ECE 484: Principles of Safe Autonomy (Fall 2025)
Lecture 10
Control (part 3)

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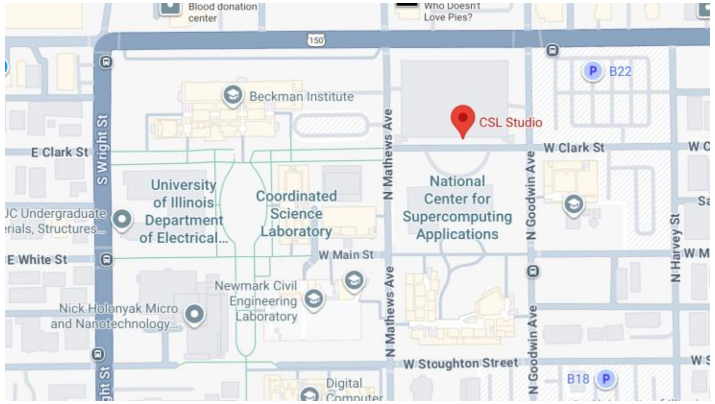
Announcements

- Field Trip to CSL Studio for F1 Tenth, GRAIC, and Drone projects (10/2, 11 am)
  - We will start in **ECEB 1015** at 11 am to see presentations, then walk to CSL studio
  - 1206 W Clark St, Urbana, IL 61801



- Project group sign up will open the day after (10/3).
  - Group limit: 4, with an exception of groups of 5 for GEM.
- Groups will be finalized the Tuesday after (10/7).
  - Students who do not sign up will be randomly assigned.
- Midterm: 10/7, covering everything in safety, perception, control
  - Go over homework questions, MP questions, and slides
  - You can bring 1-page (letter-size) handwritten note (writing on both sides ok)
  - Go to the review session hold by TAs
- Pay close attention to any announcements on Campuswire in case anything changes

Outline

- Modeling the control problem
  - Differential Equations; solutions and their properties
  - Bang-bang control
- Control design
  - PID
  - State feedback
- Linear systems ←
- Requirements
  - Stability, Asymptotic stability
  - Designing controllers for stability

Linear dynamical system

Linear system is a dynamical system $\dot{x}(t) = f(x)$ where the dynamical function f is a linear function of the state and the inputs

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

where A and B are (possibly time varying) matrices

For a given initial state $x_0 \in \mathbb{R}^n$ and piece-wise continuous input signal $u: \mathbb{R} \rightarrow \mathbb{R}^n$ the space of *solutions* $\xi(t, x_0, u)$ is a linear space

Solutions of Linear systems define a linear space

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

$u(t)$ continuous everywhere except D_x

Theorem. Let $\xi(t, x_0, u)$ be the solution

1. $\forall x_0, u, \xi(t, x_0, u): \mathbb{R} \to \mathbb{R}^n$ is continuous and differentiable $w.r.t.\ t \in \mathbb{R} \setminus D_x$
2. $\forall t, u, \xi(t, x_0, u)$ is continuous w.r.t x_0
3. $\forall t, x_0, u, \xi(t, x_0, u) = \xi(t, x_0, \mathbf{0}) + \xi(t, 0, u)$
4. $\forall t, x_{01}, x_{02} \in \mathbb{R}^n, u_1, u_2: \mathbb{R} \to \mathbb{R}^n, a_1, a_2 \in \mathbb{R},$

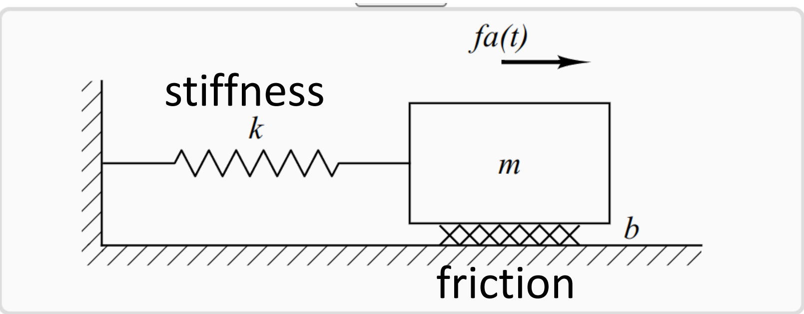
   $$\xi(t, a_1 x_{01} + a_2 x_{02}, a_1 u_1 + a_2 u_2) = a_1 \xi(t, x_{01}, u_1) + a_2 \xi(t, x_{02}, u_2)$$

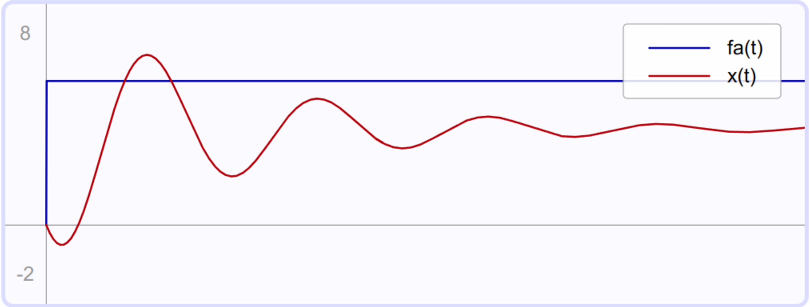
Example

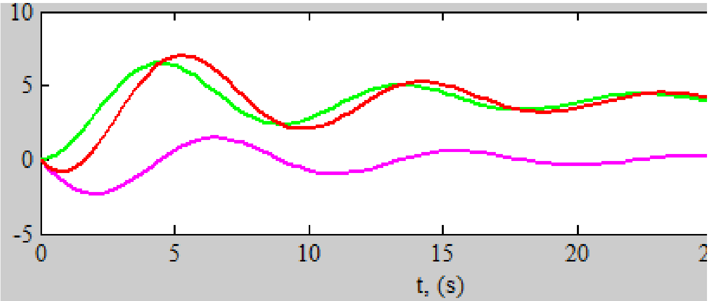
States x: postion (0m), velocity (-2m/s),
Input u(t): Force = 6 Newtons

$$m\frac{dx_2(t)}{dt} = u(t) - b\frac{dx_1(t)}{dt} - kx_1(t)$$

$$x_2(t) = \frac{dx_1(t)}{dt}$$







Zero state
Complete
Zero input

Source: https://lpsa.swarthmore.edu/Transient/TransZIZS.html

Revisit: Free swinging pendulum

$x \in \mathbb{R}^2$ x_1: angular position x_2: angular velocity

No input u; such models are called autonomous ODEs

$f: \mathbb{R}^2 \to \mathbb{R}^2$

$x_2 = \dot{x}_1$

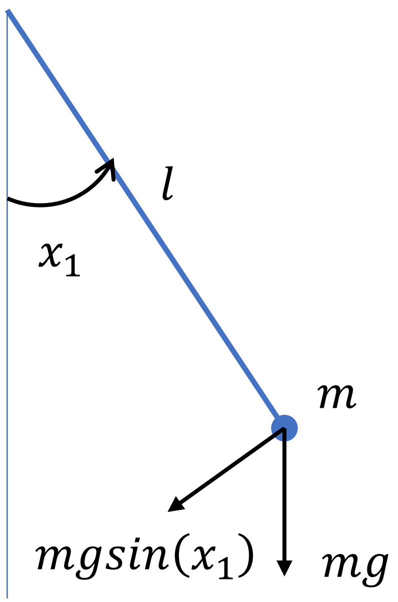
$$\dot{x}_2 = -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2$$

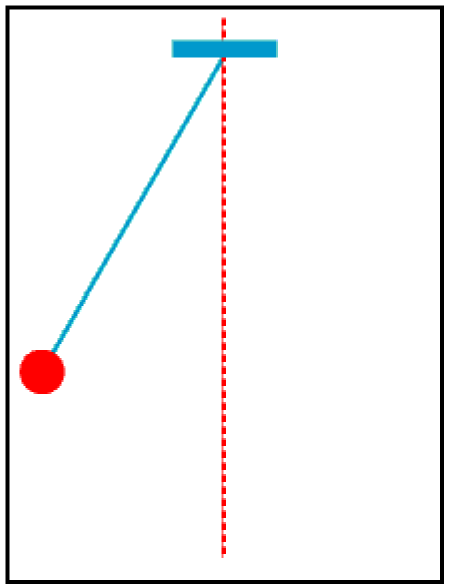
The dynamics equation can be written in vector form:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2\end{bmatrix}$$

k: friction coefficient m: mass l: length

Not a linear system





Linear dynamical system and solutions

Linear models can be obtained by **linearizing** a nonlinear model around a suitably chosen point $A_{x_0} = \frac{\partial f(x)}{\partial x}|_{x_0}$

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2 \end{bmatrix}$$

Linearize around x_0=(0,0):

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}x_1 - \frac{k}{m}x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{k}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solutions of Linear time invariant system

A Linear Time Invariant (LTI) system is a linear system with constant coefficients

$$\dot{x}(t) = Ax(t) + Bu(t)$$

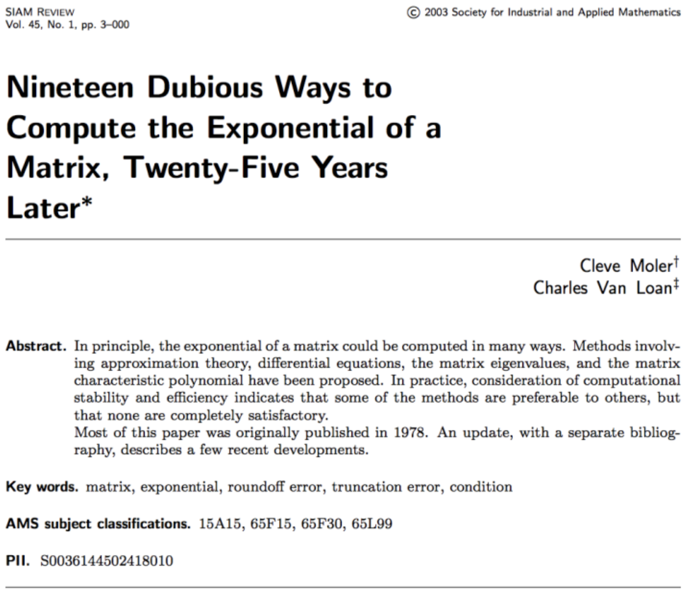
Theorem. The solution of a LTI system is given by

$$\xi(t, x_0, u) = x_0 e^{At} + \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau) d\tau$$

If there is no input then $\xi(t, x_0, \mathbf{0}) = x_0 e^{At}$

Recall the definition of the Matrix exponential:

$$e^{At} = 1 + At + \frac{1}{2!}(At)^2 + \ldots = \sum_{0}^{\infty} \frac{1}{k!}(At)^k$$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Nineteen Dubious Ways to Compute the Exponential of a Matrix, Twenty-Five Years Later*

Cleve Moler†
Charles Van Loan‡

Abstract. In principle, the exponential of a matrix could be computed in many ways. Methods involving approximation theory, differential equations, the matrix eigenvalues, and the matrix characteristic polynomial have been proposed. In practice, consideration of computational stability and efficiency indicates that some of the methods are preferable to others, but that none are completely satisfactory.
Most of this paper was originally published in 1978. An update, with a separate bibliography, describes a few recent developments.

Key words. matrix, exponential, roundoff error, truncation error, condition

AMS subject classifications. 15A15, 65F15, 65F30, 65L99

PII. S0036144502418010

Example: Simple linear model of an economy

x: national income

y: rate of consumer spending

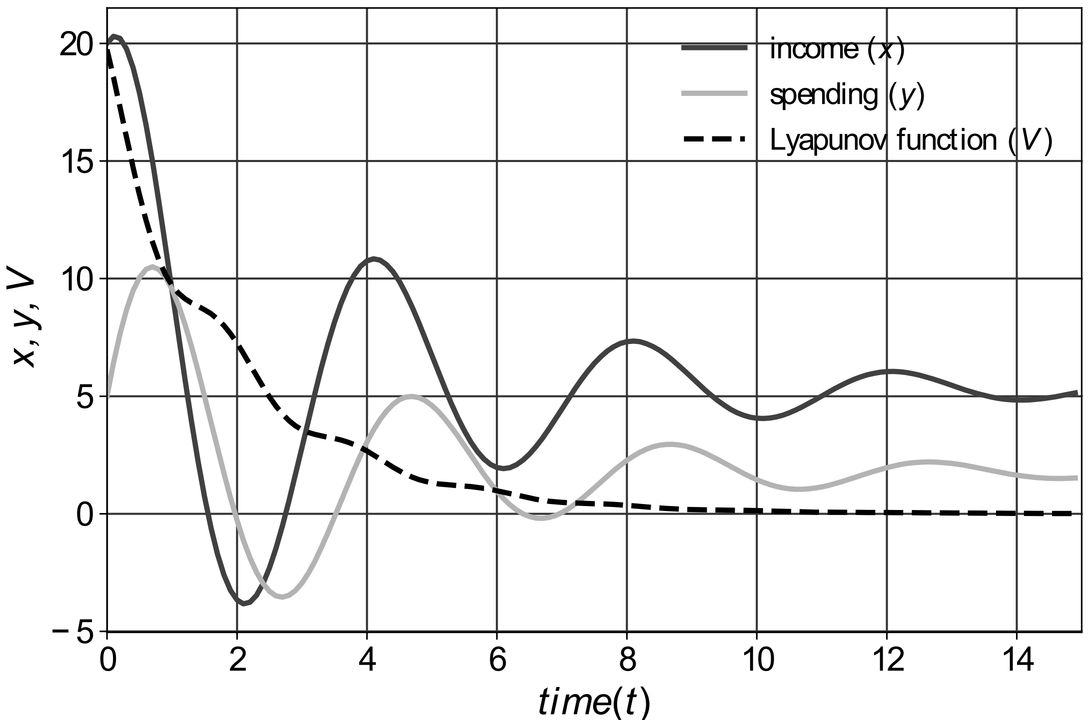
g: rate government expenditure

$$\dot{x} = x - \alpha y$$

$$\dot{y} = \beta(x - y - g)$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & -\alpha \\ \beta & -\beta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 \\ \beta \end{bmatrix} g$$

$$\dot{x}(t) = Ax(t) + Bu(t)$$



Example 2D LTI systems (assuming no control u)

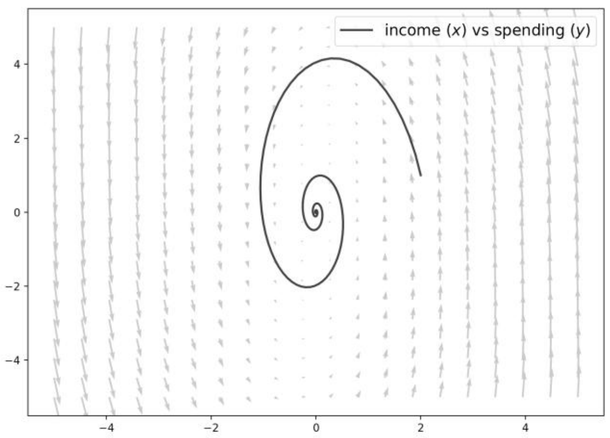
Try it yourself! https://colab.research.google.com/drive/1TXjVYI8fHWhhN72tPCEikMq5xXB2-QMI?usp=sharing

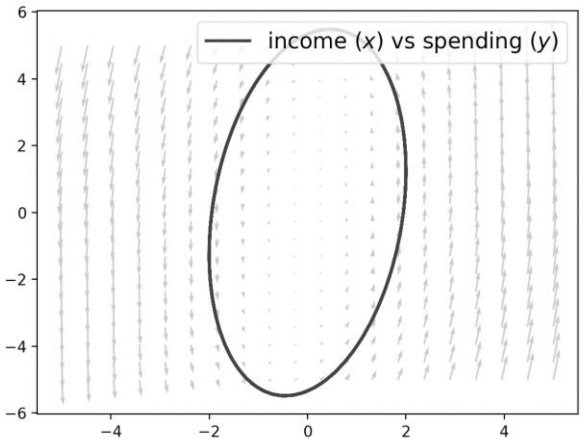
$A = \begin{bmatrix} -1/4 & -2/5 \\ 3 & -1/4 \end{bmatrix}$

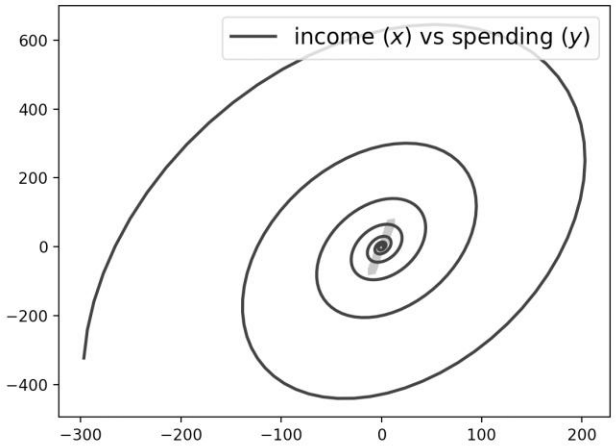
$A = \begin{bmatrix} 1/4 & -2/5 \\ 3 & -1/4 \end{bmatrix}$

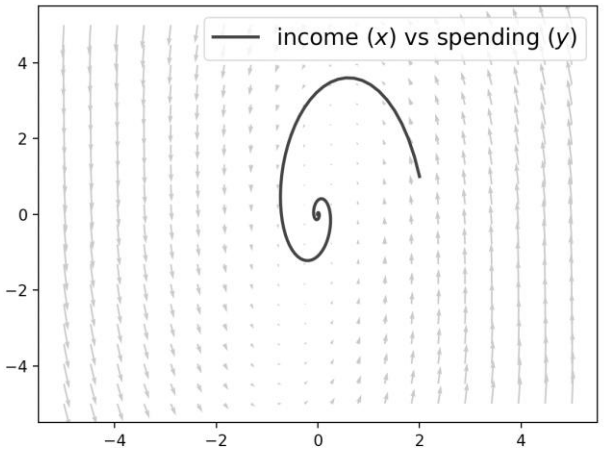
$A = \begin{bmatrix} 1/2 & -2/5 \\ 3 & -1/4 \end{bmatrix}$

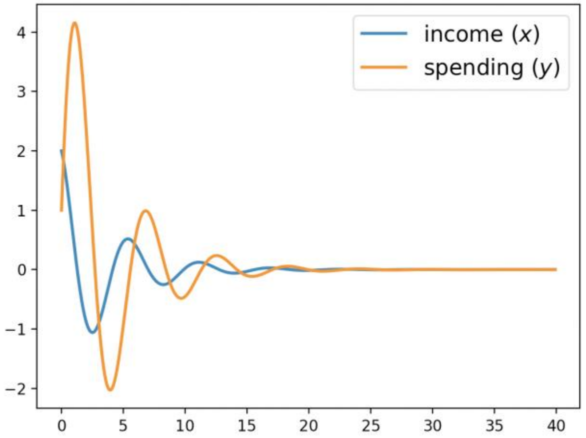
$A = \begin{bmatrix} -1/4 & -2/5 \\ 3 & -1/2 \end{bmatrix}$

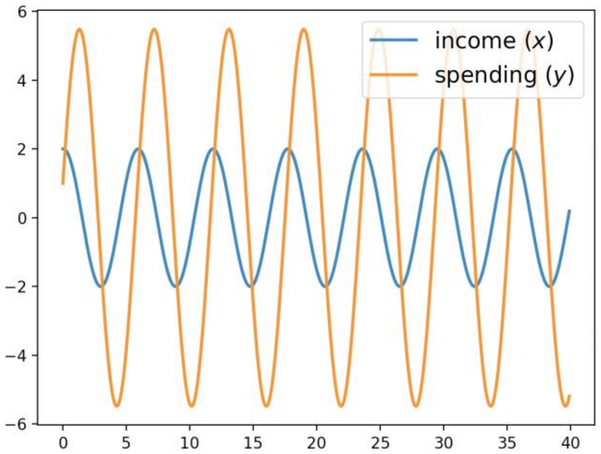


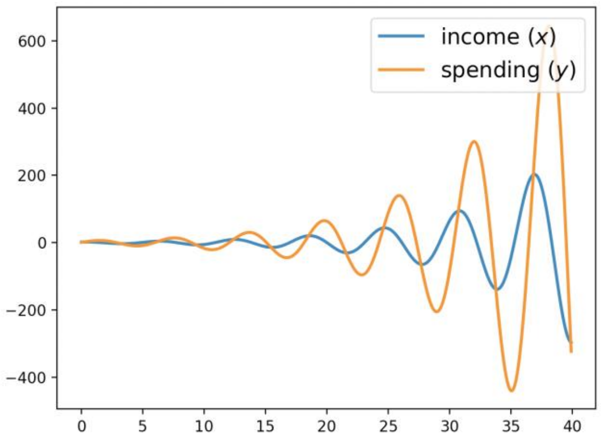


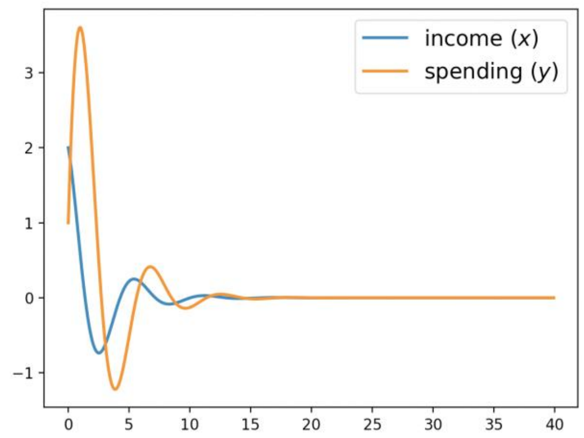












Requirements: Equilibria and Stability

Consider a LTI system (closed system, without inputs)

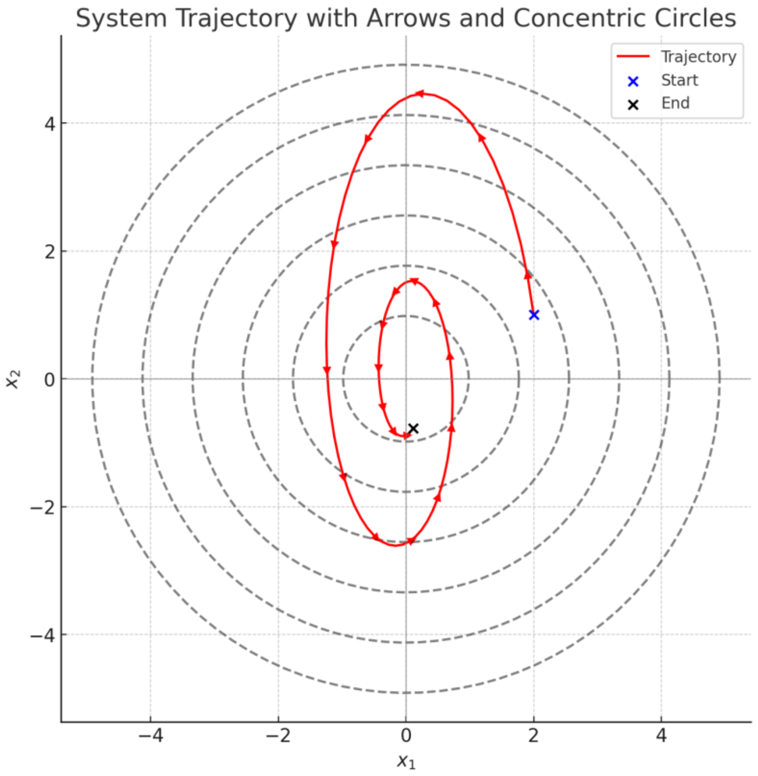
$\dot{x}(t) = f\big(x(t)\big) = Ax(t)$, suppose $x_0 \in \mathbb{R}^n, t_0 = 0$

$x^* \in \mathbb{R}^n$ is an **equilibrium point (or stationary point) if** $f(x^*) = 0$.

For analysis we will assume **0** to be an equilibrium point without loss of generality

$\xi(x_0, t)$ is the solution

$|\xi(x_0, t)|$ norm gives a measure of how far the system is from the equilibrium



Revisit: Free swinging pendulum

$x \in \mathbb{R}^2$ x_1: angular position x_2: angular velocity

No input u; such models are called autonomous ODEs

$f: \mathbb{R}^2 \to \mathbb{R}^2$

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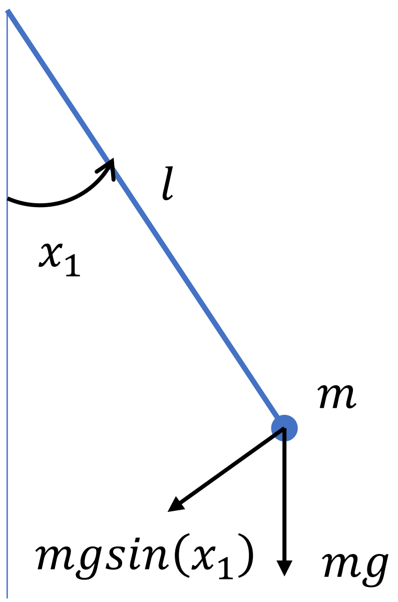
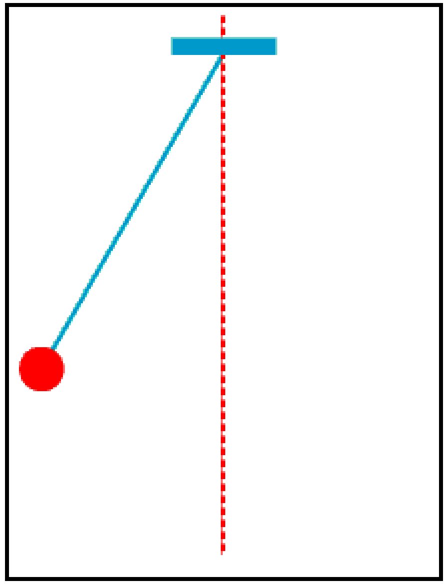
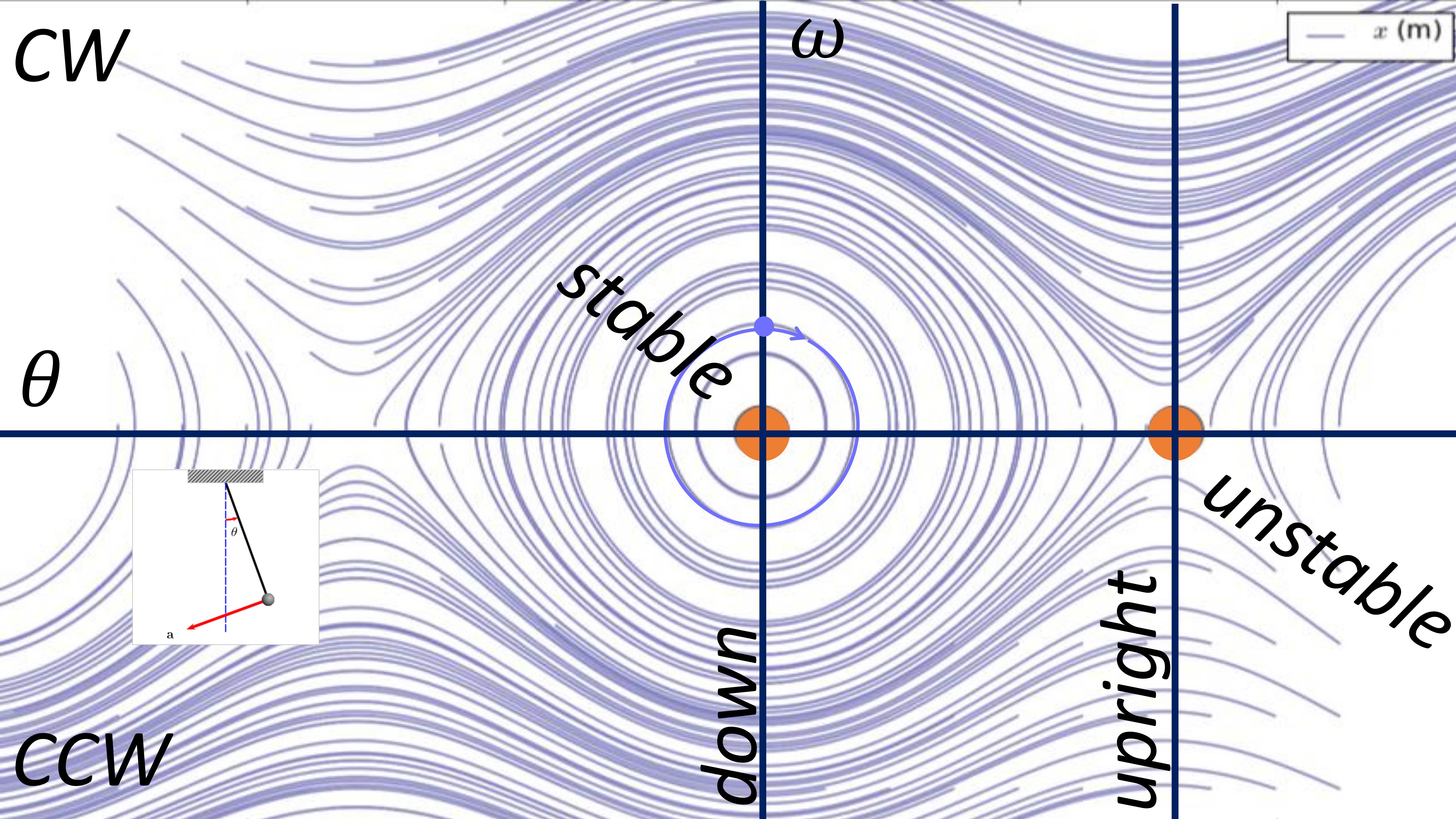
k: friction coefficient m: mass l: length





Two equilibrium points: $(0,0), (\pi, 0)$


CW
ω
x (m)
θ
stable
unstable
CCW
down
upright
θ
a

Lyapunov stability

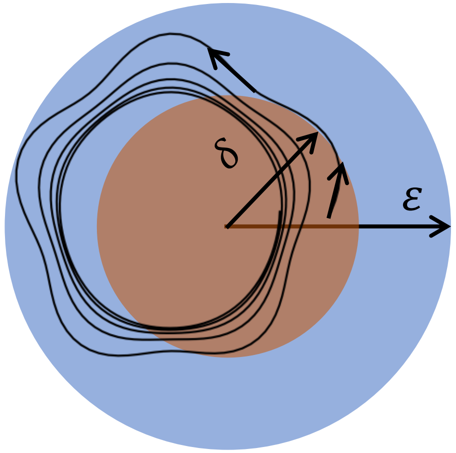
A. M. Lyapunov

An important class of requirement for control systems is to say that the state stays bounded in some small region (ε-ball)

For any ε-ball, there is a corresponding δ-ball such that if the system starts in the δ-ball then it forever stays in the ε-ball.

Lyapunov stability: The system is ***stable*** (at the origin) if

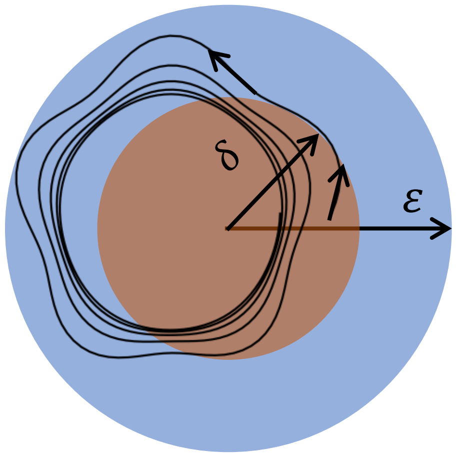
$$\forall \varepsilon > 0 \; \exists \; \delta_\varepsilon > 0 \text{ such that } |x_0| \leq \delta_\varepsilon \Rightarrow \forall \, t \geq 0, |\xi(x_0, t)| \leq \varepsilon.$$



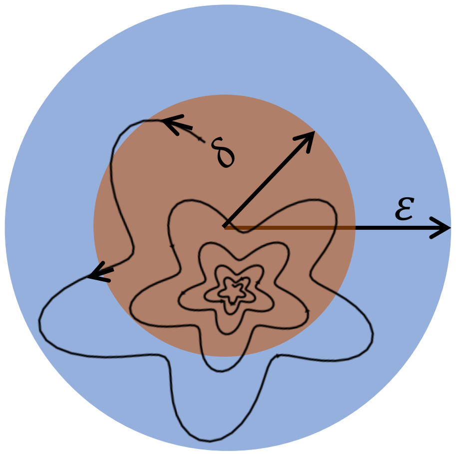
Exercise. How is Lyapunov stability related to invariants and reachable states ?

Asymptotic stability

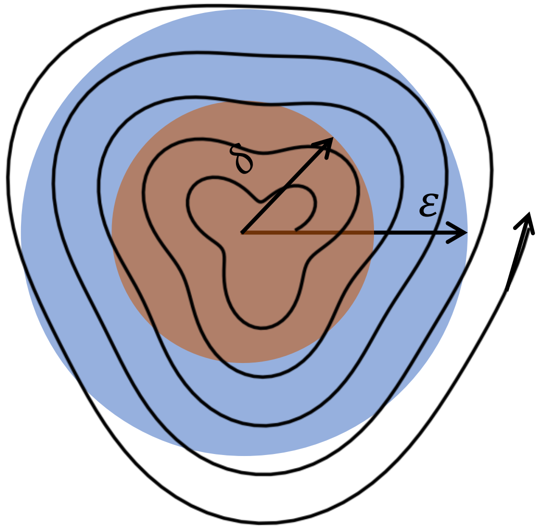
A system is ***asymptotically stable*** *(at the origin)* if it is Lyapunov stable and as t $\rightarrow \infty$, $|\xi(x_0, t)| \rightarrow \mathbf{0}$.



Lyapunov stable



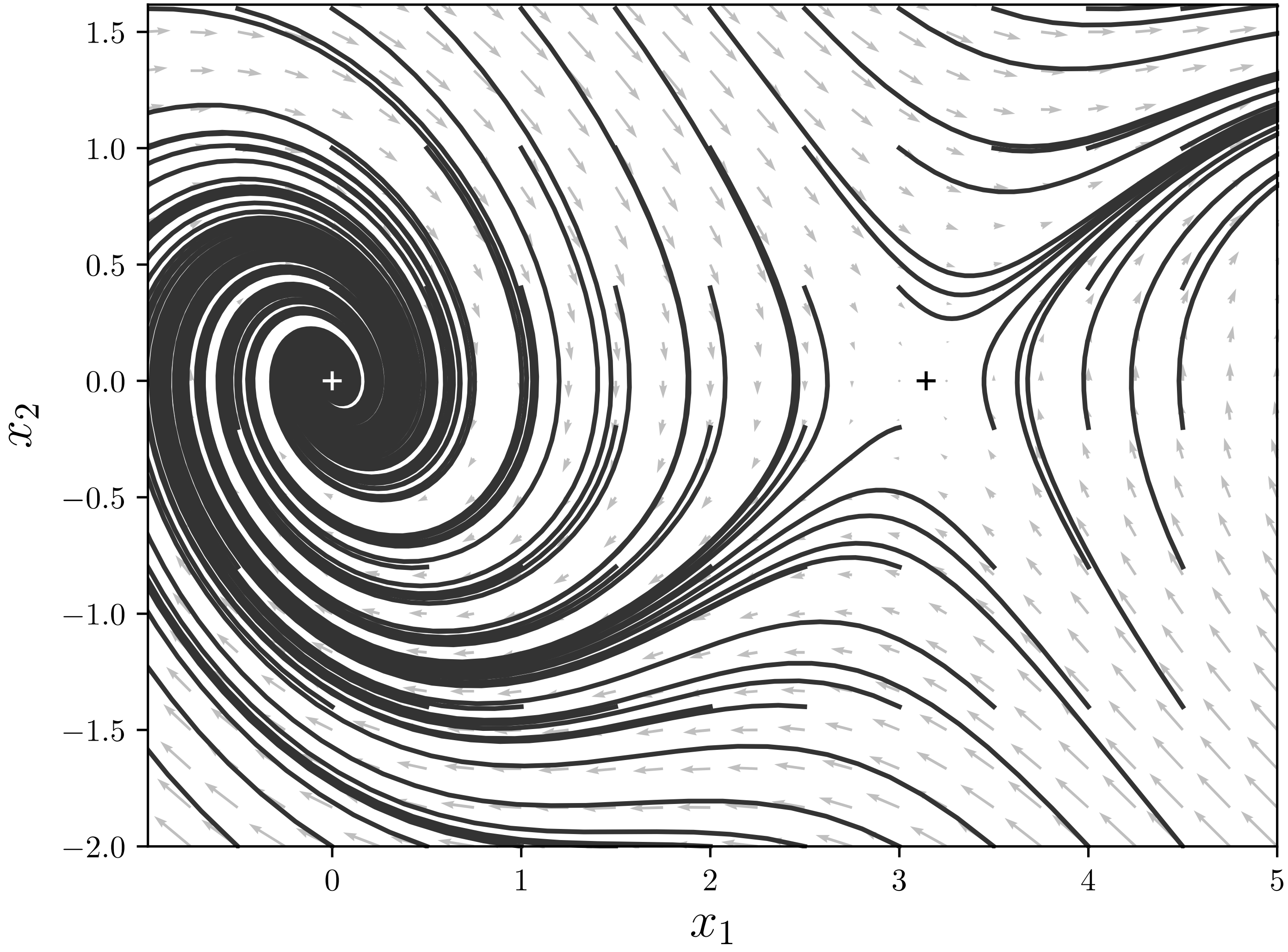
Asymptotically stable



Unstable

Phase portrait of pendulum with friction

Recall that the pendulum has two equilibria. Is it stable, asymptotically stable, or unstable?

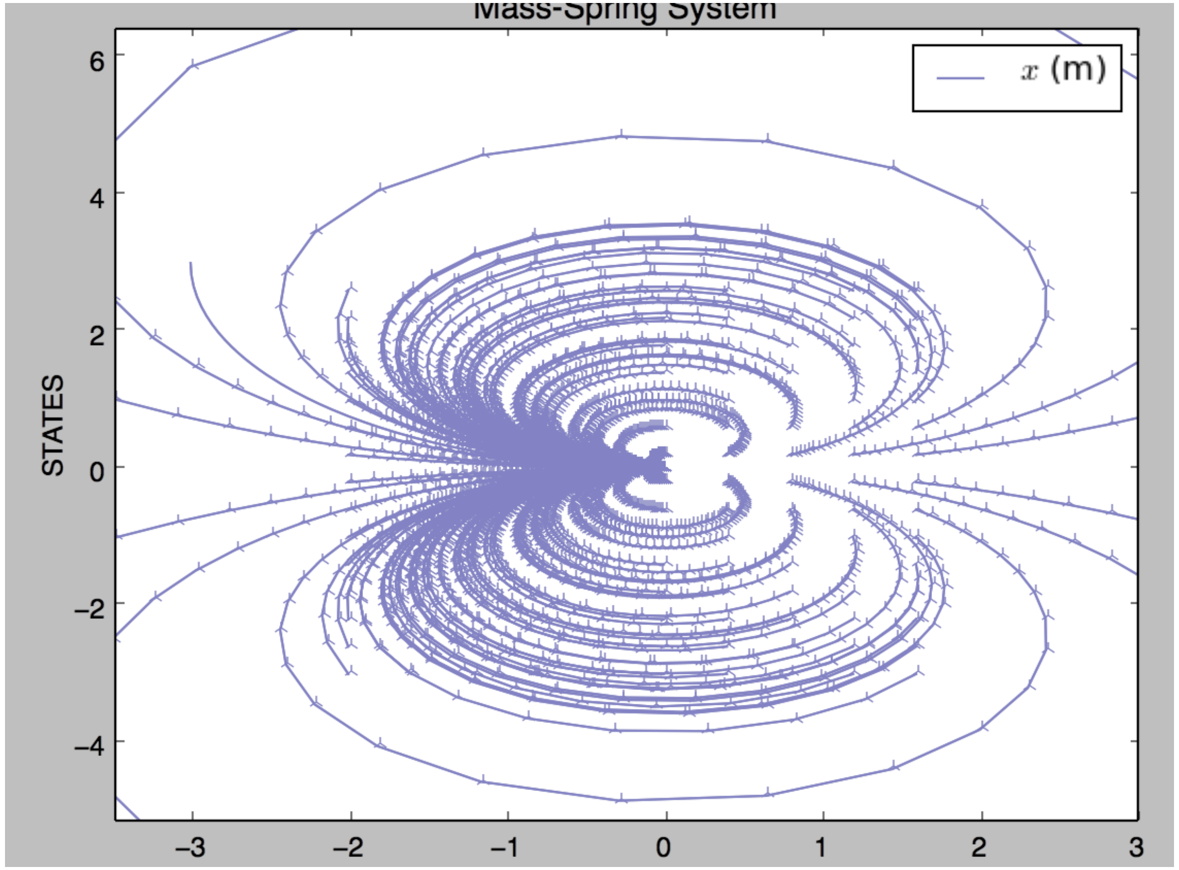


Butterfly

$$\begin{bmatrix} \dot{x_2} \\ \dot{x_1} \end{bmatrix} = \begin{bmatrix} 2x_1x_2 \\ x_1^2 - x_2^2 \end{bmatrix}$$

All solutions converge to 0 but the equilibrium point (0,0) is not Lyapunov stable

Convergence is not the same as asymptotic stability



Verifying Stability for Linear Systems

Theorem 1. (Stability of linear systems) An LTI system $\dot{x} = Ax$

1. is **asymptotically stable** iff all the eigenvalues of *A* have **strictly** negative real parts (*Hurwitz*).

2. It is Lyapunov stable iff all the eigenvalues of *A* have real parts that are either **zero** or **negative** and the *Jordan blocks* corresponding to the eigenvalues with **zero** real parts are of size 1.

Jordan decomposition*

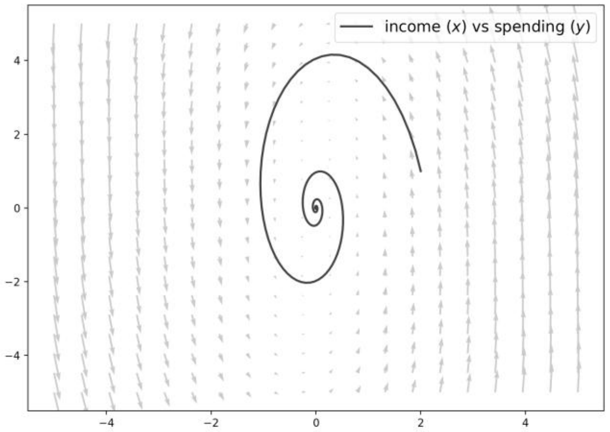
For every *n x n* matrix *A*, there exists a nonsingular *n x n* matrix *P* such that

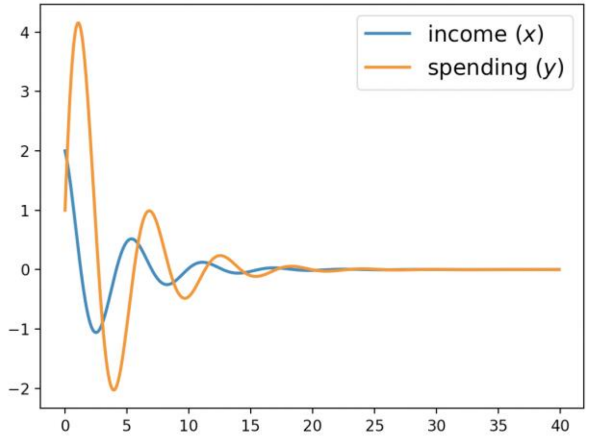
$$PAP^{-1} = J = \begin{bmatrix} J_1 & 0 & 0 & \dots & 0 \\ 0 & J_2 & 0 & \dots & 0 \\ 0 & 0 & J_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & J_\ell \end{bmatrix}, \qquad J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \dots & 0 \\ 0 & 0 & \lambda_i & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_i \end{bmatrix}.$$

where each J_i is an upper triangular matrix called a *Jordan block with diagonal elements* equal to the eigenvalue λ_i

$$\begin{bmatrix}\dot{x}_2\\ \dot{x}_1\end{bmatrix}=\begin{bmatrix}-1/4 & -2/5\\ 3 & -1/4\end{bmatrix}$$

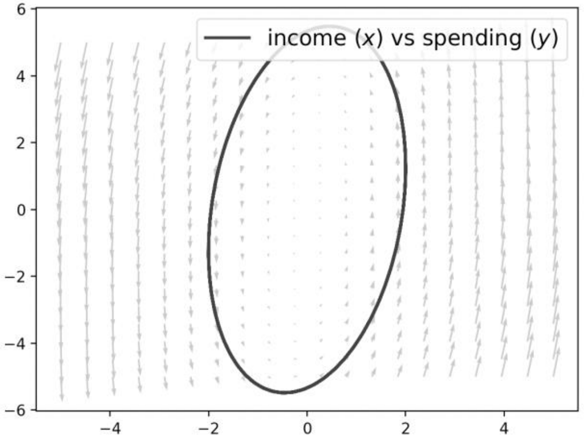
$\lambda_1=-0.25-i1.10$
$\lambda_2=-0.25+i1.10$

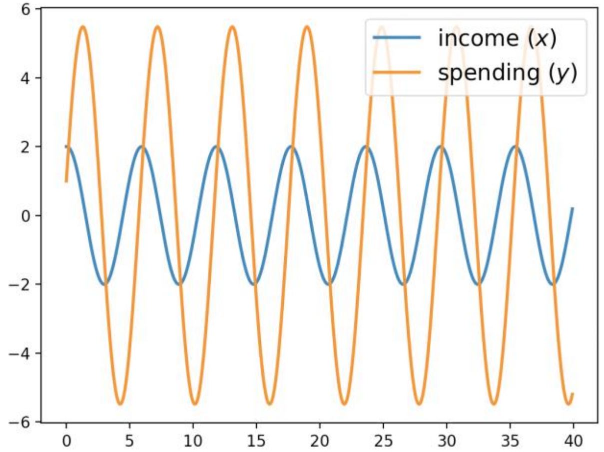




$$\begin{bmatrix}\dot{x}_2\\ \dot{x}_1\end{bmatrix}=\begin{bmatrix}1/4 & -2/5\\ 3 & -1/4\end{bmatrix}$$

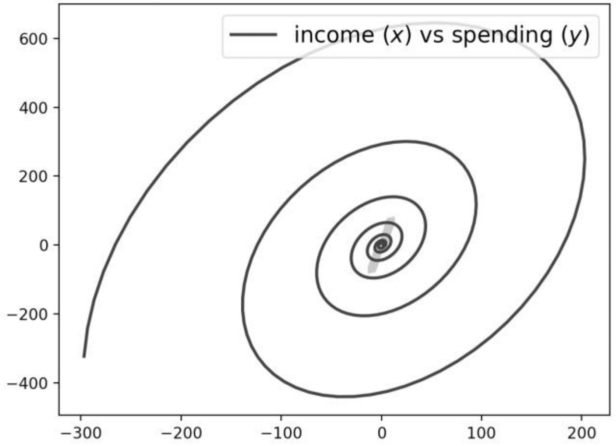
$\lambda_1=+i0.1066$
$\lambda_2=-i0.1066$
Jordan blocks of size 1

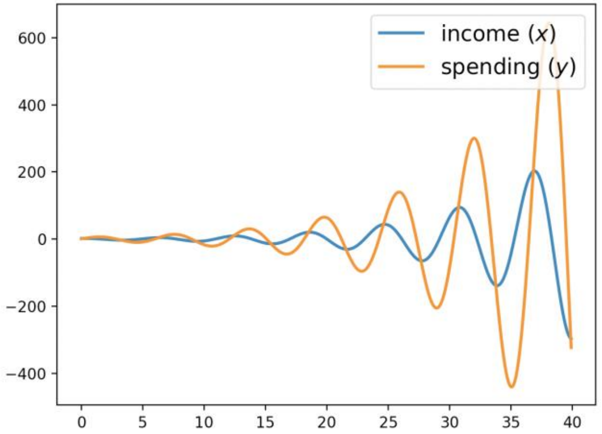




$$\begin{bmatrix}\dot{x}_2\\ \dot{x}_1\end{bmatrix}=\begin{bmatrix}1/2 & -2/5\\ 3 & -1/4\end{bmatrix}$$

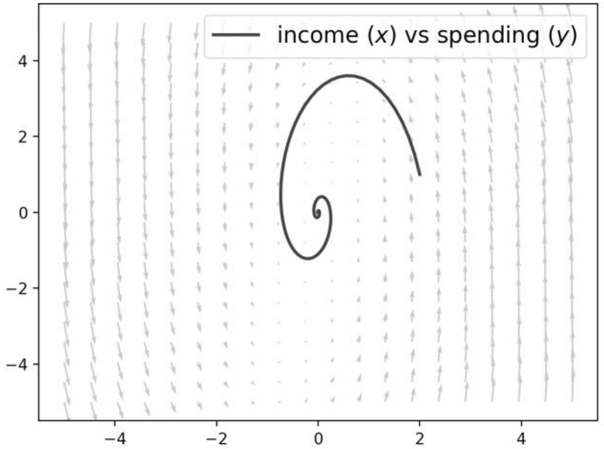
$\lambda_1=0.125+i1.029$
$\lambda_2=-0.125-i1.029$

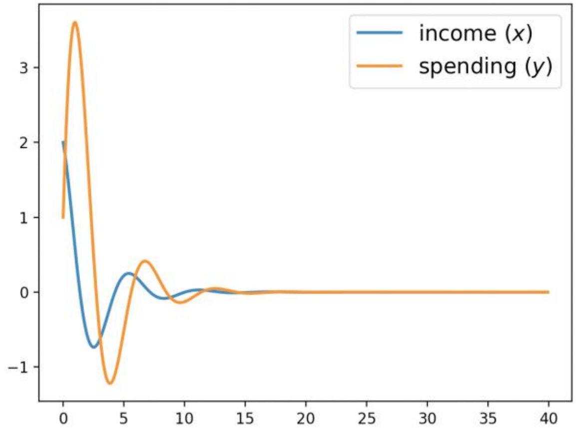




$$\begin{bmatrix}\dot{x}_2\\ \dot{x}_1\end{bmatrix}=\begin{bmatrix}-1/4 & -2/5\\ 3 & -1/2\end{bmatrix}$$

$\lambda_1=-0.375-i1.088$
$\lambda_2=-0.375+i1.088$

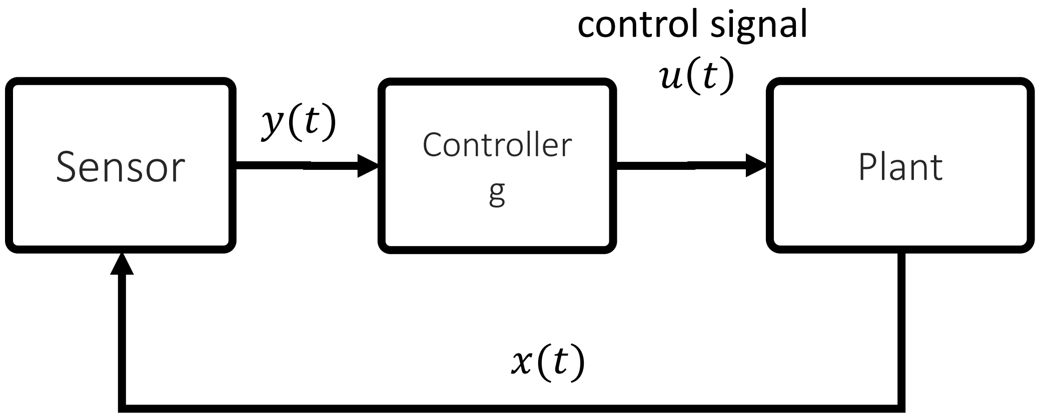


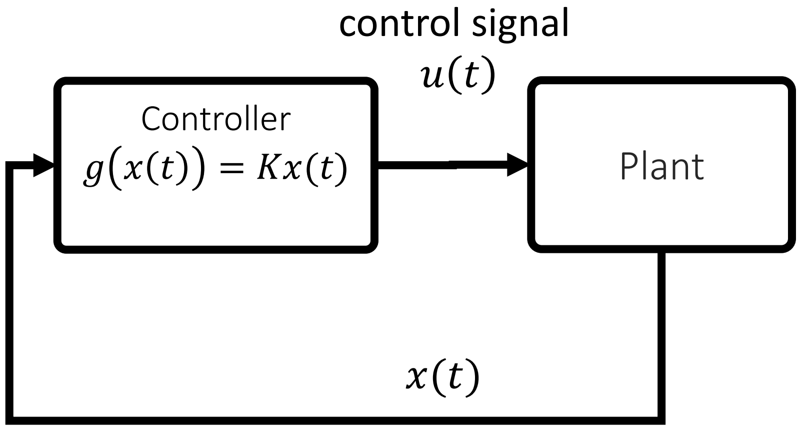


Control design by Eigenvalue placement

Problem: Given a Linear Time Invariant (LTI) system $\dot{x}(t) = Ax(t) + Bu(t)$ we would like to design a controller $u(t) = g\big(x(t)\big)$ such that the closed loop system is asymptotically stable.

We choose a **full state feedback controller** $g\big(x(t)\big) = Kx(t)$ where $K \in \mathbb{R}^{n\times n}$





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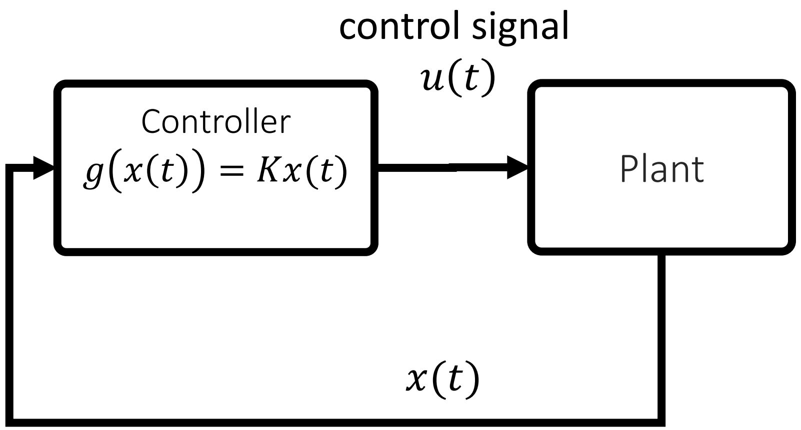
Problem: Given a Linear Time Invariant (LTI) system $\dot{x}(t) = Ax(t) + Bu(t)$ we would like to design a controller $u(t) = g\big(x(t)\big)$ such that the closed loop system is asymptotically stable.

Step 1. We choose a **full state feedback controller** $g\big(x(t)\big) = -Kx(t)$ where $K \in \mathbb{R}^{n\times n}$

$$\dot{x}(t) = Ax(t) - BKx(t) = (A - BK)x(t)$$

$$\dot{x}(t) = A_{cl}x(t) \text{ where } A_{cl} = A - BK$$

Step 2. Choose K such that $Re\big(\lambda(A_{cl})\big) < 0$.

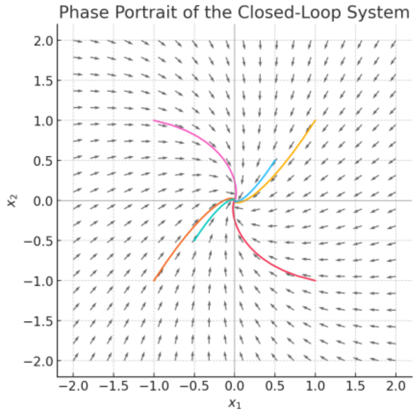


Control design by Eigenvalue placement

$A = \begin{bmatrix} 0 & v \\ 1 & \frac{1}{2} \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ $suppose\ u = \begin{bmatrix} k_{11} & 0 \\ 0 & k_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and v is a parameter > 1/3

$$A_{cl} = A - BK = \begin{bmatrix} 0 & v \\ 1 & 1/2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} k_{11} & 0 \\ 0 & k_{22} \end{bmatrix} = \begin{bmatrix} 0 & v \\ 1 & 1/2 \end{bmatrix} - \begin{bmatrix} k_{11} & 0 \\ k_{11} & k_{22} \end{bmatrix} = \begin{bmatrix} -k_{11} & v \\ 1 - k_{11} & \frac{1}{2} - k_{22} \end{bmatrix}$$

Closed loop system $\dot{x} = \begin{bmatrix} -k_{11} & v \\ 1 - k_{11} & \frac{1}{2} - k_{22} \end{bmatrix} x$



How to find eigen values of A_{cl}?

Solve the roots of **characteristic equation** $\det(\lambda I - A_{cl}) = 0$

$$det \begin{bmatrix} \lambda + k_{11} & -v \\ -1 + k_{11} & \lambda - \frac{1}{2} + k_{22} \end{bmatrix} = 0 \qquad \lambda^2 + \left(2k_{11} + k_{22} - \frac{1}{2}\right)\lambda + \left(k_{22}k_{11} - \frac{1}{2}k_{11} - v + vk_{11}\right) = 0$$

Sum of the roots of a quadratic = -b/a = $-\left(2k_{11} + k_{22} - \frac{1}{2}\right)$ and we want this to be < 0

Product of the roots of a quadratic = c/a = $\left(k_{22}k_{11} - \frac{1}{2}k_{11} - v + vk_{11}\right)$ and we want this to be > 0

For stability $-\left(2k_{11} + k_{22} - \frac{1}{2}\right) < 0$ and $\left(k_{22}k_{11} - \frac{1}{2}k_{11} - v + vk_{11}\right) > 0$ e.g. $k_{11}, k_{22} = 2$

A path following controller

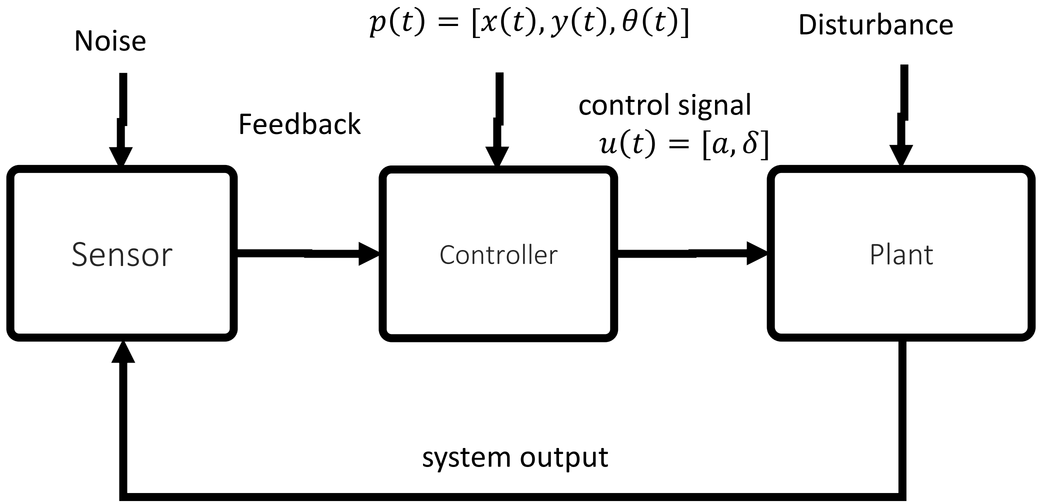
Control input is given by $u = [a, \delta]$

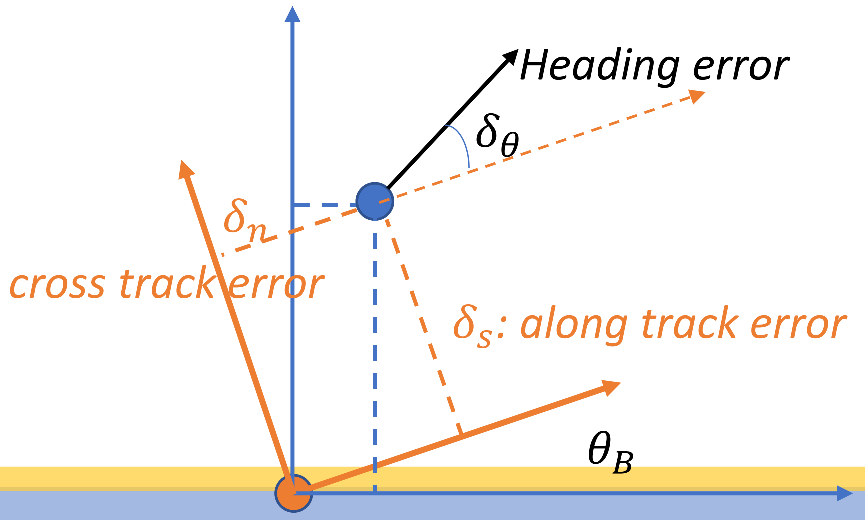
where a is the acceleration and

δ is the steering angle.

$$u = K \begin{bmatrix} \delta_s \\ \delta_n \\ \delta_\theta \\ \delta_v \end{bmatrix}$$

$$K = \begin{bmatrix} K_s & 0 & 0 & K_v \\ 0 & K_n & K_\theta & 0 \end{bmatrix}$$





Control design by eigenvalue placement

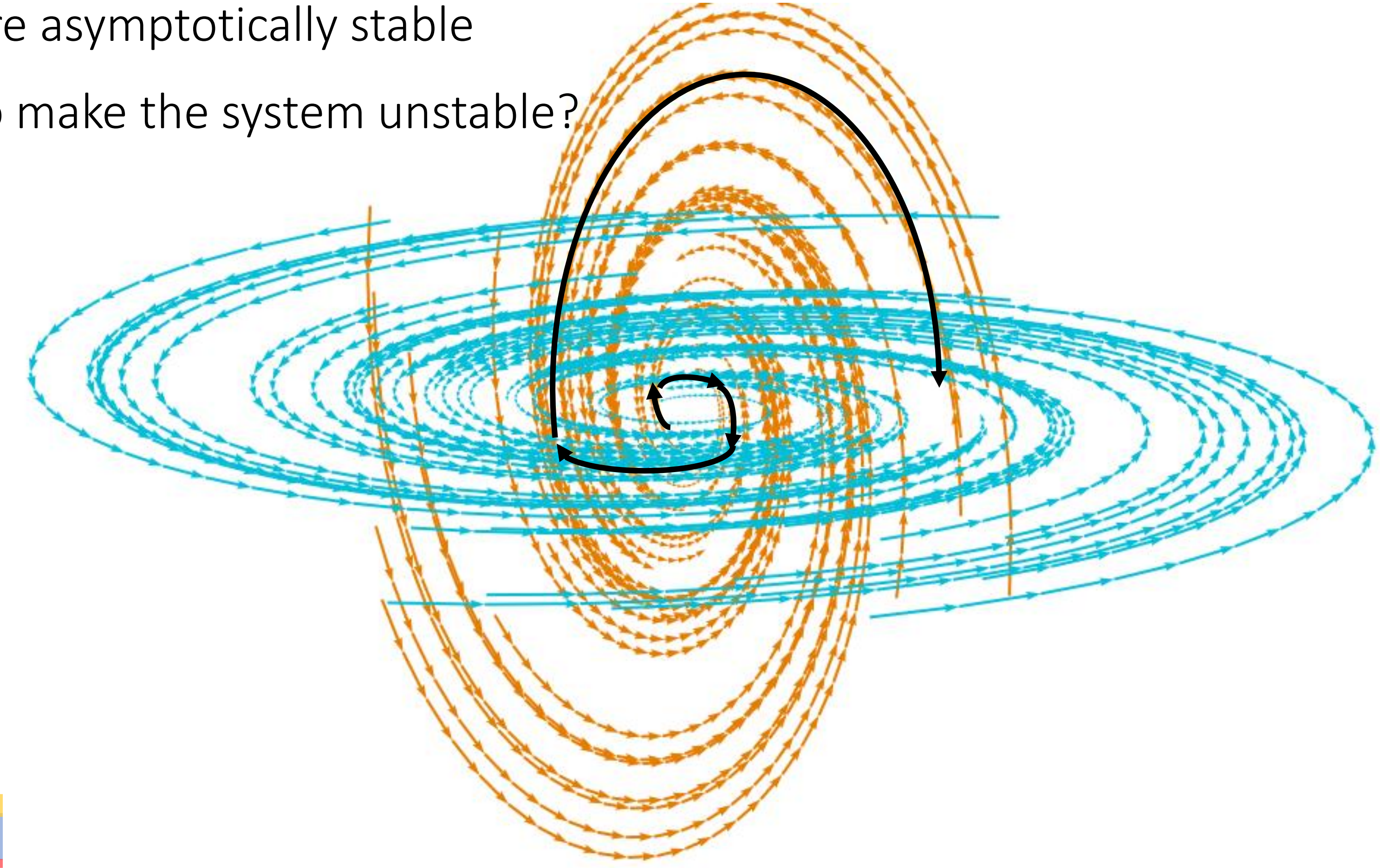
- After linearization and coordinate transformations dynamics become
- $$\begin{bmatrix} \dot{\delta s} \\ \dot{\delta n} \\ \dot{\delta \theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & v \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta s \\ \delta n \\ \delta \theta \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ \delta \end{bmatrix}$$
- State feedback control law: $\begin{bmatrix} a \\ \delta \end{bmatrix} = \begin{bmatrix} K_s & 0 & 0 \\ 0 & K_n & K_\theta \end{bmatrix} \begin{bmatrix} \delta s \\ \delta n \\ \delta \theta \end{bmatrix}$
- $$A - BK = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & v \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} K_s & 0 & 0 \\ 0 & K_n & K_\theta \end{bmatrix} = -\begin{bmatrix} K_s & 0 & 0 \\ 0 & 0 & -v \\ 0 & K_n & K_\theta \end{bmatrix}$$
- $\det(\lambda I - A + BK) = (\lambda^3 + \lambda^2(K_\theta + K_s) + \lambda(K_n v + K_s K_\theta) + K_s K_n v)$

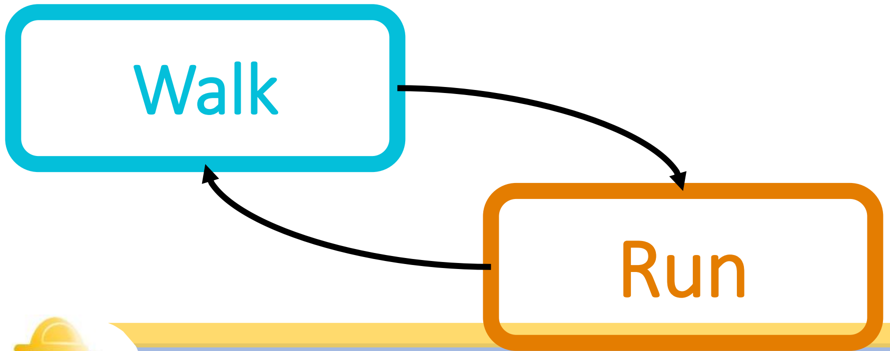
By Theorem 1 (Stability of linear systems) choose the gains so that the eigenvalues have negative real parts

Final thoughts: It is possible to make the system Unstable by Switching between two stable linear models

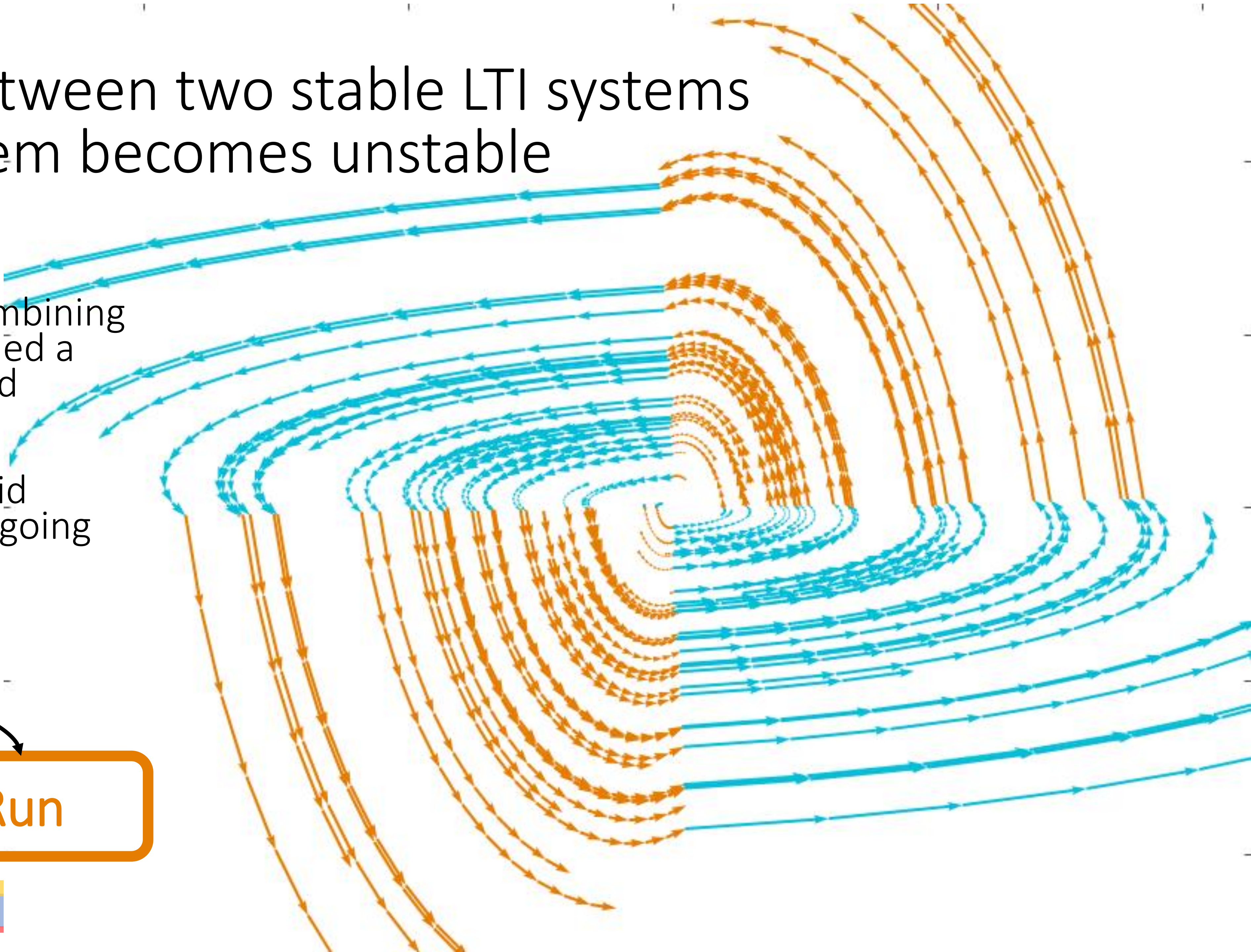
Each of the modes of a walking robot are asymptotically stable

Is it possible to switch between them to make the system unstable?



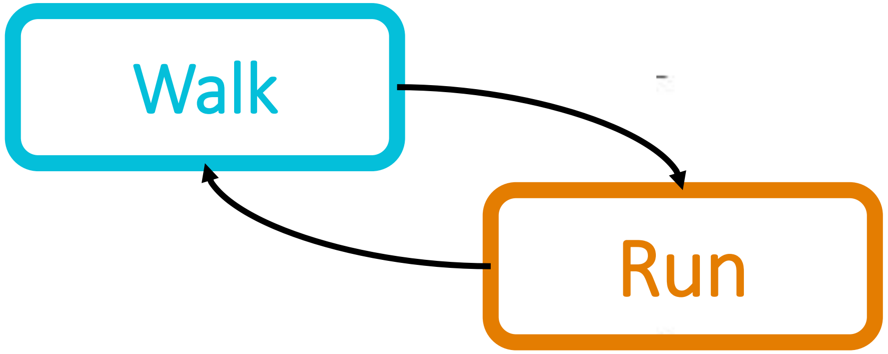


By switching between two stable LTI systems the overall system becomes unstable

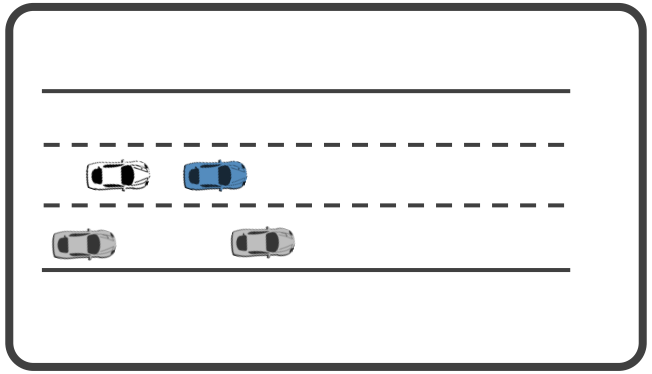


Systems obtained by combining two or more ODEs is called a hybrid system or a hybrid automaton

Stability analysis of hybrid systems in an area of ongoing research



Hybrid dynamics



Physical plant

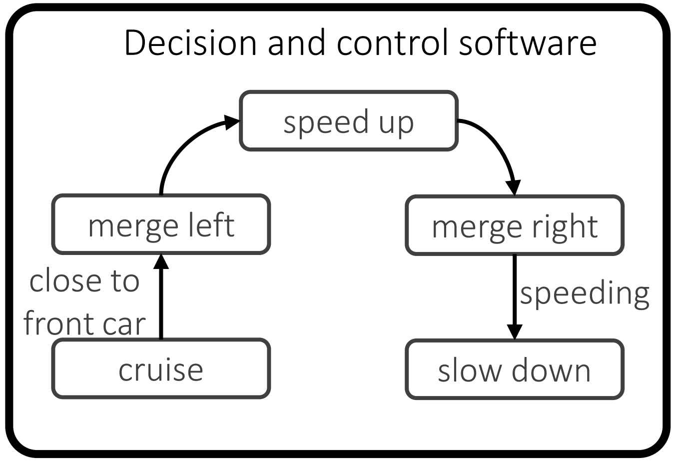
$\frac{dx}{dt} = f(x, u)$

System dynamics

$x[t+1] = f(x[t], u[t])$

$x = [v, s_x, s_y, \delta, \psi]$ State variables

$u = [a, v_\delta]$ Control inputs



Summary

- Solutions of linear systems define a linear subspace, i.e., follows superposition: $\xi(t, a_1 x_1 + a_2 x_2, a_1 u_1 + a_2 u_2) = a_1 \xi(t, x_1, u_1) + a_2 \xi(t, x_2, u_2)$
- Solution computed by calculating Matrix exponentials
- Lyapunov stability -> bounded state
  - Remember relationship to invariance and reachability
- Asymptotic stability -> bounded and convergence
- We can check stability by computing eigenvalues
- State feedback controllers can be designed by choosing the controller gains to give appropriate eigen values