Fall 2025 Principles of Safe Autonomy
ECE 484 (Sp 25)

Perception: Reconstructing 3D world from images
Lectures 5-6

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Role of Perception in Autonomy

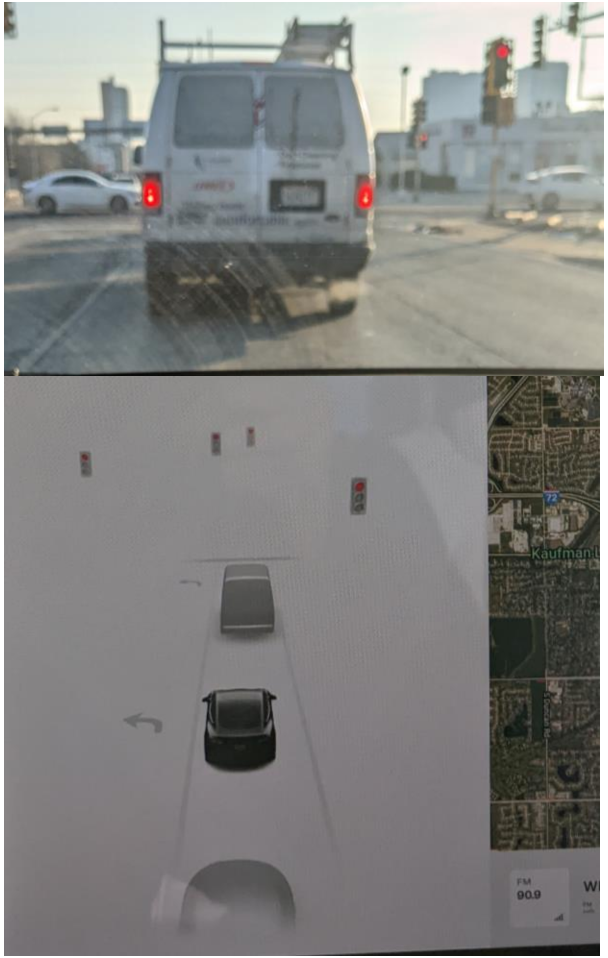
Perception module converts signals from the environment **state estimates** for the autonomous agent and its environment

Examples of state estimates:

- Type of lead vehicle, traffic sign
- Position of ego on the map, relative to the lane, distance to the leading vehicle
- Position of lead vehicle, speed, intention of the pedestrian

Types of estimates:

- Semantic: E.g., type of vehicle, sign
- Geometric: E.g., position, speed



Problem

Reconstructing the 3D structure of the scene from images

Input: image with points in pixels

Output: position of objects in millimeters in world camera frame

We will develop a method to find camera's internal and external parameters

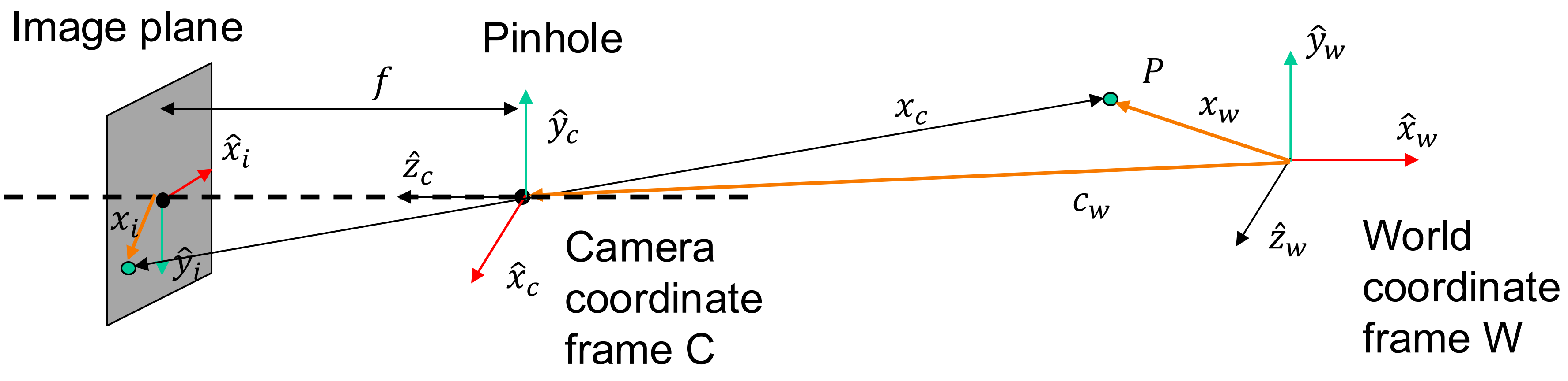
Outline:

Linear Camera Model (Projection matrix)

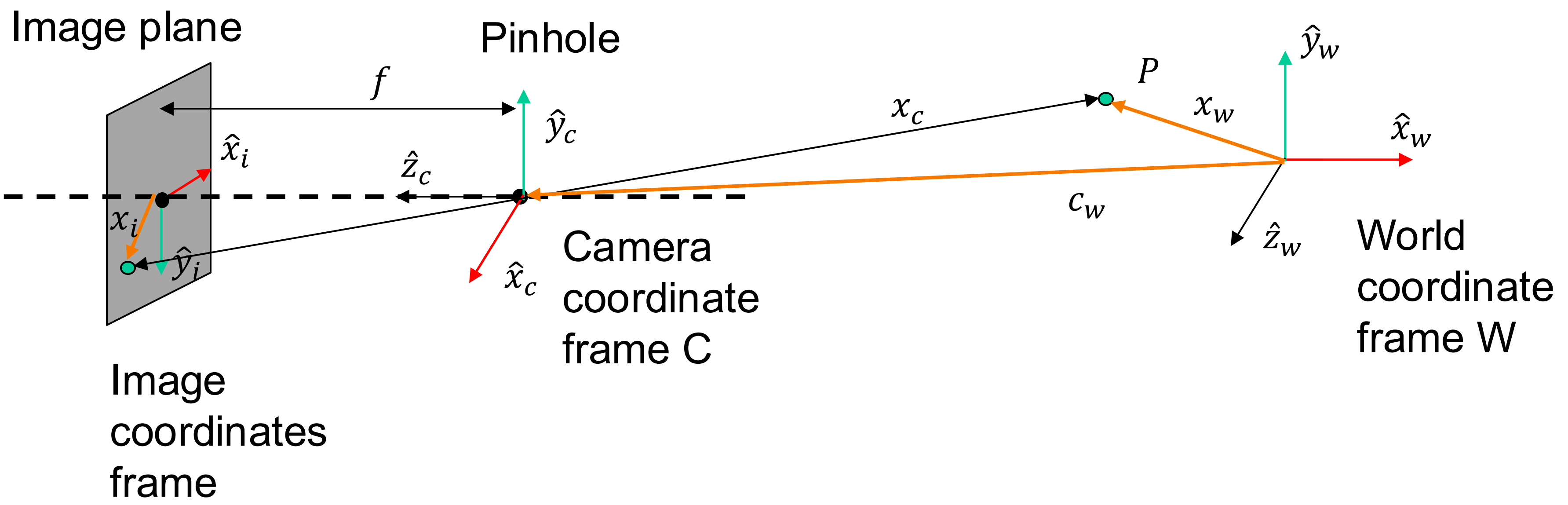
Camera calibration

Simple stereo

Forward Imaging Model: 3D to 2D



Forward Imaging Model: 3D to 2D



$$\boldsymbol{x}_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix}$$

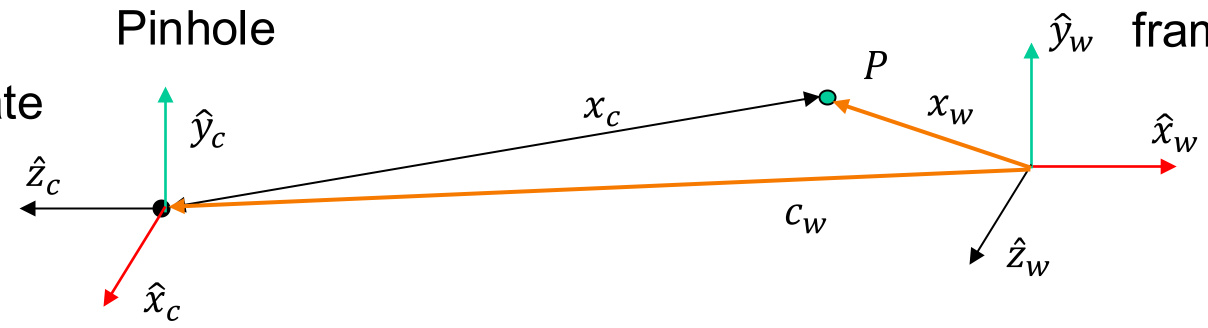
3D-2D

$$\boldsymbol{x}_c = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix}$$

3D-3D

$$\boldsymbol{x}_w = \begin{bmatrix} x_w \\ y_w \\ z_w \end{bmatrix}$$

World to camera Transformation (Extrinsic parameters)

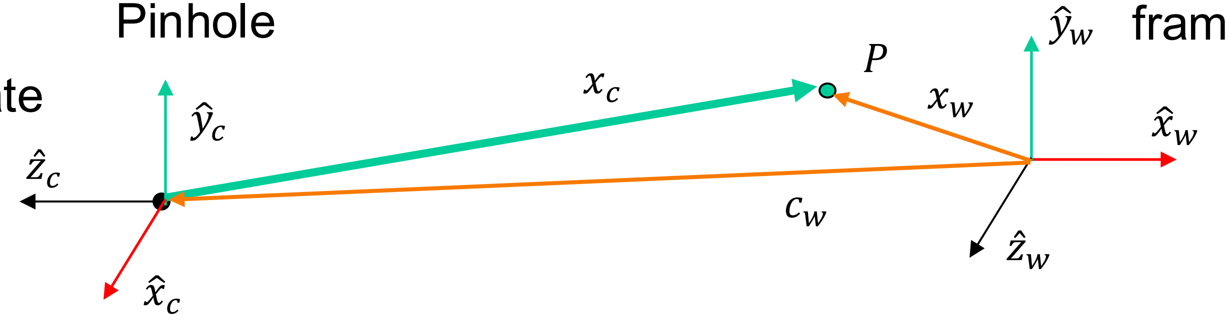


Position c_w and the orientation R of the camera in the world coordinate frame (W) are the camera's **Extrinsic Parameters**

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$ → row 1 is the direction of $\hat{x}_c$ in world coordinates, 2 for $\hat{y}_c$, …

This is an **orthonormal matrix**, i.e., the row vectors or the colum vectors are orthonormal
$R^{-1} = R^T$ i.e., $R^T R = R R^T = I$

World to camera Transformation



Position c_w and the orientation R of the camera in the world coordinate frame (W) are the camera's **Extrinsic Parameters**

Given the extrinsic parameters (R, c_w) of the camera, the camera-centric location of the point P in the world coordinate (w) is simply $(x_c)_w = x_w - c_w$

In the camera coordinate (c) $x_c = R(x_w - c_w) = Rx_w - Rc_w = Rx_w + t$

$$t = -Rc_w$$

$$x_c = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} x_w \\ y_w \\ z_w \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} \qquad x_c = Rx_w + t$$

Extrinsic Matrix

$$x_c = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} x_w \\ y_w \\ z_w \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

We have an affine transformation: $x_c = Rx_w + t$

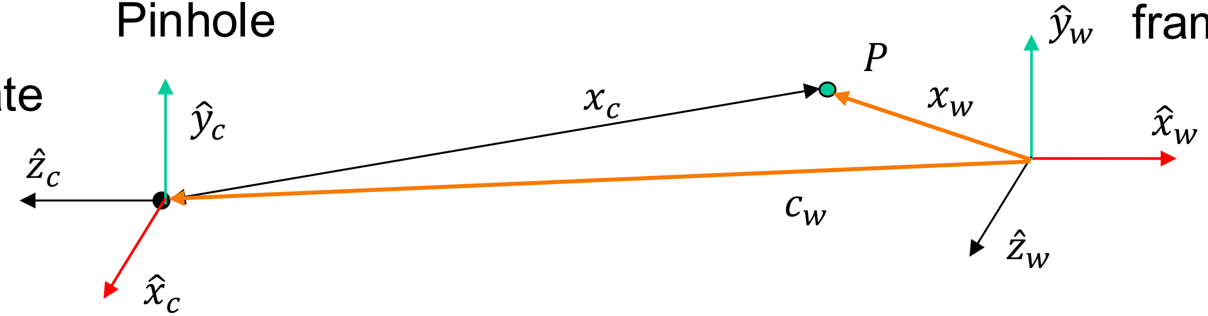
Can we represent it as $x_c = Mx_w$? No

We can introduce a new coordinate $\tilde{x}_c = [\tilde{x}, \tilde{y}, \tilde{z}, 1]^T$

Now can we represent this as a matrix multiplication $\tilde{x}_c = M\tilde{x}_w$

$$\tilde{x}_c = \begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix}$$

World to camera Transformation (Extrinsic matrix)



Given the extrinsic parameters (R, c_w) of the camera, the camera-centric location of the point P in the world coordinate is

$$x_c = R(x_w - c_w) = Rx_w - Rc_w = Rx_w + t \qquad t = -Rc_w$$

$$x_c = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} x_w \\ y_w \\ z_w \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

Using homogeneous coordinates

$$\tilde{x}_c = \begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix}$$

Extrinsic matrix M_{ext} $\tilde{x}_c = M_{eext}\, \tilde{x}_w$

Geometry of Homogeneous coordinates (for 2D)

$$x_c = \begin{bmatrix} x_c \\ y_c \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} x_w \\ y_w \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

Affine transformation: $\mathrm{x_c} = \mathrm{Rx_w} + \mathrm{t}$

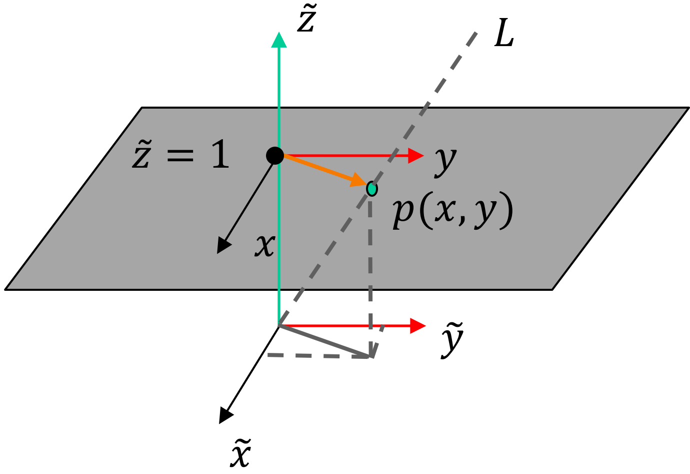
How to represent this as $\tilde{\mathrm{x}}_\mathrm{c} = \mathrm{M}\tilde{\mathrm{x}}_\mathrm{w}$

The homogeneous representation of a 2D point $p = (x, y)$ is a 3D point $\tilde{p} = (\tilde{x}, \tilde{y}, \tilde{z})$.

The third coordinate $\tilde{z} \neq 0$ is fictitious such that:

$$p = (x, y) \quad x = \frac{\tilde{x}}{\tilde{z}} \; y = \frac{\tilde{y}}{\tilde{z}}$$

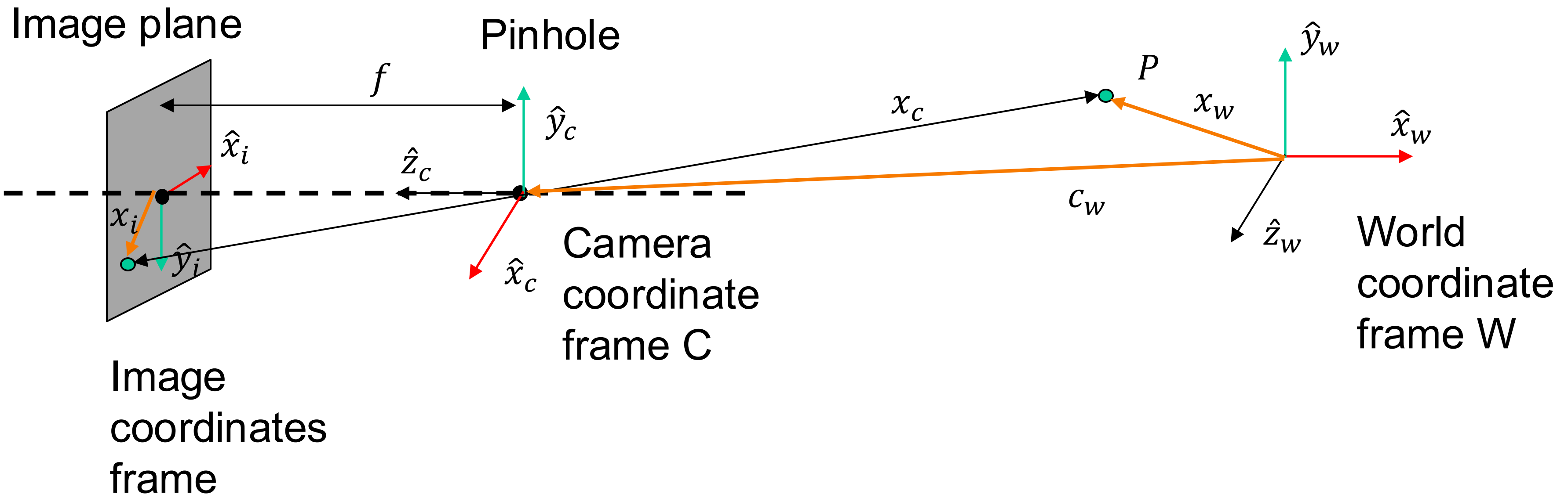
$$p \equiv \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \equiv \begin{bmatrix} \tilde{z}x \\ \tilde{z}y \\ \tilde{z} \end{bmatrix} \equiv \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{bmatrix} = \tilde{p}$$



Geometric interpretation: all points on the line L (except origin) represent homogeneous coordinate $p(x, y)$

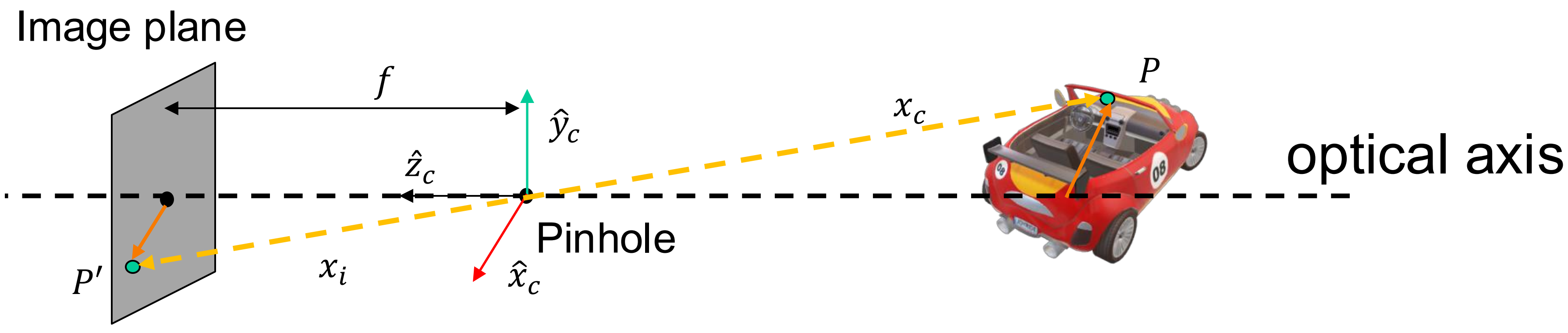
$$p \equiv \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \equiv \begin{bmatrix} wx \\ \tilde{w}y \\ \tilde{w}z \\ \tilde{w} \end{bmatrix} \equiv \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \tilde{w} \end{bmatrix} = \tilde{p}$$

Forward Imaging Model: 3D to 2D



$$\boldsymbol{x}_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix} \quad \xleftarrow{\text{3D-2D}} \quad \boldsymbol{x}_c = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} \quad \xleftarrow{\text{3D-3D}} \quad \boldsymbol{x}_w = \begin{bmatrix} x_w \\ y_w \\ z_w \end{bmatrix}$$

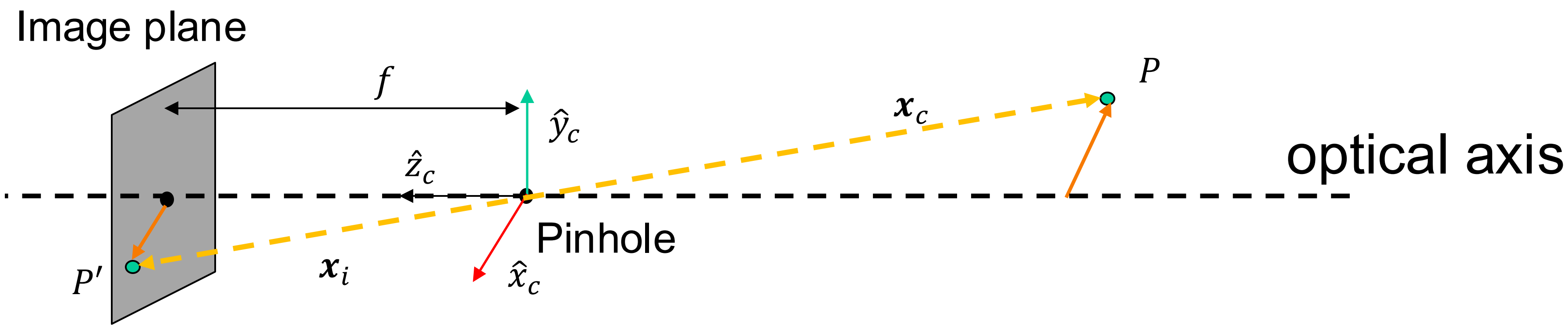
Perspective imaging with pinhole



f: Effective focal length

$$\boldsymbol{x}_c = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} \quad \boldsymbol{x}_i = \begin{bmatrix} x_i \\ y_i \\ f \end{bmatrix}$$

Perspective imaging with pinhole

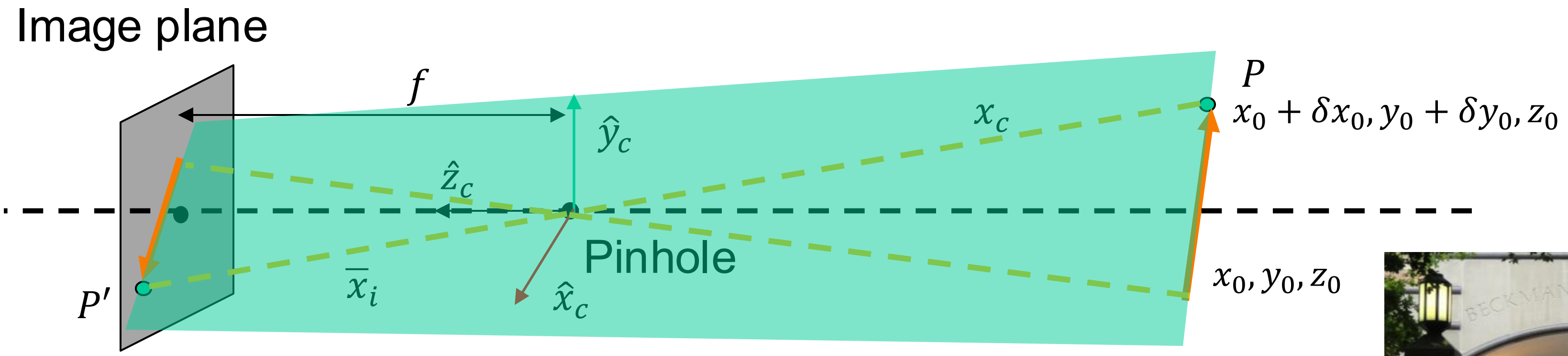


f: Effective focal length

$$\boldsymbol{x}_c = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} \quad \boldsymbol{x}_i = \begin{bmatrix} x_i \\ y_i \\ f \end{bmatrix}$$

$$\frac{\boldsymbol{x}_i}{f} = \frac{\boldsymbol{x}_c}{z_c} \Rightarrow \frac{x_i}{f} = \frac{x_c}{z_c}, \frac{y_i}{f} = \frac{y_c}{z_c}$$

Perspective projection of a line and magnification

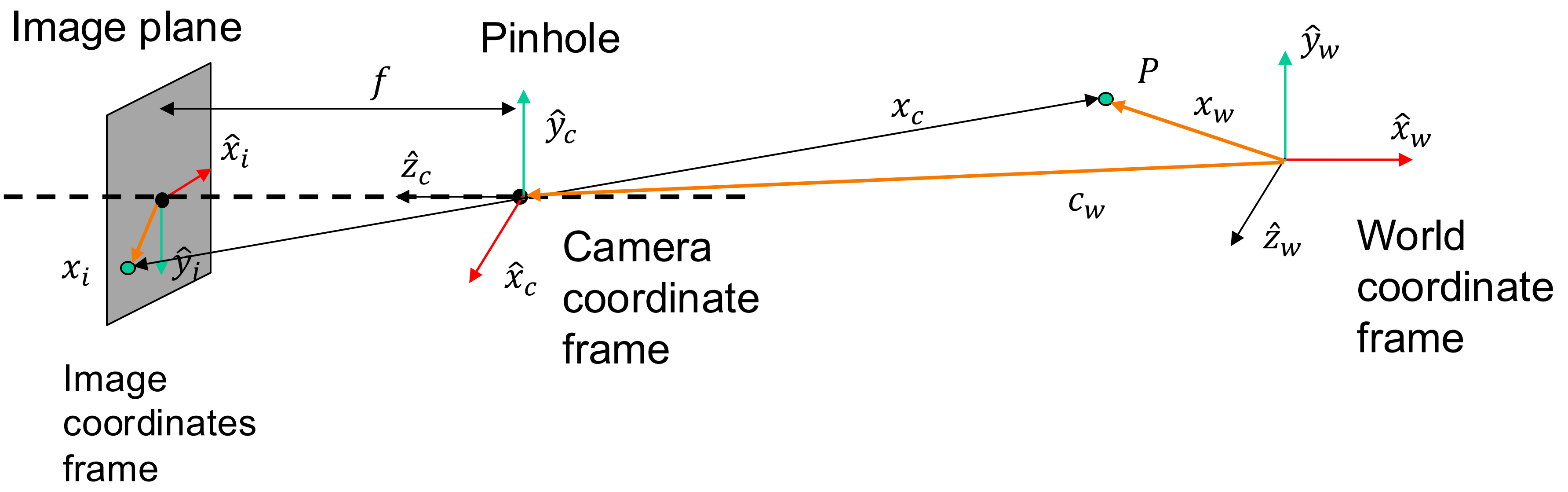


A line in 3D gets mapped to a line in the image plane

$$\frac{\bar{x}_i}{f} = \frac{x_c}{z_c} \qquad \Rightarrow \frac{x_i}{f} = \frac{x_c}{z_c}, \frac{y_i}{f} = \frac{y_c}{z_c}$$

Exercise: Show that magnification $|\mathrm{m}| = \frac{object\ length}{image\ length} = \frac{\sqrt{\delta x_i^2 + \delta y_i^2}}{\sqrt{\delta x_o^2 + \delta y_o^2}} = |\frac{f}{z_0}|$

Camera coordinates to image plane coordinates



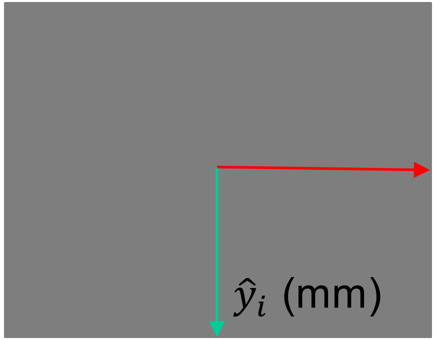
Perspective projection

$$\frac{x_i}{f} = \frac{x_c}{z_c} \text{ and } \frac{y_i}{f} = \frac{y_c}{y_c}$$

$$x_i = f\frac{x_c}{z_c} \text{ and } y_i = f\frac{y_c}{y_c}$$

Image plane to image sensor mapping

Image plane

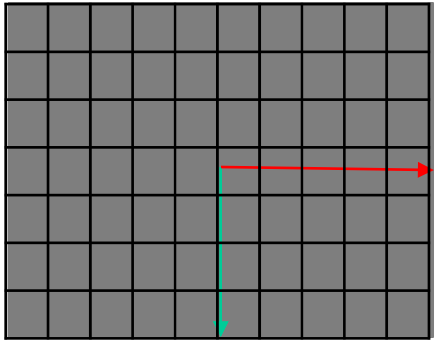


$\hat{x}_i$ (mm)

$$x_i = f\frac{x_c}{z_c} \text{ and } y_i = f\frac{y_c}{y_c}$$

$$u = m_x f\frac{x_c}{z_c} \text{ and } v = m_y f\frac{y_c}{z_c}$$

Image sensor



u (pixels)

v (pixels)

Pixels may be rectangular
Let m_x and m_y be the pixel densities (pixels/mm) in x and y directions

(o_x, o_y) Principle point

$$u = m_x f\frac{x_c}{z_c} + o_x \text{ and } v = m_y f\frac{y_c}{z_c} + o_y$$

$$u = f_x\frac{x_c}{z_c} + o_x \text{ and } v = f_y\frac{y_c}{z_c} + o_y$$

Intrinsic parameters: f_x, f_y, o_x, o_y

Nonlinear to linear model using homogeneous coordinates

$u = f_x \frac{x_c}{z_c} + o_x$ and $v = f_y \frac{y_c}{z_c} + o_y$

Use homogeneous representation of (u, v) as a 3D point $\tilde{u} = (\tilde{u}, \tilde{v}, \tilde{w})$

$uz_c = f_x x_c + o_x z_c$ and $vz_c = f_y y_c + o_y z_c$

$(uz_c, vz_c, z_c) \equiv (u, v, 1)$

$$u \equiv \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \equiv \begin{bmatrix} z_c u \\ z_c v \\ z_c \end{bmatrix} = \begin{bmatrix} f_x x_c + z_c o_x \\ f_y y_c + z_c o_y \\ z_c \end{bmatrix} = \begin{bmatrix} f_x & 0 & o_x & 0 \\ 0 & f_y & o_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix}$$

Linear model of perspective projection $\tilde{u} = [K|0]\tilde{x}_c = M_{int}\tilde{x}_c$

Intrinsic matrix (M_{int})

Calibration matrix K (upper right triangular)

Forward Camera Model

Camera to pixel

$$\begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{bmatrix} = \begin{bmatrix} f_x & 0 & o_x & 0 \\ 0 & f_y & o_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix}$$

World to camera

$$\begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix}$$

$$\tilde{u} = M_{int}\ \tilde{x}_w$$

$$\tilde{x}_c = M_{ext}\ \tilde{x}_w$$

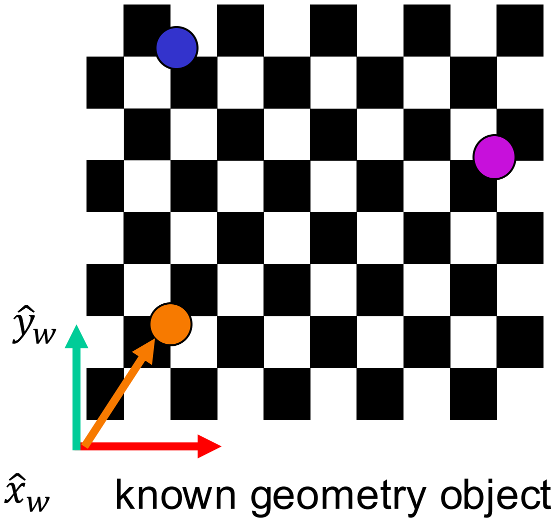
$$\tilde{u} = M_{int} M_{ext}\ \tilde{x}_w = P\ \tilde{x}_w$$

$$\begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix}$$

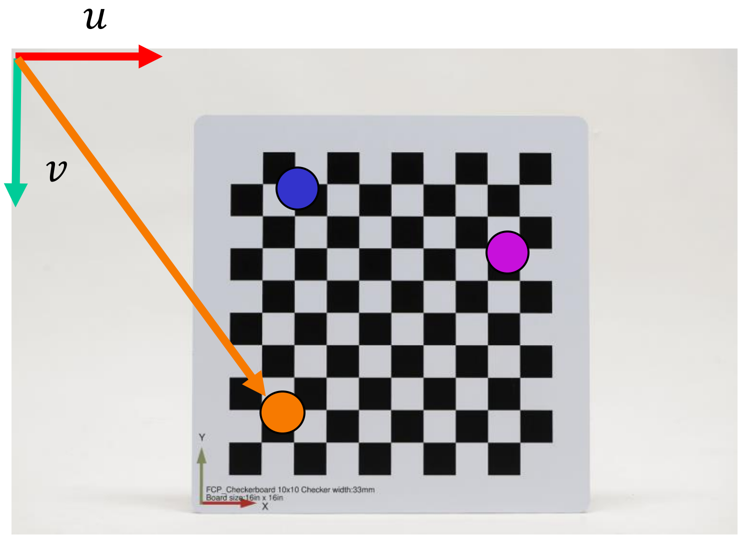
P: **Projection matrix**

Camera Calibration Procedure

Step 1. Capture image of object with known geometry



known geometry object



captured image

$$\boldsymbol{x}_W = \begin{bmatrix} x_W \\ y_W \\ z_W \end{bmatrix} \qquad \boldsymbol{u} = \begin{bmatrix} u \\ v \end{bmatrix}$$

Camera Calibration

Step 3. For each point i in the scene and the image we get a linear equation

$$\begin{bmatrix} u^{(i)} \\ v^{(i)} \\ 1 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{bmatrix} x_w^{(i)} \\ y_w^{(i)} \\ z_w^{(i)} \\ 1 \end{bmatrix}$$

Step 4. Collecting many $u^{(i)} = \frac{p_{11}x_w^{(i)} + p_{12}y_w^{(i)} + p_{13}z_w^{(i)} + p_{14}}{p_{31}x_w^{(i)} + p_{32}y_w^{(i)} + p_{33}z_w^{(i)} + p_{34}}$ points and rearranging p as a vector we get $A\boldsymbol{p} = 0$

Step 5. Solve for ***p***

Projection matrix scale

Since projection matrix works on homogeneous coordinates

$$\begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \widetilde{w} \end{bmatrix} \equiv k \begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \widetilde{w} \end{bmatrix}$$

Therefore

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix} = k \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix}$$

Therefore, Projection Matrices P and kP produce the same homogenous pixel coordinates

Projection matrix is defined only upto a scale factor

Scaling the world and the camera will produce indistinguishable images

That is , we can only find the projection matrix up to scale; we choose $||\boldsymbol{p}|| = \mathbf{1}$

Least Squares Solution for Projection Matrix

We want $A\boldsymbol{p}$ as close to 0 as possible and $||\boldsymbol{p}||^2 = 1$

$$\min_{\boldsymbol{p}} ||A\boldsymbol{p}||^{\mathbf{2}} \text{ such that } ||\boldsymbol{p}||^2 = 1$$

$$\min_{\boldsymbol{p}} \left|\left|\boldsymbol{p}^{\boldsymbol{T}} A^T A\boldsymbol{p}\right|\right|^{\mathbf{2}} \text{ such that } \boldsymbol{p}^{\boldsymbol{T}}\boldsymbol{p} = 1$$

$$L(\boldsymbol{p}, \lambda) = \boldsymbol{p}^{\boldsymbol{T}} A^T A\boldsymbol{p} - \lambda(\boldsymbol{p}^{\boldsymbol{T}}\boldsymbol{p} - 1)$$

Taking derivative $\frac{\partial L}{\partial \boldsymbol{p}} = 0$ gives $2A^T A\boldsymbol{p} - 2\lambda\boldsymbol{p} = \mathbf{0}$

$$A^T A\boldsymbol{p} = \lambda\boldsymbol{p}$$

$\boldsymbol{p}$ is the Eigenvector corresponding to the smallest eigenvalue of $A^T A$

Rearrange $\boldsymbol{p}$ to get the projection matrix **P**