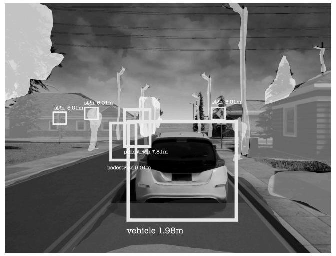
Search and Planning

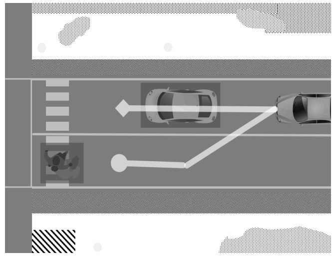
ECE 484 Fall 2023

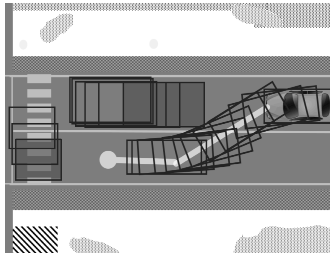
Sayan Mitra

GEM platform

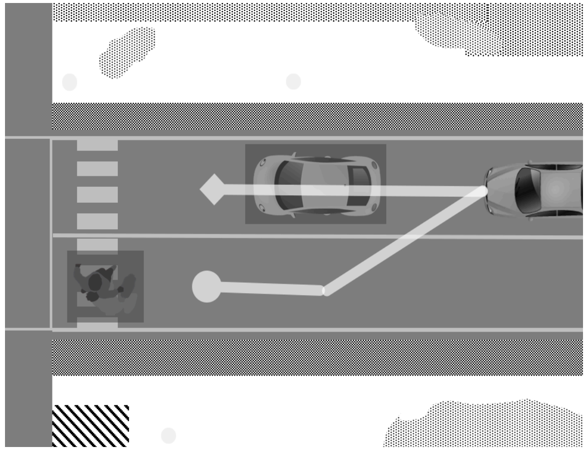
Autonomy pipeline







Sensing Physics-based models of camera, LIDAR, RADAR, GPS, etc.	**Perception** Programs for object detection, lane tracking, scene understanding, etc.	**Decisions and planning** Programs and multi-agent models of pedestrians, cars, etc.	**Control** Dynamical models of engine, powertrain, steering, tires, etc.



Decisions and planning

Programs and multi-agent models of pedestrians, cars, etc.

Search and planning problems appear in different levels of autonomy stack

Global path planner --- invoked at each new *checkpoint*

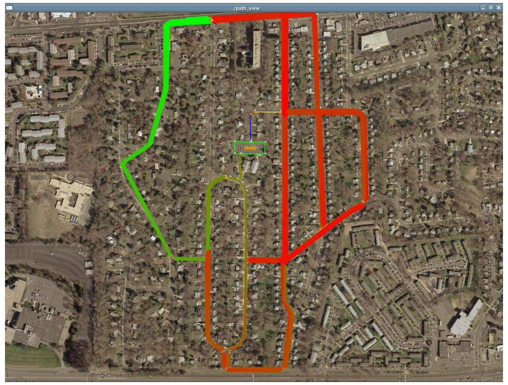
- finds paths from every point in the map to next checkpoint
- dynamic programming

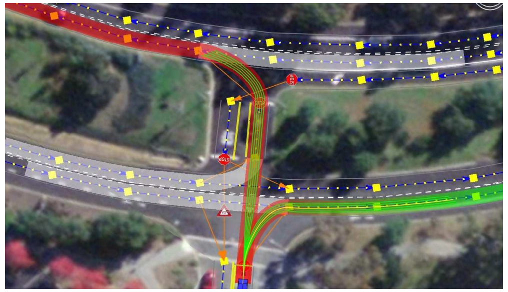
Road navigation

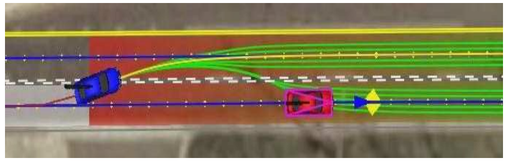
- For each path, the planner rolls out several discrete trajectories that are parallel to the smoothed center of the lane

Freeform navigation (parking lots)

- Generate arbitrary trajectories (irrespective of road structure) using modified A*







Junior: The Stanford Entry in the Urban Challenge, Thrun et al., 2008

Outline

- Uninformed search
- Informed search
- Optimal search: A, A*

Starting from uninformed graph search

Making a Drone Smarter With Motion Planning [Nicholas Rehm]

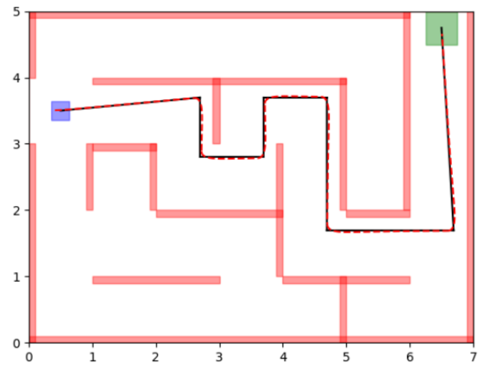
Search for collision free trajectories can be converted to graph search

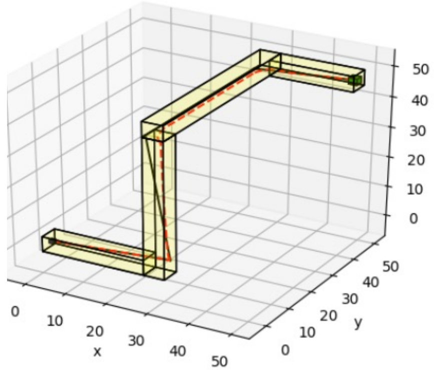
We can solve such problems using the graph search algorithms like (uninformed) Breadth-First Search and Depth-First Search

However, roadmaps are not just “generic” graphs

- Some paths are more preferable than others (e.g., shorter, faster, less costly in terms of fuel/tolls/fees, more stealthy, etc.).
- Distances have a physical meaning
- Good guesses for distances can be made, even without knowing optimal paths

Can we *utilize this information* to find efficient paths, efficiently?





https://kmmille.github.io/FACTEST/

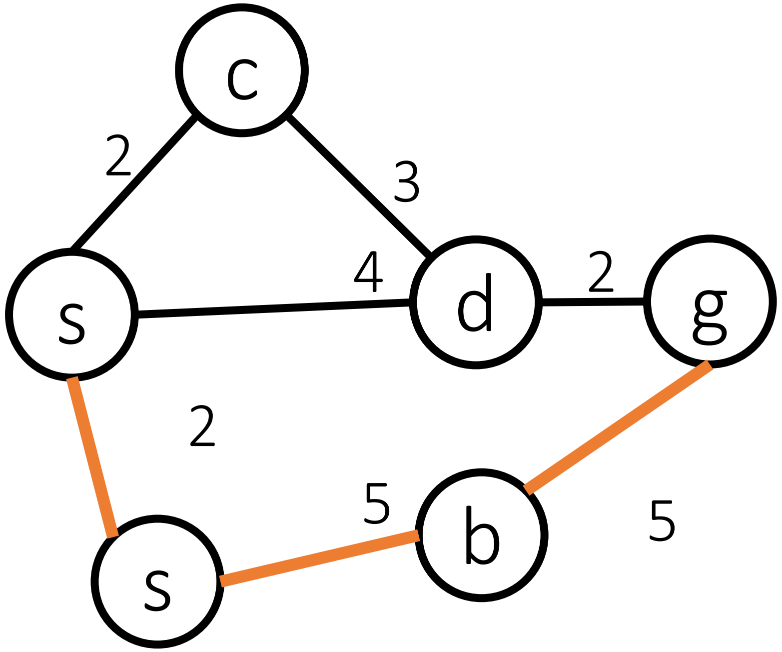
Problem statement: find shortest path

Input: $\langle V, E, w, start, goal \rangle$

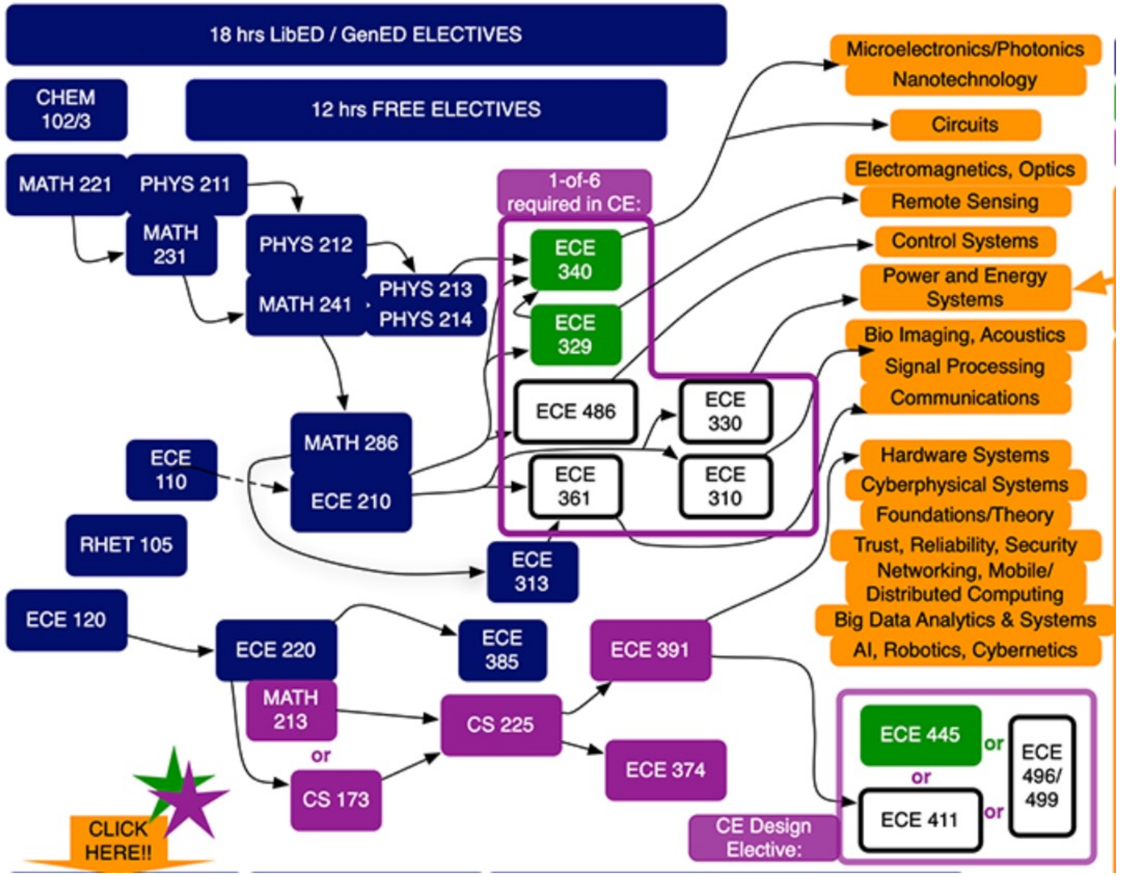
- V: (finite) set of vertices
- $E \subseteq V \times V$: (finite) set of edges
- $w : E \to \mathbb{R}_{>0}$: associates to each edge e to a strictly positive weight $w(e)$ (cost, length, time, fuel, prob. of detection)
- $start, goal \in V$: respectively, start and end vertices.

Output: $\langle P \rangle$

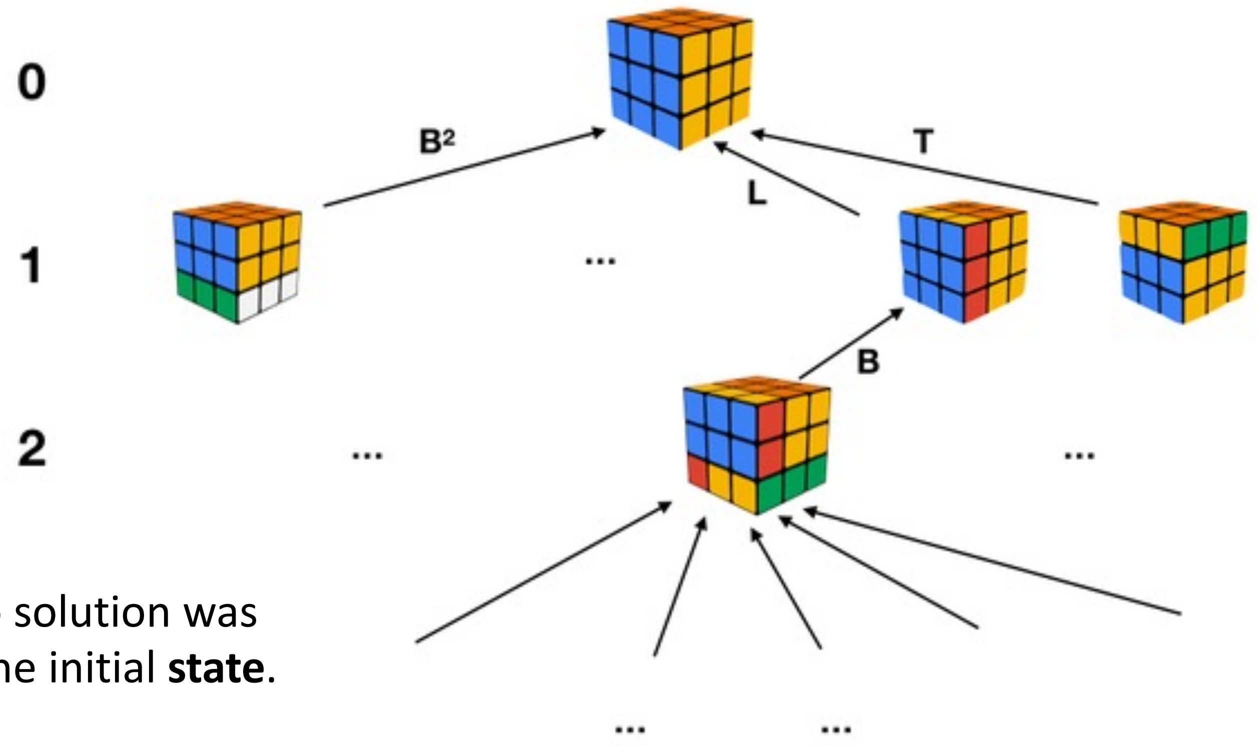
- P is a path (starting in start and ending in goal, such that its weight $w(P)$ is minimal among all such paths
- The weight of a path is the sum of the weights of its edges
- The graph may be unknown, partially known, or known



Many paths and all weights are often not known upfront



The Graph Can be Large



Number of states or vertices
43,252,003,274,489,856,000

Yet, maximum length of path to solution was shown to be 20, regardless of the initial **state**.

T. Rokicki, working with Google, proved "[God's number]" to be 20, in 2010.

Search Algorithm Performance Metrics

Soundness: when a solution is returned, is it guaranteed to be correct path?

Completeness: is the algorithm guaranteed to find a solution when one exists?

Optimality: How close is the found solution to the best solution?

Space complexity: how much memory is needed?

Time complexity: what is the running time? Can it be used for online planning?

Uniform cost search (Uninformed search)

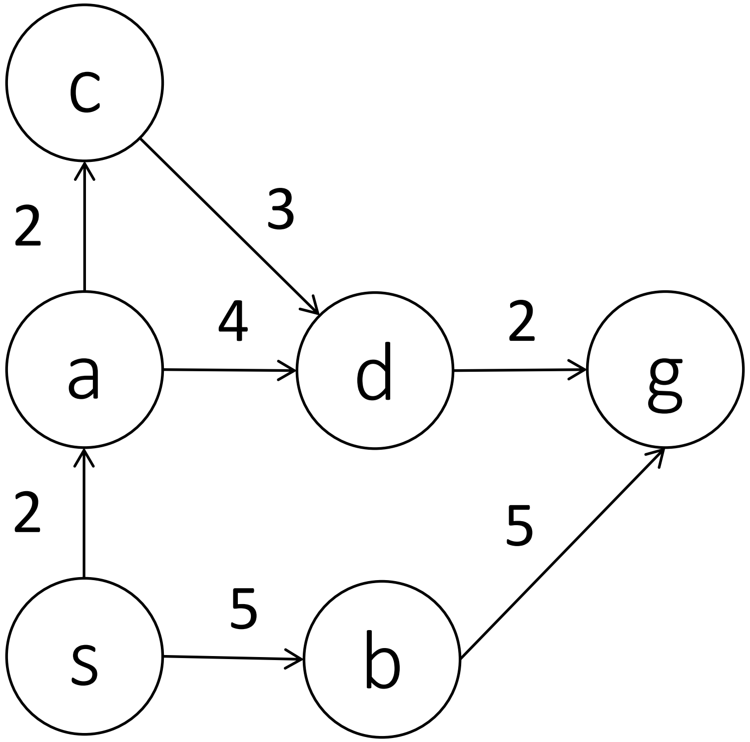
$Q \leftarrow \langle start \rangle$ // maintains paths
// initialize queue with start
while $Q \neq \emptyset$:
  pick $(and\ remove)\ the\ path\ P\ with\ lowest\ cost\ g = w(P)\ from\ Q$
  if $head(P) =\ goal$ then return P ; // Reached the goal
  foreach $vertex\ v\ such\ that\ (head(P), v) \in\ E$, do // for all neighbors
    add $\langle v, P \rangle\ to\ Q$; // Add expanded paths
return $FAILURE$; // nothing left to consider

Note no ***visited*** list; Use no information obtained from the environment

Example of Uniform-Cost Search

Q:

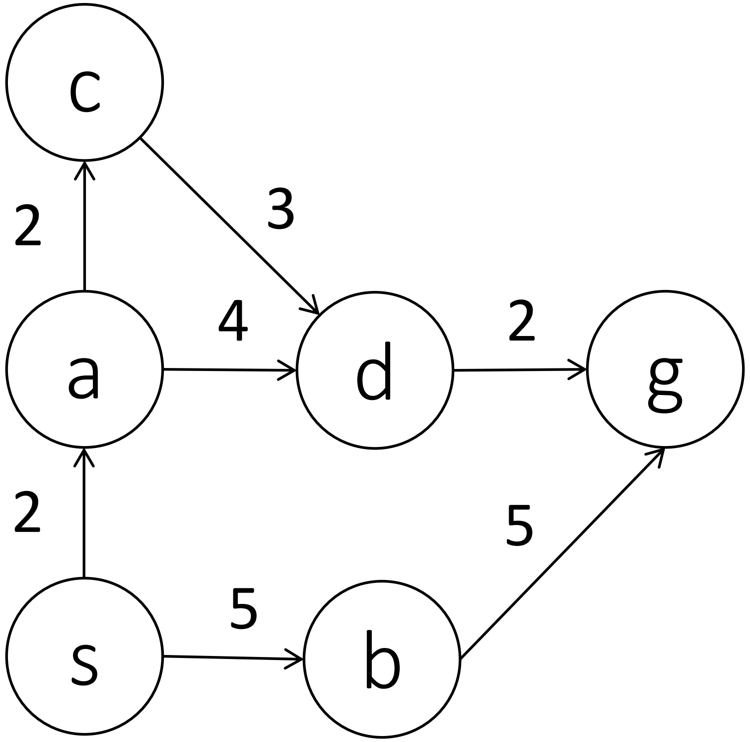
Path	Cost
$\langle s \rangle$	0



Example of Uniform-Cost Search

Q:

Path	Cost
~~$\langle a, s\rangle$~~	~~2~~
$\langle b, s\rangle$	5
$\langle c, a, s\rangle$	4
$\langle d, a, s\rangle$	6
$\langle d, c, a, s\rangle$	7



Example of Uniform-Cost Search

Q:

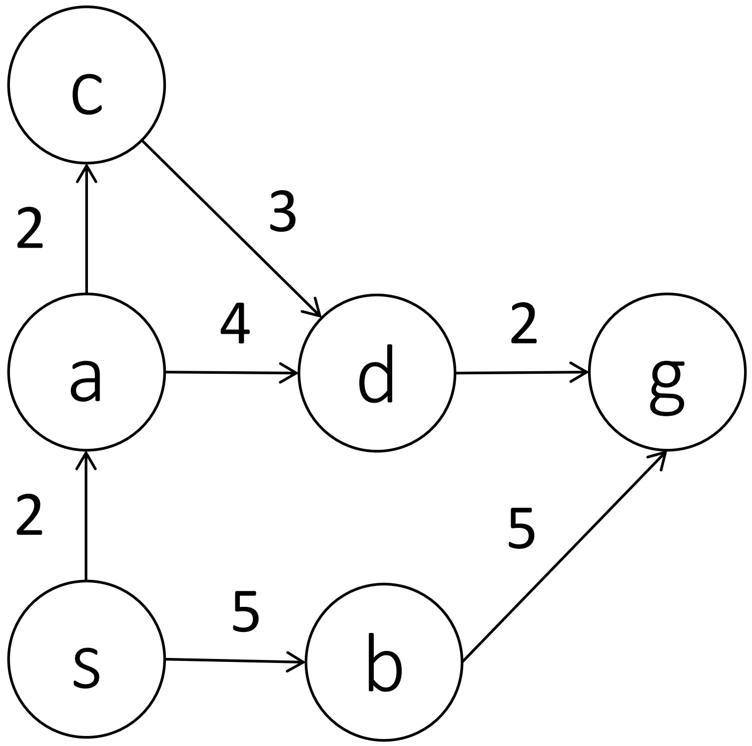
Path	Cost
<b,s>	5
<d,a,s>	6
<d,c,a,s>	7

<g,b,s> 10

<g,d,a,s> 8

<g,d,c,a,s> 9

Stops when smallest cost path has 'g' as the head



Properties of Uniform Cost Search

UCS is an extension of BFS to the weighted-graph case (UCS = BFS if all edges have the same cost)

UCS is *sound, complete* and *optimal* (assuming costs bounded away from zero)

- Exercise: Prove this

UCS is guided by path cost rather than path depth, so it may get in trouble if some edge costs are very small

Worst-case time and space complexity $O(b^{W^*/\epsilon})$, where W^* is the optimal cost, and ϵ is such that all edge weights are no smaller than

Greedy or Best-First Search

UCS explores paths in all directions, with no bias towards the goal state

What if we try to get “closer” to the goal?

We need a measure of distance to the goal

- It would be ideal to use the length of the shortest path...
- but this is exactly what we are trying to compute!

We can ***estimate*** the distance to the goal through a “heuristic function,” $h : V \to \mathbb{R}_{\geq 0}$. E.g., the Euclidean distance to the goal (as the crow flies)

- $h(v)$ is the estimate of the distance from v to goal

A reasonable strategy is to always try to move in such a way to minimize the estimated distance to the goal: this is the basic idea of the greedy (best-first) search

Greedy/Best-first search

$Q \leftarrow \langle start \rangle$ *// initialize queue with start*

while $Q \neq \emptyset$:

  pick $(and\ remove)\ the\ path\ P\ with\ lowest$ $heuristic\ cost\ h(head(P))\ from\ Q$

  if $head(P) = goal$ then return P *// Reached the goal*

  foreach $vertex\ v\ such\ that\ (head(P), v) \in E$, do *// for all neighbors*

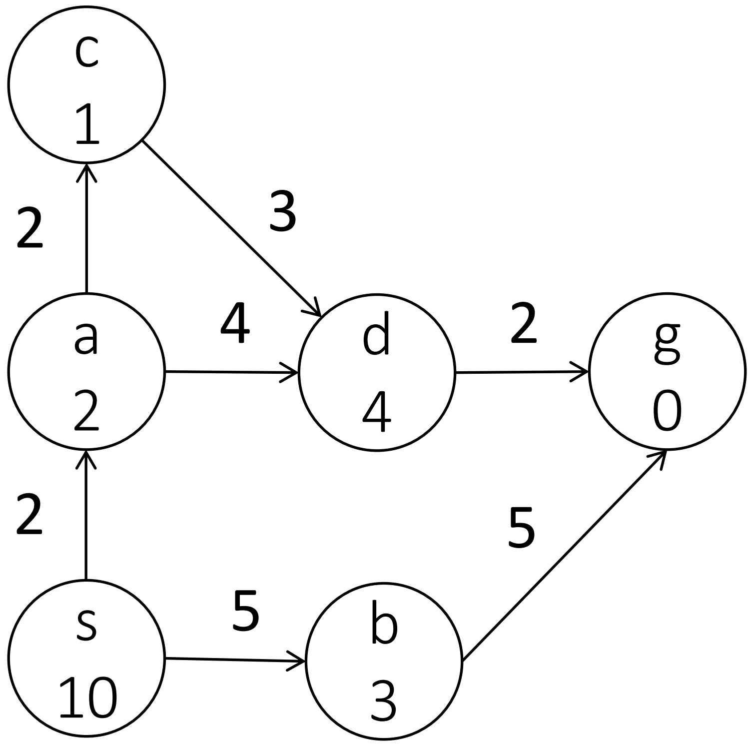
    add $\langle v, P \rangle\ to\ Q$; *// Add expanded paths*

return $FAILURE$; *// nothing left to consider*

Example of Greedy search

Q:

Path	Cost	h
$\langle s \rangle$	0	10

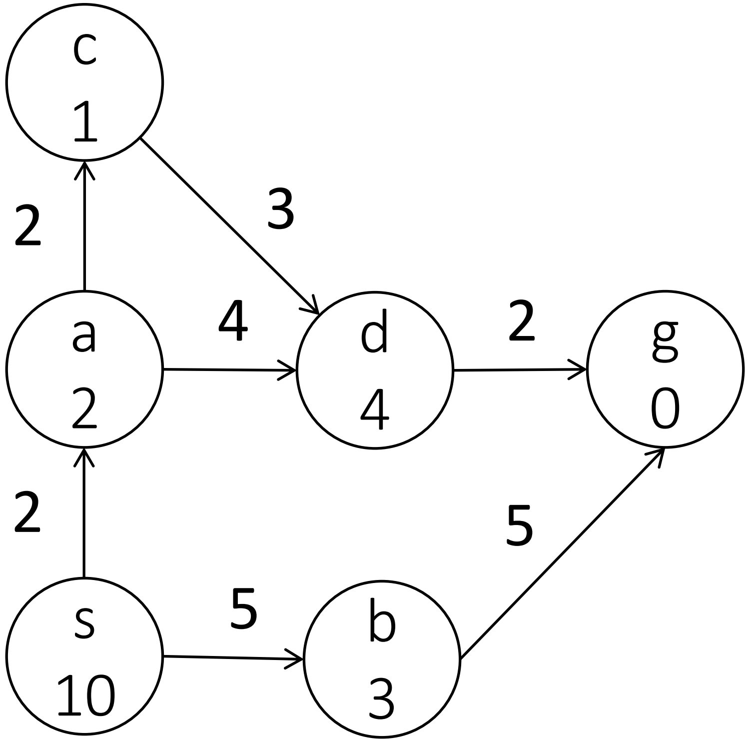


Example of Greedy search

Q:

Path	Cost	h
⟨s⟩	0	10
→ ⟨a, s⟩	2	2
→ ⟨b, s⟩	5	3
→ ⟨c, a, s⟩	4	1
⟨d, a, s⟩	6	4
⟨d, c, a, s⟩	7	4
→ ⟨g, b, s⟩	10	0

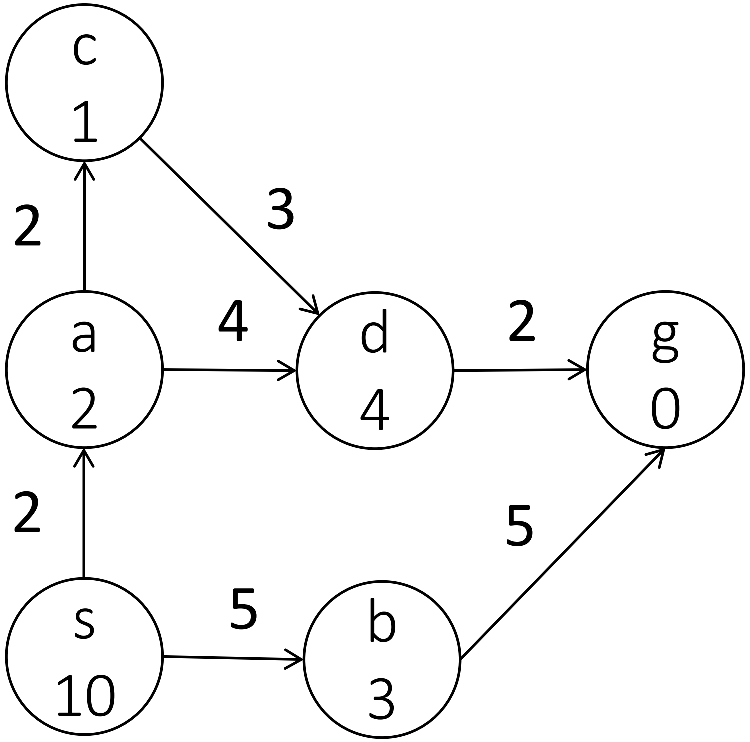
not optimal



Example of Greedy search

Q:

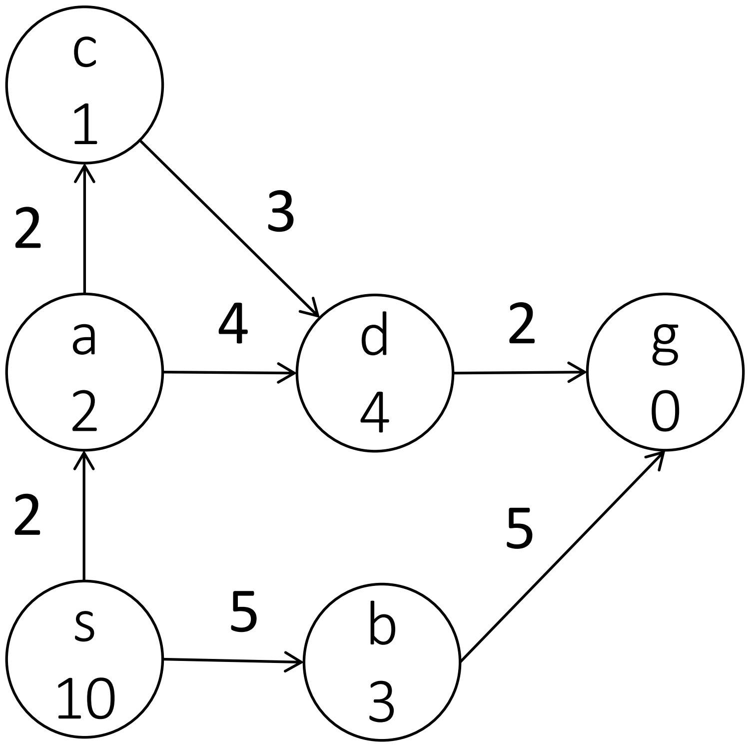
Path	Cost	h
$\langle a, s \rangle$	2	2
$\langle b, s \rangle$	5	3



Example of Greedy search

Q:

Path	Cost	h
$\langle s \rangle$	0	10



Remarks on greedy/best-first search

Greedy (Best-First) search is similar to Depth-First Search

- keeps exploring until it has to back up due to a dead end

Not complete (why?) and not optimal, but is often fast and efficient, depending on the heuristic function h

- Exercise: Find a counter-example where path exists but bad heuristic function makes the algorithm loop forever

Worst-case time and space complexity?

A search: informed search

The problems

UCS is optimal, but may wander around a lot before finding the goal

Greedy is not optimal, but can be efficient, as it is heavily biased towards moving towards the goal. The non-optimality comes from neglecting "the past."

The idea

Keep track both of the cost of the partial path to get to a vertex, say $g(v)$, and of the heuristic function estimating the cost to reach the goal from a vertex, $h(v)$

In other words, choose as a "ranking" function the sum of the two costs:

$$f(v) = g(v) + h(v)$$

$g(v)$ cost-to-come (from the start to v)

$h(v)$: cost-to-go estimate (from v to the goal)

$f(v)$: estimated cost of the path (from the start to v and then to the goal)

A search

open set and closed set

$Q \leftarrow \langle start \rangle$ // *initialize queue with start*

while $Q \neq \emptyset$:

    pick $(and\ remove)\ \ path\ P\ with\ lowest\ estimated\ cost\ f(P) = g(P) + h\big(head(P)\big)\ from\ Q$

    if $head(P) = \ goal$ then return P // *Reached the goal*

    foreach $vertex\ v\ such\ that\ (head(P), v) \in\ E$, do // *for all neighbors*

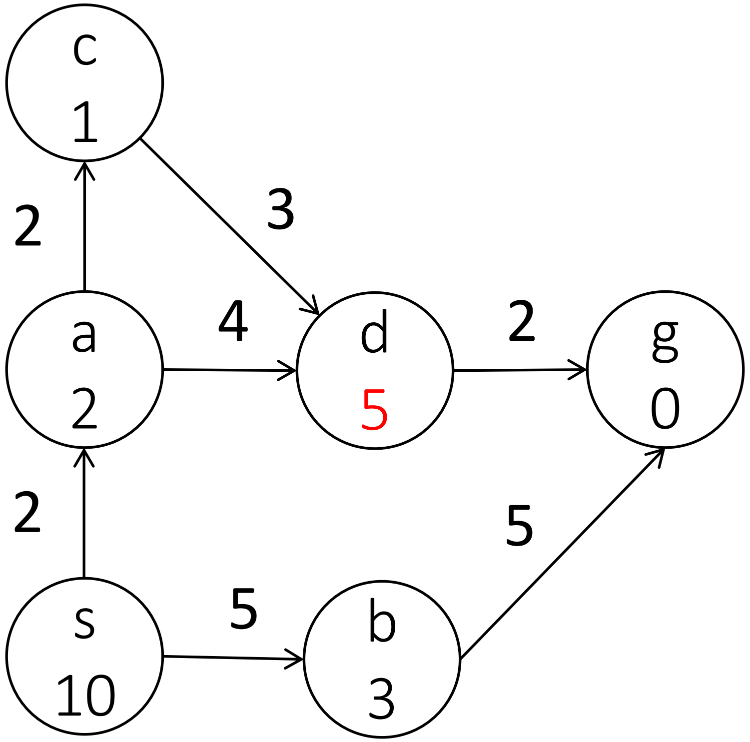
        add $\langle v, P \rangle\ to\ Q$; // *Add expanded paths*

return $FAILURE$; // ***nothing left to consider***

Example of A search

Q:

Path	g	h	f
$\langle s \rangle$	0	10	10

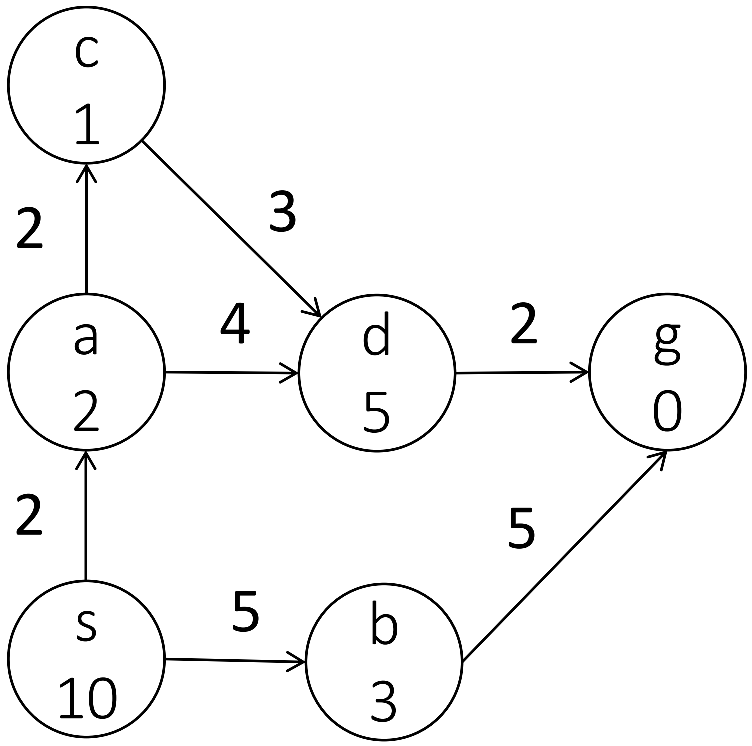


Example of A search

Q:

Path	g	h	f
~~$\langle a, s\rangle$~~	~~2~~	~~2~~	~~4~~
$\langle b, s\rangle$	5	3	8
$\langle c, a, s\rangle$	4	1	5
$\langle d, a, s\rangle$	6	5	11
$\langle d, c, a, s\rangle$	7	5	12
$\langle g, b, s\rangle$	10	0	10

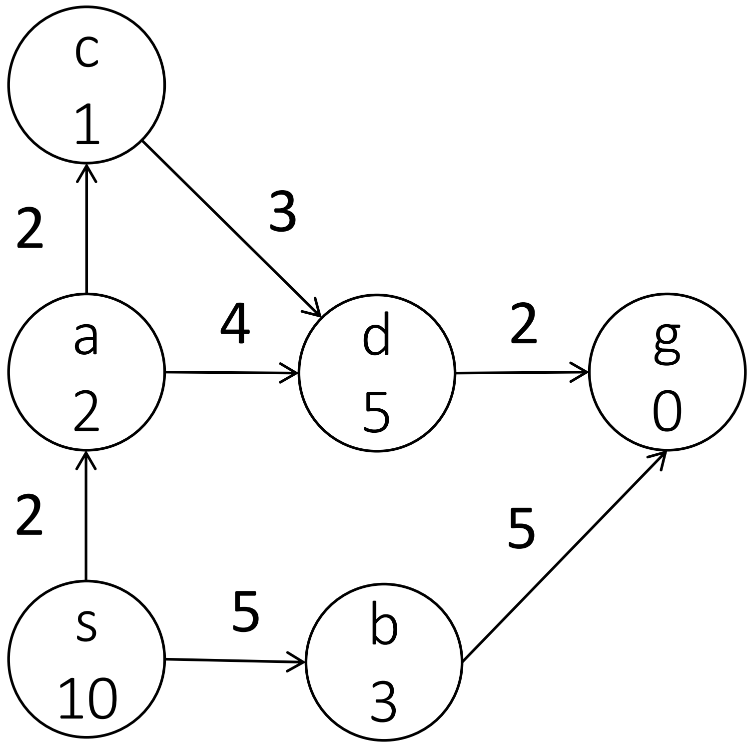
not optimal



Example of A search

Q:

Path	g	h	f
$\langle a, s \rangle$	2	2	4
$\langle b, s \rangle$	5	3	8



Remarks on A search

A search is similar to UCS, with a bias induced by the heuristic h

If h = 0, A = UCS.

The A search is complete, but is *not optimal*

- What is wrong? (Recall that if h = 0 then A = UCS, and hence optimal...)

A∗ Search

Choose an admissible heuristic, i.e., such $that\ h(v) \ \le h^*(v)$

- $h^*(v)$ is the “optimal” heuristic---perfect cost to go
- To be admissible $h(v)$ should be at most $h^*(v)$
- A search with an admissible heuristic is called A* --- guaranteed to find optimal path

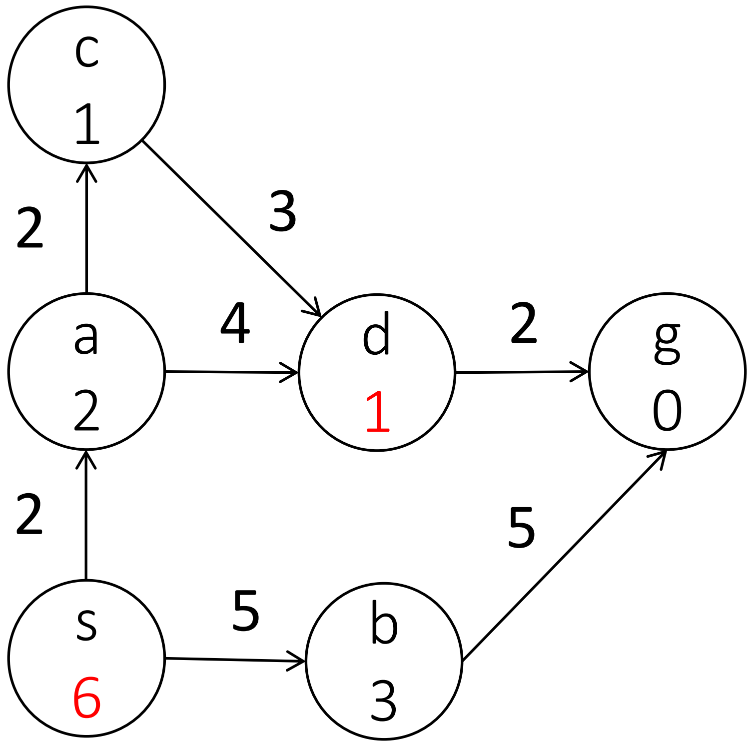
Example of A* search

Q:

Path	g	h	f
$\langle s \rangle$	0	6	6

changed h

finds <g, d, a, s>
optimal path



Proof of optimality of A*

Let w^* be the cost of the optimal path

Suppose for the sake of contradiction, that A* returns P with $w(P) > w^*$

Find the first unexpanded node on the optimal path P^*; call it n

$f(n) > w(P)$, otherwise n would have been expanded

$$
\begin{aligned}
f(n) &= g(n) + h(n) \\
&= g^*(n) + h(n) && \text{[since } n \text{ is on the optimal path]} \\
&\le g^*(n) + h^*(n) && \text{[since } h \text{ is admissible]} \\
&= f^*(n) = w^* && \text{[by def. of } f\text{, and since } w^* \text{ is the cost of the optimal path]}
\end{aligned}
$$

Hence $w^* \ge f(n) = w(P)$, which is a contradiction

Admissible heuristics

- How to find an admissible heuristic? i.e., a heuristic that never overestimates the cost-to-go.
- Examples of admissible heuristics
  - $h(v) = 0$: this always works! However, it is not very useful, A∗ = UCS
  - $h(v) = distance(v, g)$ when the vertices of the graphs are physical locations
  - $h(v) = ||v - g||_p$, when the vertices of the graph are points in a normed vector space
- A general method
  - Choose h as the optimal cost-to-go function for a relaxed problem, that is easy to compute
  - Relaxed problem: ignore some of the constraints in the original problem

Admissible heuristics for the 8-puzzle

Initial state:

1		5
2	6	3
7	4	8

Goal state:

1	2	3
4	5	6
7	8	

Which of the following are admissible heuristics?

- h = 0 — YES, always good
- h = 1 — not valid in goal state
- h = number of tiles in the wrong position — YES, “teleport” each tile to the goal in one move
- h = sum of (Manhattan) distance between tiles and their goal position — YES, move each tile to the goal ignoring other tiles.

A partial order of heuristic functions

- Some heuristics are better than others
  - h = 0 is an admissible heuristic, but is not very useful
  - h = h* is also an admissible heuristic, and it the “best” possible one (it give us the optimal path directly, no searches/backtracking)
- Partial order
  - We say that h_1 dominates h_2 if $h_1(v) \; \geq \; h_2(v)$ for all vertices v
  - h^* dominates all admissible heuristics, and 0 is dominated by all admissible heuristics
- Choosing the right heuristic
  - In general, we want a heuristic that is as close to h $*$ as possible
  - However, such a heuristic may be too complicated to compute
  - There is a tradeoff between complexity of computing h and the complexity of the search

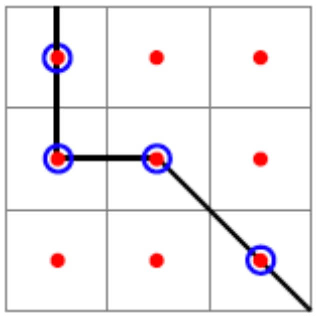
Consistent heuristics

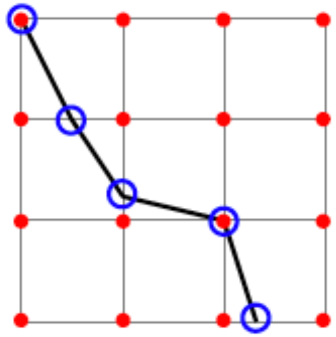
- An additional useful property for A$*$ heuristics is called consistency
  - A heuristic $h : X \to \mathbb{R}_{\geq 0}$ is said consistent if $h(u) \leq w(e = (u, v)) + h(v), \forall (u, v) \in E$
  - In other words, a consistent heuristics satisfies a triangle inequality
- If h is a consistent heuristics, then $f = g + h$ is non-decreasing along paths: $f(v) = g(v) + h(v) = g(u) + w(u, v) + h(v) \geq f(u)$
- Hence, the values of f on the sequence of nodes expanded by A$*$ is non-decreasing: the first path found to a node is also the optimal path ⇒ no need to compare costs!

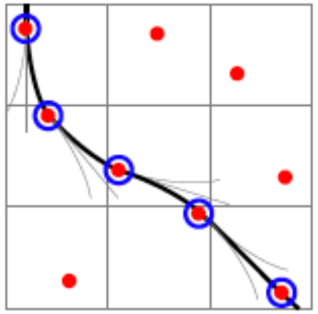
open cells: cells that are accessible from root and closed cells

Hybrid A*

(a)







- Represent vehicle state in a ~~*uniform*~~ discrete grid
  - 4D grid: $x, y, \theta\ (heading), dir$ (fwd,rev)
- If the current coordinate is $\langle x, y, \theta \rangle$ and those coordinates lie in cell c_i then the *representative continuous state* for cell c_i will be $x_i = x, y_i = y, \theta_i = \theta$
- After applying control input u to vehicle, suppose the predicted state is x', y', θ'
  - $x', y', \theta' = f(x, y, \theta, u); \dot{x} = \cdots$
  - representative for $c_j = x', y', \theta'$
  - This defines a transition from c_i to c_j
- More details in the next lecture

Summary

- A* algorithm combines cost-to-come g(v) and a heuristic function h(v) for cost-to-go to find shortest path
  - informed search
- heuristic function must be *admissible* $h(v) \leq h^*(v)$
  - Never over-estimate the actual cost to go
  - Are all $h(v)$ values needed ?
  - What if h is not admissible
  - How to find heuristics

Next: Dynamic programming/Dijkstra

- The optimality principle
  - Let $P = (s, \ldots, v, \ldots g)$ be an optimal path (from s to g).
  - Then, for any $v \in P$, the sub-path $S = (v, \ldots, g)$ is itself an optimal path (from v to g)
- Using the optimality principle
  - Essentially, optimal paths are made of optimal paths. Hence, we can construct long complex optimal paths by putting together short optimal paths, which can be easily computed. Fundamental formula in dynamic programming: $h*(u) = \min_{(u,v)\in E} [w((u, v)) + h*(v)]$. Typically, it is convenient to build optimal paths working backwards from the goal.