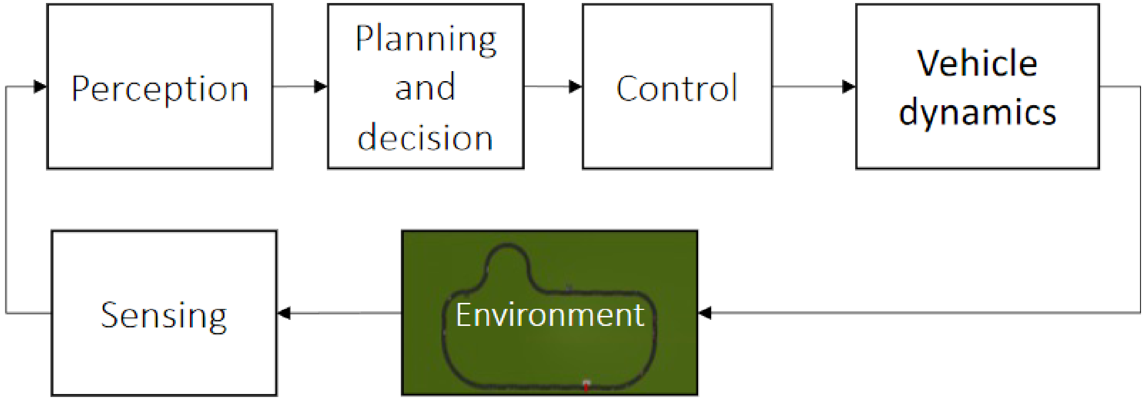
Control

Control Problem

The Control module sits between higher-level decisions (e.g., changing lanes, slowing down) and the lower-level control and actuation (e.g., steering and throttle).



env: Lidar / vision

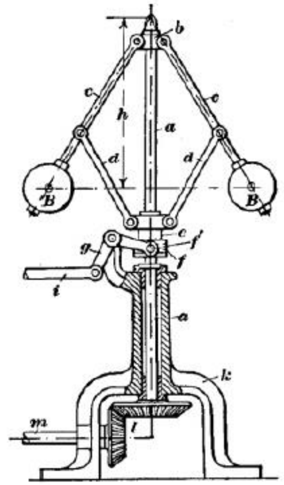
Control Theory

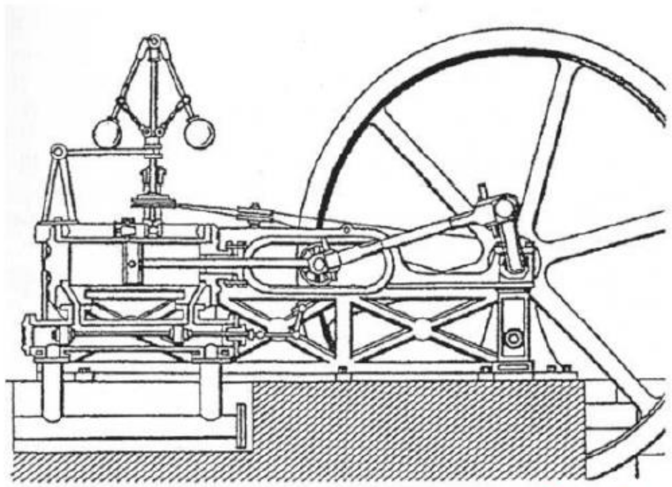
- Control theory is the art of making things do what you want them to do.
- Art describes the creation of parameterized controllers or algorithms and ways of tuning them.
- By things we mean phenomena that can be represented using differential equations.

  - we want them to follow some desired behavior, track some set-point, or follow a reference trajectory. Requirement, invariants

Example:

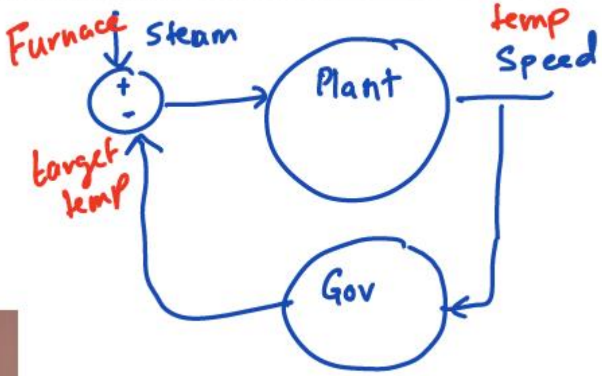
Feedback





Feed forward

-ve Feed back



(a) F/A-18 "Hornet"

(b) X-45 UCAV

Examples of Control System:

- Thermostat-heating system: the temperature is kept within a desirable range, despite the changes in occupancy and weather conditions.
- Cruise control: The car maintains a set reference speed, despite changes in slope, road conditions.
- Autosteer: The car steers to track the center of the lane markings.

Roadmap

Modeling the control problem using Differential Equations

- Solutions and their properties

Control design

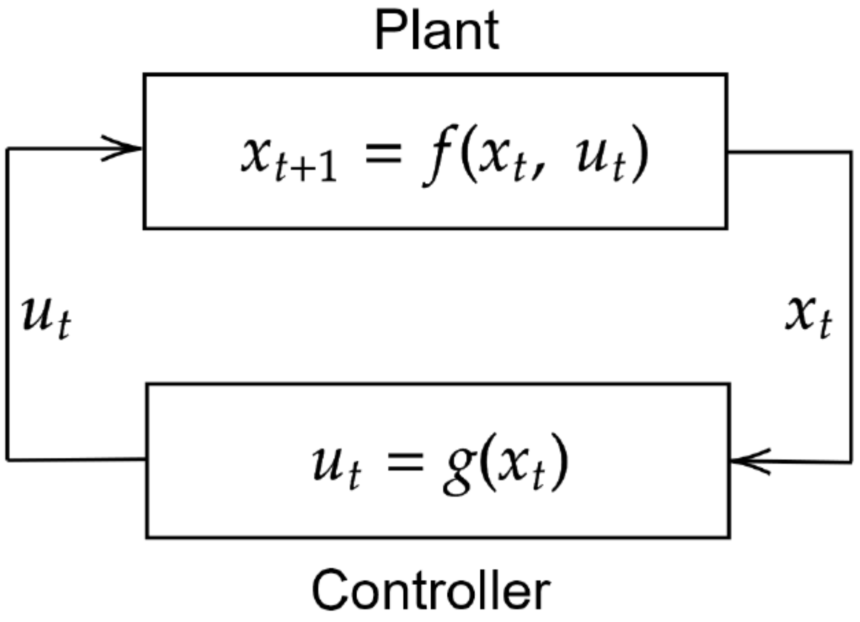
- Open vs. closed loop control
- PID control design
- State feedback

Requirements

- Stability
- Lyapunov theory and its relation to invariance

Differential Equation Model:

A control system can be modeled using two components: a plant and a controller.



Continuous Version

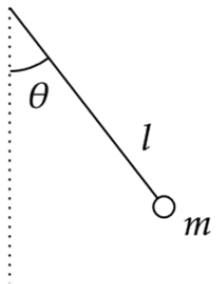
The continuous time version of this model is written using *ordinary differential equations (ODE)*:

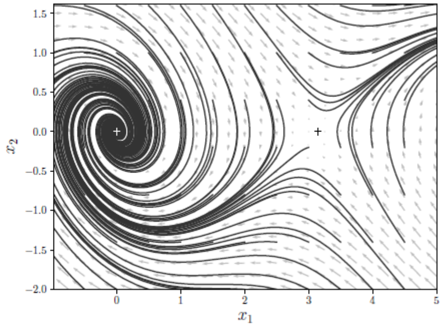
$$\dot{x}(t) = f(x(t), u(t)), or$$

$$\dot{x}(t) = f'(x(t))$$

A Solution to ODE is:

Examples of ODE





ODE describing the pendulum system is the following:

$$\dot{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \begin{bmatrix} x_2 \\ \frac{g}{l}\sin(x_1) - \frac{k}{m}x_2 \end{bmatrix}$$

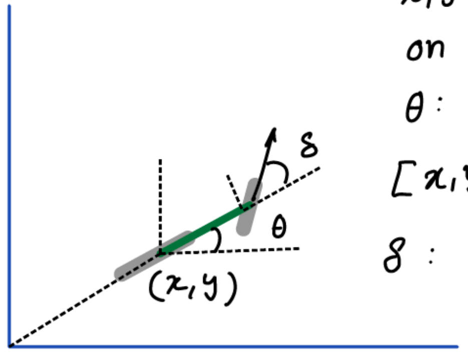
How do we find equilibria:

Vehicle Model

Example 2 Bicycle / Kinematic Vehicle model

Ref. Brian Paden et al. 2016

Survey of motion planning and Control for Self-driving



v : longitudinal velocity

x, y : position of rear wheel on the plane

θ : heading angle

$[x, y, \theta, v]$: state

δ : steering angle
control input

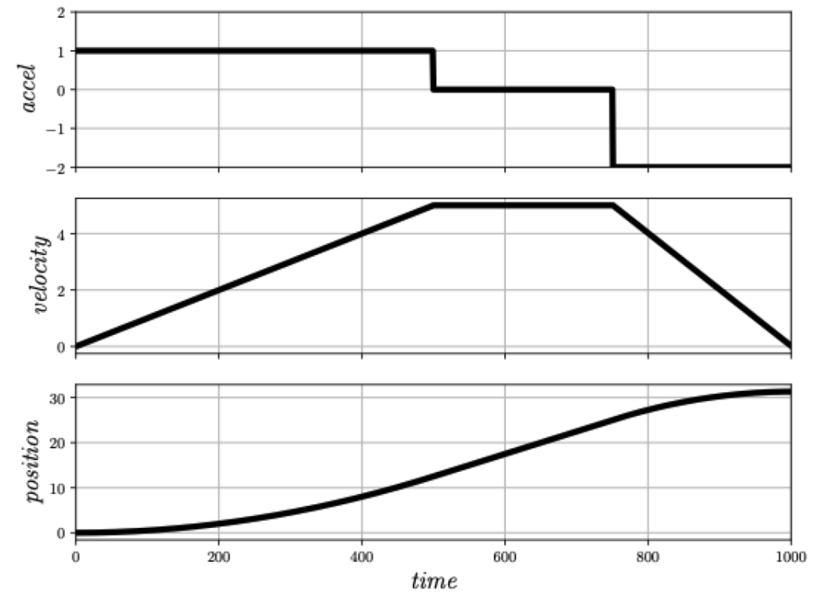
l : length of bicycle
parameter

Equations of motion derived in the above paper

$$\left.\begin{aligned} \dot{x}(t) &= v(t)\cos\theta(t) \\ \dot{y}(t) &= v(t)\sin\theta(t) \\ \dot{\theta}(t) &= \frac{v(t)}{l}\tan\delta(t) \end{aligned}\right\} \quad \dot{x} = f(x, u)$$

When do the Solutions Exist?

- If the control input u(t) is not continuous, then the solution x(t) will not be differentiable (at the points of discontinuity). See Figure below we have to be careful about the definition of solution.



When do the Solutions Exist?

<u>Example</u> $\dot{x} = x^2$ with $x(0) = 1$ $x(t)$

Check that $x(t) = \dfrac{1}{1-t}$ is a solution

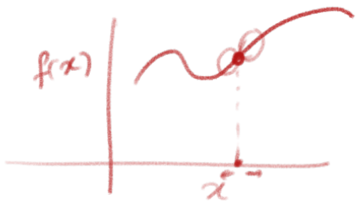
$$\frac{d\,x(t)}{dt} = -\frac{1}{(1-t)^2}(-1) = \frac{1}{(1-t)^2} = x^2(t) \checkmark$$

But as $t \to 1$ $x(t) \to \infty$ blows up

<u>Additional assumption on f</u>

f should not grow too fast

intuition



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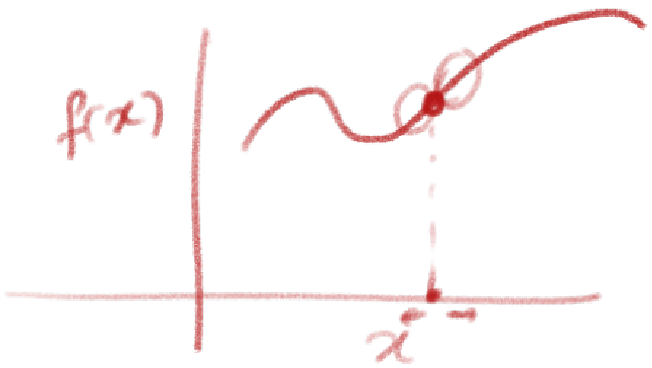


Lipschitz Continuous

Definition 5.1. A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is *Lipschitz continuous* if $\exists L > 0$ such that for any pair $x, x' \in \mathbb{R}^n$,

$$||f(x) - f(x')|| \leq L||x - x'||.$$

Lipschitz Continuous

Example . $f(x) = 6x + 4$

All differentiable functions with bounded derivative are Lipschitz Continuous

$|x|$ not differentiable but Lipschitz

x^2 not Lipschitz $\quad \|f(x) - f(x')\|$

$$= \|x_1^2 - x_2^2\|$$

Suppose $\exists L$ such that $\|x_1^2 - x_2^2\| \leq L(x_1 - x_2)$

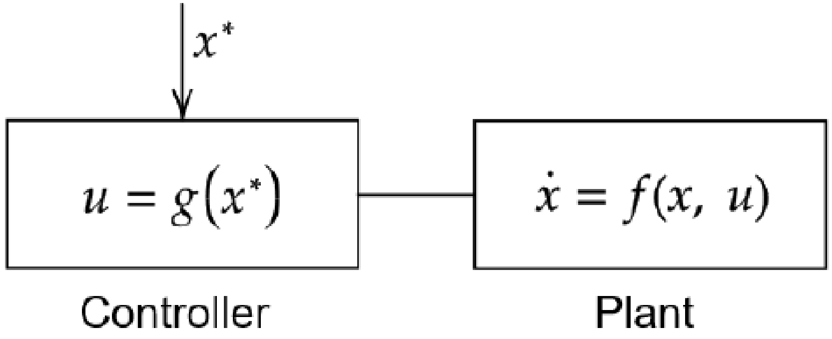
Lipschitz Continuous

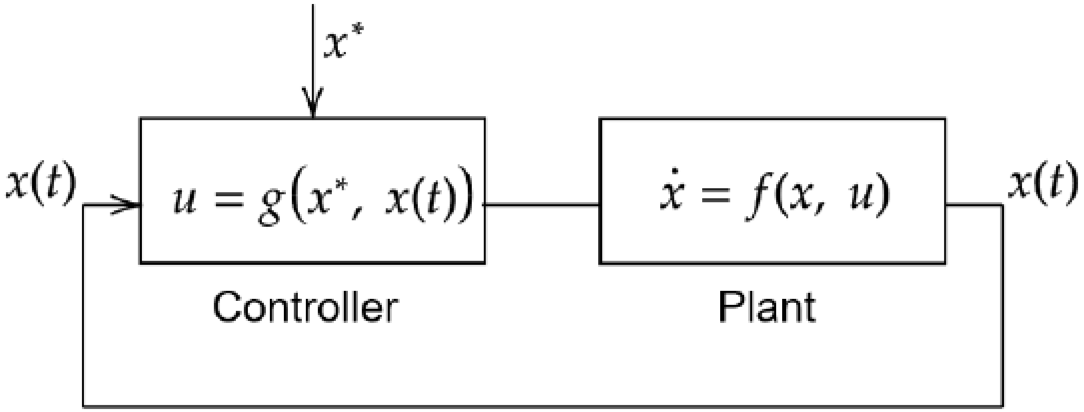
$\sqrt{x}$ is also not Lipschitz

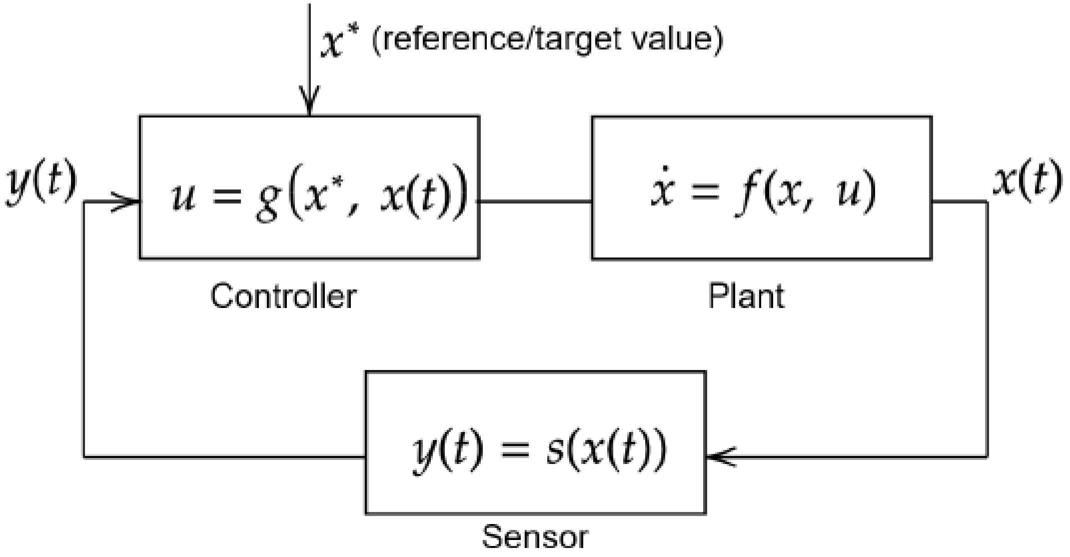
$\dot{x} = f(x) \mid \dot{x} = f(x,u)$

Thm. If $f(x,u)$ is Lipschitz Continuous in the first argument and $u(t)$ is piecewise Continuous then $\dot{x} = f(x,u)$ has unique solutions.

Open Loop and Closed Loop Control

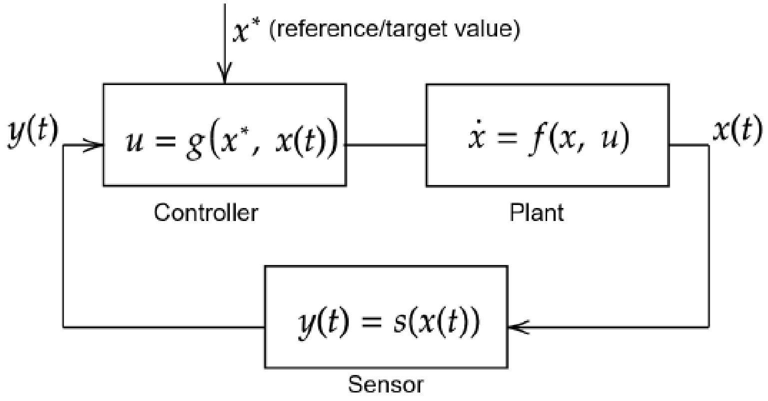






Proportional Control Example

One example is where a car driving down a lane on the road. If the car is very close to the left edge of the lane, we would want a considerable turn in the right direction. If the car is slightly to the left, we would want to move just a little bit to the right. Here the error is the deviation of the car's position x(t) from the center of the lane x ∗ . This motivates the incorporation of proportional control, where we can include a proportional gain term as so: $u(t) = g(x*, x(t)) = Kp(x* - x(t))$. This is also known as error feedback control, and to show this in our model, we add a sensor, as seen in the following Figure



PID Control

There are more ways to fine-tune the function g:

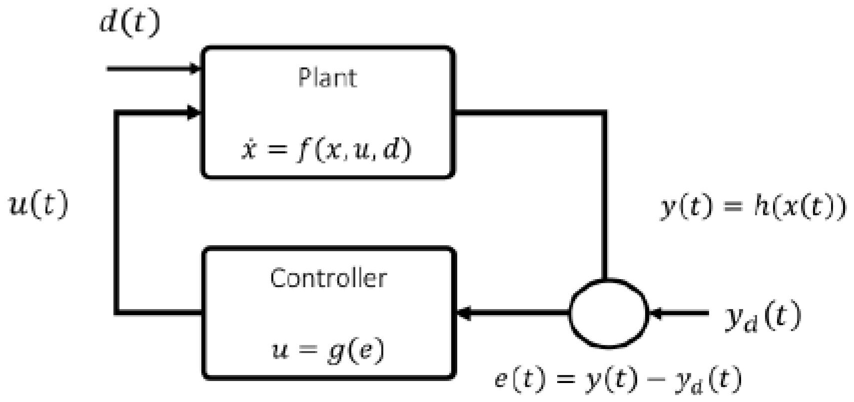
- g not only depends on the error but also the rate of change of error (derivative)
- g also depends on the history of the error (integral)

This gives the general form of the PID controller:

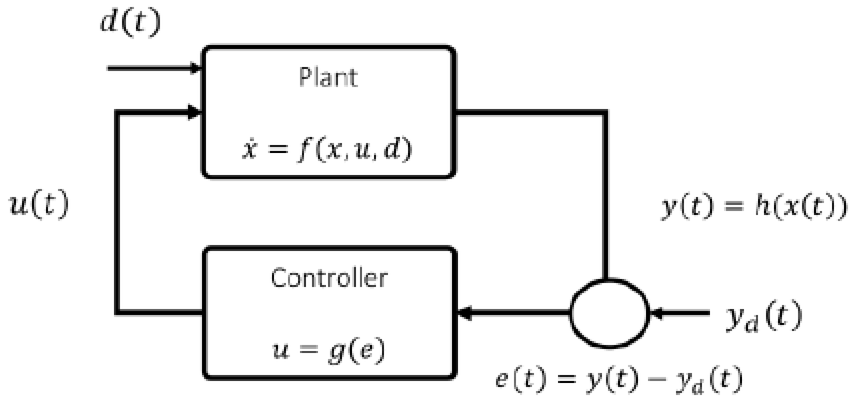
$$\begin{aligned} u(t) &= K_P e(t) + K_I \int_0^t e(\tau)d\tau + K_D \frac{de(t)}{dt} \\ &= K_P[e(t) + \frac{1}{T_I} \int_0^t e(\tau)d\tau + T_D \frac{de(t)}{dt}] \end{aligned}$$

A simple P-controller example

- $\dot{y}(t) = u(t) + d(t)$
- Using proportional (P) controller
- $u(t) = -K_P e(t) = -K_P(y(t) - y_d(t))$
- $\dot{y}(t) = -K_P y(t) + K_P y_d(t) + d(t)$
- Consider constant setpoint y_0 and disturbance d_{ss}
- $\dot{y}(t) = -K_P y(t) + K_P y_0 + d_{ss}$
- What is the steady state output?
  - Set $-K_P y(t) + K_P y_0 + d_{ss} = 0$
  - $y(t) = y_{ss} = \frac{d_{ss}}{K_P} + y_0$



A simple P-controller example



- $\dot{y}(t) = u(t) + d(t)$
- Consider constant setpoint y_0 and disturbance d_{ss}
- $\dot{y}(t) = -K_P y(t) + K_P y_0 + d_{ss}$
- Steady state output $y(t) = y_{ss} = \frac{d_{ss}}{K_P} + y_0$
- Transient behavior
  - $y(t) = y(0)e^{-t/T} + y_{ss}\left(1 - e^{-\frac{t}{T}}\right), T = 1/K_P$
- To make steady state error small we can increase K_P at