ECE484 Principles of Safe Autonomy
Lecture 9

Control 2

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Announcements

ROS supplemental lecture in ECEB 1015, 5-5:50pm **today**

Midterm **March 4** in class 80 mins

Outline

- Modeling the control problem
  - Differential Equations; solutions and their properties
  - Bang-bang control
- Control design ←
  - PID
  - State feedback
  - MPC (brief)
- Requirements
  - Stability
  - Lyapunov theory and its relation to invariance

On-off control of a room heater with a thermostat

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), u(t))$$
$$\boldsymbol{u}(t) = g(x(t))$$

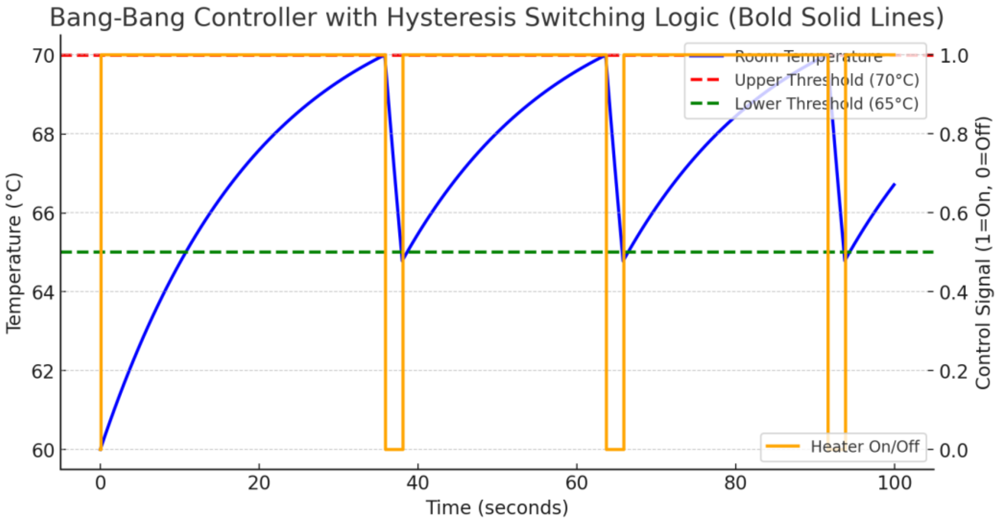
A simple thermostat controller

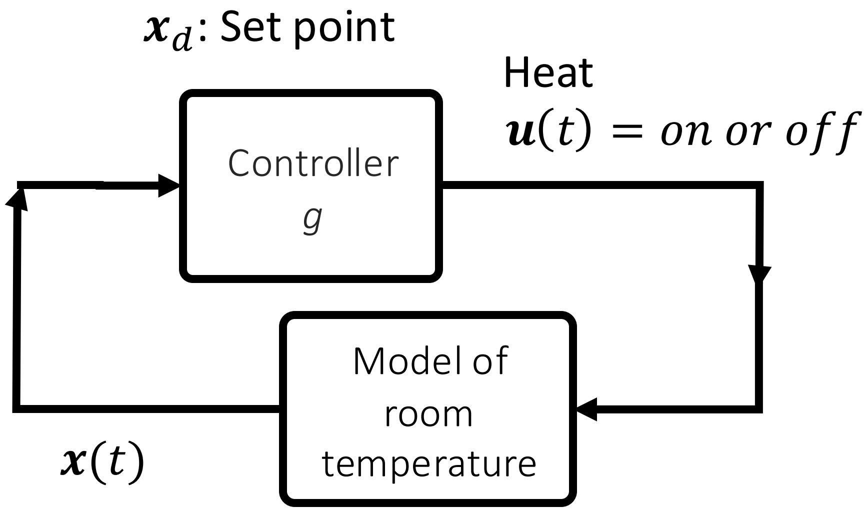
$g(x(t))$:

if $x(t) \geq x_d$ then $u(t)$ = off

else if $x(t) \leq x_d - \varepsilon$ then $u(t)$ = on

This is called bang-bang control



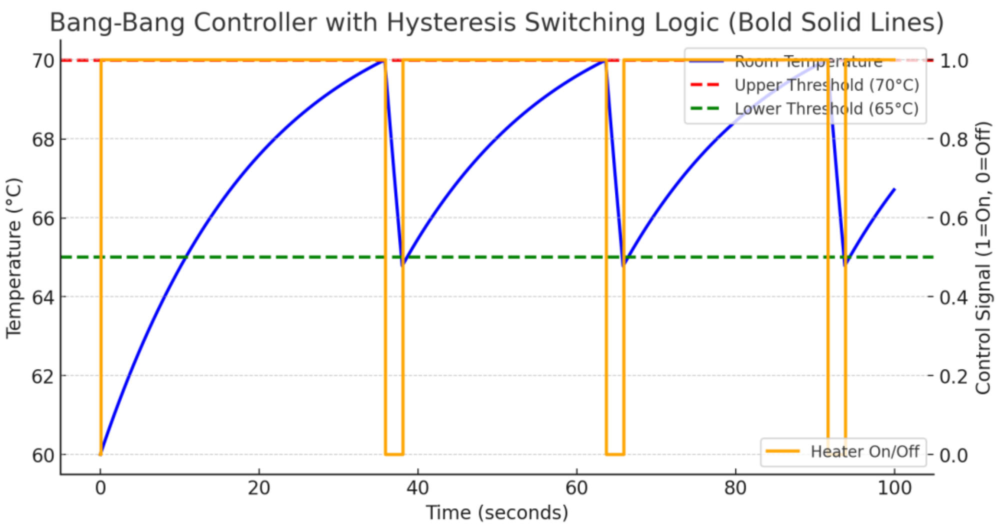


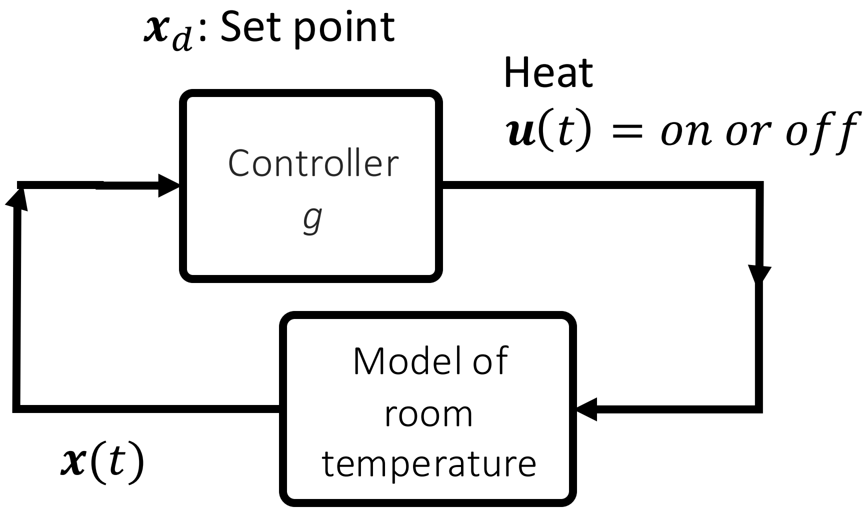
On-off control of a room heater with a thermostat

Bang-bang control is a feasible strategy when the controlled variable is observable

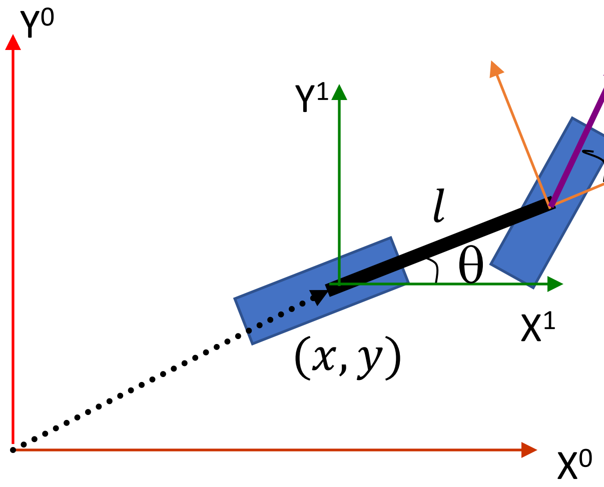
Disadvantages

- **Usually not energy efficient**
- Overshoots and undershoots because of inertia and delays
- Causes excess stress on the actuators
- Can cause the system to become unstable (to be defined later)





Review: Rear Wheel Model (Bicycle model)



Plant state: real wheel pose) = $\boldsymbol{x_B}: \mathbb{R}^3 = \begin{bmatrix} x_B \\ y_B \\ \theta_B \end{bmatrix}$

Control input: front wheel steering angle $u: \mathbb{R} = \delta_B$

Model parameters: car length (l) speed (v_B)

$f: \mathbb{R}^4 \rightarrow \mathbb{R}^3$

$\dot{\boldsymbol{x}}_B = f(\boldsymbol{x_B}, u)$

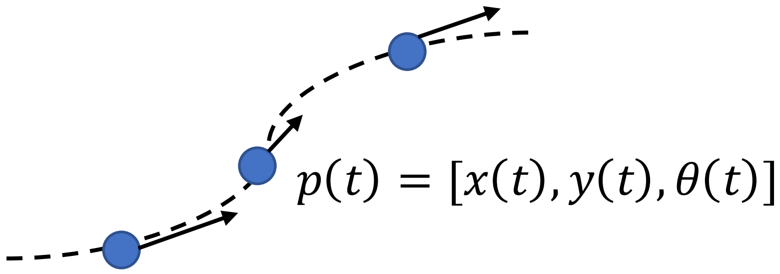
$$\begin{bmatrix} \dot{x}_B \\ \dot{y}_B \\ \dot{\theta}_B \end{bmatrix} = \begin{bmatrix} v_B \, cos\theta_B \\ v_B \, sin\theta_B \\ \frac{v_B}{l} tan\delta_B \end{bmatrix}$$

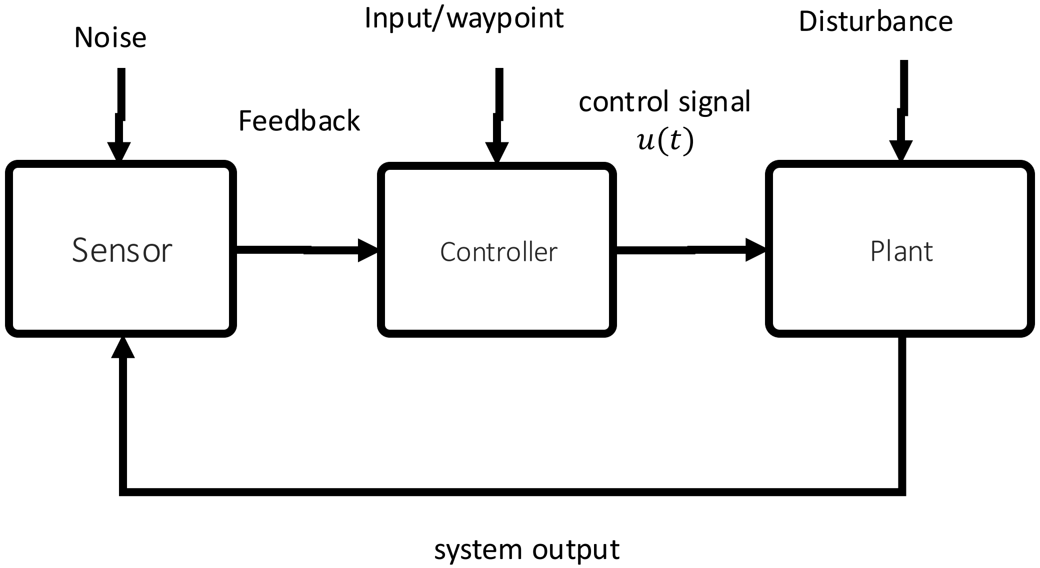
Path following control

The path **to be followed** by a robot is typically represented by a parameterized curve (e.g., parameterized by time)

This path is computed by a higher-level planner (e.g., using hybrid A*, RRT)

Each point in the path defines the desired instantaneous pose $p(t)$ of the vehicle

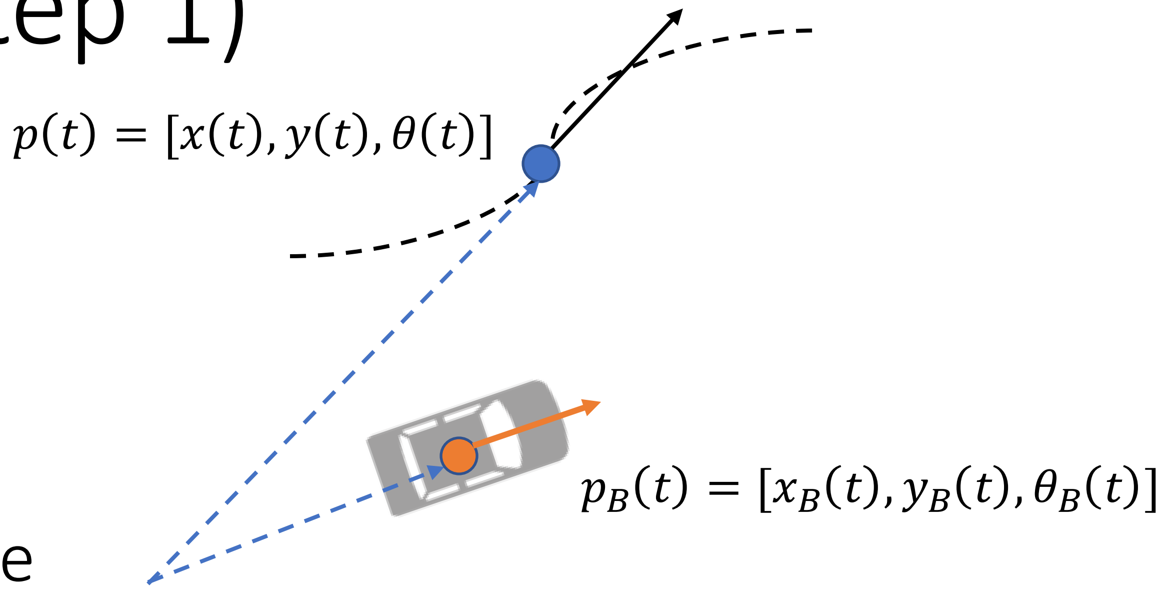




Path following control (Step 1)

Desired instantaneous pose $p(t)$

How to define error between actual pose $p_B(t)$ and desired pose $p(t)$ in the form of $x_d(t) - x(t)$ *so that then we can develop a control law*



A path following control

Desired instantaneous pose $p(t)$

The error vector measured vehicle coordinates
$\mathrm{e}(t) = [\delta_s(t), \delta_n(t), \delta_\theta(t), \delta_v(t)]$

$[\delta_s, \delta_n]$ define the coordinate errors in the vehicle's reference frame: along track error and cross track error

- Along track error: distance ahead or behind the target in the instantaneous direction of motion.

$$\delta_s = \cos(\theta_B(t))\,(x(t) - x_B(t)) + sin(\theta_B(t))\,(y(t) - y_B(t))$$

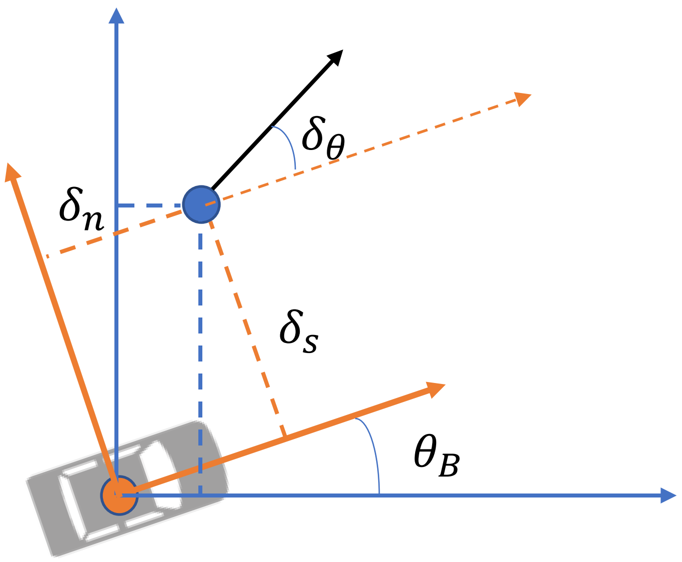
- Cross track error: portion of the position error orthogonal to the intended direction of motion

$$\delta_n = -\sin(\theta_B(t))\,(x(t) - x_B(t)) + cos(\theta_B(t))\,(y(t) - y_B(t))$$

- Heading error

$$\delta_\theta = \theta(t) - \theta_B(t)$$
$$\delta_v = v(t) - v_B(t)$$

$p(t) = [x(t), y(t), \theta(t), v(t)]$



$p_B(t) = [x_B(t), y_B(t), \theta_B(t), v_B(t)]$

Each of these errors match the form $x_d(t) - x(t)$ [From L8]

Bang bang controller for bicycle model (Step 2)

Dynamics $\begin{bmatrix} \dot{x}_B \\ \dot{y}_B \\ \dot{\theta}_B \end{bmatrix} = \begin{bmatrix} v\, cos\theta_B \\ v\, sin\theta_B \\ \frac{v}{l} tan\delta_B \end{bmatrix}$

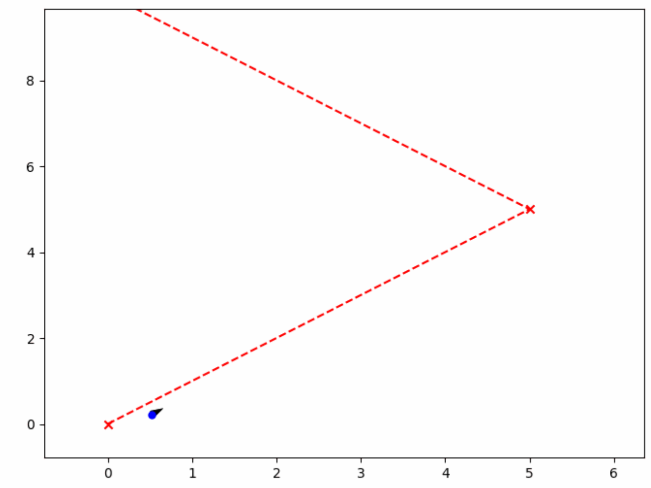
Heading error: $e_h = \theta_B - \theta$

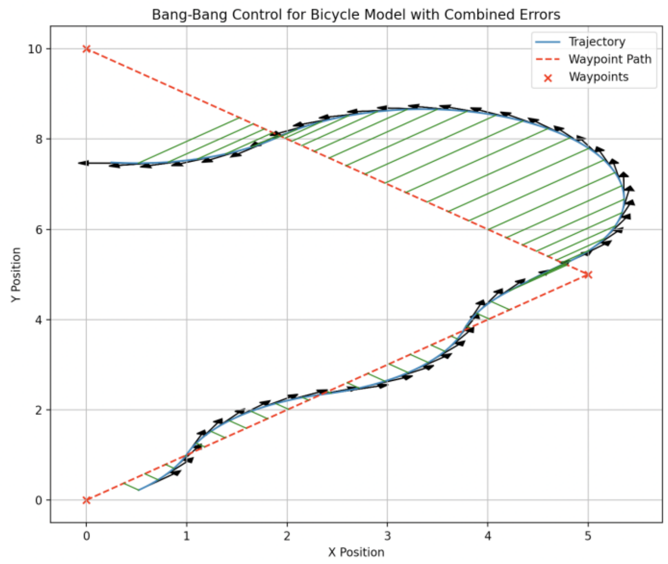
Cross track error: $\mathrm{e}_d = ||(x_B, y_B) - (x, y)||$

Combined error: $e = e_h + \alpha e_d$

Bang-bang controller:

if $e > 0$ then $\delta = \delta_{max}$ else $\delta = -\delta_{max}$





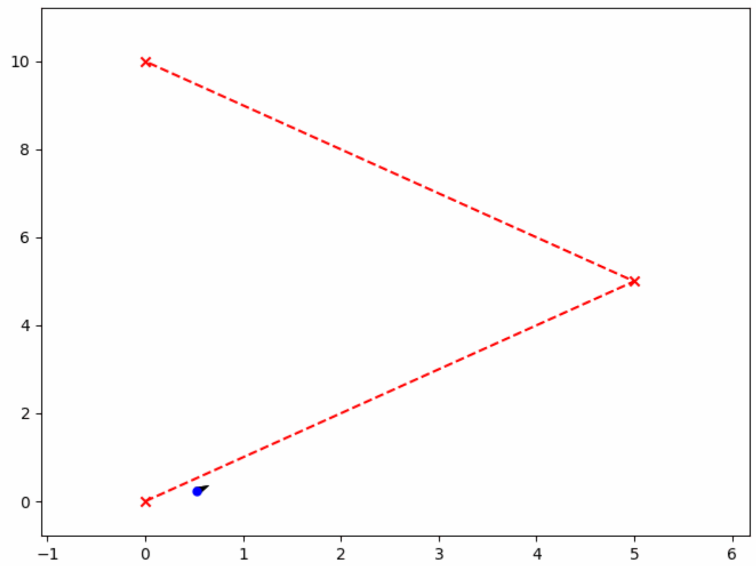
Proportional control

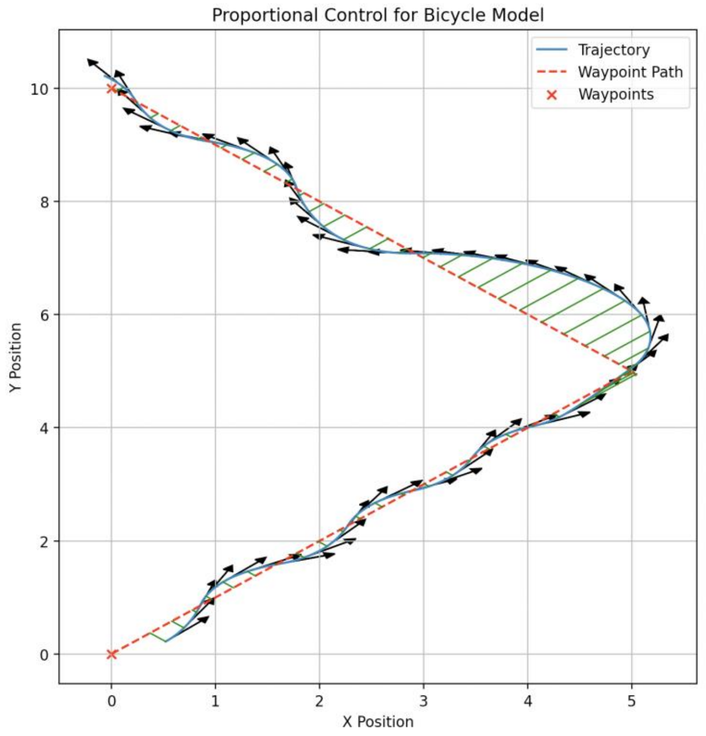
Dynamics $\begin{bmatrix} \dot{x}_B \\ \dot{y}_B \\ \dot{\theta}_B \end{bmatrix} = \begin{bmatrix} v\ cos\theta_B \\ v\ sin\theta_B \\ \frac{v}{l} tan\delta_B \end{bmatrix}$

Heading error: $e_h = \theta_B - \theta$

Cross track error: $\mathrm{e}_d = ||(x_B, y_B) - (x, y)||$

Proportional controller $\delta = -K_h e_h + -K_d e_d$





A Proportional controller

Plant $\dot{x}(t) = u(t) + d(t)$, where $d(t)$ is a small disturbance signal

The goal is the drive the plant state to a target steady state value, say $x_d = 70°$

Idea: Make the control input negatively proportional to the error: **Negative feedback**

Error: $e(t) = x(t) - x_d$

Proportional controller: $u(t) = -K_p e(t)$, the constant K_p is called **controller gain**

Using proportional (P) **negative feedback**

$$u(t) = -K_P e(t) = -K_P(x(t) - x_d)$$

$$\dot{x}(t) = -K_P x(t) + K_P x_d + d(t)$$

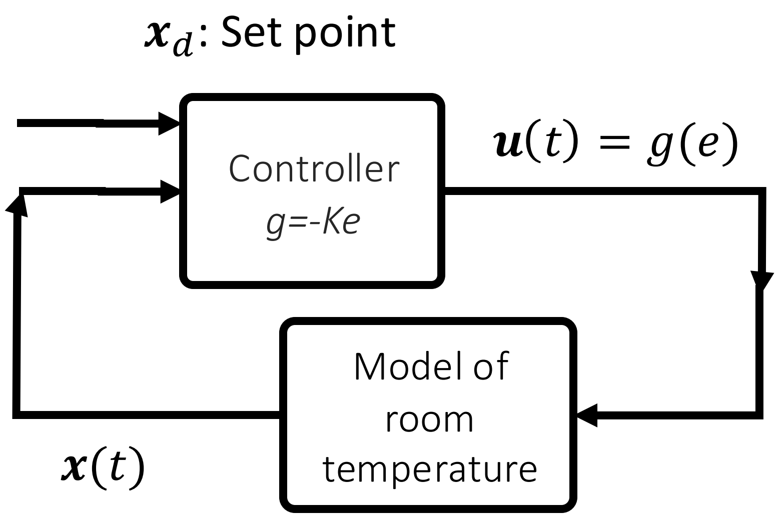
Consider a constant disturbance d_{ss}

$$\dot{x}(t) = -K_P x(t) + K_P x_d + d_{ss}$$

What is the steady state value? Trick: set RHS = 0

Set $-K_P x(t) + K_P x_d + d_{ss} = 0$

$$x(t) = x_{ss} = \frac{d_{ss}}{K_P} + y_d$$



Proportional controller example

With constant disturbance d_{ss} we rewrite the ODE

$$\dot{x}(t) = -K_P x(t) + K_P x_d + d_{ss} \text{ with } x_{ss} = \frac{d_{ss}}{K_P} + x_d$$

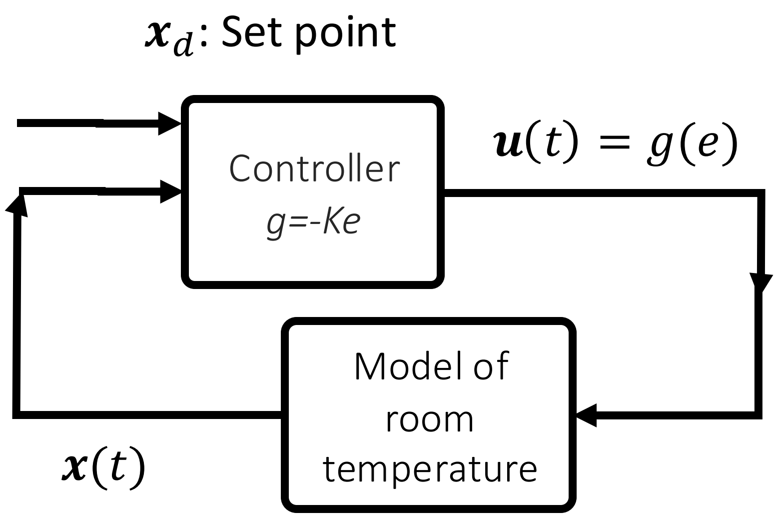
$$\dot{x}(t) = -K_P(x_{ss} - x(t))$$

The solution of this ODE

$$x(t) = x_{ss} + (x(0) - x_{ss})e^{-tK_p}$$

Transient behavior

$$x(t) = x(0)e^{-tK_p} + x_{ss}\left(1 - e^{-tK_p}\right)$$



General solution of first-order linear DE

$x(t) = x_{ss} + Ce^{-K_p t}$

Setting t=0

$x(0) = x_{ss} + C$

Proportional Controller

Transient behavior of the control system

$$x(t) = x(0)e^{-tK_p} + x_{ss}\left(1 - e^{-tK_p}\right); x_{ss} = \frac{d_{ss}}{K_P} + x_d$$

The proportional controller uses negative feedback to track the desired setpoint smoothly

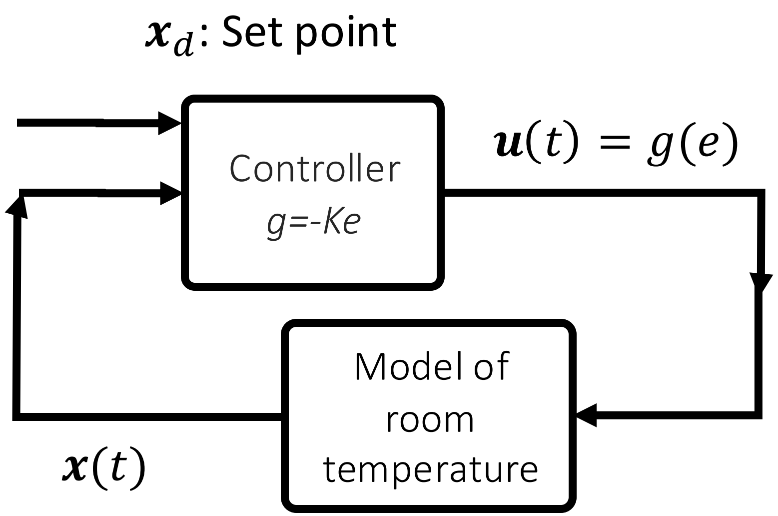
Steady state error may not be 0

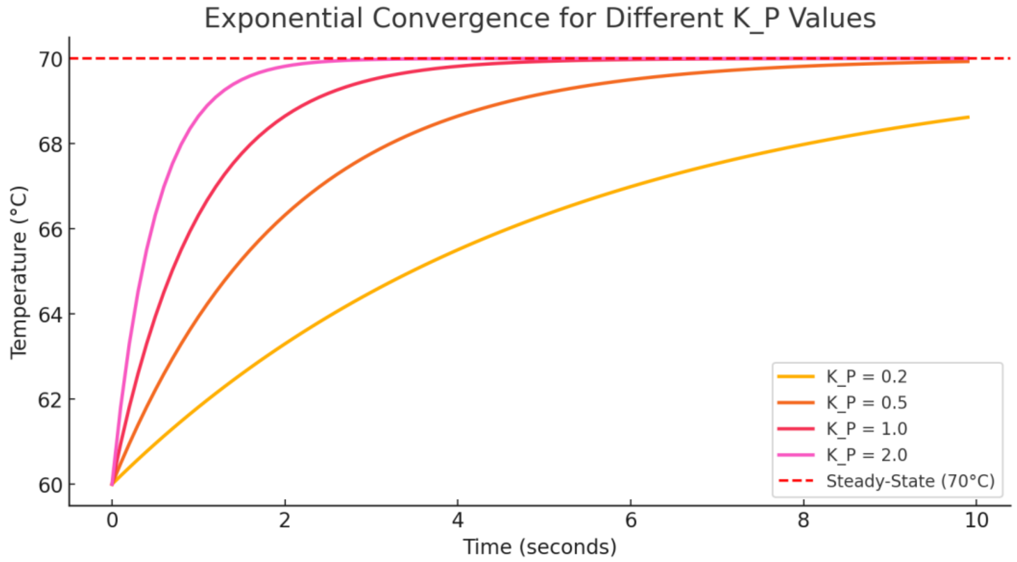
Larger proportional gain K_P more reactive the controller and faster the system converges to the target state K_P

Larger K_P implies smaller steady state tracking error

For systems with delays and inertia high proportional gain can cause oscillations or overshoots

There may be actuator limits that prevent $u(t) = -K_P e(t) = -K_P(x(t) - x_d)$ to be a feasible control input





The PID controller

Error: difference of desired and measured $e(t) = x_d - x(t)$

$$u(t) = K_P e(t) + K_I \int_0^t e(\tau)d\tau + K_D \frac{de(t)}{dt}$$

Tune K_P, K_I, K_d for the required performance

P (Proportional): Corrects based on the current error

Reacts to errors quickly (may lead to oscillations)

I (Integral): Corrects based on the accumulated past error.

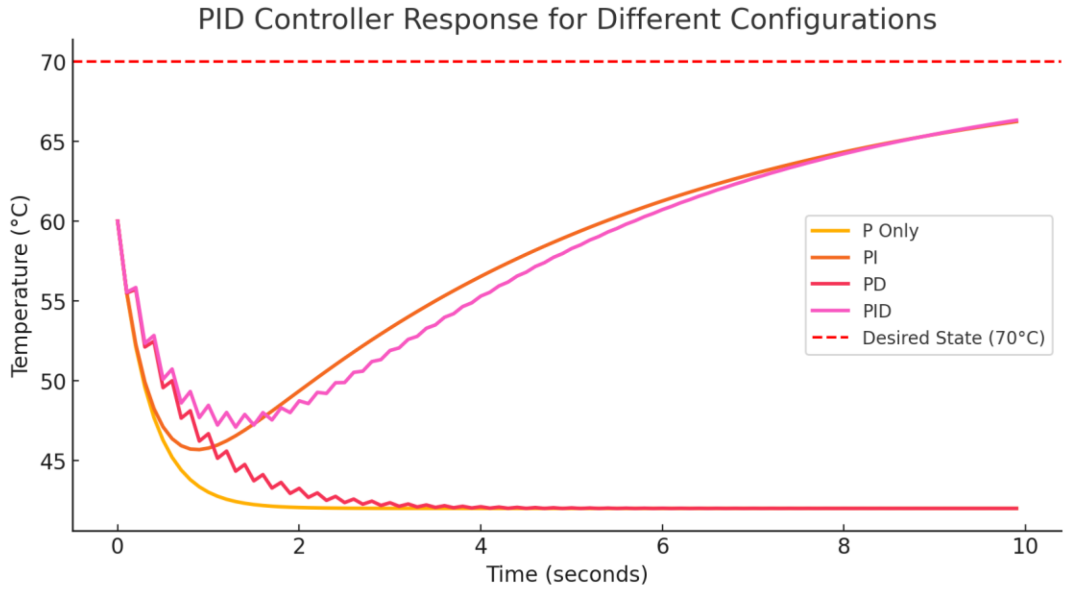
Removes steady state error

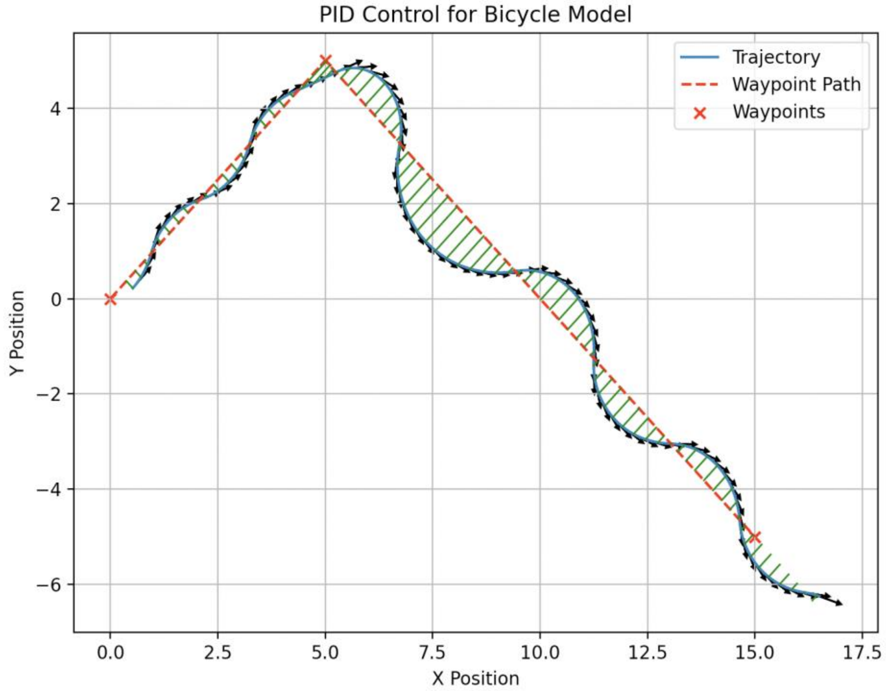
D (Derivative): Predicts future error based on the rate of change.

Dampens oscillations

PD control: $K_I = 0$

PI control: $K_D = 0$





Linear systems

Linear dynamical system and solutions

Linear system is a dynamical system where the dynamical function f is a linear function of the state and the inputs

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

where A and B are time varying matrices

For a given initial state $x_0 \in \mathbb{R}^n, t_0 \in \mathbb{R}\ and\ u(.) \in PC(\mathbb{R}, \mathbb{R}^n)$ the *solution* is a function $\xi(., t_0, x_0, u): \mathbb{R} \to \mathbb{R}^n$

Example: Simple linear model of an economy

x: national income

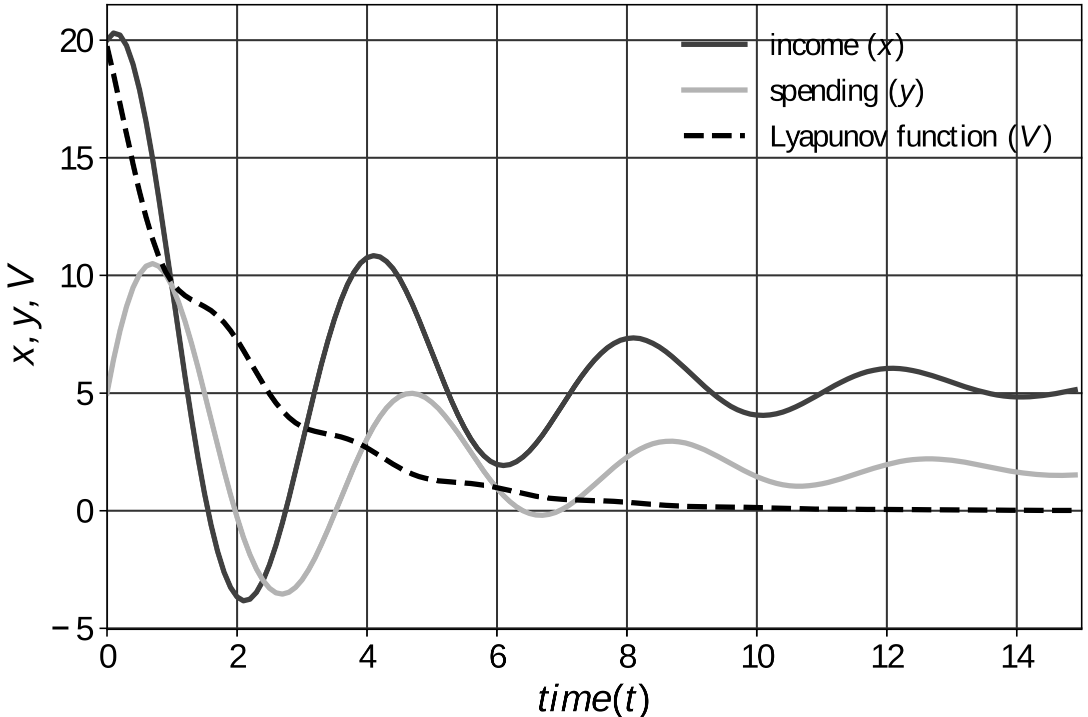
y: rate of consumer spending

g: rate government expenditure

$$\dot{x} = x - \alpha y$$

$$\dot{y} = \beta(x - y - g)$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & -\alpha \\ \beta & -\beta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 \\ \beta \end{bmatrix} g$$



Solutions of Linear systems define a linear space

$\dot{x}(t) = A(t)x(t) + B(t)u(t)$ --- (2)
$y(t) = C(t)x(t) + D(t)u(t)$

$u(t)$ continuous everywhere except D_x

Theorem*. Let $\xi(t, t_0, x_0, u)$ be the solution for (2) with points of discontinuity , D_x

1. $\forall t_0 \in \mathbb{R}, x_0 \in \mathbb{R}^n, u \in PC(\mathbb{R}, \mathbb{R}^m), \xi(\cdot, t_0, x_0, u): \mathbb{R} \to \mathbb{R}^n$ is continuous and differentiable $\forall\ t \in \mathbb{R} \setminus D_x$
2. $\forall t, t_0 \in \mathbb{R}, u \in PC(\mathbb{R}, \mathbb{R}^m), \xi(t, t_0, \cdot, u): \mathbb{R}^n \to \mathbb{R}^n$ is continuous
3. $\forall t, t_0 \in \mathbb{R}, x_{01}, x_{02} \in \mathbb{R}^n, u_1, u_2 \in PC(\mathbb{R}, \mathbb{R}^m), a_1, a_2 \in \mathbb{R}, \xi(t, t_0, a_1 x_{01} + a_2 x_{02}, a_1 u_1 + a_2 u_2) = a_1 \xi(t, t_0, x_{01}, u_1) + a_2 \xi(t, t_0, x_{02}, u_2)$
4. $\forall t, t_0 \in \mathbb{R}, x_0 \in \mathbb{R}^n, u \in PC(\mathbb{R}, \mathbb{R}^m), \xi(t, t_0, x_0, u) = \xi(t, t_0, x_0, \mathbf{0}) + \xi(t, t_0, 0, u)$

Linear time invariant system

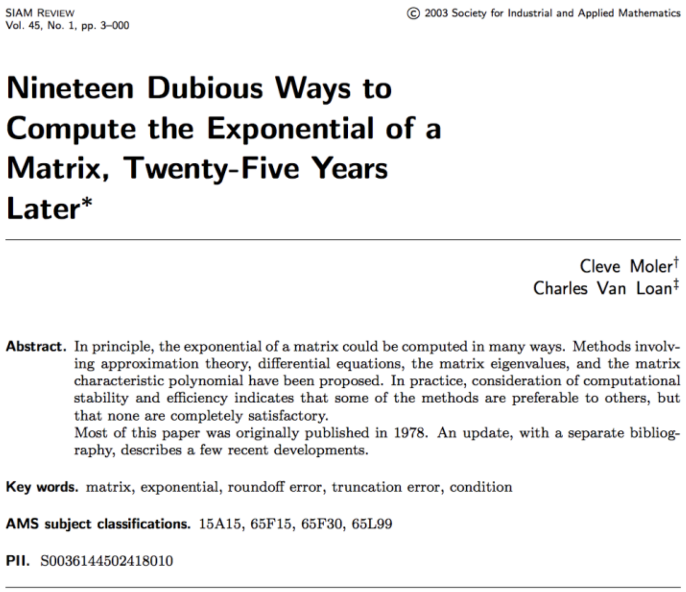
$$\dot{x}(t) = Ax(t) + Bu(t)$$

Matrix exponential:

$$e^{At} = 1 + At + \frac{1}{2!}(At)^2 + \ \dots = \sum_{0}^{\infty} \frac{1}{k!}(At)^k$$

Theorem. (Solution of linear systems)

$$\xi(t, t_0, x_0, u) = x_0 \mathrm{e}^{A(t-t_0)} + \int_{t_0}^{t} \mathrm{e}^{A(t-\tau)} Bu(\tau) d\tau$$



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Nineteen Dubious Ways to Compute the Exponential of a Matrix, Twenty-Five Years Later*

Cleve Moler†
Charles Van Loan‡

Abstract. In principle, the exponential of a matrix could be computed in many ways. Methods involving approximation theory, differential equations, the matrix eigenvalues, and the matrix characteristic polynomial have been proposed. In practice, consideration of computational stability and efficiency indicates that some of the methods are preferable to others, but that none are completely satisfactory.
Most of this paper was originally published in 1978. An update, with a separate bibliography, describes a few recent developments.

Key words. matrix, exponential, roundoff error, truncation error, condition

AMS subject classifications. 15A15, 65F15, 65F30, 65L99

PII. S0036144502418010

Requirements of dynamical systems: Stability

Consider a linear time invariant autonomous system (closed systems, systems without inputs)

- $\dot{x}(t) = f\big(x(t)\big)$, suppose $x_0 \in \mathbb{R}^n, t_0 = 0$
- $\xi(t)$ is the solution
- $|\xi(t)|$ norm
- $x^* \in \mathbb{R}^n$ is an **equilibrium point** if $f(x^*) = 0$.
- For analysis we will assume **0** to be an equilibrium point without loss of generality

Example (continued): Simple model of an economy

- x: national income y: rate of consumer spending; g: rate government expenditure
  - $\dot{x} = x \ - \alpha y$
  - $\dot{y} = \beta(x - y - g)$
  - Suppose $g = g_0 + kx$ α, β, k are positive constants
- What is the equilibrium?
  - $x^* = \frac{g_0 \alpha}{\alpha - 1 - k\alpha}$ $y^* = \frac{g_0}{\alpha - 1 - k\alpha}$
- Dynamics (now closed system):

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & -\alpha \\ \beta(1-k) & -\beta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Example: Pendulum

Pendulum equation

$x_1 = \theta \;\; x_2 = \dot{\theta}$

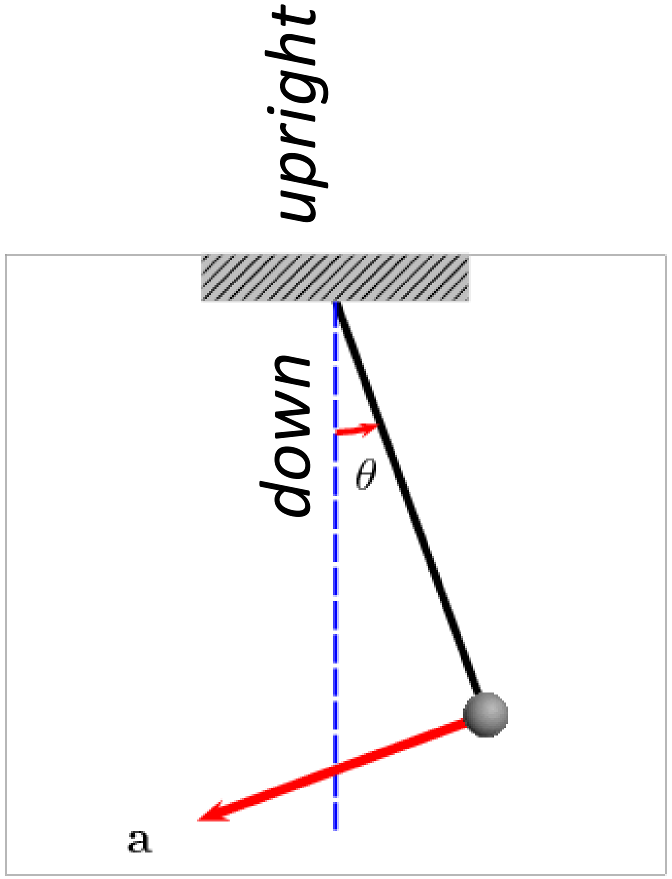
$x_2 = \dot{x}_1$

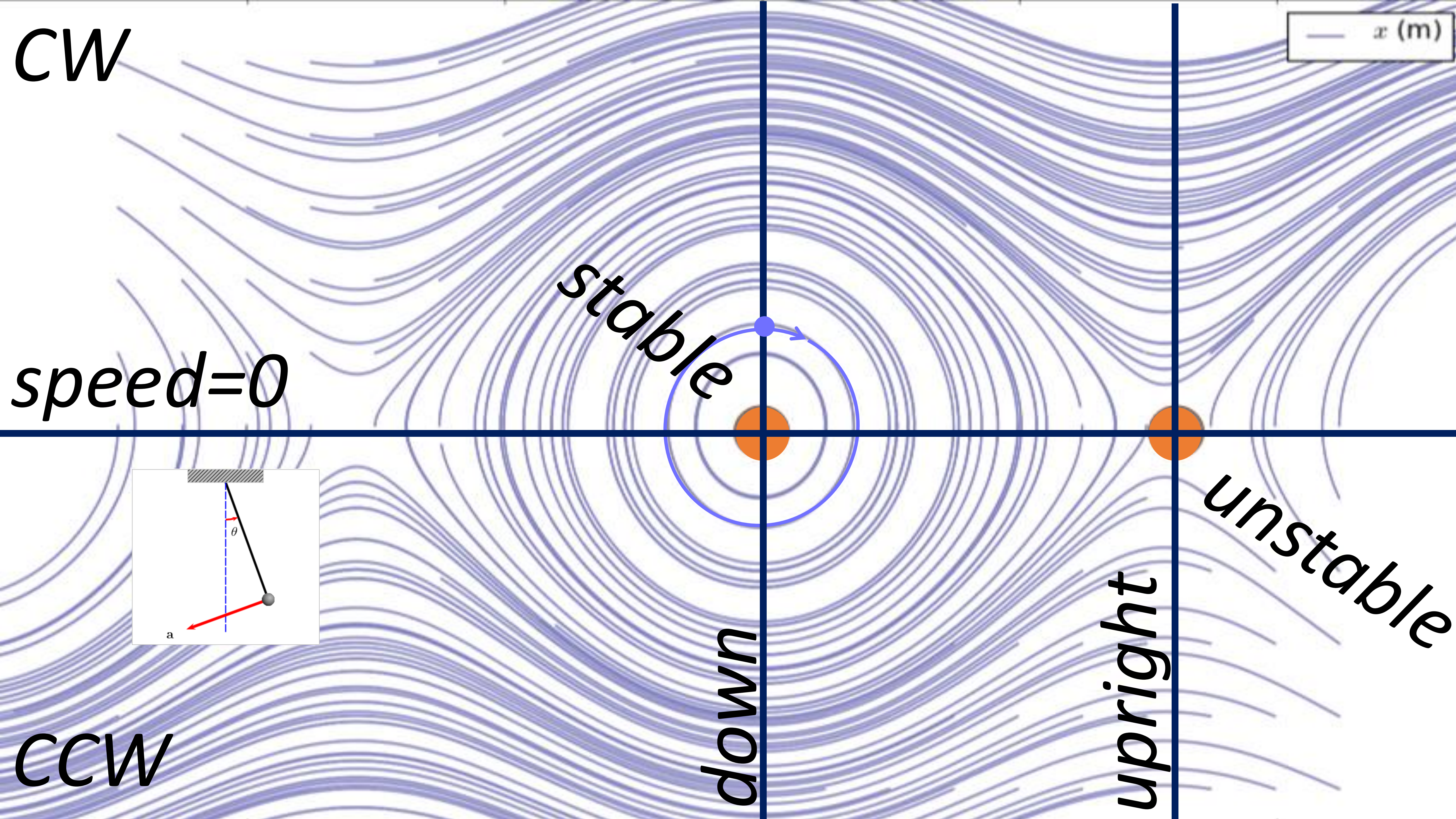
$\dot{x}_2 = -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2$

$\begin{bmatrix} \dot{x_2} \\ \dot{x_1} \end{bmatrix} = \begin{bmatrix} -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2 \\ x_2 \end{bmatrix}$

k: friction coefficient

Two equilibrium points: $(0,0), (\pi, 0)$




CW
x (m)
stable
speed=0
θ
a
unstable
upright
down
CCW

Aleksandr M. Lyapunov

Defined stability of ordinary differential equations and gave conditions for proving stability

A. M. Lyapunov

Linear system and solutions

$\dot{x}(t) = A(t)x(t) + B(t)u(t)$

$\dot{x}(t) = Ax(t) + Bu(t)$: Linear time invariant system

A, B:matrices, x: state vector, u: input vector

For a given initial state $x_0 \in \mathbb{R}^n, t_0 \in \mathbb{R} and\ u(.) \in PC(\mathbb{R}, \mathbb{R}^n)$ the *solution* is a function $\xi(., t_0, x_0, u): \mathbb{R} \to \mathbb{R}^n$

We studied several properties of ξ in the last lecture: continuity with respect to first and third argument, linearity, decomposition

Lyapunov stability

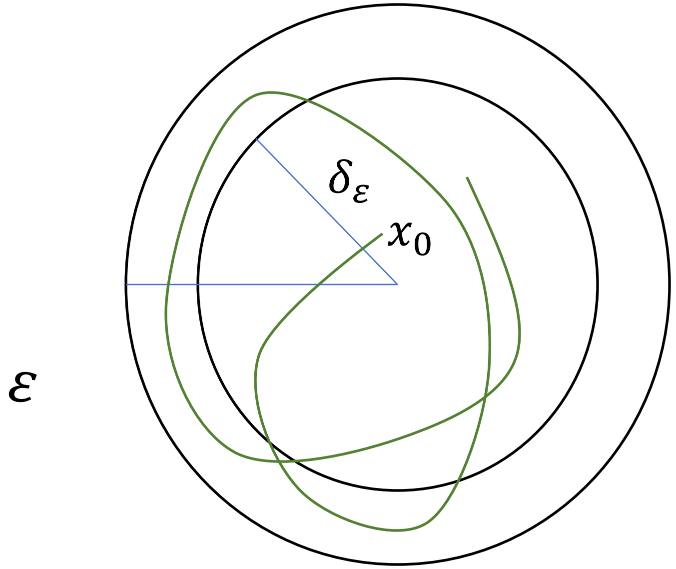
Lyapunov stability: The system (1) is said to be ***Lyapunov stable*** (at the origin) if

$\forall \varepsilon > 0 \; \exists \; \delta_\varepsilon > 0$ such that $|x_0| \leq \delta_\varepsilon \Rightarrow \forall \; \mathrm{t} \geq 0, |\xi(x_0, t)| \leq \varepsilon$.

How is this related to

invariants and

reachable states ?

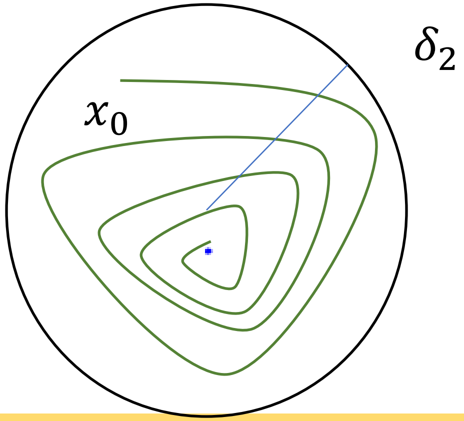


Asymptotically stability

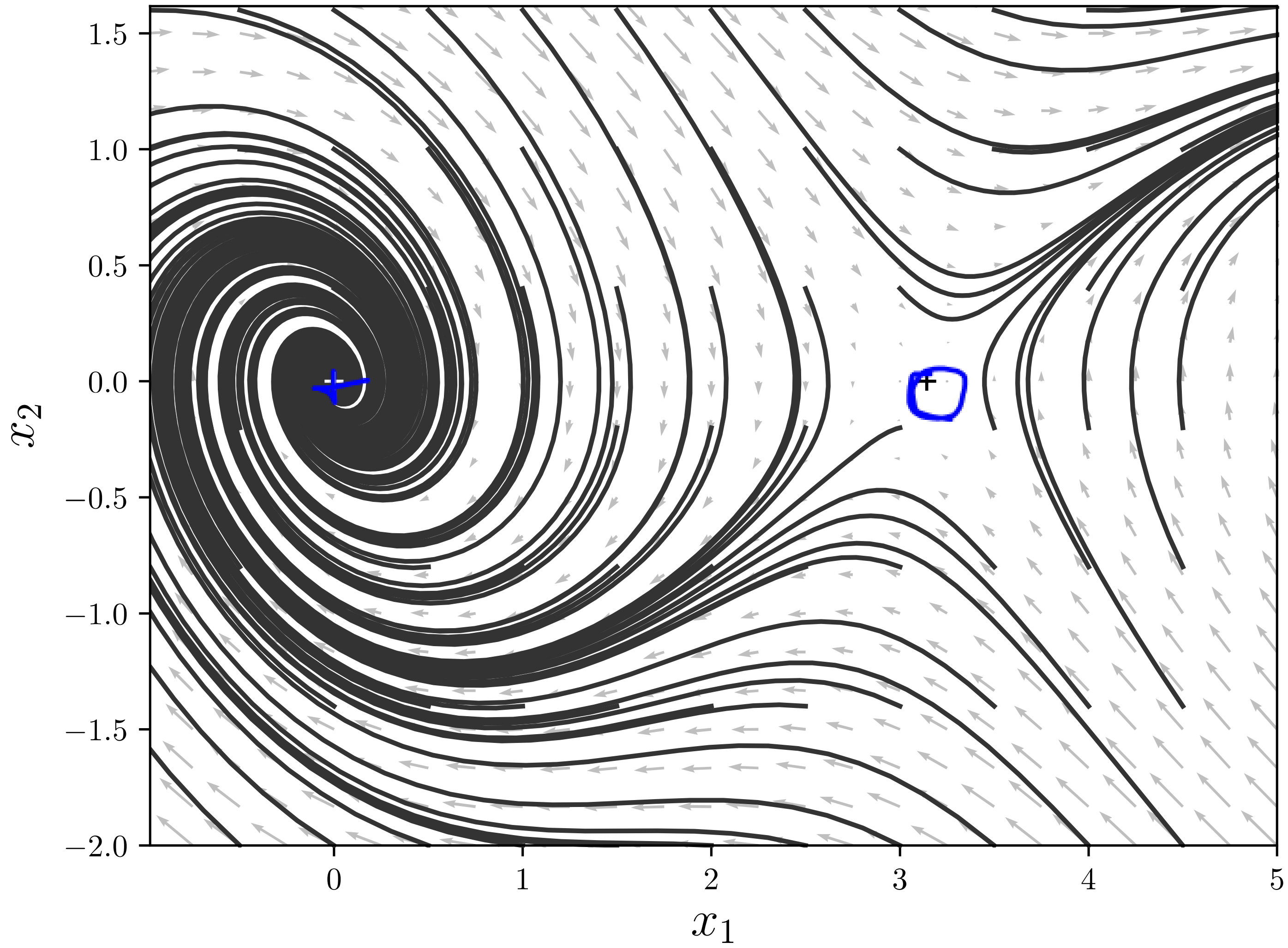
The system (1) is said to be ***Asymptotically stable*** *(at the origin)* if it is Lyapunov stable and

$\exists \delta_2 > 0$ such that $\forall |x_0| \leq \delta_2$ as t $\rightarrow \infty, |\xi(x_0, t)| \rightarrow \mathbf{0}$.

If the property holds for any δ_2 then **Globally Asymptotically Stable**



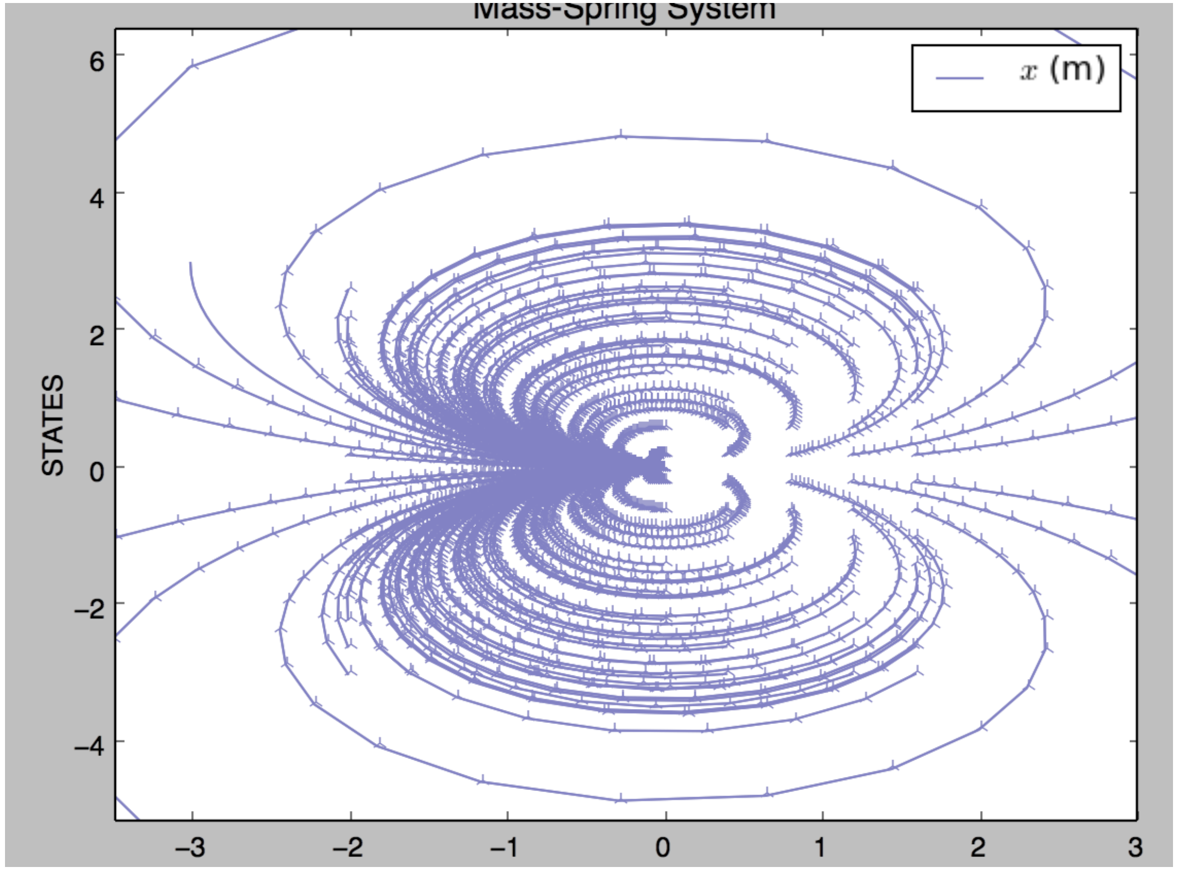
Phase portrait of pendulum with friction



Butterfly*

$$\begin{bmatrix} \dot{x_2} \\ \dot{x_1} \end{bmatrix} = \begin{bmatrix} 2x_1x_2 \\ x_1^2 - x_2^2 \end{bmatrix}$$

All solutions converge to 0 but the equilibrium point (0,0) is not Lyapunov stable



*Not discussed in class

Summary

- For ODE solutions to be well-defined we need the model to be Lipschitz continuous and the control input to be piece-wise continuous
- Steady state values of variables can be obtained by setting the RHS of the ODE to be zero
- Bang-bang controller does not require knowledge of the plant, does not give precise tracking, and may be energy inefficient
- Proportional controller uses negative feedback, gives smooth tracking performance, but can lead to overshoots

A general view of the controller

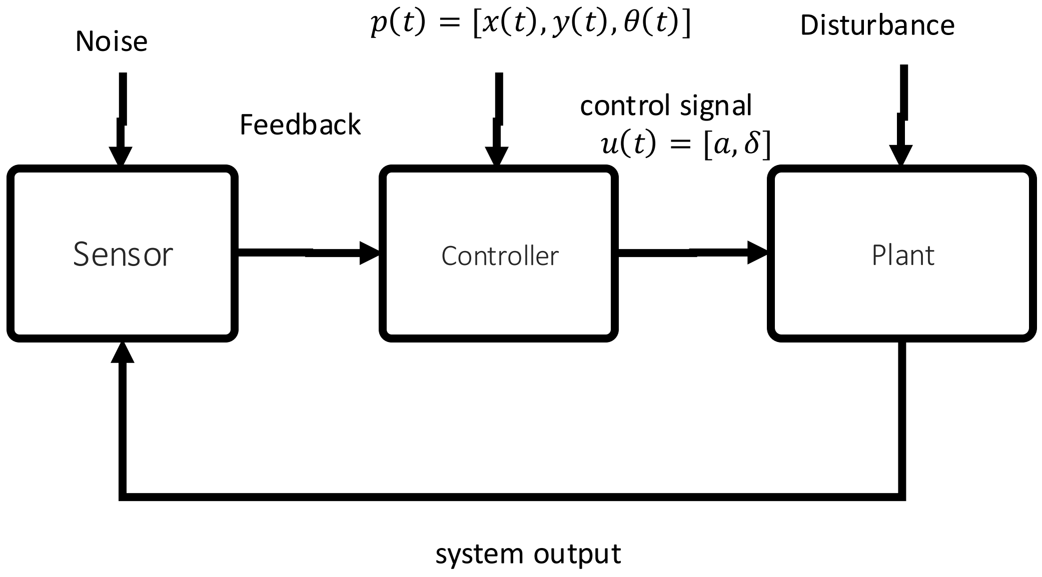
Control input is given by $u = [a, \delta]$

where a is the acceleration and

δ is the steering angle.

$$u = K \begin{bmatrix} \delta_s \\ \delta_n \\ \delta_\theta \\ \delta_v \end{bmatrix}$$

$$K = \begin{bmatrix} K_s & 0 & 0 & K_v \\ 0 & K_n & K_\theta & 0 \end{bmatrix}$$



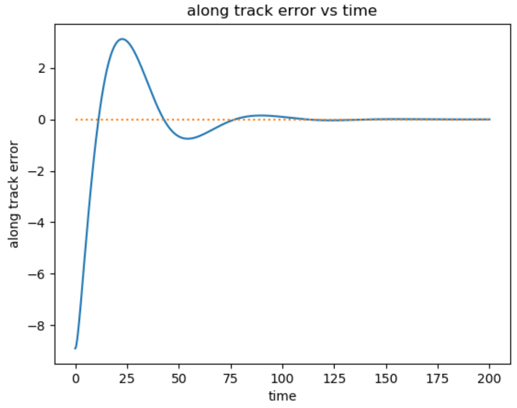
Control Law

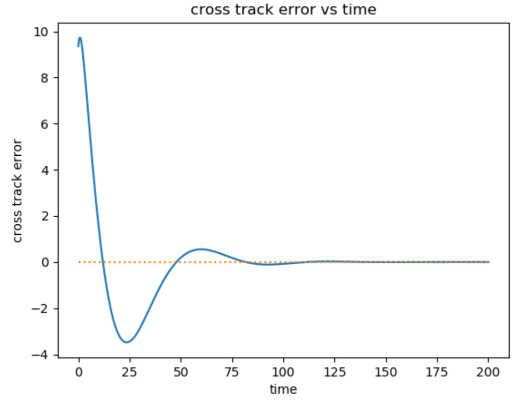
$$K = \begin{bmatrix} K_s & 0 & 0 & K_v \\ 0 & K_n & K_\theta & 0 \end{bmatrix}$$

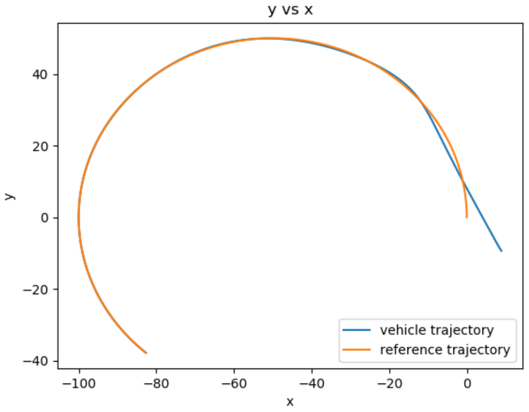
A pure-pursuit controller

produced by this gain matrix performs a PD-control. It uses a PD-controller to correct **along-track error**.

The control on curvature is also a PD-controller for **cross-track error** because δ_θ is related to the derivative of δ_n.







Verifying Stability for Linear Systems

Consider a linear system $\dot{x} = Ax$

Theorem 1. (Stability of linear systems)

1. It is asymptotically stable iff all the eigenvalues of *A* have **strictly** negative real parts (*Hurwitz*).

2. It is Lyapunov stable iff all the eigenvalues of *A* have real parts that are either zero or negative and the *Jordan blocks* corresponding to the eigenvalues with zero real parts are of size 1.

Jordan decomposition*

For every *n x n* matrix *A*, there exists a nonsingular *n x n* matrix *P* such that

$$PAP^{-1} = J = \begin{bmatrix} J_1 & 0 & 0 & \dots & 0 \\ 0 & J_2 & 0 & \dots & 0 \\ 0 & 0 & J_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & J_\ell \end{bmatrix},$$

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \dots & 0 \\ 0 & 0 & \lambda_i & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_i \end{bmatrix}.$$

where each J_i is an upper triangular matrix called a *Jordan block*

Examples

$$\begin{bmatrix} \dot{x_2} \\ \dot{x_1} \end{bmatrix} = \begin{bmatrix} -1/4 & -2/5 \\ 3 & -1/4 \end{bmatrix}$$

λ_1=−0.25−i1.10å
λ_2=−0.25+i1.10

$$\begin{bmatrix} \dot{x_2} \\ \dot{x_1} \end{bmatrix} = \begin{bmatrix} 1/4 & -2/5 \\ 3 & -1/4 \end{bmatrix}$$

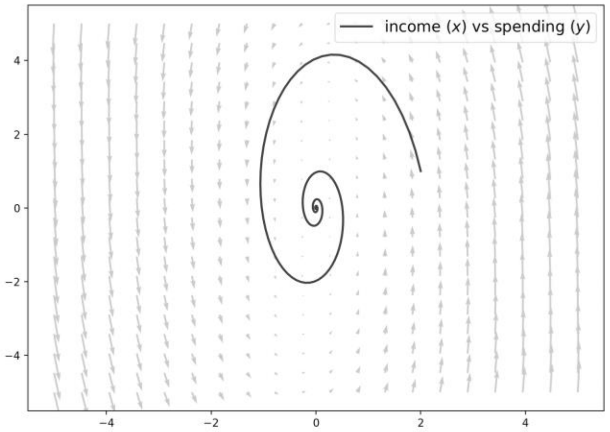
λ_1=+i0.1066
λ_2=-i0.1066

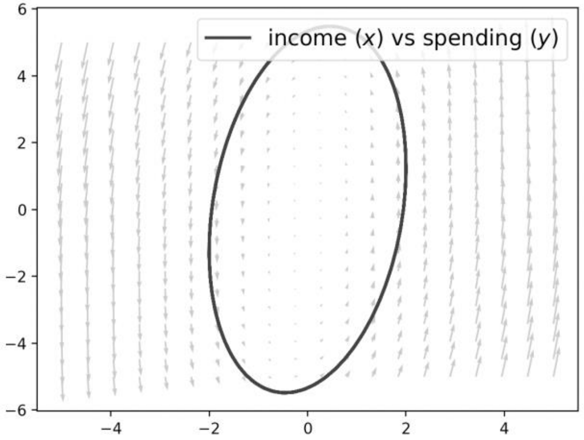
$$\begin{bmatrix} \dot{x_2} \\ \dot{x_1} \end{bmatrix} = \begin{bmatrix} 1/2 & -2/5 \\ 3 & -1/4 \end{bmatrix}$$

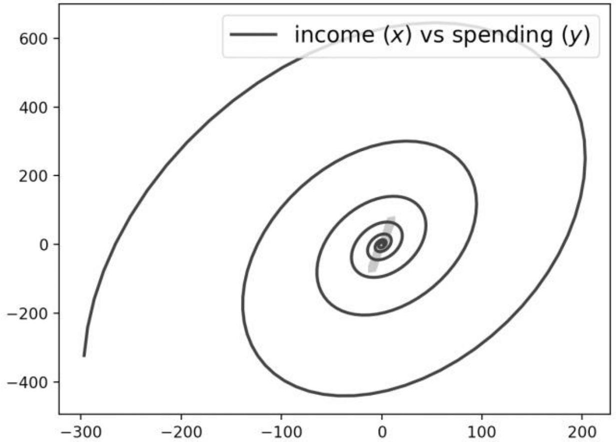
λ_1=0.125+i1.029
λ_2=-0.125-i1.029

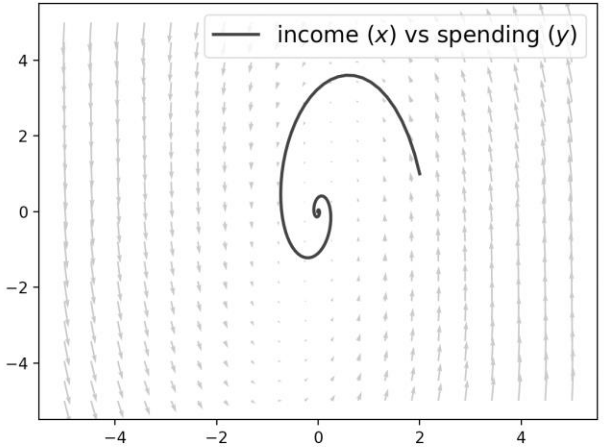
$$\begin{bmatrix} \dot{x_2} \\ \dot{x_1} \end{bmatrix} = \begin{bmatrix} -1/4 & -2/5 \\ 3 & -1/2 \end{bmatrix}$$

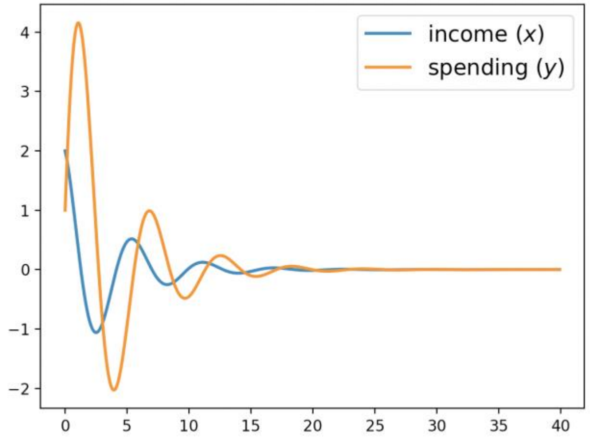
λ_1=−0.375−i1.088
λ_2=−0.375+i1.088

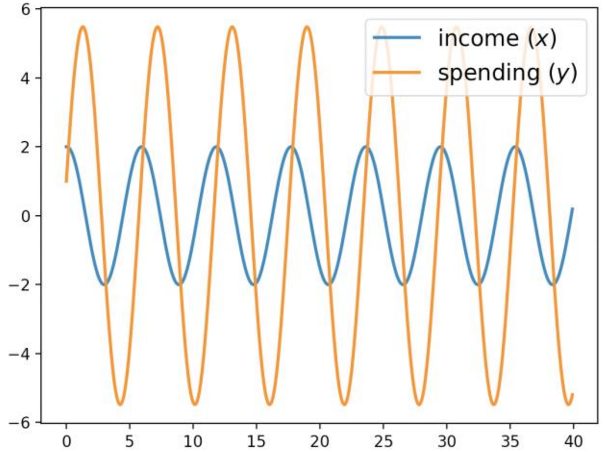


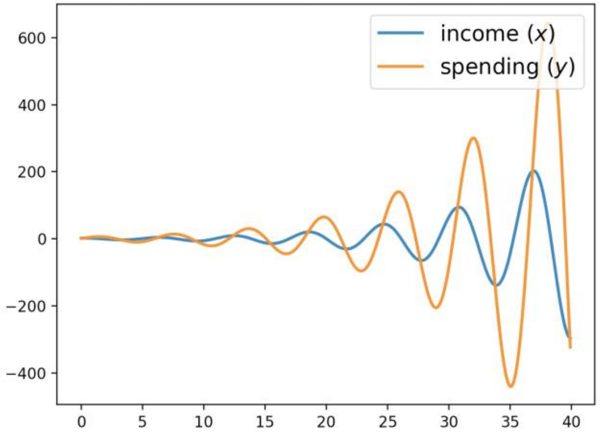


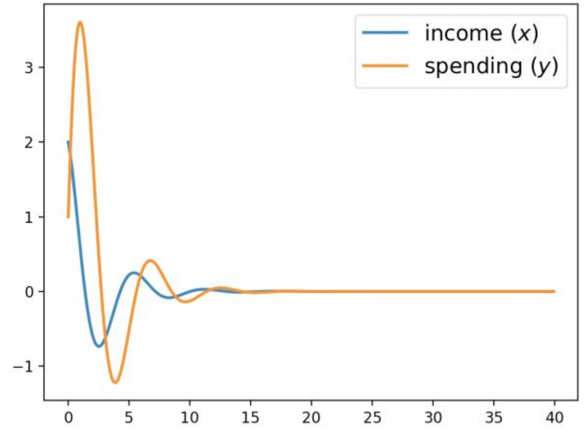








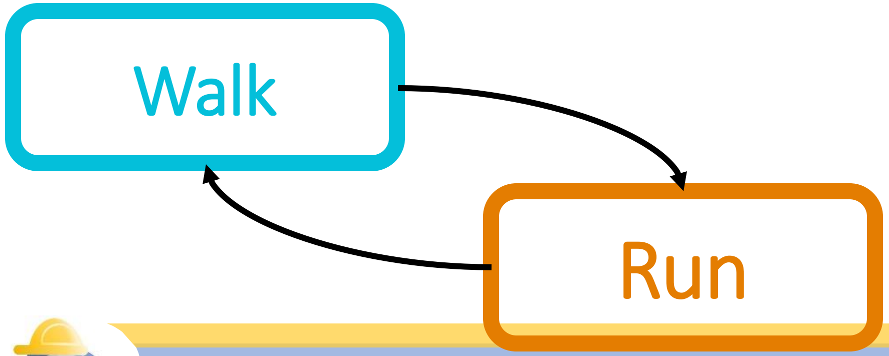


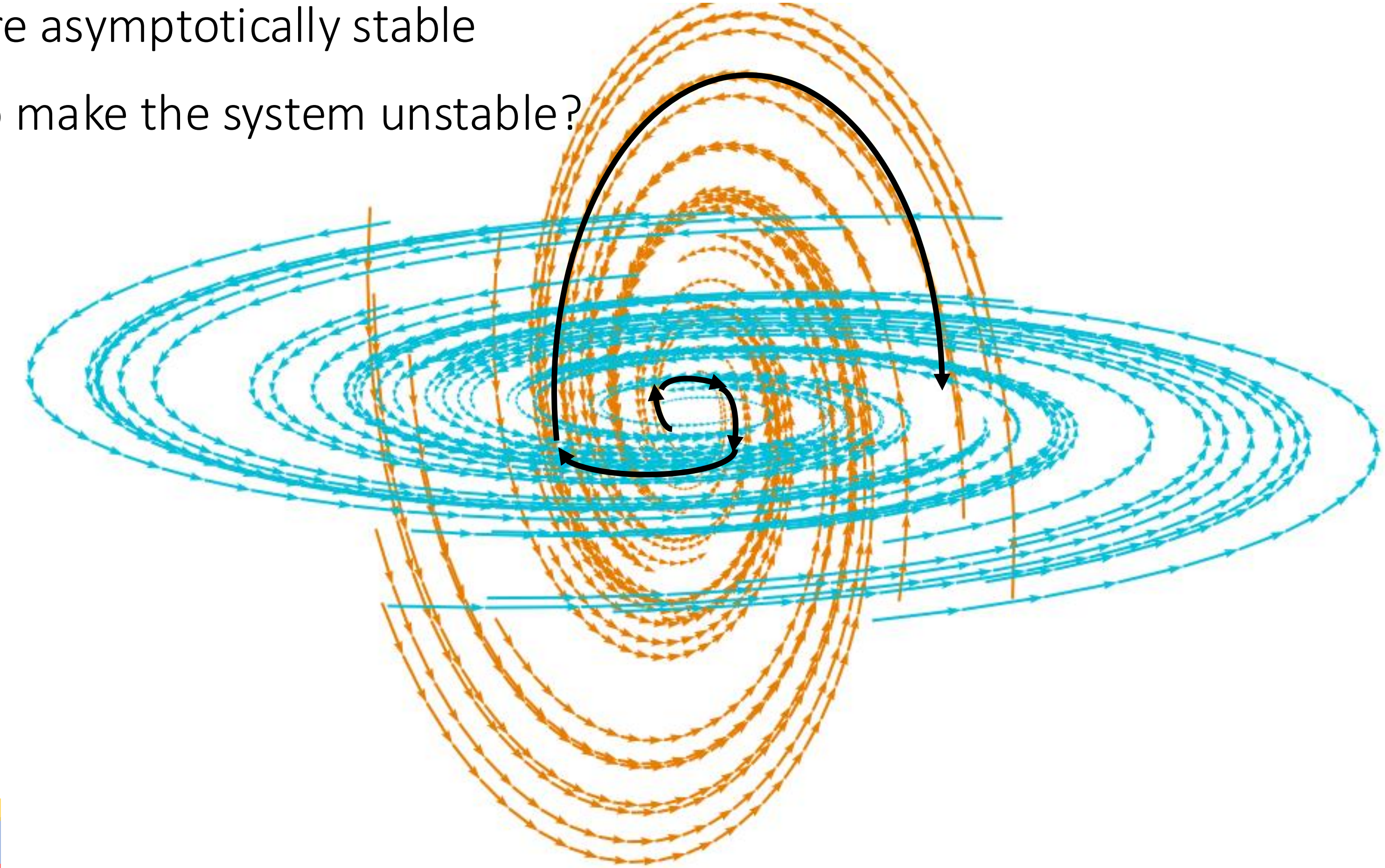


Hybrid Instability: Switching between two stable linear models

Each of the modes of a walking robot are asymptotically stable

Is it possible to switch between them to make the system unstable?





Yes! By switching between them the system becomes unstable