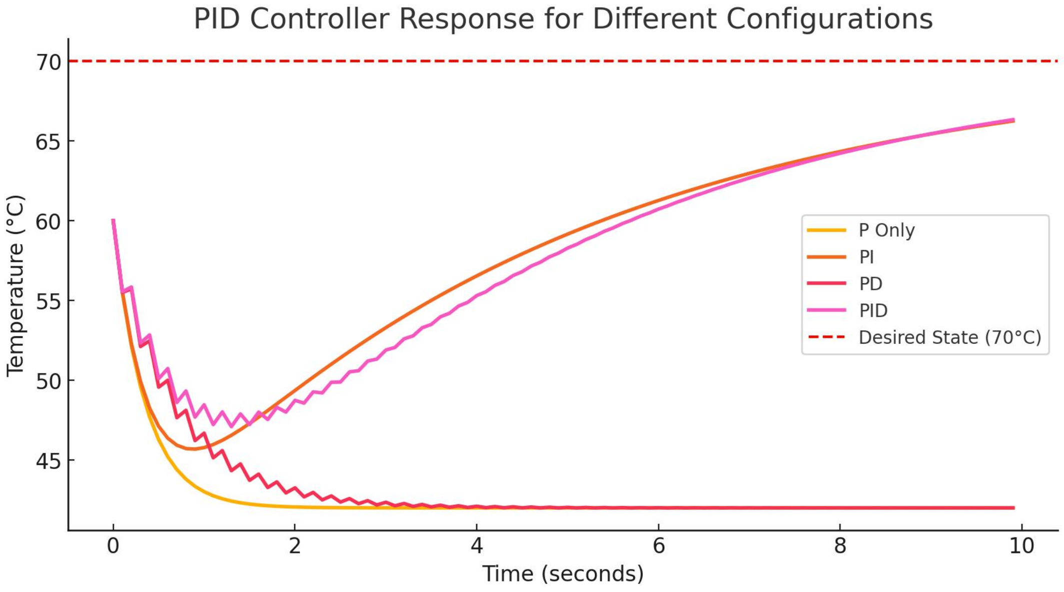


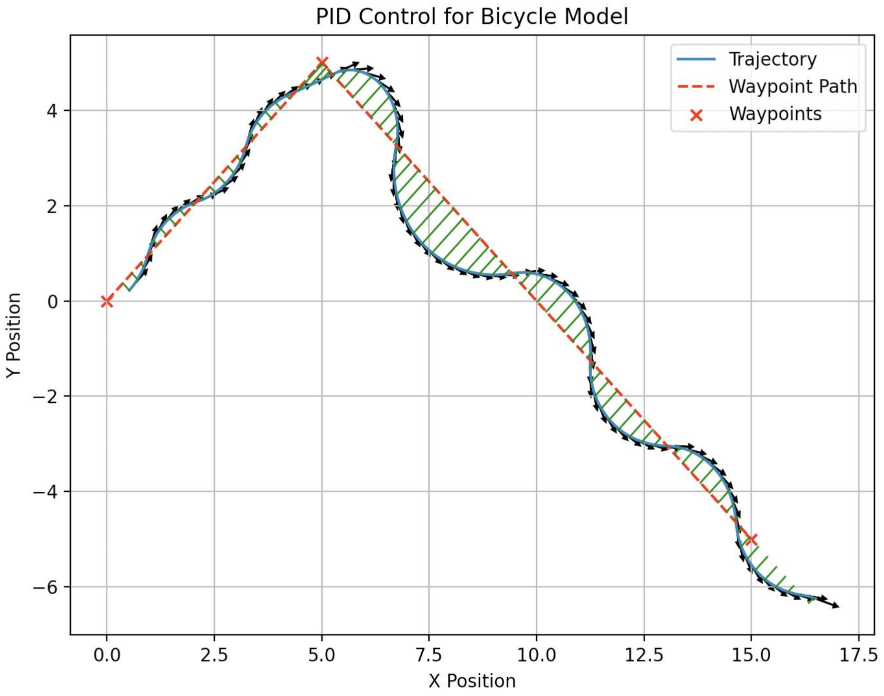
https://www.realpars.com/blog/pid-vs-advanced-control-methods



https://www.yokogawa.com/library/resources/white-papers/pid-tuning-in-distributed-control-systems/

PID controller tuning





Linear systems

Linear dynamical system and solutions

Linear system is a dynamical system where the dynamical function f is a linear function of the state and the inputs
$\dot{x}(t) = A(t)x(t) + B(t)u(t)$

where A and B are (possibly time varying) matrices

Linear models can be obtained by **linearizing** a nonlinear model around a suitably chosen point $A_{x_0} = \frac{\partial f}{\partial x}|_{x_0}$

Linear systems are amenable to efficient control design and analysis

For a given initial state $x_0 \in \mathbb{R}^n, t_0 \in \mathbb{R}\ and\ u(.) \in PC(\mathbb{R}, \mathbb{R}^n)$ the *solution* is a function $\xi(., t_0, x_0, u): \mathbb{R} \to \mathbb{R}^n$

Example: Simple linear model of an economy

x: national income

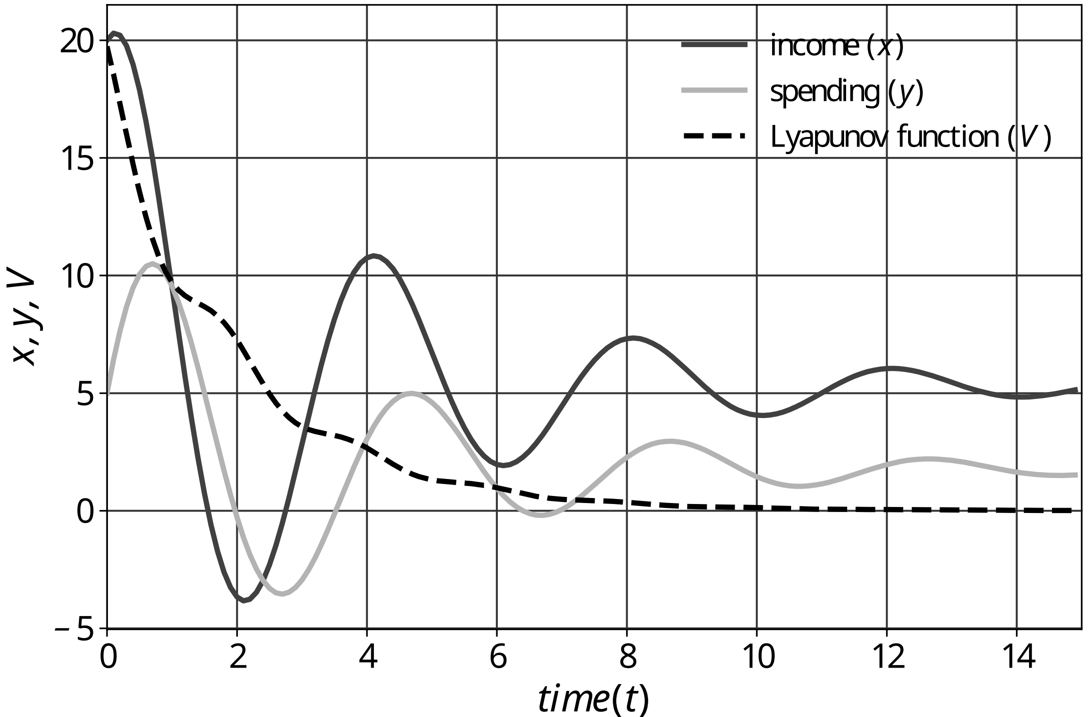
y: rate of consumer spending

g: rate government expenditure

$$\dot{x} = x - \alpha y$$

$$\dot{y} = \beta(x - y - g)$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & -\alpha \\ \beta & -\beta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 \\ \beta \end{bmatrix} g$$



Solutions of Linear systems define a **linear** space

$\dot{x}(t) = A(t)x(t) + B(t)u(t)$ --- Eq. (2)

$u(t)$ continuous everywhere except D_x

Theorem*. Let $\xi(t, t_0, x_0, u)$ be the solution for (2) with points of discontinuity , D_x

1. $\forall t_0 \in \mathbb{R}, x_0 \in \mathbb{R}^n, u \in PC(\mathbb{R}, \mathbb{R}^m), \xi(\cdot, t_0, x_0, u): \mathbb{R} \to \mathbb{R}^n$ is continuous and differentiable $\forall\, t \in \mathbb{R} \setminus D_x$
2. $\forall t, t_0 \in \mathbb{R},\ u \in PC(\mathbb{R}, \mathbb{R}^m), \xi(t, t_0, \cdot, u): \mathbb{R}^n \to \mathbb{R}^n$ is continuous
3. linearity/superposition: $\forall t, t_0 \in \mathbb{R}, x_{01}, x_{02} \in \mathbb{R}^n, u_1, u_2 \in PC(\mathbb{R}, \mathbb{R}^m), a_1, a_2 \in \mathbb{R},\ \xi(t, t_0, a_1 x_{01} + a_2 x_{02}, a_1 u_1 + a_2 u_2) = a_1 \xi(t, t_0, x_{01}, u_1) + a_2 \xi(t, t_0, x_{02}, u_2)$
4. decomposition: $\forall t, t_0 \in \mathbb{R},\ x_0 \in \mathbb{R}^n, u \in PC(\mathbb{R}, \mathbb{R}^m),\ \xi(t, t_0, x_0, u) = \xi(t, t_0, x_0, \mathbf{0}) + \xi(t, t_0, 0, u)$

Zero input Zero state

Linear time invariant system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

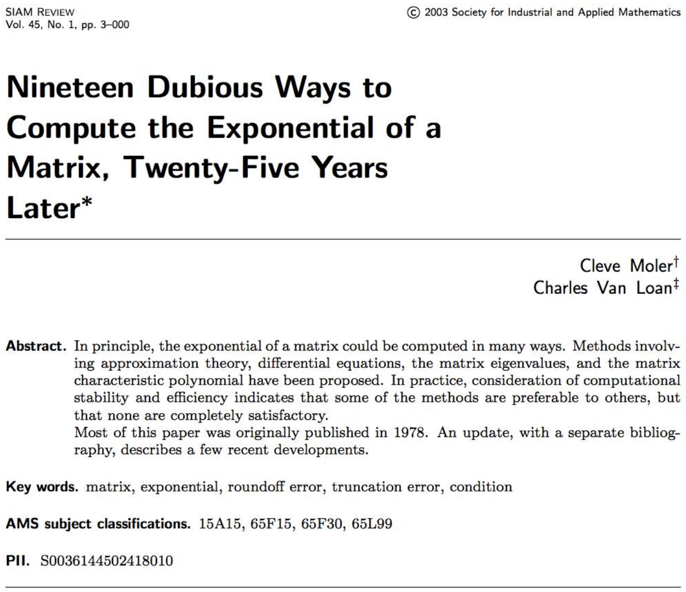
Matrix exponential:

$$e^{At} = 1 + At + \frac{1}{2!}(At)^2 + \ \dots = \sum_{0}^{\infty} \frac{1}{k!}(At)^k$$

Theorem. (Solution of linear systems)

$$\xi(t, t_0, x_0, u) = x_0 e^{A(t-t_0)} + \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau) d\tau$$

Zero input   Zero state


SIAM Review
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Nineteen Dubious Ways to Compute the Exponential of a Matrix, Twenty-Five Years Later*

Cleve Moler†
Charles Van Loan‡

Abstract. In principle, the exponential of a matrix could be computed in many ways. Methods involving approximation theory, differential equations, the matrix eigenvalues, and the matrix characteristic polynomial have been proposed. In practice, consideration of computational stability and efficiency indicates that some of the methods are preferable to others, but that none are completely satisfactory.
Most of this paper was originally published in 1978. An update, with a separate bibliography, describes a few recent developments.

Key words. matrix, exponential, roundoff error, truncation error, condition

AMS subject classifications. 15A15, 65F15, 65F30, 65L99

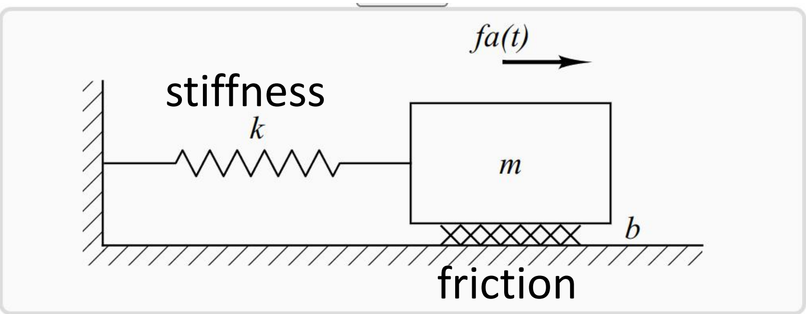
PII. S0036144502418010

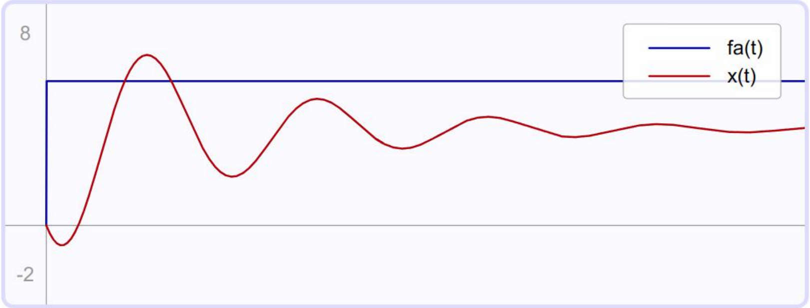
Example

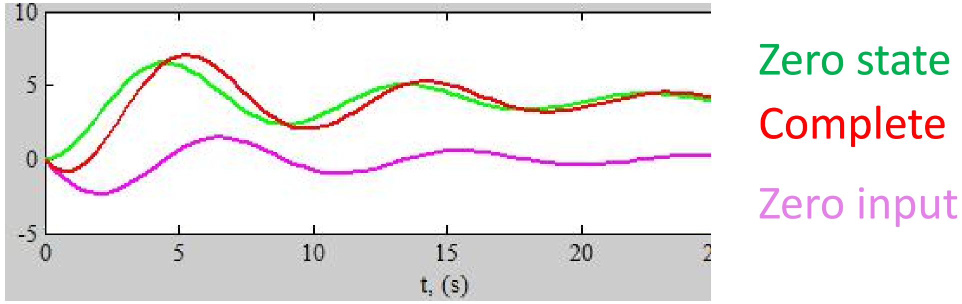
States x: postion (0m), velocity (-2m/s),
Input u(t): Force $f_a(t)$ = 6 Newtons

$$m\frac{dx_2(t)}{dt} = u(t) - b\frac{dx_1(t)}{dt} - kx_1(t)$$

$$x_2(t) = \frac{dx_1(t)}{dt}$$







Source: https://lpsa.swarthmore.edu/Transient/TransZIZS.html

Deriving the solution of Linear time invariant system*

$$\dot{x}(t) = Ax(t) + Bu(t)$$

A and B are not function of t.

Solution of the system ξ can be explicitly derived. How to do that?

Consider the decomposition property, we solve two problems:

$$\forall t \in \mathbb{R},\ x_0 \in \mathbb{R}^n, u \in PC(\mathbb{R}, \mathbb{R}^m),\ \xi(t,\ x_0,\ u) = \xi(t,\ x_0,\ \mathbf{0}) + \xi(t, 0,\ u)$$

We assume t_0=0 for simplicity, so no t_0 term anymore

Deriving the solution of Linear time invariant system*

First set input u(t) to 0 (we do this due to the decomposition)

$$\frac{dx(t)}{dt} = A\,x(t),\; x(t=0) = x_0$$

Due to linearity, the solution is in this form:

$$x(t) = \phi(t)x_0 = (E + \phi_1 t + \phi_2 t^2 + \ldots + \phi_n t^n + \ldots)x_0$$

Taylor expansion of $\Phi(t)$

Substitute into the differential equation:

$$\frac{d}{dt}(\phi(t)x_0) = A\,\phi(t)\,x_0$$

$$(\phi_1 + 2\,\phi_2\,t + \ldots + n\,\phi_n\,t^{n-1} + \ldots)x_0 = (A + A\phi_1 t + A\phi_2 t^2 + \ldots + A\phi_n t^n + \ldots)\,x_0$$

Deriving the solution of Linear time invariant system*

$$\frac{d}{dt}(\phi(t)x_0) = A\,\phi(t)\,x_0$$
$$(\phi_1 + 2\,\phi_2\,t + \ldots + n\,\phi_n\,t^{n-1} + \ldots)x_0 = (A + A\phi_1 t + A\phi_2 t^2 + \ldots + A\phi_n t^n + \ldots)\,x_0$$

Now we want to solve Φ(t), by comparing the terms:

$$\phi_1 = A$$
$$\phi_2 = \frac{1}{2}A\,\phi_1 = \frac{1}{2!}A^2$$
$$\phi_3 = \frac{1}{3}A\,\phi_2 = \frac{1}{3!}A^3$$
$$\ldots$$
$$\phi_n = \frac{1}{n!}A^n.$$

Deriving the solution of Linear time invariant system*

$$\phi_1 = A$$

$$\phi_2 = \frac{1}{2} A \phi_1 = \frac{1}{2!} A^2$$

$$\phi_3 = \frac{1}{3} A \phi_2 = \frac{1}{3!} A^3$$

$$\cdots$$

$$\phi_n = \frac{1}{n!} A^n.$$

$$\phi(t) = e^{At} = E + At + \frac{1}{2!} A^2 t^2 + \frac{1}{3!} A^3 t^3 + \ldots + \frac{1}{n!} A^n t^n + \ldots$$

$$\forall t \in \mathbb{R},\ x_0 \in \mathbb{R}^n, u \in PC(\mathbb{R}, \mathbb{R}^m),\ \xi(t,\ x_0,\ u) = \boxed{\xi(t,\ x_0,\ \mathbf{0})} + \xi(t, 0,\ u)$$

This part done

Deriving the solution of Linear time invariant system*

Consider the decomposition property, we solve two problems:

$$\forall t \in \mathbb{R},\ x_0 \in \mathbb{R}^n, u \in PC(\mathbb{R}, \mathbb{R}^m),\ \xi(t,\ x_0,\ u) = \xi(t,\ x_0,\ \mathbf{0}) + \xi(t, 0,\ u)$$

Now the second part

Now for $\xi(t, 0,\ u)$, assume $x_0 = 0$, solve

$$\frac{dx(t)}{dt} = Ax(t)\ +\ Bu(t)$$

Rearrange:

$$\frac{dx(t)}{dt} - Ax(t) =\ \ Bu(t)$$

Multiply a common factory:

$$e^{-At}\frac{dx(t)}{dt} - e^{-At}Ax(t) =\ e^{-At}\,Bu(t)$$

Note the perfect differential:

$$\frac{d}{dt}(e^{-At}x(t)) =\ e^{-At}\,Bu(t)$$

Deriving the solution of Linear time invariant system*

$$\frac{d}{dt}(e^{-At}x(t)) = e^{-At}Bu(t)$$

Integration on both sides:

$$\int_0^t \frac{d}{d\tau}(e^{-A\tau}x(\tau)) = \int_0^t e^{-A\tau}Bu(\tau)\,d\tau$$

$$e^{-At}x(t) - e^{A0}x(0) = \int_0^t e^{-A\tau}Bu(\tau)\,d\tau$$

Since x(0) = 0:

$$x(t) = e^{At}\int_0^t e^{-A\tau}Bu(\tau)\,d\tau$$

$$x(t) = \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

Deriving the solution of Linear time invariant system*

$$\dot{x}(t) = Ax(t) + Bu(t)$$

Define Matrix exponential:

$$e^{At} = 1 + At + \frac{1}{2!}(At)^2 + \ \dots = \sum_{0}^{\infty} \frac{1}{k!}(At)^k$$

Theorem. $\xi(t, x_0, u) = \Phi(t)x_0 + \int_0^t \mathrm{e}^{A(t-\tau)}Bu(\tau)d\tau$

Zero input Zero state

Here $\Phi(t) := e^{At}$ is the **state-transition** matrix