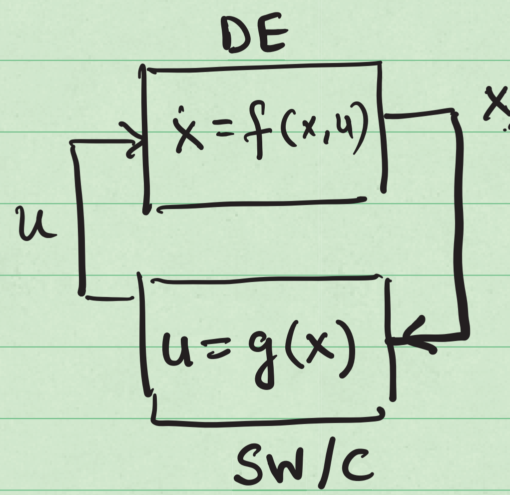
State-feedback control design

$$\frac{dx(t)}{dt} = f(x(t)) \qquad \dot{x} = f(x)$$

$$\dot{x}(t) = f(x(t), u(t))$$

Used for designing

Vehicles, Circuits, satellites, aircraft
Biological processes, Brain-machine interfaces,
Medical devices



$$\dot{x} = f(x, g(x))$$

State feedback control

Consider a linear time invariant system

$\dot{x} = Ax \qquad x \in \mathbb{R}^n \qquad A \in \mathbb{R}^{n \times n} \qquad x(0) \in \mathbb{R}^n$

Solution of this LTI system is given by

$x(t) = x(0)\, e^{At} \leftarrow$ matrix exponential
(Review how this is defined)

$x(t+1) = A^{t+1} x(0)$ in case of discrete time

$x(t+1) = A\,x(t) \qquad x(t+2) = A \cdot A\,x(t) \qquad x(t) = A^t \cdot x(0)$

Often the system we study will model the "error". E.g. the tracking error for path following

We would like the error to go to 0 as $t \to \infty$

$\lim_{t \to \infty} A^t x(0) = 0$

$\forall x(0) \in \mathbb{R}^n \cdot \lim_{t \to \infty} x(t) = \lim_{t \to \infty} x(0) e^{At} \overset{?}{=} 0$

Theorem. This holds iff all eigenvalues of A have strictly negative real parts
A is said to be Hurwitz.
(see website)

Examples : Two dimensional system

Ex 1

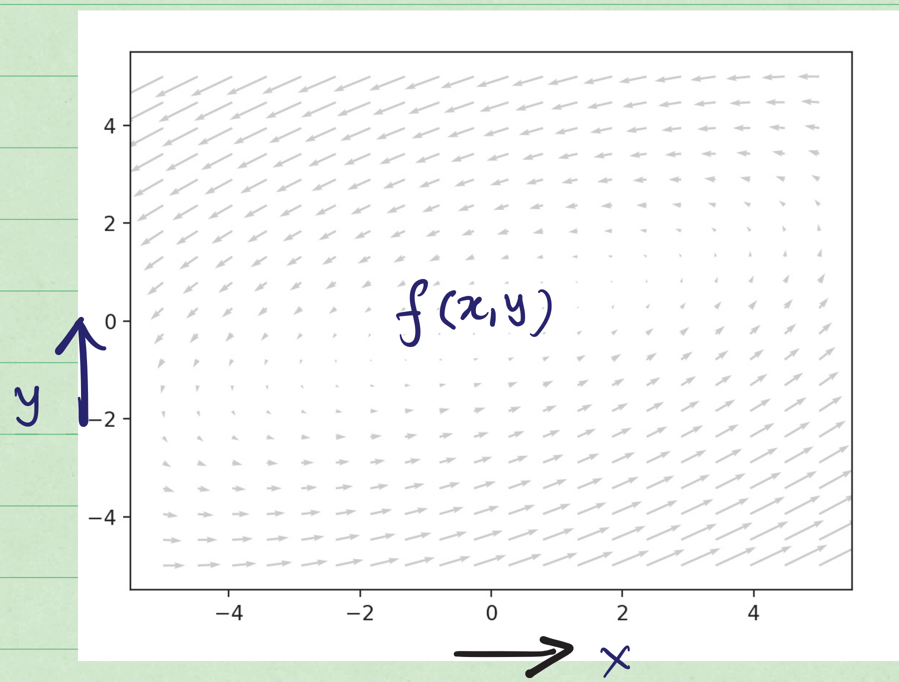
x: national income
y: rate of consumer spend
g: rate of govt expenditur

$$\begin{rcases} \dot{x} = x - \alpha y \\ \dot{y} = \beta(x - y - g) \end{rcases} \text{RHS } f$$

$$g = g_0 + kx$$

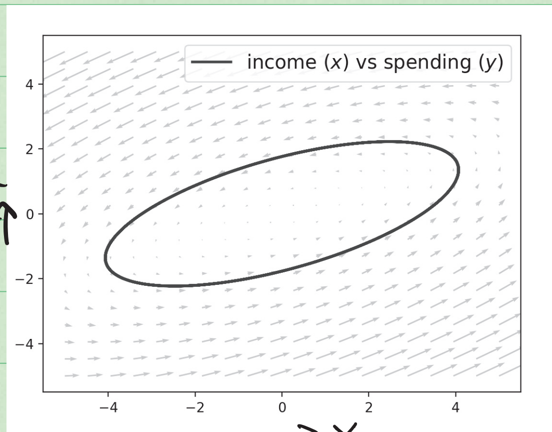
$$\dot{y} = \beta[(1-k)x - y - g_0]$$

We can view the ODE as a vector field



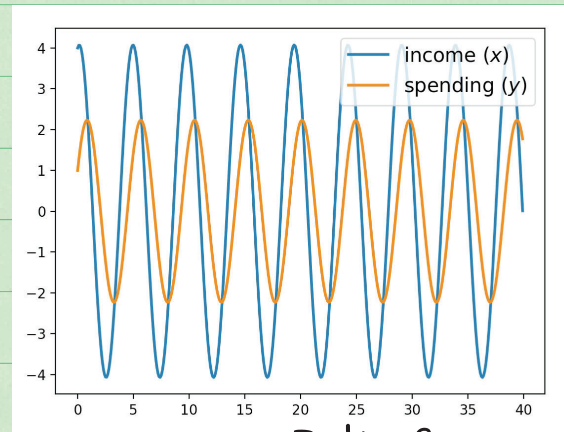
$f(x)$

Ax

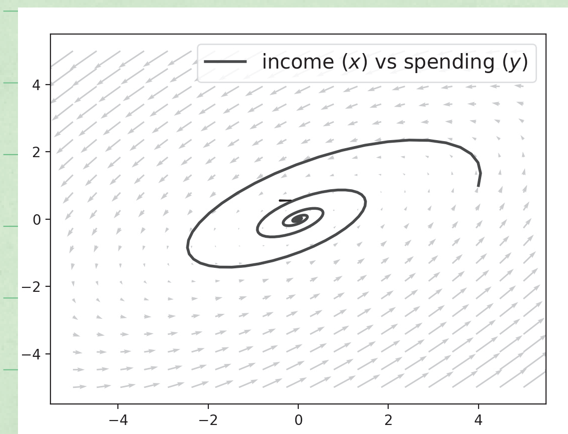

income (x) vs spending (y)
−4
−2
0
2
4

y

x


income (x)
spending (y)
0
5
10
15
20
25
30
35
40

time


income (x) vs spending (y)
−4
−2
0
2
4

Recall, $x \in \mathbb{R}^n$ and λ are eigenvalue and eigenvector of A if

$$Ax = \lambda x$$

i.e. $(A - \lambda I)x = 0$

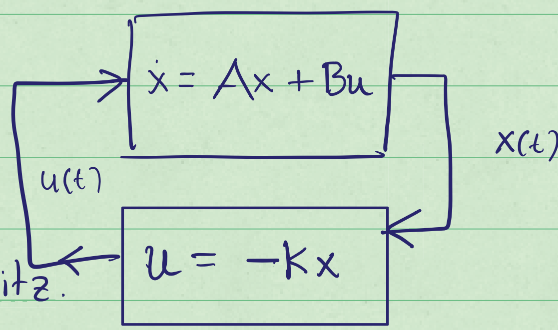
How to find the eigenvalues: solve

$\det(\lambda I - A) = 0$ characteristic equation

How can this be used for control design?

$$\dot{x} = Ax - BKx$$
$$= (A - BK)x$$

Choose K (gain matrix) such that A' is Hurwitz.



Example (Path tracking problem continued)

Linearizing

$$\underbrace{\dot{\begin{bmatrix} \delta s \\ \delta n \\ \delta \theta \end{bmatrix}}}_{X} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & v \\ 0 & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} \delta s \\ \delta n \\ \delta \theta \end{bmatrix} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} \delta v \\ \delta k \end{bmatrix}}_{u}$$

State feedback control $u = \cancel{KBx} = Kx$

$$\begin{bmatrix} \delta v \\ \delta \kappa \end{bmatrix} = \begin{bmatrix} k_s & 0 & 0 \\ 0 & k_n & k_\theta \end{bmatrix} \begin{bmatrix} \delta s \\ \delta n \\ \delta \theta \end{bmatrix}$$

$$A - BK = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & v \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} k_s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & k_n & k_\theta \end{bmatrix}$$

$$= - \begin{bmatrix} k_s & 0 & 0 \\ 0 & 0 & -v \\ 0 & k_n & k_\theta \end{bmatrix}$$

$\det(A - BK - \lambda I) = 0 \leftarrow$ Characteristic Equation

- polynomial in λ with k_θ, k_n, k_s
- Find the roots of the polynomial

- Choose k_θ, k_n, k_s to make real parts of the roots negative

Example $\quad \dot{x} = Ax + Bu \quad x \in \mathbb{R}^2$

Choose $\quad u = -Kx \quad$ Control is some linear function of state

Closed loop system

$$\dot{x} = Ax - BKx = \underbrace{(A - BK)}_{A'}x$$

Choose K to make A' Hurwitz

Say $\quad A = \begin{bmatrix} 0 & \vartheta \\ 1 & 1/2 \end{bmatrix} \quad B = \cancel{\begin{bmatrix} 0 & 1 \end{bmatrix}} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$$-K = -\begin{bmatrix} K_{11} & 0 \\ 0 & K_{22} \end{bmatrix}$$

$$A - BK = \begin{bmatrix} 0 & \vartheta \\ 1 & 1/2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & K_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \vartheta \\ 1 & 1/2 - K_{22} \end{bmatrix}$$

$$\lambda I - (A - BK) = \begin{bmatrix} \lambda & -v \\ -1 & \lambda - 1/2 + k_{22} \end{bmatrix}$$

$\det(\ ) = 0$

Characteristic equation

$$\lambda^2 + \lambda(k_{22} - 1/2) - v = 0$$

Roots

$$\lambda_1, \lambda_2 = \frac{-(k_{22} - 1/2) \pm \sqrt{(k_{22} - 1/2)^2 + 4v}}{2}$$

$k_{11} = 0$

When do we have $Re(\lambda_1) < 0$ $Re(\lambda_2) < 0$

If $(k_{22} - 1/2)^2 + 4v < 0$ then

k_{22} must be $> 1/2$

Otherwise $(k_{22} - 1/2)^2 + 4v < (k_{22} - 1/2)^2$

i.e v has to be < 0

Summary

- PID control

  Need not use detailed models

  heuristics for tuning parameters

- State-feedback control

  Uses model Linearized

  Algorithm for finding parameters (gain matrix) using Hurwitz condition

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\boxed{\dot{x}_1(t) = \sin t \, x_1(t) + t^2 x_2(t)}$$

$$A(t) = \underbrace{\begin{bmatrix} \sin t & 1 \\ t^2 & 0 \end{bmatrix}}$$

A

Solution

$$x(t) = e^{At}$$

$$\dot{x}(t) = A(t)\, x(t)$$

$$\begin{bmatrix} \dot{x}_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$