# Lecture 10: Control II

Professor Katie Driggs-Campbell

February 20, 2024

ECE484: Principles of Safe Autonomy

# Administrivia

- Team formation due this week
- Upcoming due dates:
  - HW1 and MP1 due Friday 2/23
  - HW2 and MP2 due Friday 3/01
  - Project Pitches in class 3/05 and 3/07
- Bonus MP in April

# Today's Plan

- Review some differential equations and linear algebra
- Take a look at PID controllers
- Build up waypoint following using the models discussed previously

# Dynamical Systems Model

Describe behavior in terms of instantaneous laws:

$$\frac{dx(t)}{dt} = \dot{x}(t) = f\big(x(t), u(t)\big) \qquad // \; x[t+1] = \bar{f}(x[t], u[t])$$

where $t \in \mathbb{R}, x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m$, and $f: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ gives the dynamics / transition function

$\dot{x} = Ax + Bu$ if $u = -Kx$

$\rightarrow \dot{x} = \underbrace{(A - kB)}_{\tilde{A}}x$

# Recall (1)

$\dot{x} = Ax \leadsto x(t) = e^{At} x_0$

what does it mean when we say a fn $x(\cdot)$ satisfies a diff eq?

① take fn $x(\cdot)$
② differentiate it
③ plug into $f(x(t))$
④ does it match $\forall t$?

ex | $x(t) = e^{at}$

does it satisfy $\dot{x} = ax$?

② $\dot{x} = a e^{at}$

③ $f(x(t)) = a x(t)$
$= a e^{at}$

④ ✓

# Recall (2)

system is <u>stable</u> if the real parts of the eigenvalues are negative

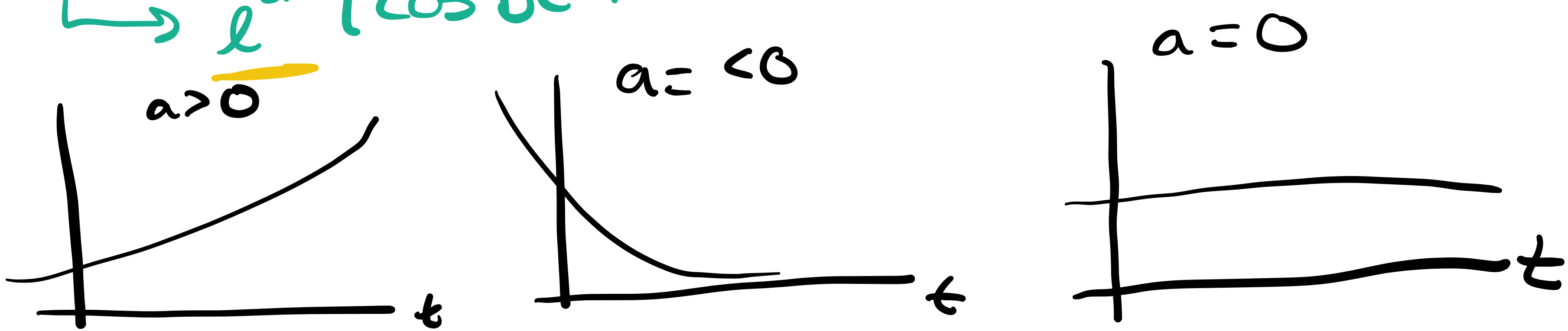
$\rightarrow$ unstable if $\exists\, x_0$ s.t. $\lim_{t\to\infty} \|x(t)\| = \infty$

$e^{\lambda t}$ where $\lambda = a + bi \in \mathbb{C}$

$\hookrightarrow e^{at}(\cos bt + i \sin bt)$

find $\lambda$ w/ characteristic polynomial:

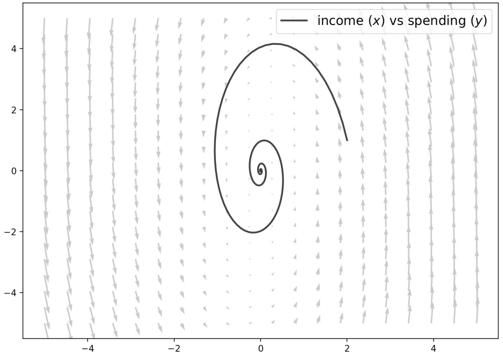
$\det(A - \lambda I) = 0$

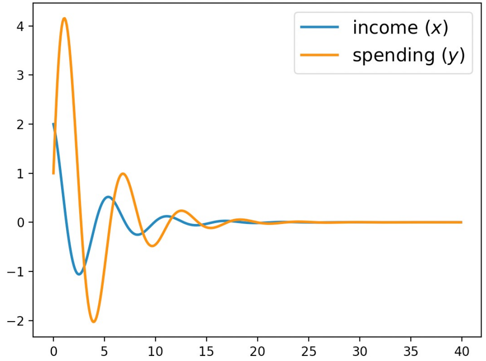


# Examples $\dot{x} = Ax$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-1/4 & -2/5\\ 3 & -1/4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$\lambda_1 = -0.25 - i1.10$
$\lambda_2 = -0.25 + i1.10$



# Examples

$$\begin{bmatrix} \dot{x_2} \\ \dot{x_1} \end{bmatrix} = \begin{bmatrix} -1/4 & -2/5 \\ 3 & -1/4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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$\lambda_1$=+i0.1066
$\lambda_2$=-i0.1066

$$\begin{bmatrix} \dot{x_2} \\ \dot{x_1} \end{bmatrix} = \begin{bmatrix} 1/2 & -2/5 \\ 3 & -1/4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\lambda_1$=0.125+i1.029
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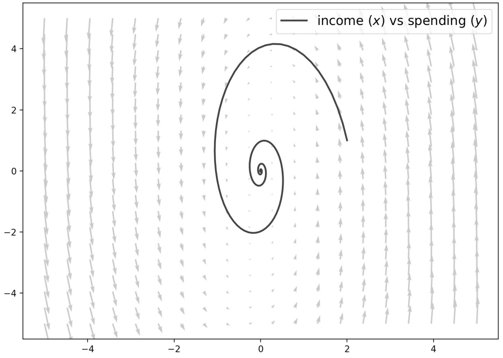


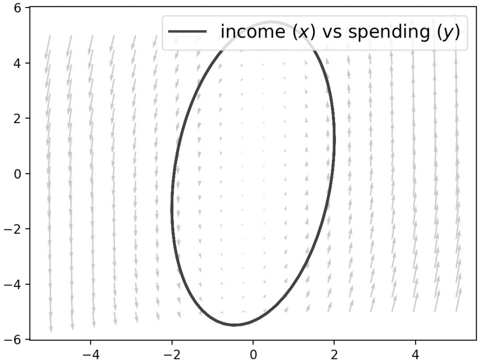


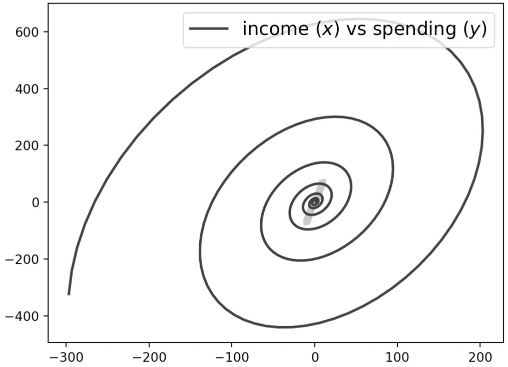


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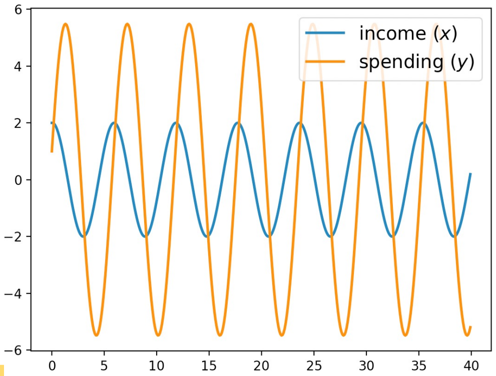


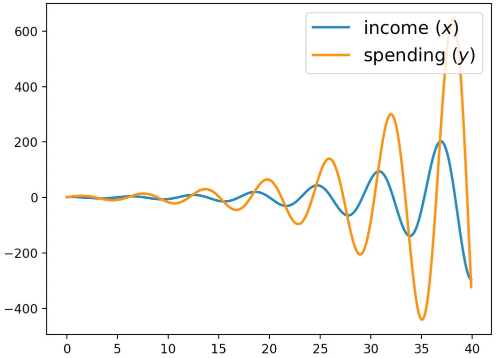

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$\lambda_1$=−0.375−i1.088
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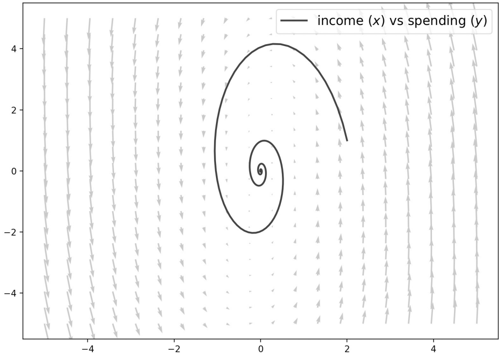


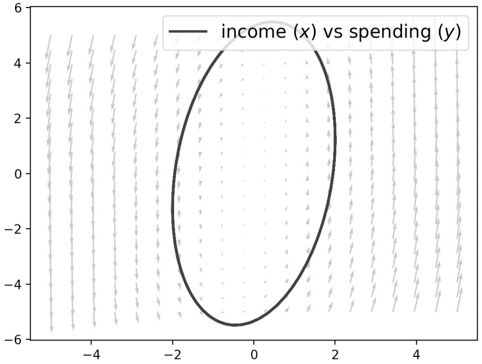


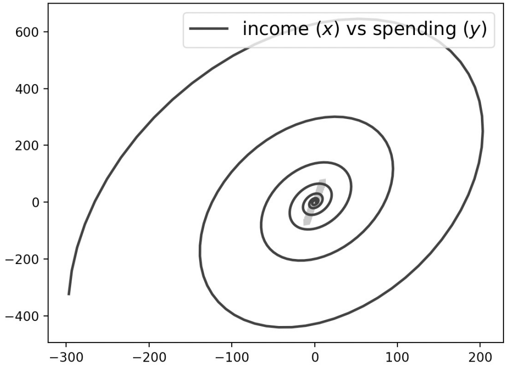


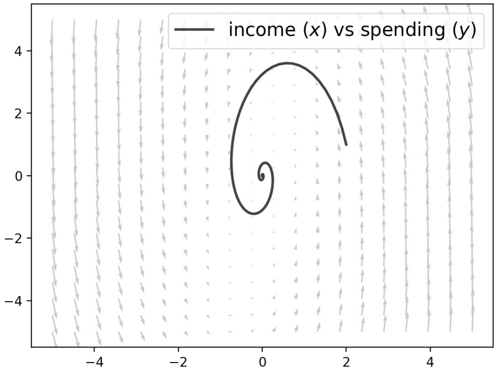



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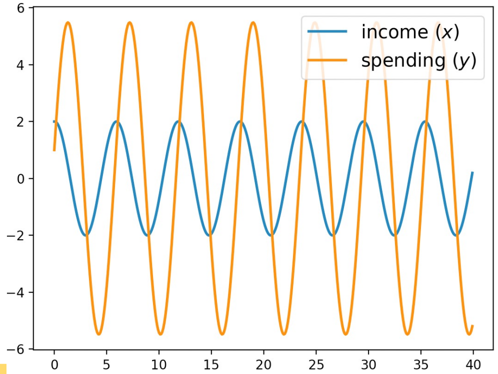
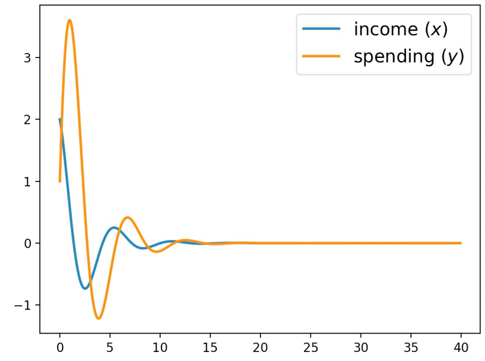
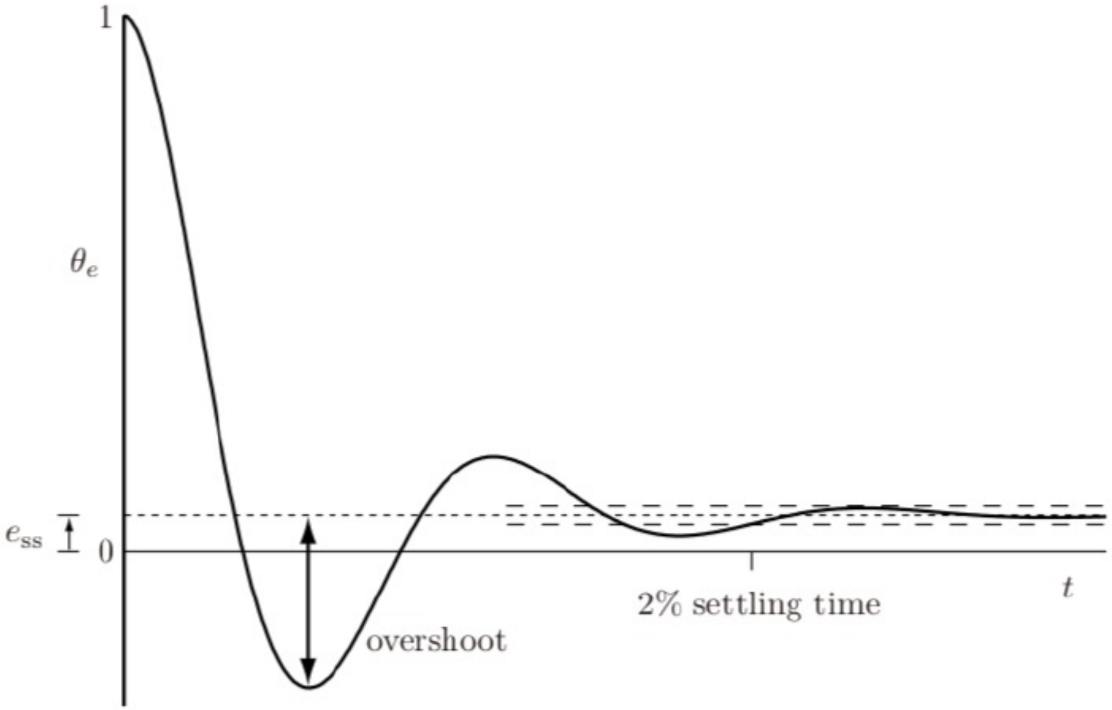


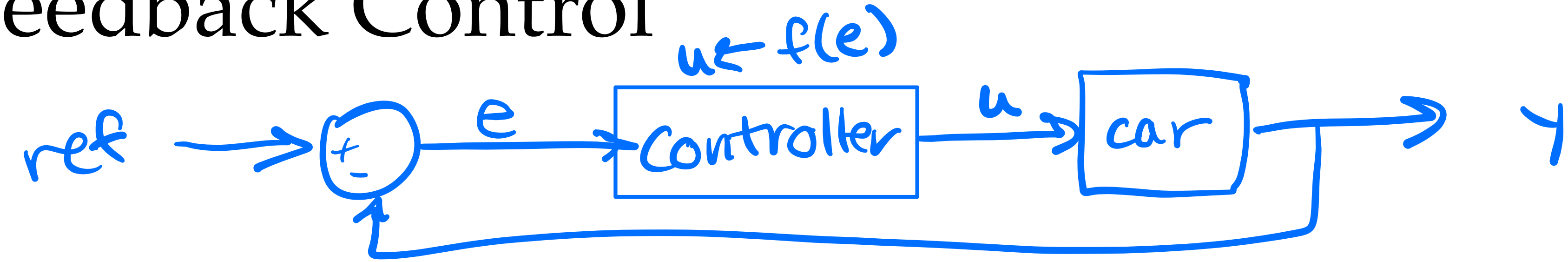




# Error Dynamics



# Feedback Control

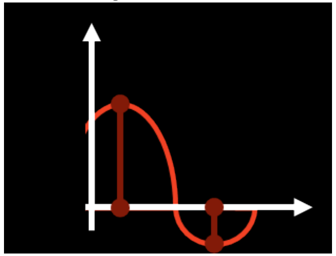



## PID controller

$$u = \underbrace{k_p e}_{\text{reduce pos error}} + \underbrace{k_d \dot{e}}_{\text{dampen vel error}} + \underbrace{k_i \int e(\tau)\,d\tau}_{\text{remove steady state or accumulated error}}$$

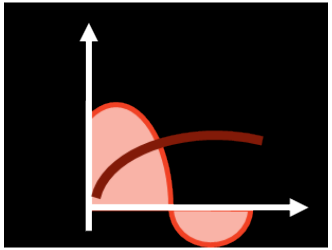
# PID Controllers

- Proportional



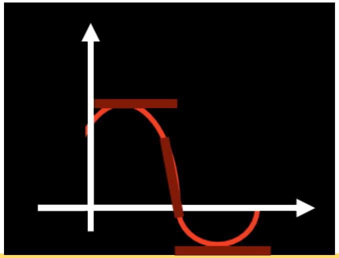
$$u = k_p e$$

- Integral

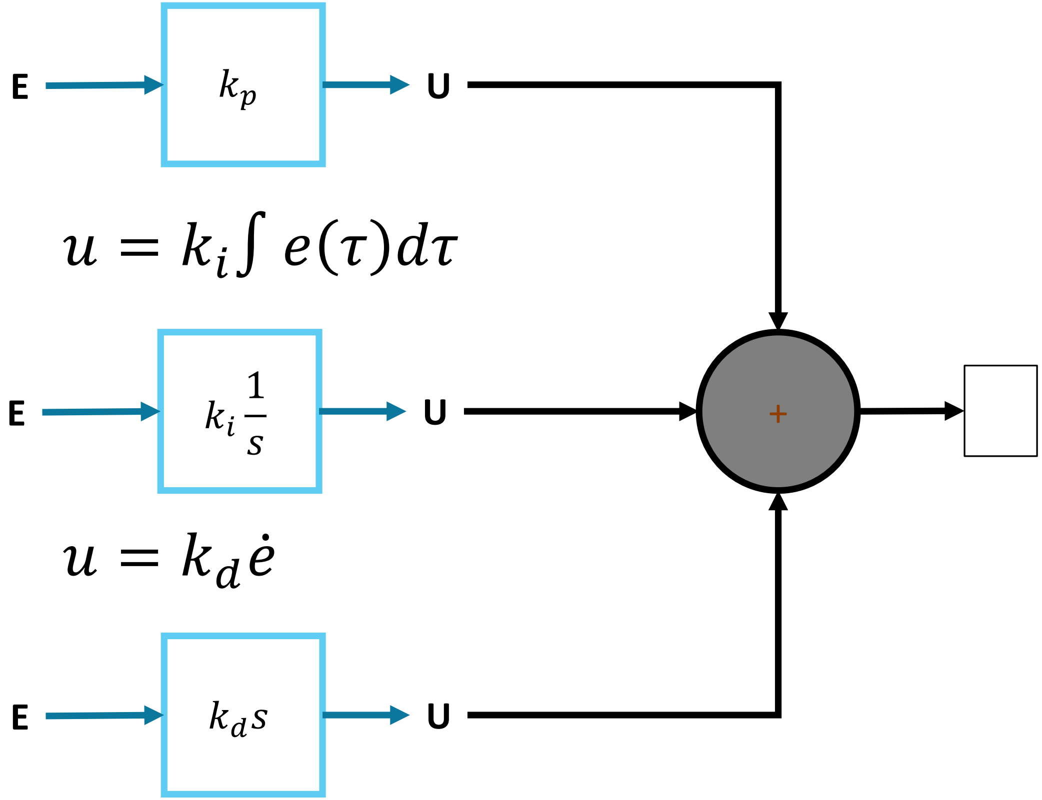


$$u = k_i \int e(\tau) d\tau$$

- Derivative



$$u = k_d \dot{e}$$



# Linear Error Dynamics

$$a_p e^{(p)} + a_{p-1} e^{(p-1)} + \cdots + a_1 \dot{e} + a_0 e = 0$$

$$x_1 = e, \quad x_2 = \dot{e} = \dot{x}_1, \quad x_3 = \ddot{e} = \dot{x}_2, \quad \cdots$$

$$x_p = -a_0/a_p \, x_1 - a_1/a_p \, x_2 - \cdots$$

$$\downarrow$$

$$\dot{x} = Ax, \quad \text{where } x = [x_1 \cdots x_p]^T$$

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_p \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ & & & & \\ -a_0/a_p & \cdots & & & -a_{p-1}/a_p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_p \end{bmatrix}$$

# Viewing as a Second Order System

- The second order system is: $\ddot{e} + c_1\dot{e} + c_2 e = 0$
- In standard form, we write:

$$\ddot{e}(t) + 2\xi\omega_n\dot{e}(t) + \omega_n^2 e(t) = 0$$

where $\xi$ is the *damping ratio* and $\omega_n$ is the *natural frequency*

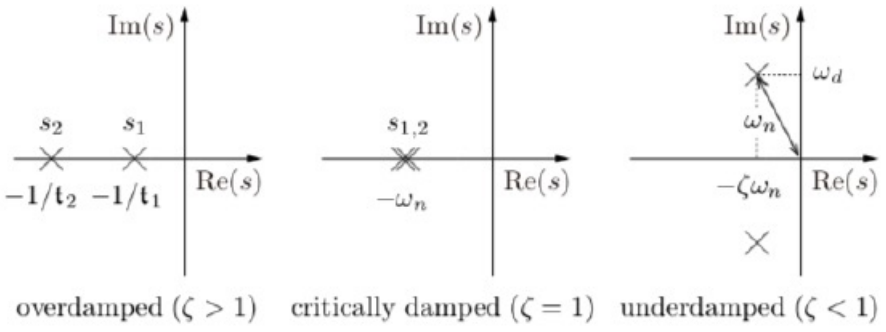
- The eigenvalues are given as:

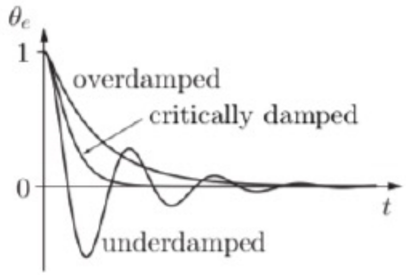
$$\lambda_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

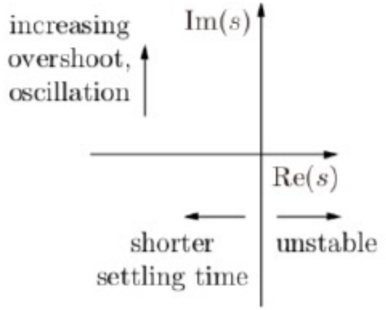
- Note that the system is stable iff $\omega_n$ and $\xi$ are positive

# Second Order Dynamics: Cases

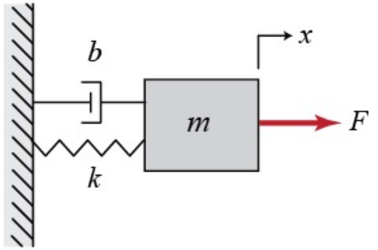
- Overdamped: $\zeta > 1$
  - Roots $s_1$ and $s_2$ are distinct
  - $\theta_e(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$
  - Time constant is the less negative root
- Critically damped: $\zeta = 1$
  - Roots $s_1$ and $s_2$ are equal and real
  - $\theta_e(t) = (c_1 + c_2 t)e^{-\omega_n t}$
  - Time constant is given by $1/\omega_n$
- Underdamped: $\zeta < 1$
  - Roots are complex conjugates:
    $$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$
  - $\theta_e(t) = (c_1 \cos\omega_d t + c_2 \sin\omega_d t)e^{-\zeta\omega_n t}$



# Simple Damped Spring System




$$m\,\ddot{x} + b\dot{x} + kx = F$$

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = u$$

$$\ddot{x} + 2\xi\omega_0\dot{x} + \omega_0^2 x = u$$

$\xi$ damping ratio

$\omega_0$ natural frequency

$$m\,\ddot{x} + b\dot{x} + kx = u$$

$$\mathcal{L}\{m\ddot{x} + b\dot{x} + kx\} =$$

$$ms^2X(s) + bsX(s) + kX(s)$$

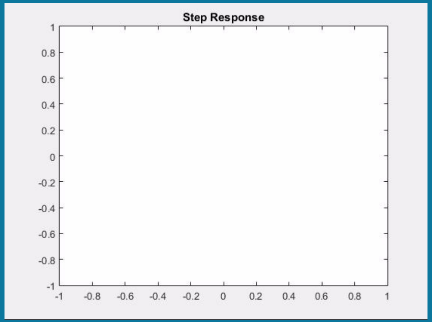
Transfer Function:

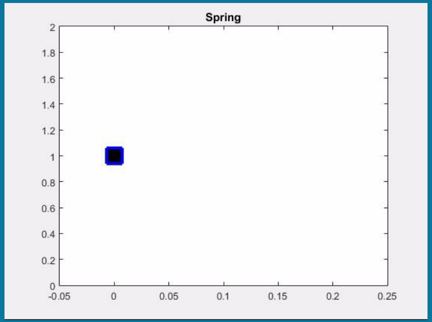
$$\frac{X(s)}{U(s)} = \frac{1}{ms^2 + bs + k}$$

Poles:

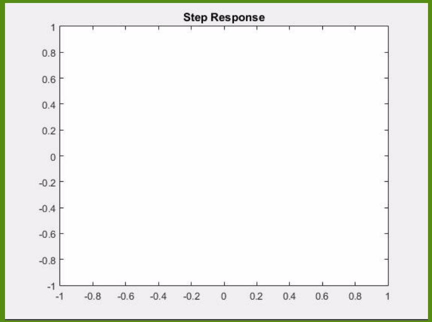
$$s = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

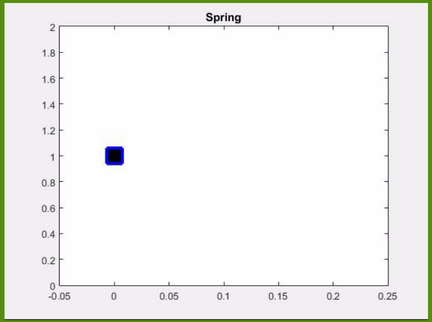
## Undamped Case: $b = 0$



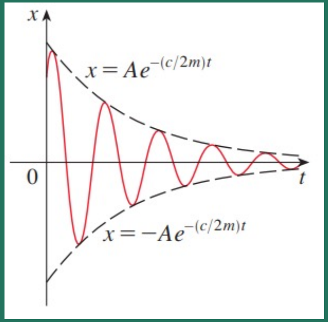



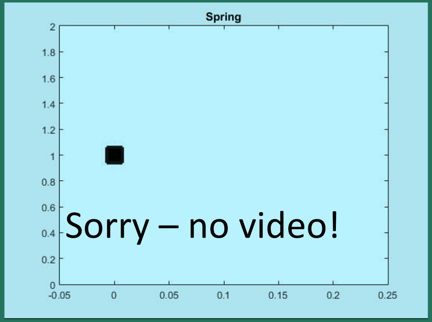

## Overdamped Case: $b^2 - 4mk > 0$



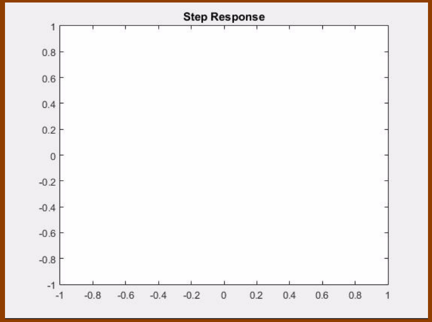



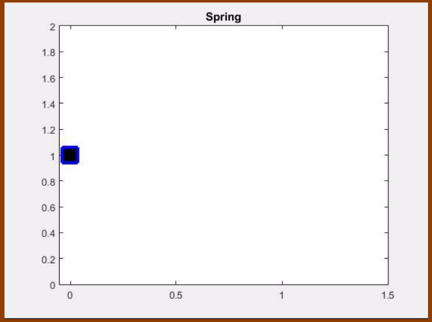

## Underdamped Case: $b^2 - 4mk < 0$




## With Feedback Control

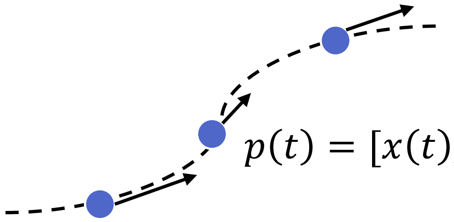


# On to PID for path following

# Path following control

- The path followed by a robot can be represented by a *trajectory or path* parameterized by time
  - → from a higher-level planner, map, or perception system
- Defines the desired instantaneous pose $p(t)$



$$p(t) = [x(t), y(t), \theta(t)]$$

# Open-loop waypoint following

- We can write an open-loop controller for a robot that is naturally controlled via angular velocity, such as a differential-drive robot:

$$u_{\omega,OL}(t) = \begin{bmatrix} v(t) \\ \omega(t) \end{bmatrix} = \begin{bmatrix} \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2} \\ \dot{\theta}(t) \end{bmatrix}$$

- We can write an open-loop controller for a robot with car-like steering:

$$u_{\kappa,OL}(t) = \begin{bmatrix} v(t) \\ \kappa(t) \end{bmatrix} = \begin{bmatrix} \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2} \\ \dfrac{\dot{\theta}(t)}{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}} \end{bmatrix}$$

# Path following control

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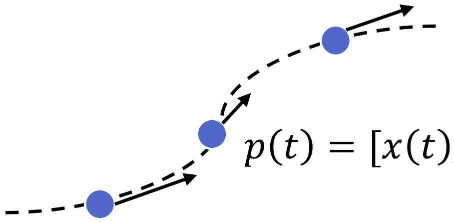
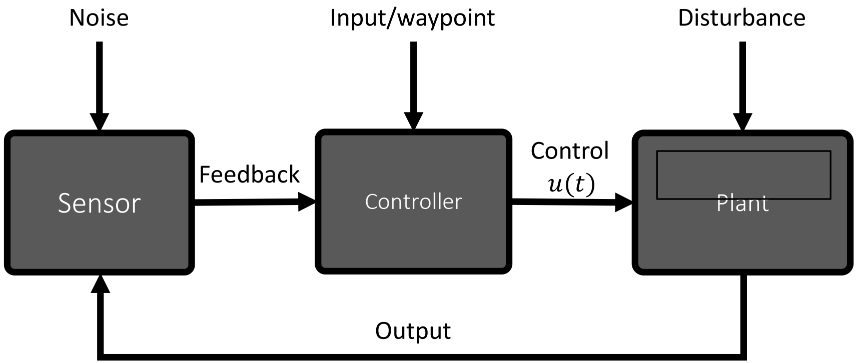


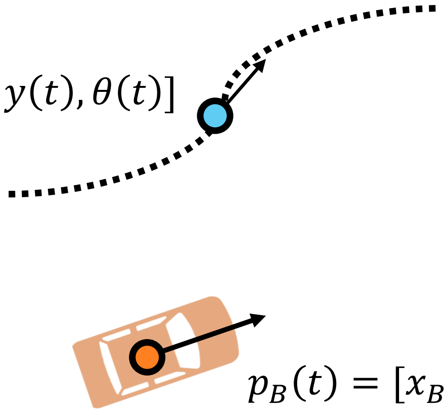
$$p(t) = [x(t), y(t), \theta(t)]$$



# Path following control

- Desired instantaneous pose $p(t)$
- How to define error between actual pose $p_B(t)$ and desired pose $p(t)$ in the form of $y_d(t) - y(t)$?

$p(t) = [x(t), y(t), \theta(t)]$



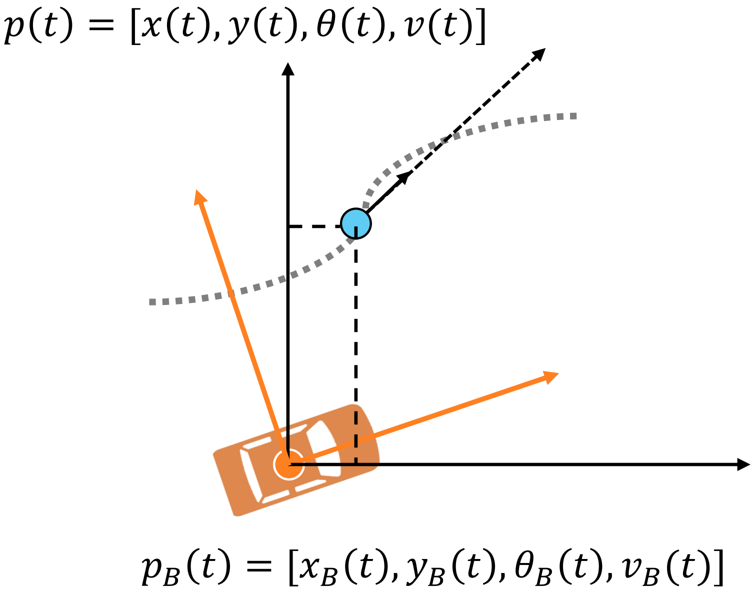
$p_B(t) = [x_B(t), y_B(t), \theta_B(t)]$

# Path following control

The error vector measured vehicle coordinates

$$e(t) = [\delta_s(t), \delta_n(t), \delta_\theta(t), \delta_v(t)]$$

$[\delta_s, \delta_n]$ define the coordinate errors in the vehicle's reference frame: along track error and cross track error



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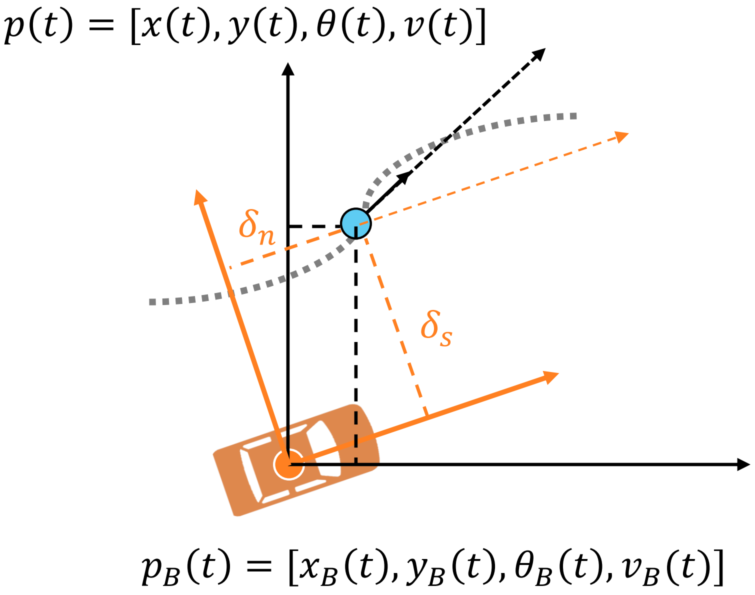
- Along track error: distance ahead or behind the target in the instantaneous direction of motion.
  $$\delta_s = \cos(\theta_B(t))\,(x(t) - x_B(t)) + sin(\theta_B(t))\,(y(t) - y_B(t))$$
- Cross track error: portion of the position error orthogonal to the intended direction of motion
  $$\delta_n = -\sin(\theta_B(t))\,(x(t) - x_B(t)) + cos(\theta_B(t))\,(y(t) - y_B(t))$$



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- Along track error: distance ahead or behind the target in the instantaneous direction of motion.

$$\delta_s = \cos\big(\theta_B(t)\big)\big(x(t) - x_B(t)\big) + sin\big(\theta_B(t)\big)\big(y(t) - y_B(t)\big)$$

- Cross track error: portion of the position error orthogonal to the intended direction of motion

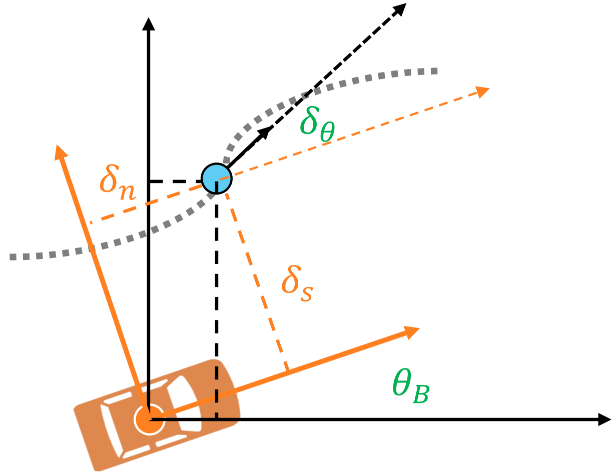
$$\delta_n = -\sin\big(\theta_B(t)\big)\big(x(t) - x_B(t)\big) + cos\big(\theta_B(t)\big)\big(y(t) - y_B(t)\big)$$

- Heading error: difference between desired and actual orientation and direction

$$\delta_\theta = \theta(t) - \theta_B(t)$$
$$\delta_v = v(t) - v_B(t)$$

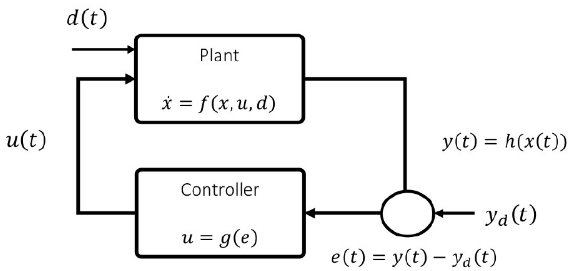
→ Each of these errors match the form $y_d(t) - y(t)$

$p(t) = [x(t), y(t), \theta(t), v(t)]$




$p_B(t) = [x_B(t), y_B(t), \theta_B(t), v_B(t)]$

# A simple P-controller example




- Given a simple system: $\dot{y}(t) = u(t) + d(t)$
- Using proportional (P) controller:

$$u(t) = -K_P e(t) = -K_P(y(t) - y_d(t))$$

$$\dot{y}(t) = -K_P y(t) + K_P y_d(t) + d(t)$$

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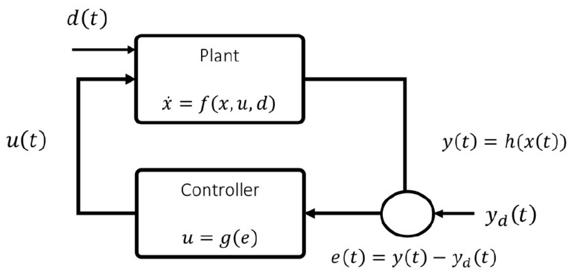


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- Consider constant setpoint $y_0$ and disturbance $d_{ss}$

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- What is the steady state output?
  - Set: $-K_P y(t) + K_P y_0 + d_{ss} = 0$
  - Solve for $y_{ss}$: $y(t) = \frac{d_{ss}}{K_P} + y_0$

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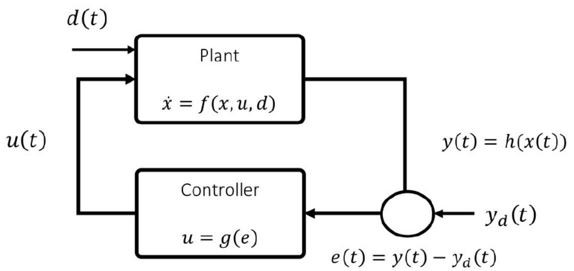


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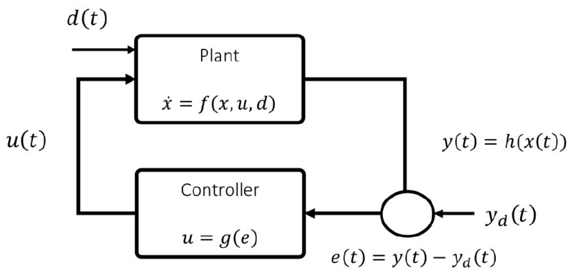


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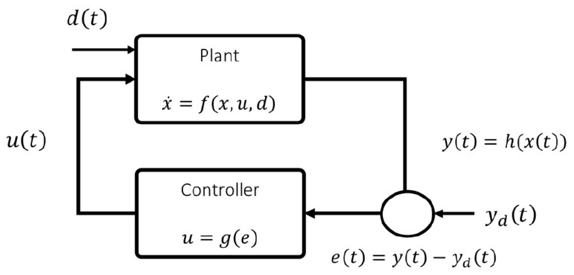


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- Steady state output $y_{ss} = \frac{d_{ss}}{K_P} + y_0$
- Transient behavior:

$$y(t) = y_0 e^{-t/T} + y_{ss}\left(1 - e^{-t/T}\right), T = 1/K_P$$

- To make steady state error small, we can increase $K_P$ at the expense of longer transients

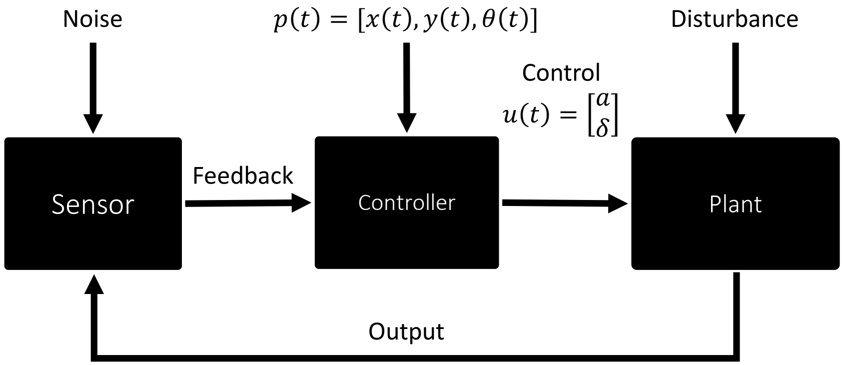
# Control Law

Control input is given by $u = [a, \delta]^{\mathsf{T}}$

where $a$ is the acceleration and $\delta$ is the steering angle

$$u = K \begin{bmatrix} \delta_s \\ \delta_n \\ \delta_\theta \\ \delta_v \end{bmatrix}$$

$$K = \begin{bmatrix} K_s & 0 & 0 & K_v \\ 0 & K_n & K_\theta & 0 \end{bmatrix}$$

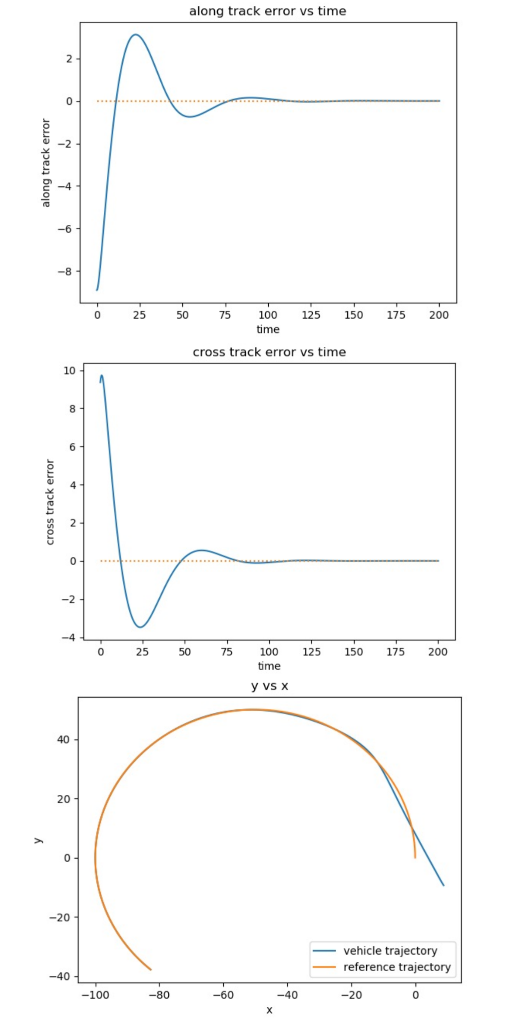


# Control Law

$$K = \begin{bmatrix} K_s & 0 & 0 & K_v \\ 0 & K_n & K_\theta & 0 \end{bmatrix}$$

The pure-pursuit controller produced by this gain matrix performs a PD-control. It uses a PD-controller to correct along-track error.

The control on curvature is also a PD-controller for cross-track error because $\delta_\theta$ is related to the derivative of $\delta_n$.



# Summary

- Reviewed linear systems and stability of differential equations
- Looked at PID controllers as a way to regulate systems using state feedback
- Derived a waypoint following error dynamics
  - →This will be needed for MP2!
- *Next time:* Advanced Control Topics!