Sampling-based Planning and Control
Lecture 18

Principles of Safe Autonomy ECE484

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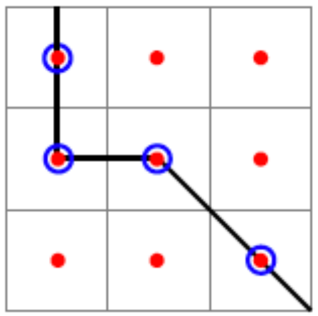
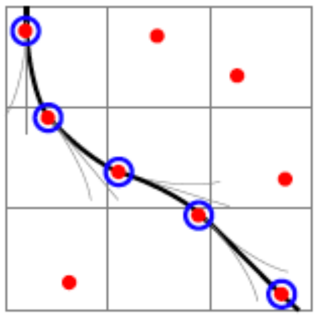
Announcements

- Guest lecture: Nov 9th Dr. Qinru Li, Waymo, ECE Alumni (must attend)
- Intermediate check-in with TAs
  - How much progress have you made?
  - How well is the team working?
- Midterm 2: Nov 14th
  - Filtering and planning
  - Midterm 2 Review: Nov 7th

Motion planning problem

- Get from point A to point B avoiding obstacles
- Last 2 lectures we saw how to search for collision free trajectories can be converted to graph search
  - Each vertex represents a region of the gridded state space; edges between centers
    - Paths may not be realizable
  - Hybrid A* constructs dynamically feasible paths
    - edges between arbitrary points in grid regions
    - Not guaranteed to be complete
  - Grid/discretization does not scale to high-dimensional state spaces
- Today: sampling-based motion planning
  - Can directly incorporate dynamical constraints
  - Scales to higher dimensions
  - Cons: Probabilistic completeness

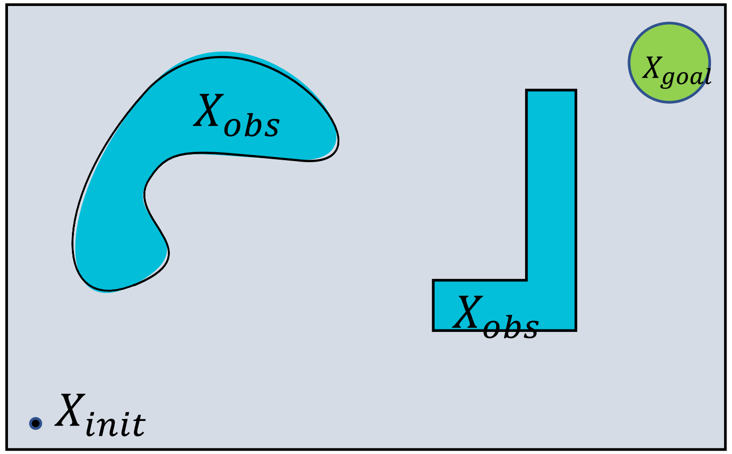




Motion planning problem

Consider a dynamical control system defined by an ODE of the form $\frac{dx}{dt} = f\,(x, u),\ \ x(0) = x_{init}.\quad (1)$

where x is the state, u is the control. Given an obstacle set $X_{obst} \subset R^d$, and a goal set $X_{goal} \subset R^d$, the objective of the motion planning problem is to find, if it exists, a control signal u such that the solution of (1) satisfies



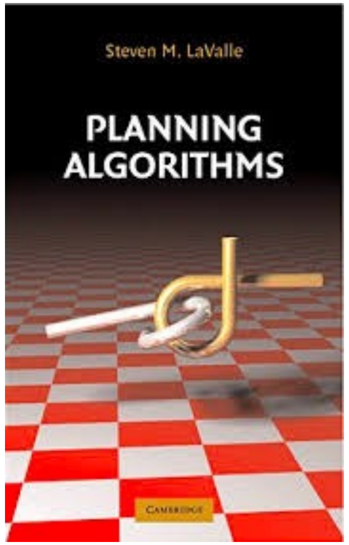
- for all $t \in R_{\geq 0}, x(t) \notin Xobs$, and
- for some finite T ≥ 0, for all t > T , $x(t) \in X_{goal}$, 0
- Return failure if no such control signal exists.

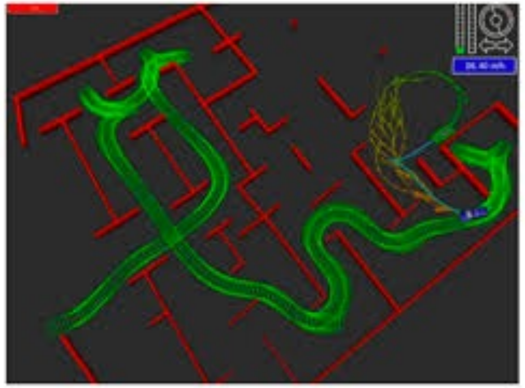
Canonical problem

Basic problem in robotics

- Autonomous vehicles
- Puzzles

Provably very hard: a basic version (the Generalized Piano Mover's problem) is known to be PSPACE-hard [Reif, '79].





Types of planners

Discretization + graph search: Analytic/grid-based methods do not scale well to high dimensions.

- $A*$, $D*$, etc. can be sensitive to graph size. Resolution complete.

Algebraic planners: Explicit representation of obstacles.

- Use complicated algebra (visibility computations/projections) to find the path. Complete, but often impractical.

Potential fields/navigation functions: Virtual attractive forces towards the goal, repulsive forces away from the obstacles.

- No completeness guarantees, unless “navigation functions” are available—very hard to compute in general.

Sampling-based algorithms

Solutions are computed based on samples from some distribution.

Retain some form of completeness, e.g., probabilistic completeness

Incremental sampling methods

- Lend themselves to real-time, on-line implementations
- Can work with very general dynamics
- Do not require explicit constraints

Outline

Probabilistic Roadmaps

Rapidly expanding random trees (RRT)

RRG

Probabilistic RoadMaps (PRM)

Introduced by Kavraki and Latombe in 1994

Mainly geared towards "multi-query" motion planning problems

Idea: build (offline) a graph (i.e., the roadmap) representing the "connectivity" of the environment; use this roadmap to figure out paths quickly at run time.

Learning/pre-processing phase:

- Sample n points from $X_{free} = [0, 1]^d \setminus X_{obs}$
- Try to connect these points using a fast "local planner"
- If connection is successful (i.e., no collisions), add an edge between the points.

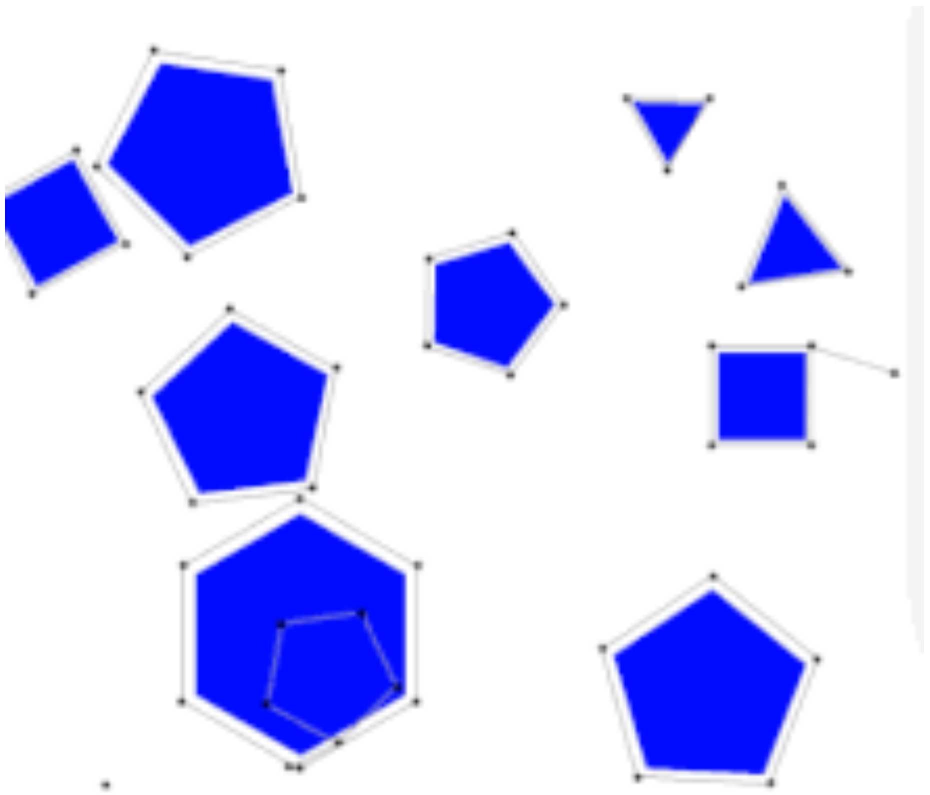
At run time:

- Connect the start and end goal to the closest nodes in the roadmap
- Find a path on the roadmap, e.g., using BFS, DFS, A*

First planner ever to demonstrate the ability to solve general planning problems in > 4-5 dimensions!

PRM in action

Kavraki, L. E.; Svestka, P.; Latombe, J.-C.; Overmars, M. H. (1996), "Probabilistic roadmaps for path planning in high-dimensional configuration spaces", IEEE Transactions on Robotics and Automation, **12** (4): 566–580



Picture from Wikipedia.org
https://en.wikipedia.org/wiki/Probabilistic_roadmap

Simple PRM construction

$V \leftarrow \{x_{init}\} \cup \{\text{SampleFreei}\}_{i=1,\dots,N-1}$

$E \leftarrow \emptyset$

foreach $v \in V$ **do**

  $U \leftarrow \text{Near}(G = (V, E), v, r) \setminus \{v\}$

    foreach $u \in U$ **do**

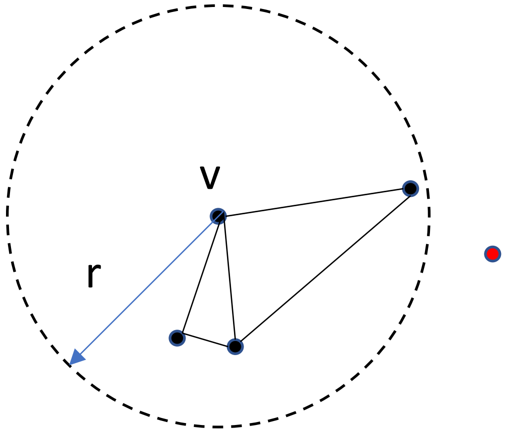
      if CollisionFree(v, u) **then**

        $E \leftarrow E \cup \{(v, u),(u, v)\}$

return $G = (V, E)$

Near(G, v, r): Finds the subset of vertices in G that are within r distance of v

CollisionFree(v, u): checks whether there is a path from u to v that does not collide with the obstacles



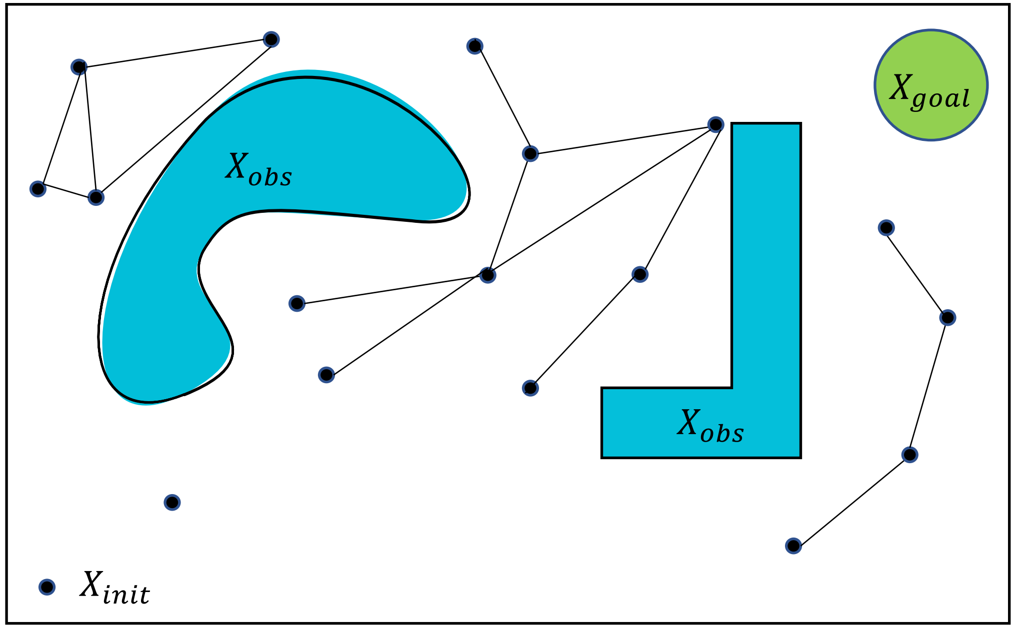
Probabilistic RoadMap

Connect points within a radius r, starting from “closest” ones

Do not attempt to connect points already on the same connected component of PRM

What properties does this algorithm have?

- Will it find a solution if one exists?
- Is this an optimal solution?
- What is the complexity?



Robustness and Probabilistic completeness

Definition. A motion planning problem $P = (X_{free}, x_{init}, X_{goal})$ is ***robustly feasible*** if there exists some small δ>0 such that a solution remains a solution if obstacles are “dilated” by δ.

Definition. An algorithm ALG is ***probabilistically complete*** if, for any *robustly feasible* motion planning problem defined by $P = (X_{free}, x_{init}, X_{goal})$, $\lim_{N \to \infty} \Pr(ALG\ returns\ a\ solution\ to\ P) = 1.$

- Applicable to motion planning problems with a robust solution.

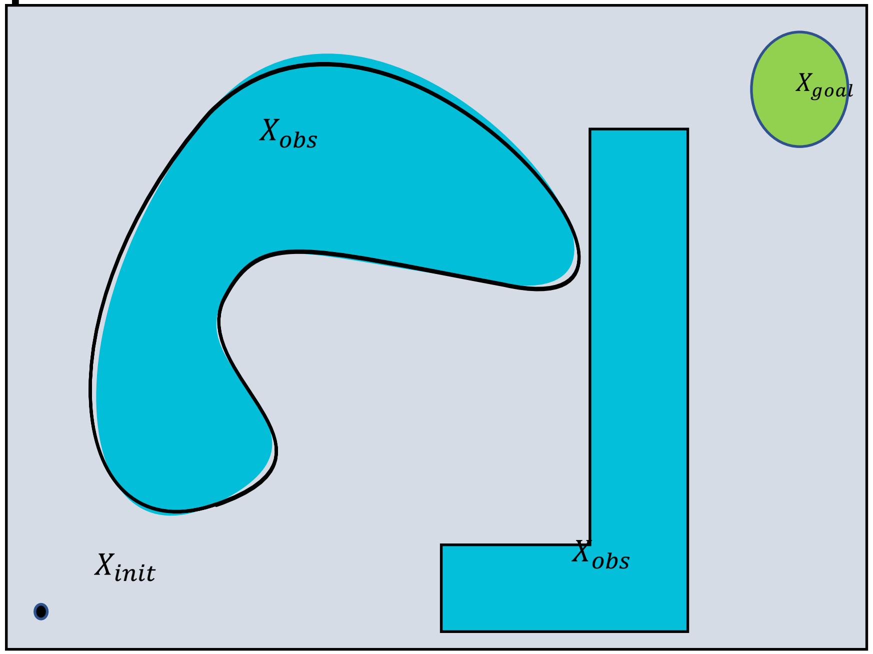


Fig. not robustly feasible.

Paper: Sampling-based Algorithms for Optimal Motion Planning, Sertac Karaman Emilio Frazzoli

Asymptotic optimality of sampling-based algorithms

Suppose we have a cost function c that associates to each path $\boldsymbol{\sigma}$ a non-negative cost $c(\sigma)$, e.g., $c(\sigma) = \int_{\sigma} \chi(s)\, ds$.

$Y_i^{ALG} = c(\sigma_i)$ *Cost of the output path* $\boldsymbol{\sigma_i}$ *from ALG with i samples*

Definition. An algorithm ALG is *asymptotically optimal* if, for any motion planning problem $P = (X_{free}, x_{init}, X_{goal})$ and cost function c that admits a robust optimal solution with finite cost c^*,

$$\boldsymbol{P}\left(\left\{\lim_{i\to\infty} Y_i^{ALG} = c^*\right\}\right) = 1\text{, } \textit{where}$$

Properties of PRM

The simplified version of the PRM (sPRM) algorithm has been shown to be probabilistically complete. (No proofs available for the “real” PRM!)

Moreover, the probability of success goes to 1 exponentially fast, if the environment satisfies certain “good visibility” conditions.

But, NOT asymptotically optimal

- Edges make unnecessary connections in a connected component
- Set of optimal paths has measure 0

New key concept: combinatorial complexity vs. “visibility”

Complexity of Sampling-based Algorithms

How can we measure complexity for an algorithm that does not necessarily terminate?

- Treat the number of samples as "the size of the input." (Everything else stays the same)
- Complexity per sample: how much work (time/memory) is needed to process one sample.
- Useful for comparison of sampling-based algorithms. Not for deterministic, complete algorithms.

Complexity of PRM for N samples $\Theta(N^2)$

Practical complexity reduction tricks

- k-nearest neighbors: connect to the k nearest neighbors. Complexity Θ(N log N). (Finding nearest neighbors takes log N time.)
- Bounded degree: connect at most k neighbors among those within radius r.
- Variable radius: change the connection radius r as a function of N. How?

Rapidly Exploring Random Trees (RRT)

Introduced by LaValle and Kuffner in 1998

Appropriate for single-query planning problems

Idea: build (online) a tree, exploring the region of the state space that can be reached from the initial condition.

At each step: sample one point from X_{free}, and try to connect it to the closest vertex in the tree.

Very effective in practice

RRT

LaValle, Steven M.; Kuffner Jr., James J. (2001). "Randomized Kinodynamic Planning" (PDF). *The International Journal of Robotics Research (IJRR)*. **20** (5): 378–400. doi:10.1177/02783640122067453. S2CID 4047 9452.

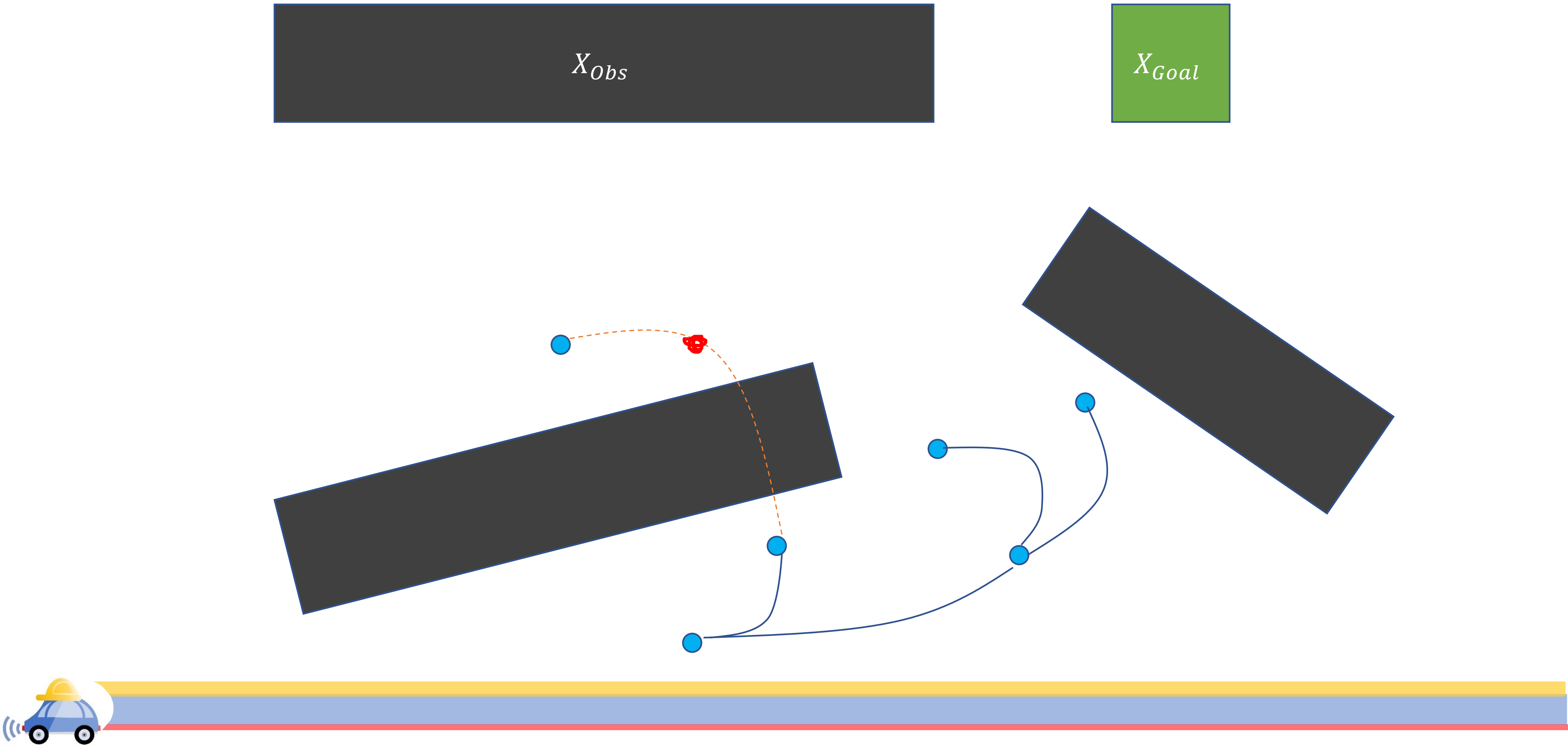
RRT

$V \leftarrow \{x_{init}\}$; $E \leftarrow \emptyset$
for $i = 1, \ldots, N$ **do**
  $x_{rand} \leftarrow \text{SampleFree}_i$
  $x_{nearest} \leftarrow \text{Nearest}(G = (V, E), x_{rand})$
  $x_{new} \leftarrow \text{Steer}(x_{nearest}, x_{rand})$
  if ObtacleFree($x_{nearest}$, x_{new}) **then**
    $V \leftarrow V \cup \{x_{new}\}$
    $E \leftarrow E \cup \{(x_{nearest}, x_{new})\}$
return $G = (V, E)$

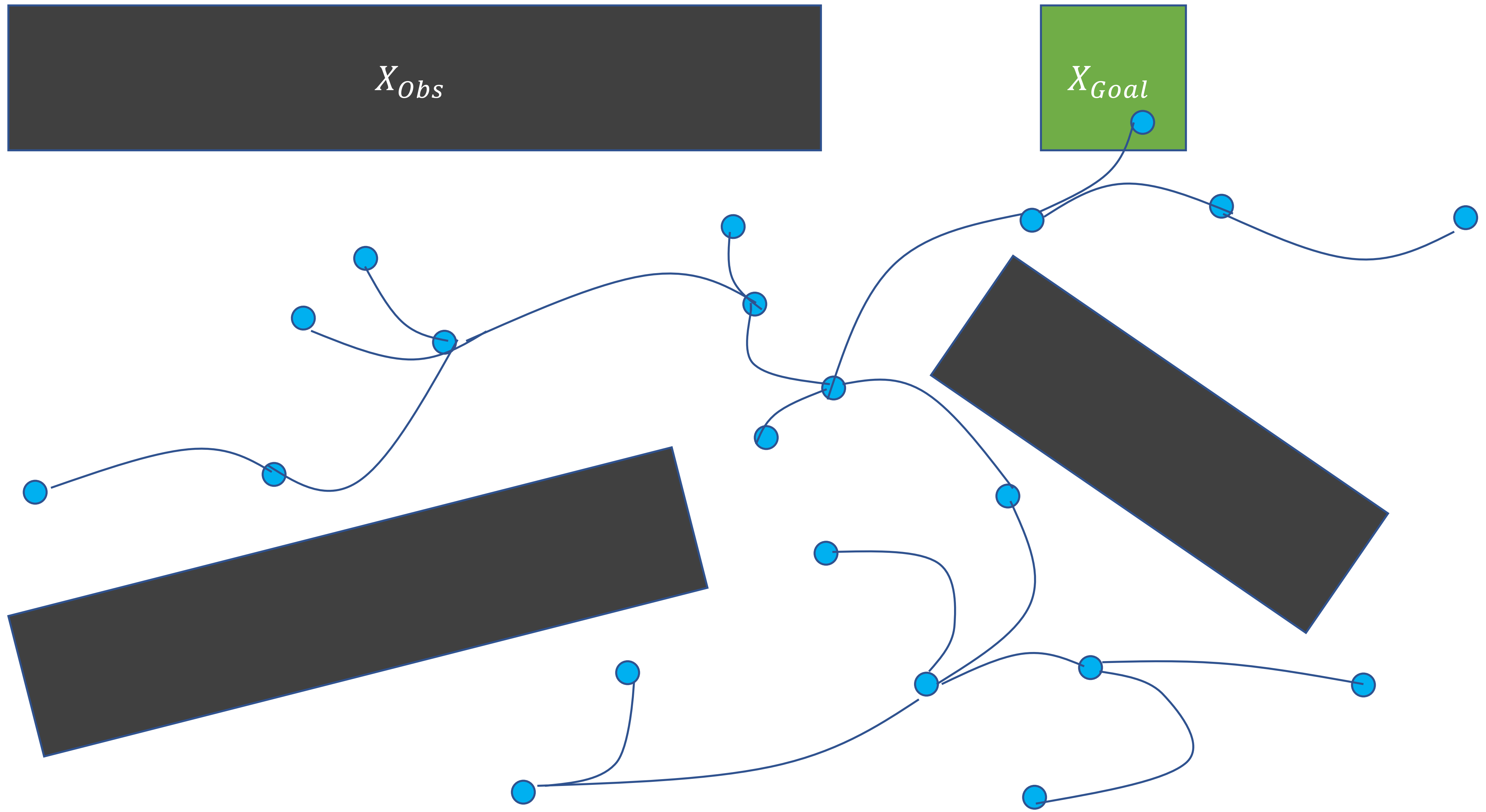
Nearest(G, v, r): Finds the nearest vertex in G from x_{rand}

Steer(x_0, x_g): Tries to drive the robot from x_0 to x_g and returns the point nearest to x_g that it could reach

ObstacleFree(x_0, x_g): Checks whether the path from x_0 to x_g is obstacle free



X_{Obs}
X_{Goal}



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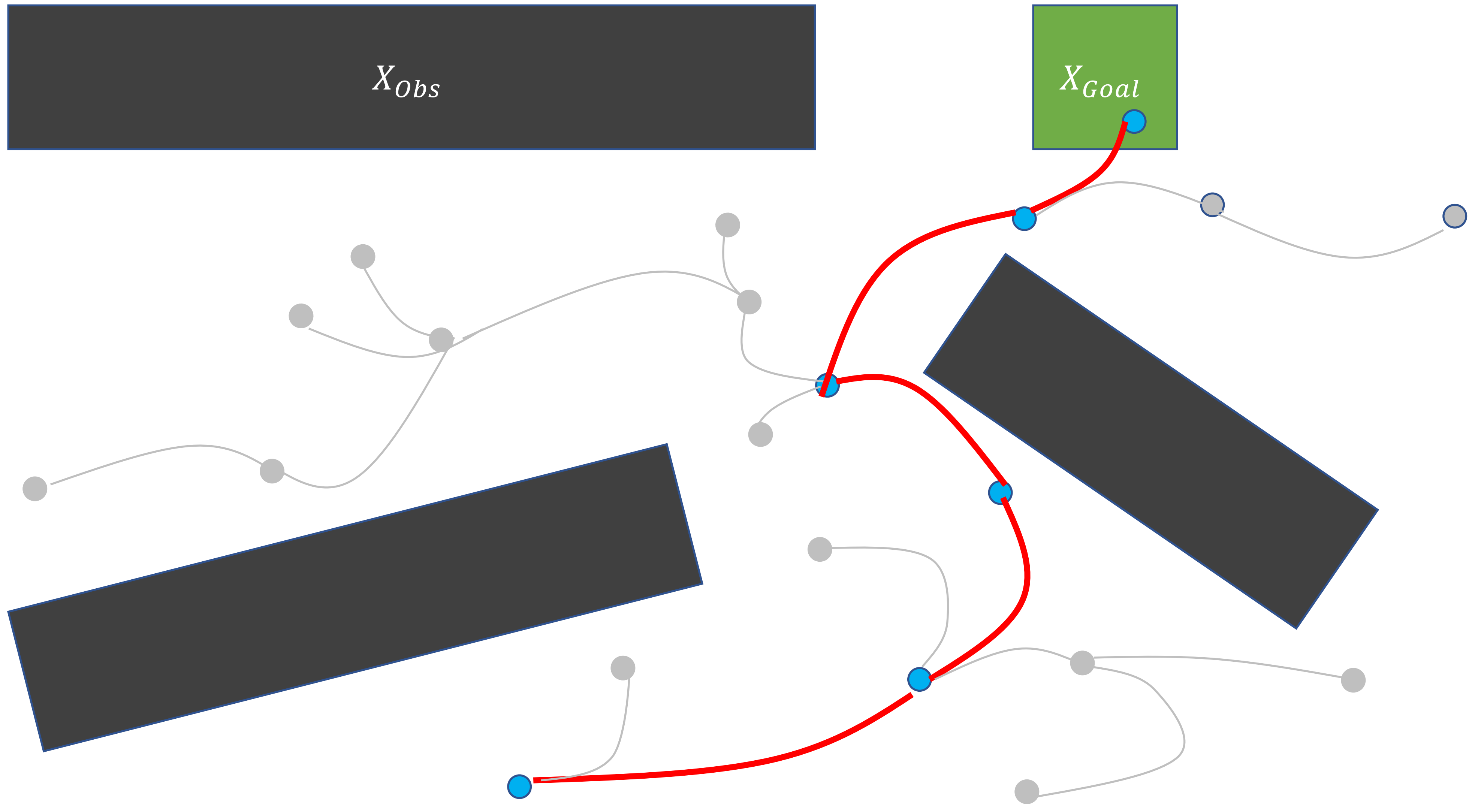
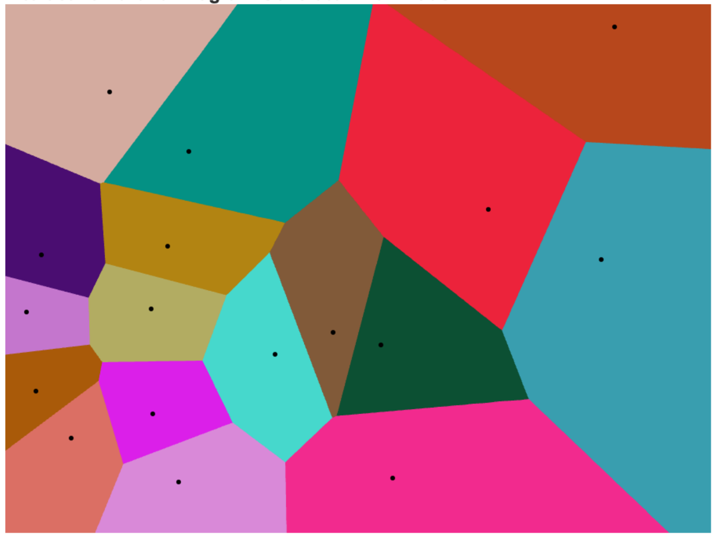

X_{Obs}
X_{Goal}

Voronoi bias



Given n points in d dimensions, the ***Voronoi diagram*** of the points is a partition of R^d into regions, one region per point, such that all points in the interior of each region lie closer to that regions site than to any other site.

Try it: http://alexbeutel.com/webgl/voronoi.html

Voronoi bias. Vertices of the RRT that are more "isolated" (e.g., in unexplored areas, or at the boundary of the explored area) have larger Voronoi regions—and are more likely to be selected for extension.

RRT in action [Frazzoli]

- Talos, the MIT entry to the 2007 DARPA Urban Challenge, relied on an“RRT-like” algorithm for real-time motion planning and control.
- The devil is in the details: provisions needed for, e.g.,
  - Real-time, on-line planning for a safety-critical vehicle with substantial momentum.
  - Uncertain, dynamic environment with limited/faulty sensors.
- Main innovations [Kuwata, et al. ’09]
  - Closed-loop planning: plan reference trajectories for a closed-loop model of the vehicle under a stabilizing feedback
  - Safety invariance: Always maintain the ability to stop safely within the sensing region.
  - Lazy evaluation: the actual trajectory may deviate from the planned one,need to efficiently re-check the tree for feasibility.
- The RRT-based P+C system performed flawlessly throughout the race.
- https://journals.sagepub.com/doi/abs/10.1177/0278364911406761

Limitations

The MIT DARPA Urban Challenge code, as well as other incremental sampling methods, suffer from the following limitations:

- No characterization of the quality (e.g., “cost”) of the trajectories returned by the algorithm.
- Keep running the RRT even after the first solution has been obtained, for as long as possible (given the real-time constraints), hoping to find a better path than that already available.
- No systematic method for imposing temporal/logical constraints, such as, e.g., the rules of the road, complicated mission objectives, ethical/deontic code.
- In the DARPA Urban Challenge, all logics for, e.g., intersection handling, had to be hand-coded, at a huge cost in terms of debugging effort/reliability of the code.

RRTs and Asymptotic Optimality

- RRTs are great at finding feasible trajectories quickly, however, RRTs are apparently terrible at finding good trajectories. Why?
- Let Y^{RRT}_n be the cost of the best path in the RRT at the end of iteration n.
- It is easy to show that Y^{RRT}_n converges (to a random variable), $\lim_{n \to \infty} Y_n^{RRT} = Y_\infty^{RRT}$

where Y_∞^{RRT} is sampled from a distribution with zero mass at the optimum

Theorem [Karaman & Frizzoli`10] (Almost sure *sub-optimality* of RRTs) If the set of sampled optimal paths has measure zero, the sampling distribution is absolutely continuous with positive density in X_{free}, and d ≥ 2, then the best path in the RRT converges to a sub-optimal solution almost surely, i.e.,

$$\Pr[Y_\infty^{RRT} > c^*] = 1.$$

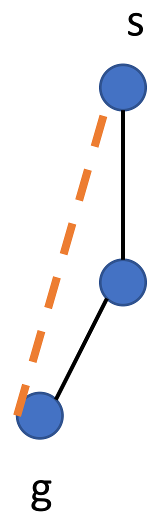
Why is RRT not asymptotically optimal?

Root node has infinitely many subtrees that extend at least a distance ϵ away from x_{init}.

The RRT algorithm "traps" itself by disallowing new better paths to emerge. Why?

Heuristics such as running the RRT multiple times, running multiple trees concurrently etc., work better than the standard RRT, but also result in almost-sure sub-optimality.

A careful rethinking of the RRT algorithm is required for (asymptotic) optimality.



RRT will not connect s to g

Rapidly Exploring Random Graphs (possibly cyclic)

$V \leftarrow \{x_{init}\}$; $E \leftarrow \emptyset$;

for i = 1, . . . , N **do**

$x_{rand} \leftarrow$ SampleFreei;

$x_{nearest} \leftarrow$ Nearest(G = (V, E), x_{rand});

$x_{new} \leftarrow$ Steer($x_{nearest}$, x_{rand}) ;

if ObtacleFree($x_{nearest}$, x_{new}) **then**

$X_{near} \leftarrow$ Near(G = (V, E), x_{new}, min{γ_{RRG}(log(card V)/ card V)$^{1/d}$, η}) ;

$V \leftarrow V \cup \{x_{new}\}$; $E \leftarrow E \cup \{(x_{nearest}, x_{new}),(x_{new}, x_{nearest})\}$;

foreach $x_{near} \in X_{near}$ **do**

if CollisionFree(x_{near}, x_{new}) **then** $E \leftarrow E \cup \{(x_{near}, x_{new}),(x_{new}, x_{near})\}$

return G = (V, E);

RRG tries to connect the new sample x_{new} to all vertices in a ball of radius r centered at it. (Or just default to the nearest one if such b all is empty.)

$$r(card\ (V\ ))$$

$$= \min\{\gamma_{RRG} \left(\frac{\log(card(V))}{card(V)}\right)^{\frac{1}{d}}, \eta\}$$

The RRT graph is a subgraph of the RRG graph (which may have cycles)

Theorems [not required for exam]

Probabilistic completeness. Since $V_n^{RRG} = V_n^{RRT}$, for all n RRG has the same completeness properties as RRT, i.e.,

$$Pr\left[V_n^{RRG} \cap X_{goal} = \emptyset\right] = O\left(e^{-bn}\right).$$

Asymptotic optimality. If the **Near** procedure returns all nodes in V within a ball of volume $Vol = \frac{\gamma \log n}{n}, \gamma > 2^d\left(1 + \frac{1}{d}\right)$, under some additional technical assumptions (e.g., on the sampling distribution, on the ϵ clearance of the optimal path, and on the continuity of the cost function), the best path in the RRG converges to an optimal solution almost surely, i.e.,

$$\Pr[Y_\infty^{RRG} = c^*] = 1.$$

Remarks on RRG

- What is the additional computational load?
  - O(log n) extra calls to ObstacleFree compared to RRT
- Key idea in RRG/RRT*:
  - Combine optimality and computational efficiency, it is necessary to attempt connection to $\Theta(\log N)$ nodes at each iteration.
  - Reduce volume of the "connection ball" as log(N)/N;
  - Increase the number of connections as log(N).
- These principles can be used to obtain "optimal" versions of PRM, etc.

Summary and future directions

- State-of-the-art algorithms such as RRT converge to a NON-optimal solution almost-surely
- new algorithms (RRG and the RRT$*$), which almost-surely converge to optimal solutions while incurring no significant cost overhead
- Bibliographical reference: S. Karaman and E. Frazzoli. Sampling-based algorithms for optimal motion planning. Int. Journal of Robotics Research, 2011. TAlso available at http://arxiv.org/abs/1105.1186.
- research directions:
  - Optimal motion planning with temporal/logic constraints
  - Anytime solution of differential games
  - Stochastic optimal motion planning (process + sensor noise)
  - Multi-agent problems.

Algorithm	Prob. Completeness	Asymptotic Optimality	Complexity
sPRM	Yes	Yes	O(N)
k-nearest sPRM	No	No	O(log N)
RRT	Yes	No	O(log N)
PRM*	Yes	Yes	O(log N)
k-nearest PRM*	Yes	Yes	O(log N)
RRG	Yes	Yes	O(log N)
k-nearest RRG	Yes	Yes	O(log N)
RRT*	Yes	Yes	O(log N)
k-nearest RRT*	Yes	Yes	O(log N)