Previously on control

- ODE models, solutions, equilibria
- Designing controller, PID, State feedback $\lim_{t \to \infty} x(t) = 0$
- Hurwitz condition for linear systems $\dot{x} = Ax$

Today
- Requirements
  Stability
  Asymptotic stability
- General method for proving stability
- Relationship to invariants
- Balls, sub-level sets

What are requirements for a control system

(0) State variables stay bounded

(1) Invariance $\forall x(0) \in \Theta \ \forall t \ x(t) \in I_\Theta$ ← Some nice set

(2) Sets of invariants

(3) Convergence to equilibria

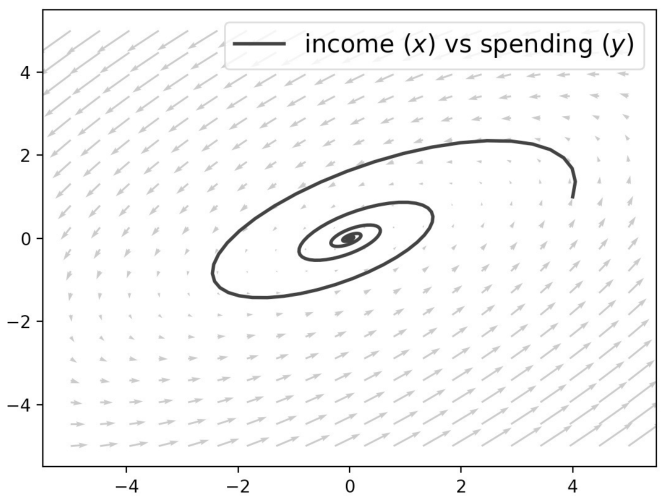
(4) Small inputs to $\dot{x} = f(x, u)$ produce small outputs BIBO

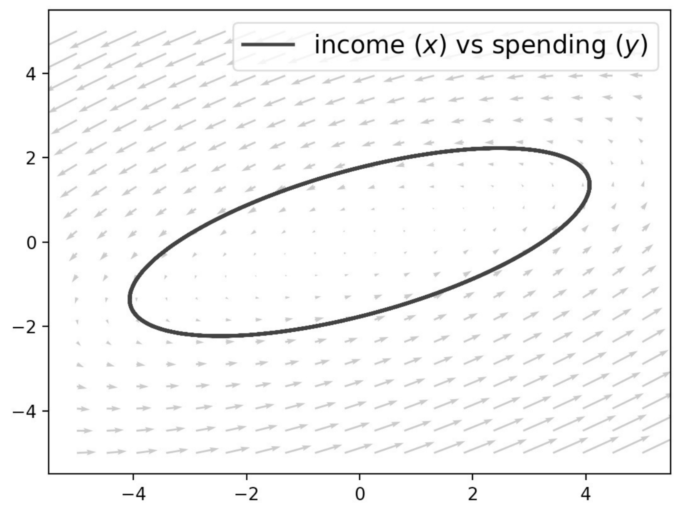
Setup. We are studying an ODE $\dot{x} = f(x)$
$x \in \mathbb{R}^n \quad f : \mathbb{R}^n \to \mathbb{R}^n$

Assume that the origin $\vec{0} \in \mathbb{R}^n$ is an equilibrium. i.e. $f(\vec{0}) = \vec{0}$
if not apply appropriate coordinte transform to f.

You have aready seen that for linear systems
$$f(x) = Ax$$
$\dot{x} = Ax$ is asymptotically stable at $\vec{0}$ iff for every eigenvalue λ_i of A $\quad Re(\lambda_i) < 0$
Hurwitz criterion.

Today: General nonlinear systems.



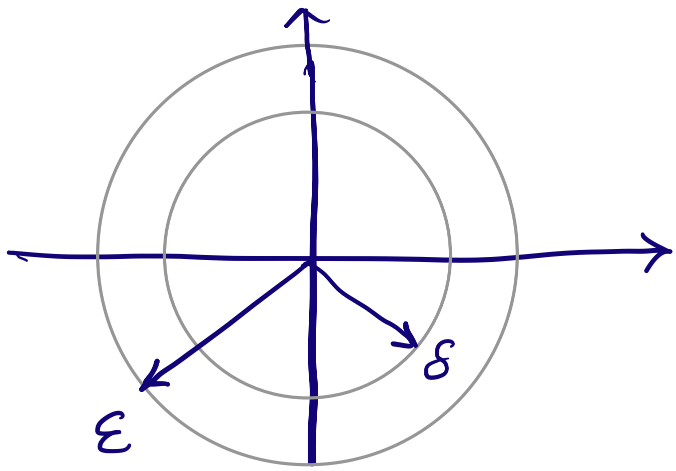


Def. $B_r \subseteq \mathbb{R}^n$ is the ball of radius r centered at $\vec{0}$
i.e. $\quad B_r = \{ x \mid \ |x| \leq r \}$
Some norm

Def. Lyapunov Stability

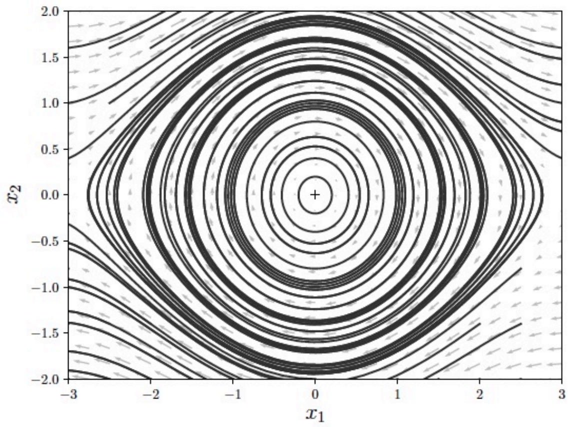
The system $\dot{x} = f(x)$ is Lyapunov stable (Lyapunov stable)
if $\forall \varepsilon > 0 \quad \exists \delta > 0$ such that
if $x(0) \in B_\delta$ then $\forall t \quad x(t) \in B_\epsilon$.

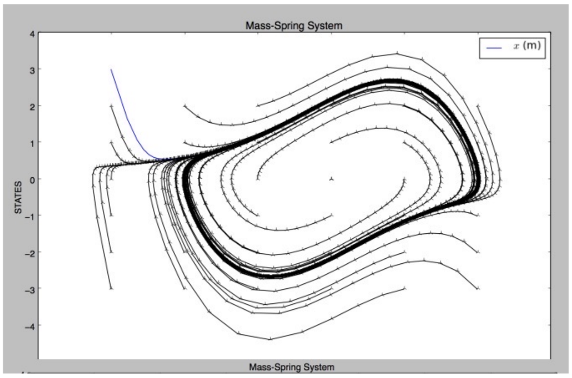
Otherwise the system is Unstable

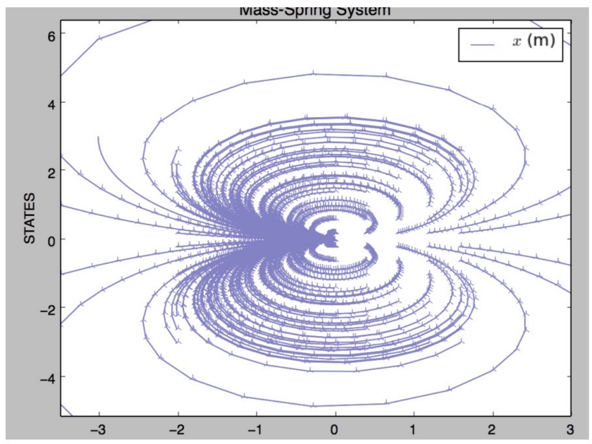


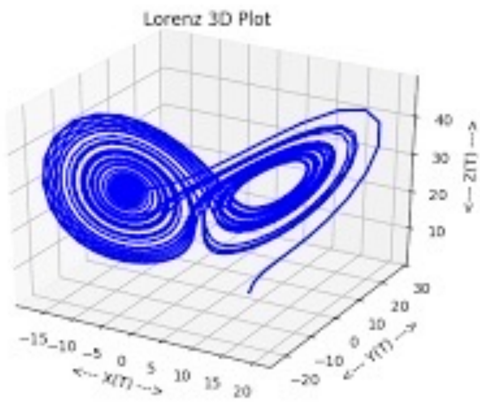
Def. The system is asymptotically stable if
it is Lyapunov stable and $\lim_{t \to \infty} x(t) \to 0$.

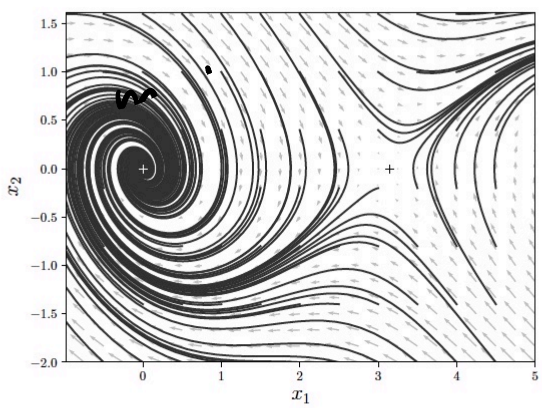
Stable / Unstable / Asymptotically stable?







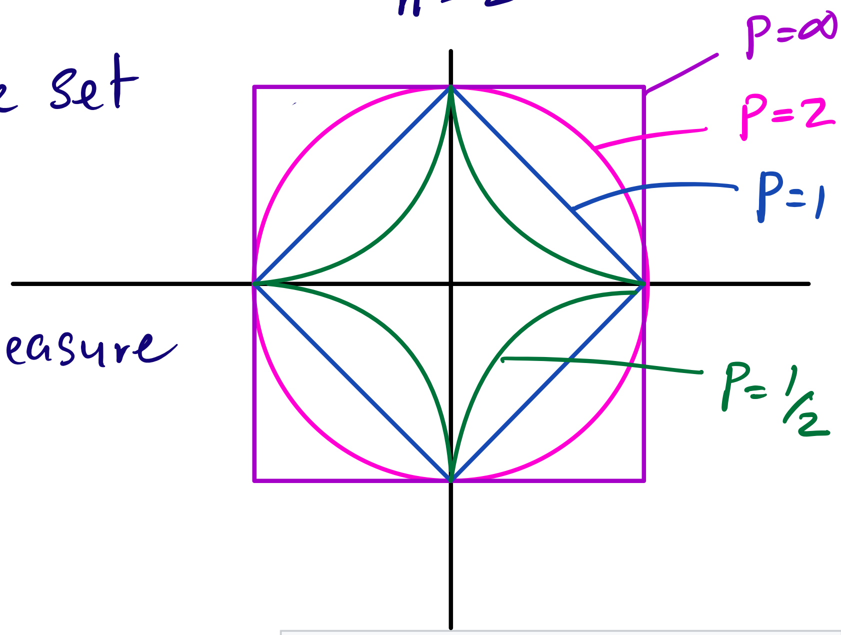




$$B(x, r) = \{ x' \in \mathbb{R}^n \mid |x - x'|_p \leq r \}$$

If $r_2 \geq r_1$ then $B(x, r_1) \subseteq B(x, r_2)$

$n = 2$



Parameters defining the set

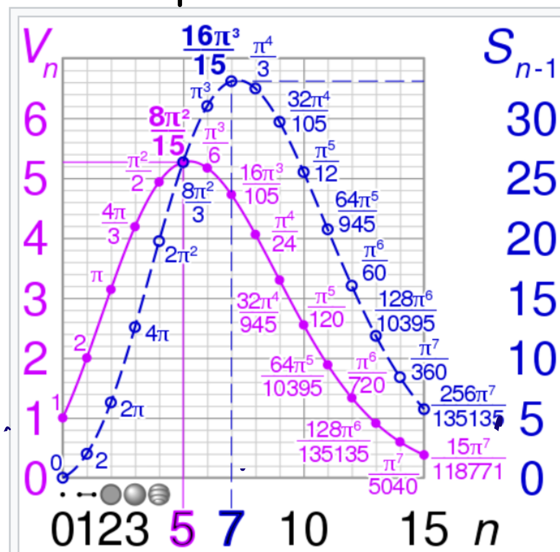
x : Center

r : radius

p : norm / distance measure

n : dimension

Factoid



Graphs of volumes (V) and surface areas (S) of n-balls of radius 1. In the SVG file, hover over a point to highlight it and its value.

for $r = 1$

what is the volume of an n-dimensional sphere of unit radius

$n = 1$	2
2	π
3	$\frac{4}{3}\pi$
$\vdots$	

Relating Invariance & Stability

Recall. $\mathcal{I} \subseteq \mathbb{R}^n$ is an invariant of $x_{t+1} = f(x_t)$
provided $X_0 \subseteq \mathcal{I}$ and $f(\mathcal{I}) \subseteq \mathcal{I}$

How is Lyapunov stability related to invariance ?

How to prove stability?

How do you prove aprogram like this terminates?

$$\text{While } (x \geq 0) \quad x = f(x)$$

Do you think this is easy?

Example

$$\text{While } (n > 1)$$

$$f(n) = \begin{cases} n/2 & \text{if } n \equiv 0 \pmod 2 \\ 3n+1 & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

Does this always end with $n = 1$?

Generally proving termination is undecidable but we try to find a ranking function $R : X \to \mathbb{N}$
energy function

Such that $\forall x \quad R(x) \geq 0$
and $R(f(x)) < R(x)$

Similar idea for ODEs

<u>Thm</u>. Suppose there exists a positive definite, radially unbounded, continuous function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ Such that

$\dot{V} \leq 0$ then $\dot{x} = f(x)$ is Lyapunov stable.

$\dot{V} < 0$ then $\dot{x} = f(x)$ is asymptotically stable.

Positive definite : $\forall x \neq 0 \quad f(0) > 0$
Radially unbounded : $x \to \infty \quad f(x) \to \infty$

$\dot{V}$? $\quad V : X \to \mathbb{R} \quad V(x)$

V is a function of the state $V(x)$
and x state is a function of time
So, really $V(x(t))$

$$\dot{V} = \frac{d}{dt} V(x(t)) = \frac{\partial V}{\partial x} \cdot \frac{dx(t)}{dt} = \frac{\partial V}{\partial x} \cdot f(x)$$

Example

$$\dot{x} = -a\sin^2(x) = f(x)$$
$$V(x) = x^3 + 6$$

Note. we never solved the ODE!

V: Lyapunov function

Ex. $\dot{x} = -x \qquad \dot{y} = -y$

Candidate $V = \dfrac{x^2 + y^2}{2}$

$$\dot{V} = \begin{bmatrix} \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{bmatrix} \begin{bmatrix} -x \\ -y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} -x \\ -y \end{bmatrix} = -(x^2+y^2)$$

$\vec{0}$ is globally asymptotically stable

<u>EX</u> $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -x + y^2 \\ -2y + 3x^2 \end{bmatrix}$ How many equilibria?

Consider $V(x,y) = \dfrac{x^2}{2} + \dfrac{y^2}{4}$ P.D. & R.U.

$$\dot{V}(x,y) = \begin{bmatrix} x & y/2 \end{bmatrix} \begin{bmatrix} -x + y^2 \\ -2y + 3x^2 \end{bmatrix}$$

$$= -x^2 + xy^2 - y^2 + \frac{3}{2}x^2 y$$

$$= -x^2\left(1 - \frac{3}{2}y\right) - y^2(1 - x)$$

This is $-$ve $\forall$ $x < 1$ $y < 2/3$

$\therefore$ origin is asymptotically stable but not globally

Level sets and Sublevel-Sets

for any function $f: \mathbb{R}^n \to \mathbb{R}$

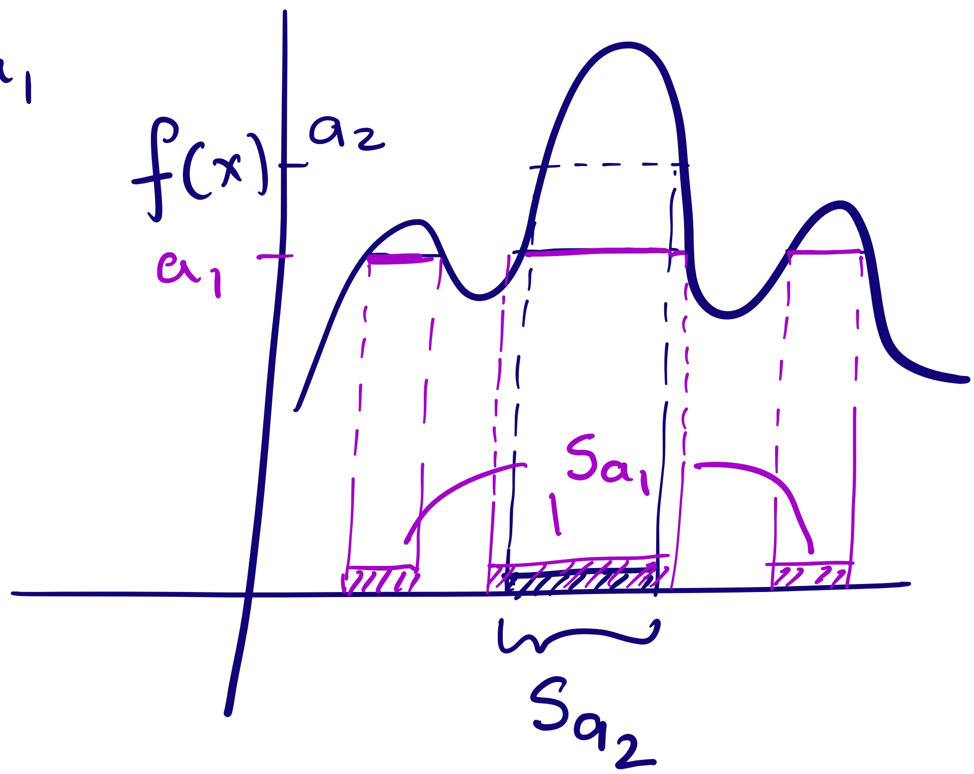
and $a \in \mathbb{R}$, the <u>a-level set</u>

of f is the set $L_a = \{x \in \mathbb{R}^n \mid f(x) = a\}$

a-Sublevel set $S_a = \{x \in \mathbb{R}^n \mid f(x) \leq a\}$

Obviously $a_1 \leq a_2 \Rightarrow$

$S_{a_2} \subseteq S_{a_1}$



$B(0,r) = S_r$
for $|x|$

Corollary. Consider a nonlinear system $\dot{x} = f(x)$ and suppose V is a Lyapunov function. Then every sublevel set L_a containing $x(0)$ is an invariant.

How to find Lyapunov functions?

Guess a template (e.g. quadratic)

$$V(x) = a_1 x_1^2 + a_2 x_1 x_2 + a_3 x_2^2 + a_4 x_1 + a_5 x_2 + a_6$$

Then solve for parameters $a_1 \ldots a_6$ with constraints

$$\frac{\partial V}{\partial x} \cdot f(x) \leq 0$$

Can often be solved using Convex optimization

For linear systems $\dot{x} = Ax \quad x \in \mathbb{R}^n$

Thm there is always a quadratic Lyapunov function if the system is stable.

Quadratic function $\quad x^T x = [x_1 \;\; x_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$x^T P x \;, \; P \in \mathbb{R}^{n \times n} \qquad = x_1^2 + x_2^2$

general quadratic form (unknown P)

$V(x) := x^T P x \qquad$ Guessed form

$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} \qquad$ [Chain rule]

$= (Ax)^T P x + x^T P (Ax)$

$= x^T (A^T P + P A) x$

Set this to

$= - x^T Q x$

that is Fix Q to be positive definite and symmetric matrix $\mathbb{R}^{n \times n}$

e.g. $\quad Q = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & & & \end{bmatrix} = I$

Solve for P in $A^T P + P A = -Q$

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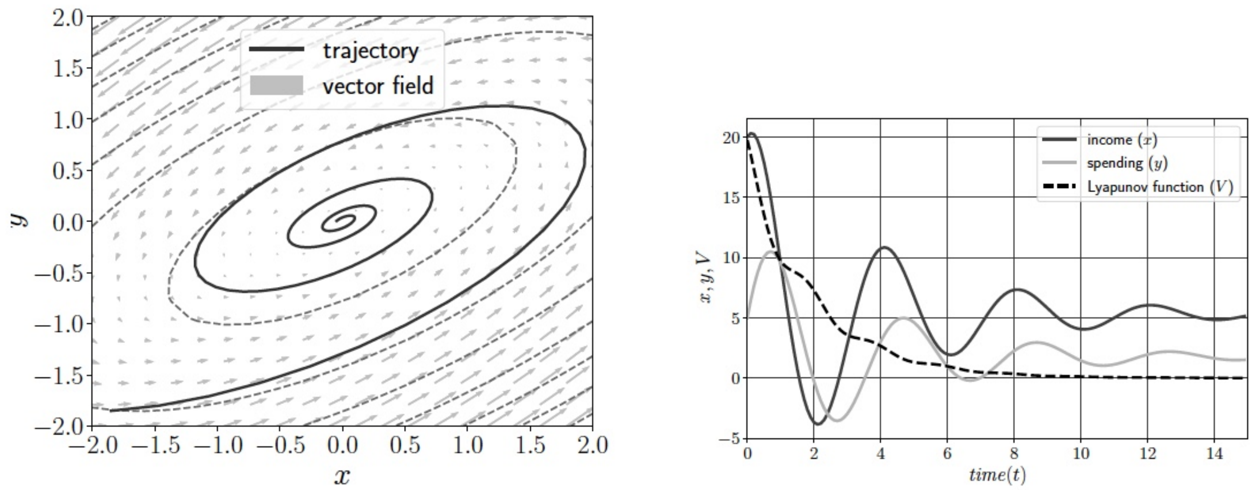

Figure 3.8
Trajectories of the simple economy model. *Left:* Phase portrait for the model of Equation (3.24) for $\alpha = 3, \beta = 1.5, k = .1$, and $g_0 = 3$. the sublevel sets of the Lyapunov function are shown in dotted lines. *Right:* Evolution of the solutions over time.

Summary

- Definitions of Stability and Asymptotic stability
- Lyapunov's method for proving stability of Nonlinear systems
- Relationship to invariants & Sub-level sets

Additional reading

Proof of Lyapunov's Method

Proof. (a) Fix $\varepsilon > 0$. We have to find $\delta_\varepsilon > 0$ such that for any $x_0 \in \mathbb{R}^n$ if $|x_0| \leq \delta_\varepsilon$ then $\forall t > 0$ $|\xi(x_0, t)| \leq \varepsilon$.

Pick $a > 0$ such that
$S_a \subseteq B(0, \varepsilon)$ [why $\exists a$? by continuity of V]

Pick δ_ε such that $B(0, \delta_\varepsilon) \subseteq S_a$

Now consider any $|x_0| \leq \delta_\varepsilon$

$$V(\xi(x_0,t)) \leq V(\xi(x_0,0)) = V(x_0) \leq a$$

By def of S_a $\xi(x_0,t) \in S_a \subseteq B(0,\delta_\varepsilon)$

i.e. $|\xi(x_0,t)| \leq \varepsilon$

(b) Consider any ∞ traj with $|\xi(x_0,0)| \leq \delta_\varepsilon$

As $V(\xi(t)) \geq 0$ and decreasing as $t \to \infty$
$\exists$ a limit $c \geq 0$.

$$\lim_{t\to\infty} V(\xi(x_0,t)) = c$$

IF $c = 0$ then $\lim_{t\to\infty} V(\xi(x_0,t)) = 0$

and since $V(x) = 0 \iff x = 0$
$\Rightarrow \lim_{t\to\infty} \xi(x_0,t) = 0$ A.S.

Else $c > 0$. —— (#)
That is system evolves in the donut $O := B(0,\varepsilon) \setminus B(0,r)$ for some small but +ve r.
As this set is compact

Let $d = \max_{x \in O} \dot{V}(x) < 0$ [Slowest rate of decrease]
Thus $\forall t$ $V(\xi(x_0,t)) \leq V(\xi(x_0,0)) + d.t$

But then for any such d we can find a t so that $V(\xi(x_0,t)) < c$
which contradicts (#).