What is control theory?

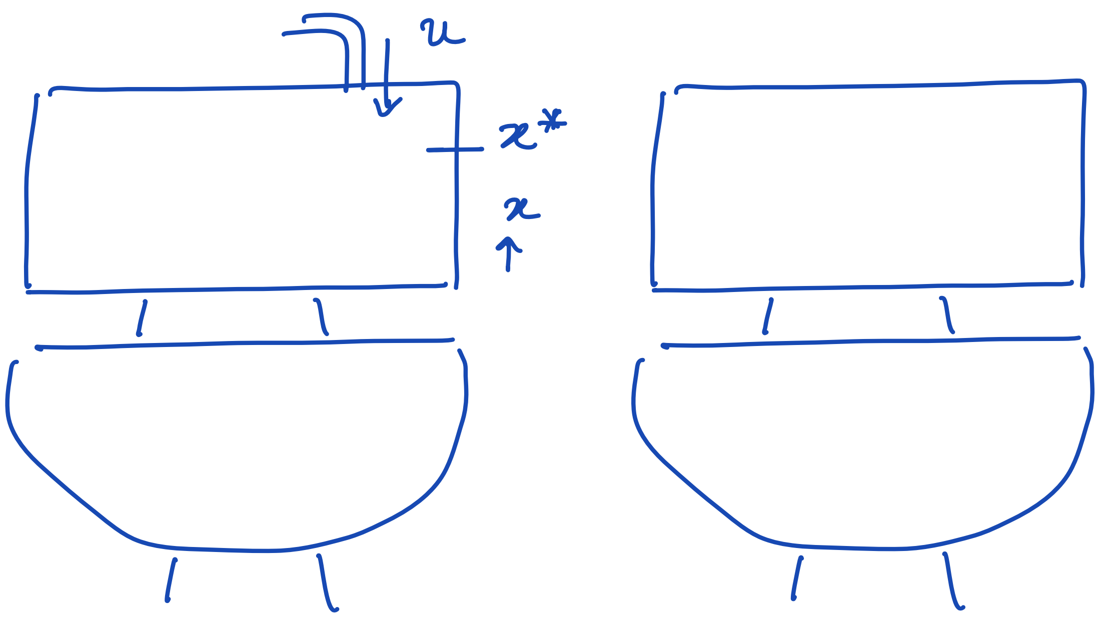
The art of making things do what you want

art : design parameterized controllers
ways of tuning parameters

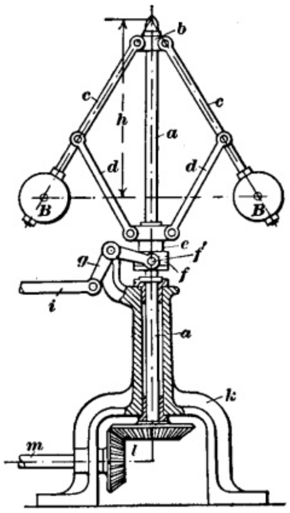
things : Physical processes cars, circuits,
represented by ordinary differential equations
ODEs

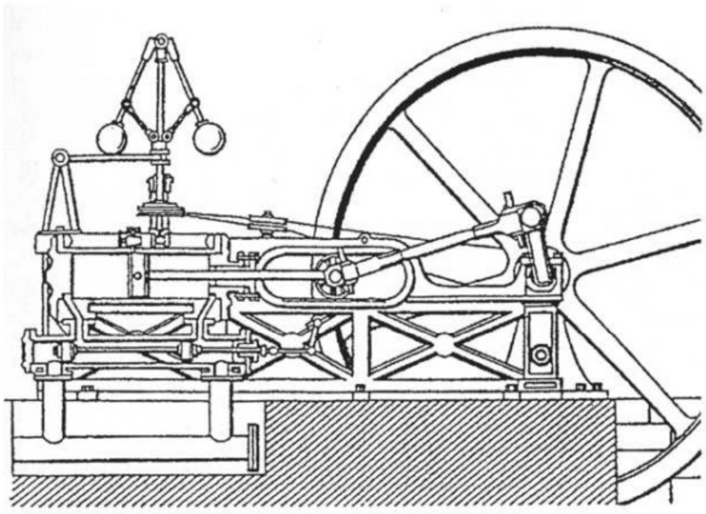
what you want : Requirements, invariants
follow a path, stay in lanes

Example



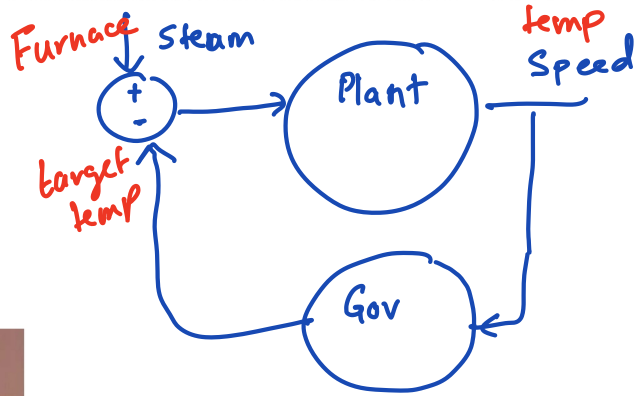
Feedback





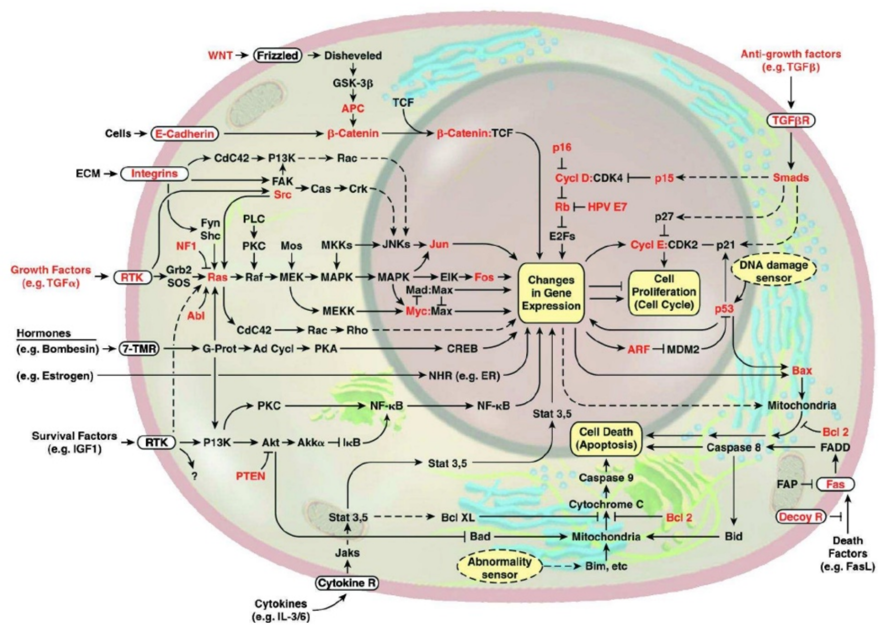
Feed forward

-ve Feed back



(a) F/A-18 "Hornet"

(b) X-45 UCAV



Modeling a system

$x_{t+1} = x_t + u_t$

State, Control input

Discrete time model

Recall deterministic automata

more generally $x_{t+1} = f(x_t, u_t)$

Plant model

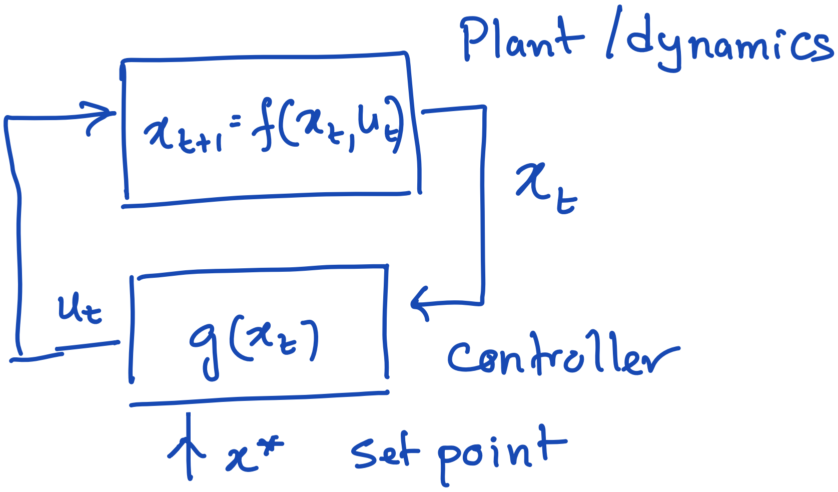
$x \in \mathbb{R}^n \quad u \in \mathbb{R}^m \qquad f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$

The control u, is decided based on the current state $u_t = g(x_t)$

Controller $\quad g: \mathbb{R}^n \to \mathbb{R}^m$

E.g. $g(x_t) =$ if $x \geq x^*$ then 0
else $u = 1$

Closed-loop system



$$x_{t+1} = f(x_t, g(x_t))$$

Continuous time model

$$\frac{dx}{dt} = f(x,u) \qquad u = g(x)$$

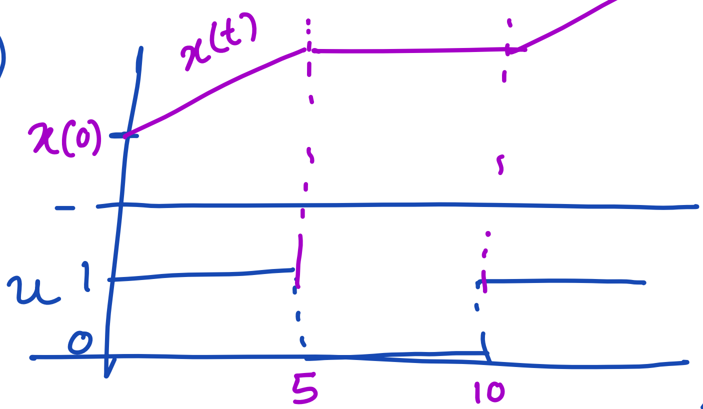
Example

$$\frac{dx}{dt} = u$$

Really this is relating the rate of change of x w.r.t time. Here $x : \mathbb{R} \to \mathbb{R}^n$ is a function of time

$$\frac{dx(t)}{dt} = f(x(t), u(t))$$

$$u(t) = g(x(t), x^*)$$

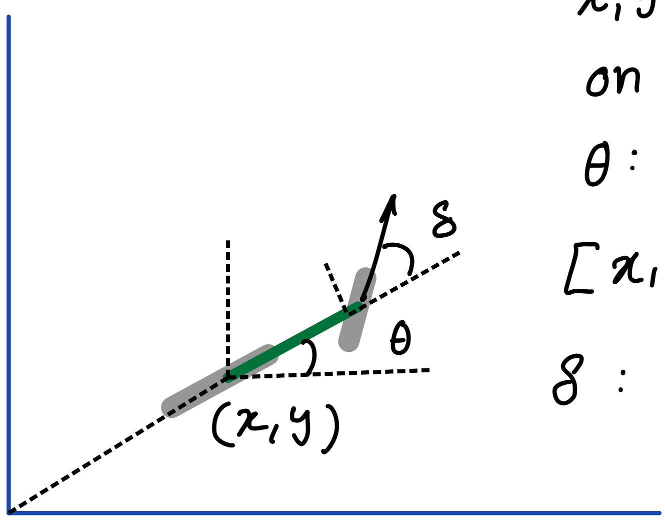


This is written in short as $\dot{x} = f(x,u)$ $\qquad \dot{x} = f(x, g(x))$ $\qquad \dot{x} = f'(x)$

Example 2 Bicycle / Kinematic Vehicle model

Ref. Brian Paden et al. 2016

Survey of motion planning and Control for Self-driving



v : longitudinal velocity

x, y : position of rear wheel on the plane

θ : heading angle

$[x, y, \theta, v]$: state

δ : steering angle
control input

ℓ : length of bicycle
parameter

Equations of motion derived in the above paper

$$\left.\begin{aligned} \dot{x}(t) &= v(t) \cos \theta(t) \\ \dot{y}(t) &= v(t) \sin \theta(t) \\ \dot{\theta}(t) &= \frac{v(t)}{\ell} \tan \delta(t) \end{aligned}\right\} \quad \dot{x} = f(x, u)$$

When do solutions exist? When unique?

Recall a solution for $\dot{x} = f(x)$ is $\xi : \mathbb{R} \to \mathbb{R}^n$

Can continue forward in time indefinitely

<u>Example</u> $\quad \dot{x} = x^2 \quad$ with $x(0) = 1 \quad x(t)$

Check that $x(t) = \dfrac{1}{1-t}$ is a solution ✓

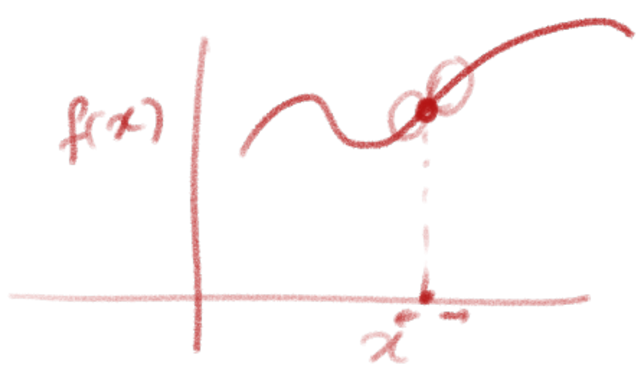
$$\frac{d\,x(t)}{dt} = -\frac{1}{(1-t)^2}(-1) = \frac{1}{(1-t)^2} = x^2(t)$$

But as $t \to 1 \quad x(t) \to \infty$ blows up

<u>Additional assumption on f</u>

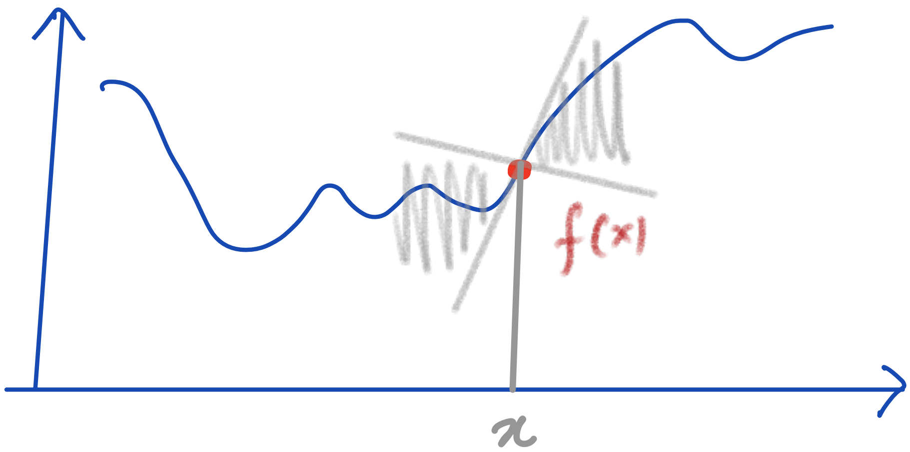
f should not grow too fast

intuition



Def. $f: \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz Continuous if $\exists L > 0$ such that for any $x, x' \in \mathbb{R}^n$

$$\| f(x) - f(x') \| \leq L \| x - x' \|$$



Example. $f(x) = 6x + 4$

All differentiable functions with bounded derivative are Lipschitz Continuous

$|x|$ not differentiable but Lipschitz

x^2 not Lipschitz $\quad \| f(x) - f(x') \|$

$$= \| x_1^2 - x_2^2 \|$$

Suppose $\exists L$ such that $\| x_1^2 - x_2^2 \| \leq L (x_1 - x_2)$

$x_1 + x_2 \leq L$ Contradiction

$\sqrt{x}$ is also not Lipschitz

$\dot{x} = f(x) \mid \dot{x} = f(x,u)$

Thm. If $f(x,u)$ is Lipschitz Continuous in the first argument and $u(t)$ is piecewise Continuous then $\dot{x} = f(x,u)$ has unique solutions.

Specifications for ODEs / Control Systems

Given a closed-loop system $\dot{x}(t) = f(x(t))$ and an initial set of states $X_0 \subseteq \mathbb{R}^n$ a Solution is any function $x : \mathbb{R} \to \mathbb{R}^n$ such tha

$$x(0) \in X_0 \text{ and } \forall t \quad \frac{d\, x(t)}{dt} = f(x(t))$$

Note the parallels with the definition of an execution $\alpha = x_0 x_1 \ldots$ of an automaton.

To show the dependence on $x_0 \in X_0$ Sometimes we write the solution as

$$\xi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \qquad \xi(x_0, t) = x(t) \quad \forall x_0 \in X_0$$

A specification for an ODE defines properties of the Solutions

(1) Is the solution Safe? $\forall x_0 \in X_0, t \geq 0 \quad \xi(x_0, t) \in S$

(2) Does it track target "waypoints"?

(3) Does it converge to an "equilibrium"?

$\forall x_0 \in X_0 \text{ as } t \to \infty \quad \xi(x_0, t) \to x^*$

Verification Problem

How can we check that a given System satisfies such properties

Synthesis / Design problem

How to design controller function $g: \mathbb{R}^n \to \mathbb{R}^m$ so that the closed loop system satisfies the property?

Equilibrium : Points that do not move

Def. A point $x^* \in \mathbb{R}^n$ is an equilibrium of $\dot{x} = f(x)$ if

$$f(x^*) = 0 \qquad \dot{x} = f(x^*) = 0$$

Linear ODEs

$\dot{x} = f(x, u)$ where f is linear

Any linear function $y = f(x)$ $\quad x, y \in \mathbb{R}^n$
can be represented by a matrix $A \in \mathbb{R}^{n \times n}$

$$y = Ax$$

$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$ Linear Time Varying System

$\forall t \quad A(t) \in \mathbb{R}^{n \times n} \qquad B(t) \in \mathbb{R}^{m \times n}$

If $A(t)$ and $B(t)$ are independent of time

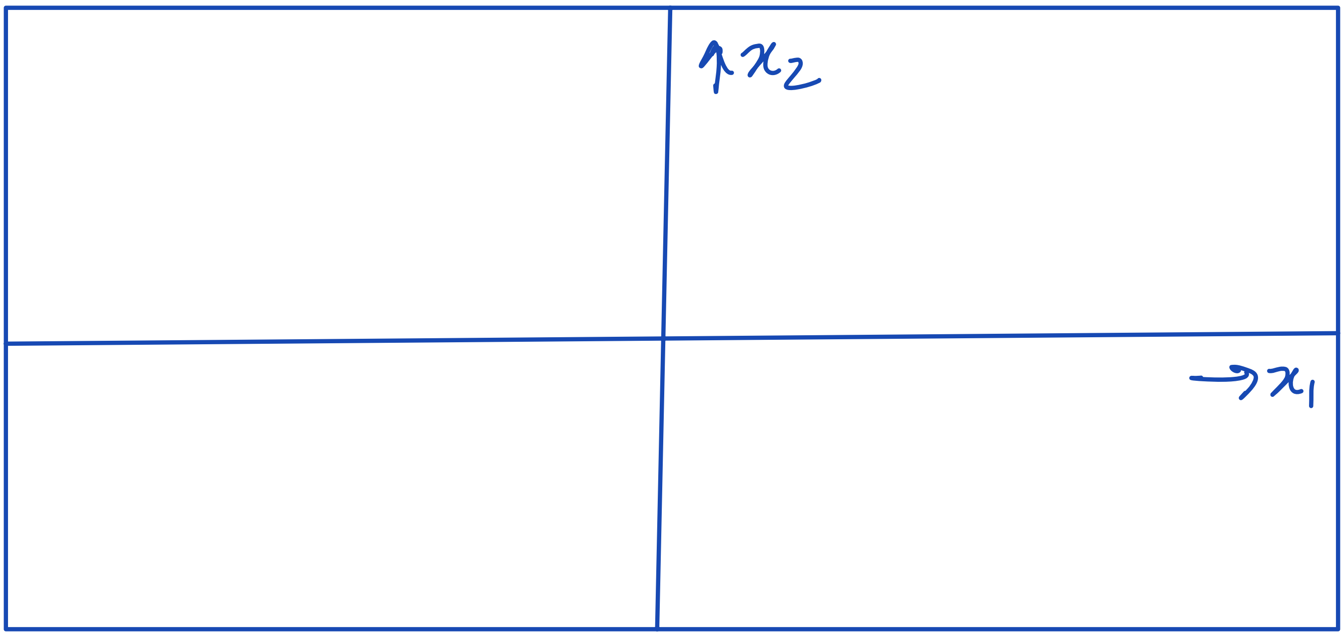
$\dot{x} = Ax + Bu$ Linear Time Invariant System (LTI)

Equilibria of LTI $= \{x \mid Ax = 0\}$

<u>Example</u>. Double Integrator (MA 180)

$$\frac{dx_1}{dt} = x_2 \qquad \frac{dx_2}{dt} = u$$

Openloop dynamics $u=0$



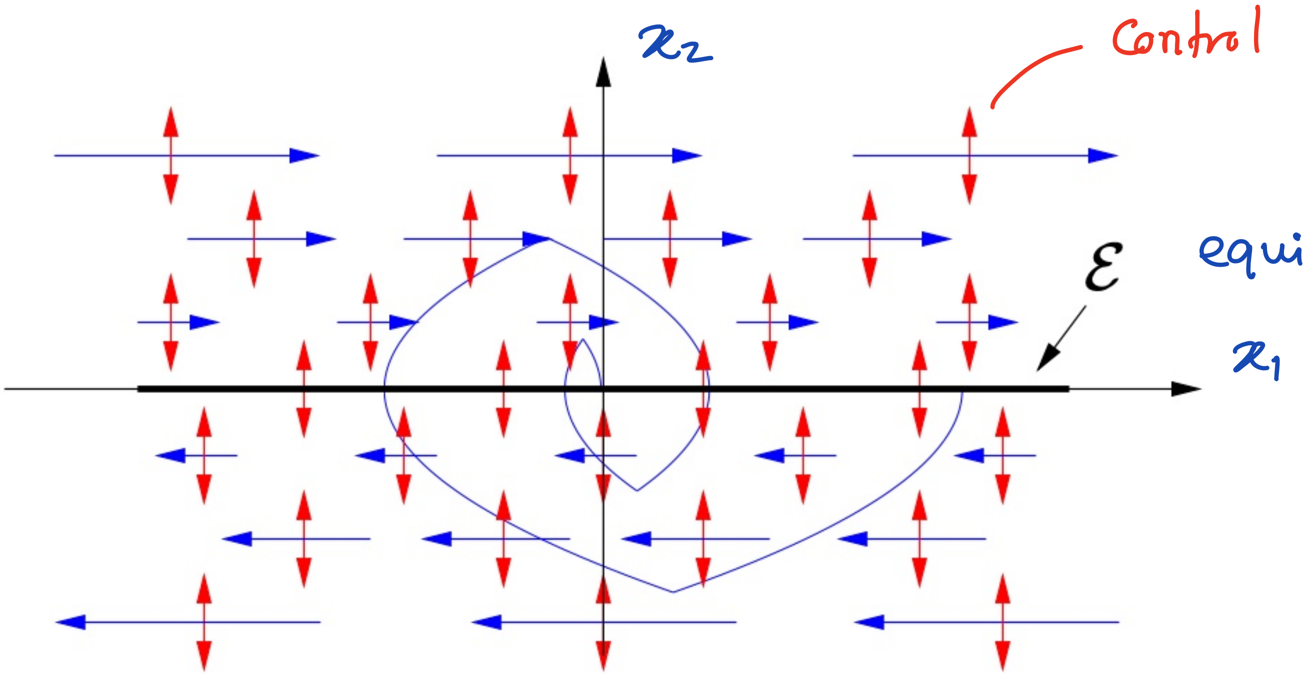
What are the equilibria?

$$\begin{bmatrix} \dot{x}_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

With $u=0$ the set of equilibria

$$\begin{bmatrix} x_2^* \\ 0 \end{bmatrix} = 0 \quad \text{The } x\text{-axis}$$

How to drive the system to 0?



Phase portrait