# Sampling-based Planning and Control
# Lecture 18

Principles of Safe Autonomy ECE484

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# Motion planning problem

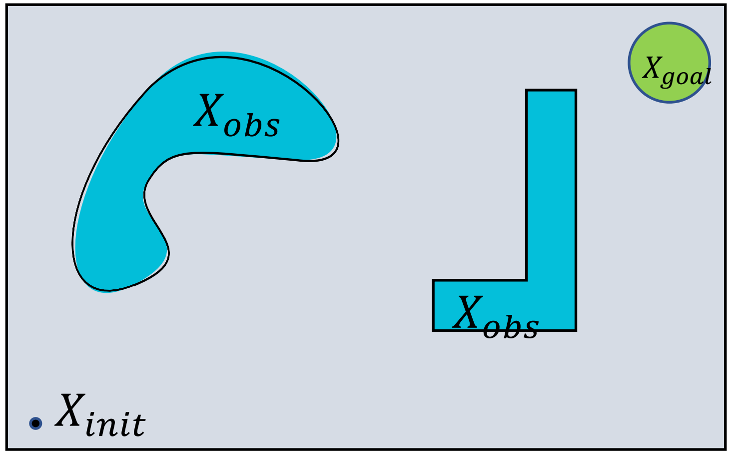
- Get from point A to point B avoiding obstacles
- Last few lectures we saw how to search for collision free trajectories can be converted to graph search
  - Hybrid A* aims to add dynamical constraints to make paths realizable
- Now we will study sampling-based motion planning/search algorithms that can directly incorporate dynamical constraints

# Motion planning problem

Consider a dynamical control system defined by an ODE of the form $\frac{dx}{dt} = f\,(x, u), \;\; x(0) = x_{init}. \quad (1)$

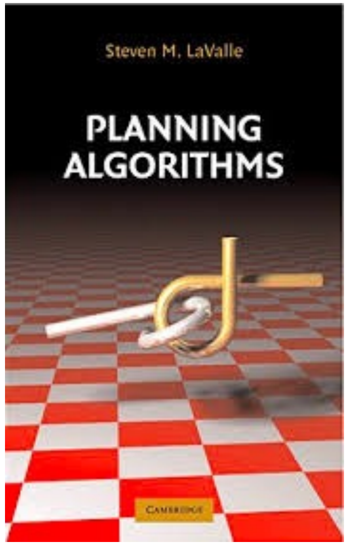
where $x$ is the state, $u$ is the control. Given an obstacle set $X_{obst} \subset R^d$ , and a goal set $X_{goal} \subset R^d$, the objective of the motion planning problem is to find, if it exists, a control signal $u$ such that the solution of (1) satisfies

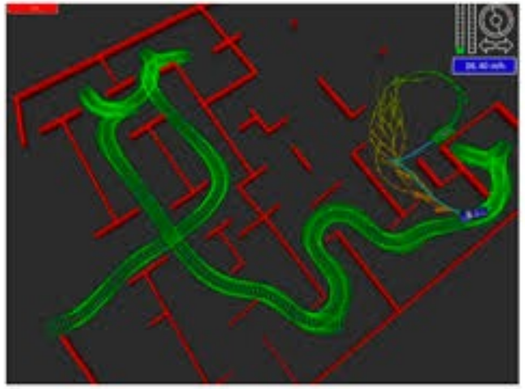
- for all $t \in R_{\geq 0}, x(t) \notin Xobs$ , and
- for some finite T ≥ 0, for all t > T , $x(t) \in X_{goal}$, 0
- Return failure if no such control signal exists.



# Canonical problem

- Basic problem in robotics
  - Autonomous vehicles
  - Puzzles
- Provably very hard: a basic version (the Generalized Piano Mover's problem) is known to be PSPACE-hard [Reif, '79].
- But evolution has found ingenious solutions: E.g., Tunicate is a simple organism capable of basic planning to finds a suitable rock to cement itself in place.
  - Once it becomes stationary, it digests its own cerebral ganglion, or "eats its own brain" and develops a thick covering, a "tunic" for self defense. [S. Soatto, 2010, R. Bajcsy, 1988] [http://en.wikipedia.org/wiki/File:Sea-tulip.jpg](http://en.wikipedia.org/wiki/File:Sea-tulip.jpg).



# Types of planners

- **Discretization + graph search**: Analytic/grid-based methods do not scale well to high dimensions.
  - $A*$ , $D*$ , etc. can be sensitive to graph size. Resolution complete.
- **Algebraic planners**: Explicit representation of obstacles.
  - Use complicated algebra (visibility computations/projections) to find the path. Complete, but often impractical.
- **Potential fields/navigation functions**: Virtual attractive forces towards the goal, repulsive forces away from the obstacles.
  - No completeness guarantees, unless "navigation functions" are available—very hard to compute in general.

# Sampling-based algorithms

Solutions are computed based on samples from some distribution.

Retain some form of completeness, e.g., probabilistic completeness

Incremental sampling methods

- Lend themselves to real-time, on-line implementations
- Can work with very general dynamics
- Do not require explicit constraints

# Outline

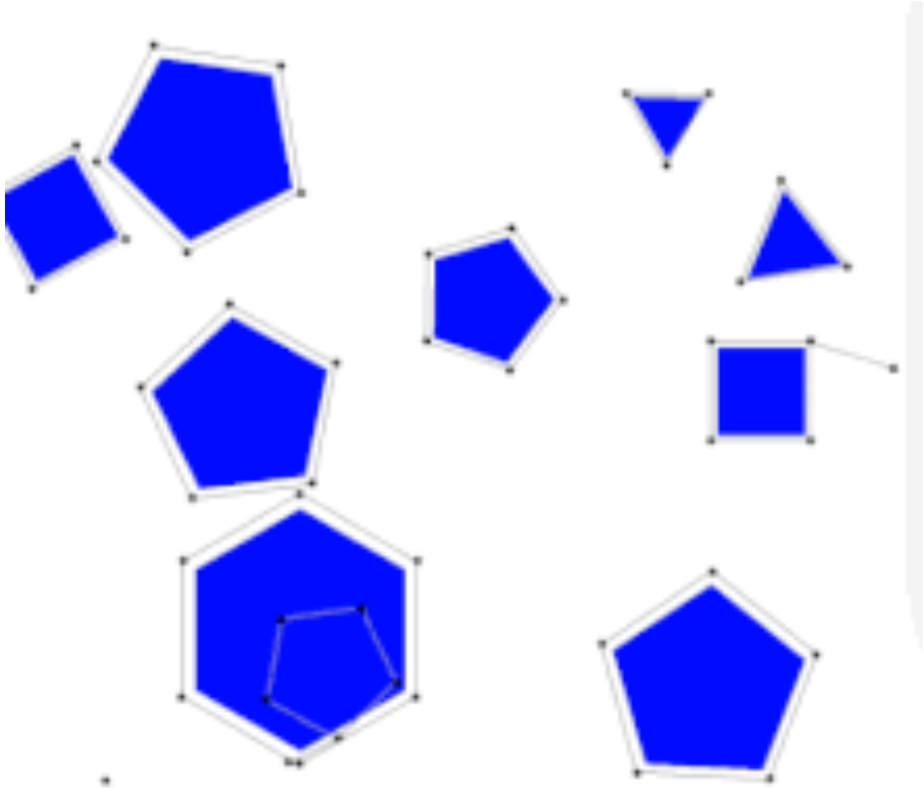
- Probabilistic Roadmaps
- Rapidly expanding random trees (RRT)
- RRG

# Probabilistic RoadMaps (PRM)

- Introduced by Kavraki and Latombe in 1994
- Mainly geared towards “multi-query” motion planning problems
- Idea: build (offline) a graph (i.e., the roadmap) representing the “connectivity” of the environment; use this roadmap to figure out paths quickly at run time.
- Learning/pre-processing phase:
  - Sample n points from $X_{free} = [0,1]^d \setminus X_{obs}$
  - Try to connect these points using a fast “local planner” (e.g., ignore obstacles).
  - If connection is successful (i.e., no collisions), add an edge between the points.
- At run time:
  - Connect the start and end goal to the closest nodes in the roadmap
  - Find a path on the roadmap
- First planner ever to demonstrate the ability to solve general planning problems in > 4-5 dimensions!

# PRM in action

[Kavraki, L. E.](); Svestka, P.; [Latombe, J.-C.](); [Overmars, M. H.]() *(1996), "Probabilistic roadmaps for path planning in high-dimensional configuration spaces", IEEE Transactions on Robotics and Automation,* ***12*** *(4): 566–580*



Picture from Wikipedia.org
https://en.wikipedia.org/wiki/Probabilistic_roadmap

# Simple PRM construction

$V \leftarrow \{x_{init}\} \cup \{\text{SampleFreei}\}_{i=1,\ldots,N-1}$

$E \leftarrow \emptyset$

**foreach** $v \in V$ **do**

  $U \leftarrow \text{Near}(G = (V, E), v, r) \setminus \{v\}$

    **foreach** $u \in U$ **do**

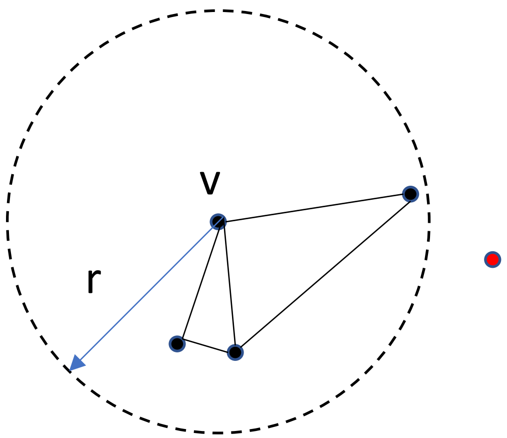
      if CollisionFree(v, u) **then**

        $E \leftarrow E \cup \{(v, u), (u, v)\}$

**return** $G = (V, E)$

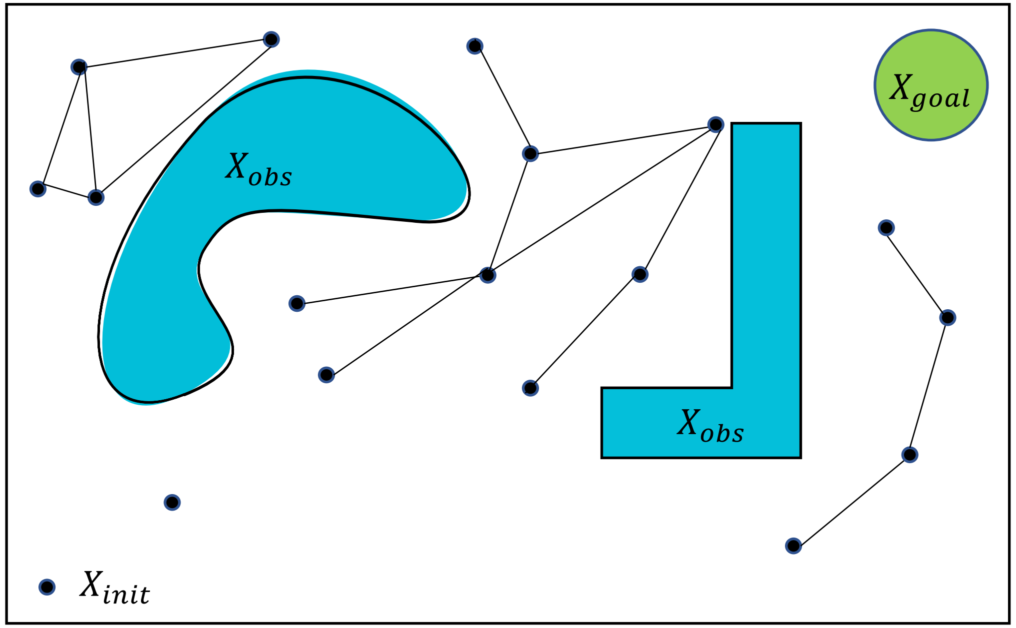
Near(G, v, r): Finds the subset of vertices in G that are within r distance of v

CollisionFree(v, u): checks whether there is a path from u to v that does not collide with the obstacles



# Probabilistic RoadMap

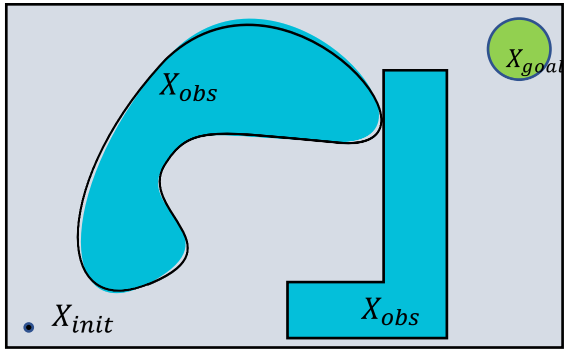
- Connect points within a radius r, starting from "closest" ones
- Do not attempt to connect points already on the same connected component of PRM
- What properties does this algorithm have?
  - Will it find a solution if one exists?
  - Is this an optimal solution?
  - What is the complexity?



# Probabilistic completeness

**Definition.** A motion planning problem $P = (X_{free}, x_{init}, X_{goal})$ is ***robustly feasible*** if there exists some small δ>0 such that a solution remains a solution if obstacles are "dilated" by δ.

**Fig. not robust.**




**Definition.** An algorithm ALG is ***probabilistically complete*** if, for any *robustly feasible* motion planning problem defined by $P = (X_{free}, x_{init}, X_{goal})$, $\lim_{N\to\infty} \Pr(ALG\ returns\ a\ solution\ to\ P) = 1$.

- Applicable to motion planning problems with a robust solution.

# Asymptotic optimality

Suppose we have a cost function $c$ that associates to each path $\sigma$ a non-negative cost $c(\sigma)$, e.g., $c(\sigma) = \int_\sigma \chi(s)\, ds$.

**Definition.** An algorithm ALG is *asymptotically optimal* if, for any motion planning problem $P = (X_{free}, x_{init}, X_{goal})$ and cost function $c$ that admit a robust optimal solution with finite cost $c^*$,

$$\boldsymbol{P}\left(\left\{\lim_{i\to\infty} Y_i^{ALG} = c^*\right\}\right) = 1$$

# Remarks on PRM

- The simplified version of the PRM (sPRM) algorithm has been shown to be probabilistically complete. (No proofs available for the "real" PRM!)
- Moreover, the probability of success goes to 1 exponentially fast, if the environment satisfies certain "good visibility" conditions.
- But, NOT asymptotically optimal
- New key concept: combinatorial complexity vs. "visibility"

# Complexity of Sampling-based Algorithms

- How can we measure complexity for an algorithm that does not necessarily terminate?
  - Treat the number of samples as "the size of the input." (Everything else stays the same)
  - Complexity per sample: how much work (time/memory) is needed to process one sample.
  - Useful for comparison of sampling-based algorithms. Not for deterministic, complete algorithms.
- Complexity of PRM for N samples $\Theta(N^2)$
- Practical complexity reduction tricks
  - k-nearest neighbors: connect to the k nearest neighbors. Complexity Θ(N log N). (Finding nearest neighbors takes log N time.)
  - Bounded degree: connect at most k neighbors among those within radius r.
  - Variable radius: change the connection radius r as a function of N. How?

# Rapidly Exploring Random Trees (RRT)

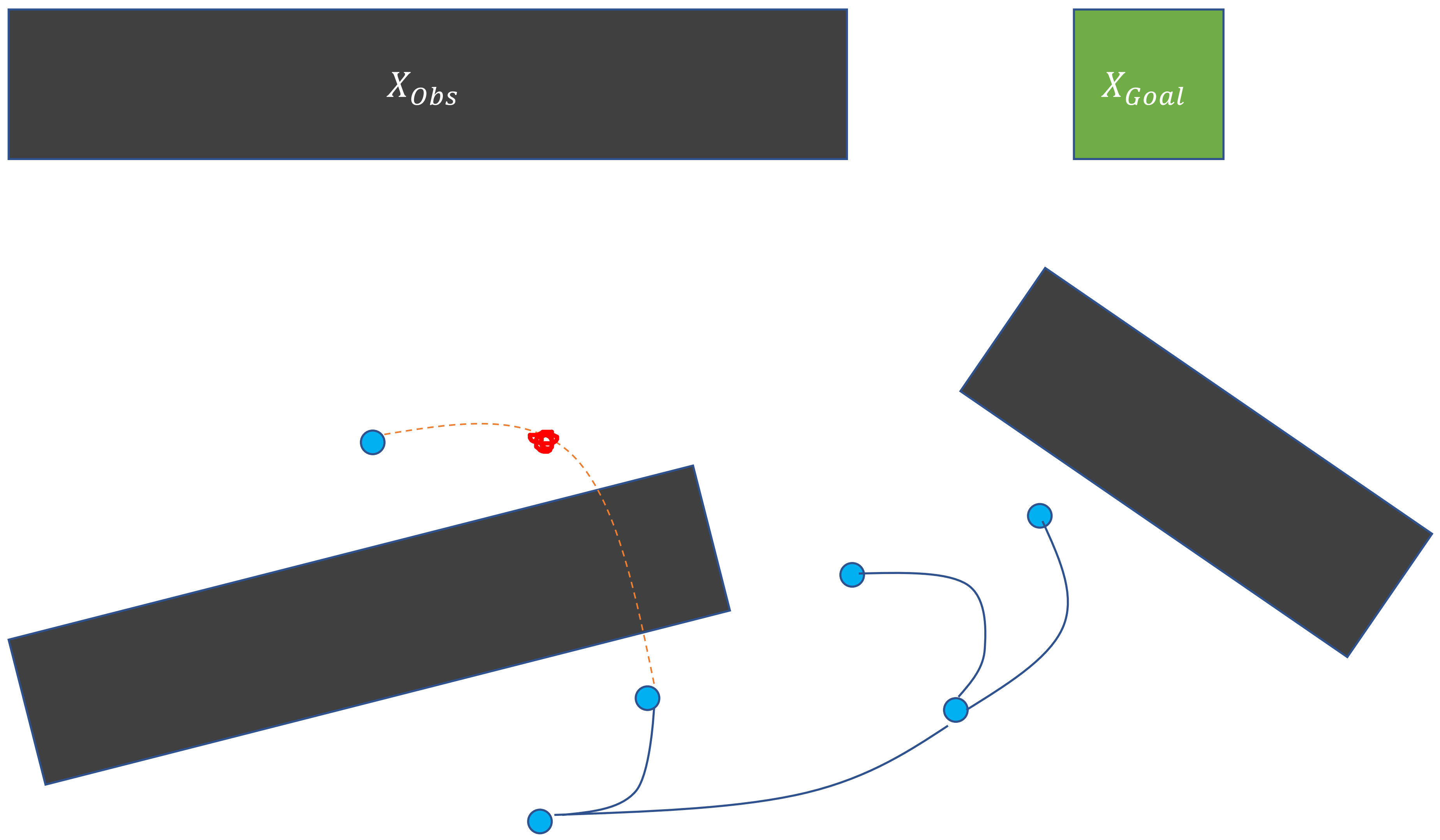
- Introduced by LaValle and Kuffner in 1998
- Appropriate for single-query planning problems
- Idea: build (online) a tree, exploring the region of the state space that can be reached from the initial condition.
- At each step: sample one point from $X_{free}$, and try to connect it to the closest vertex in the tree.
- Very effective in practice

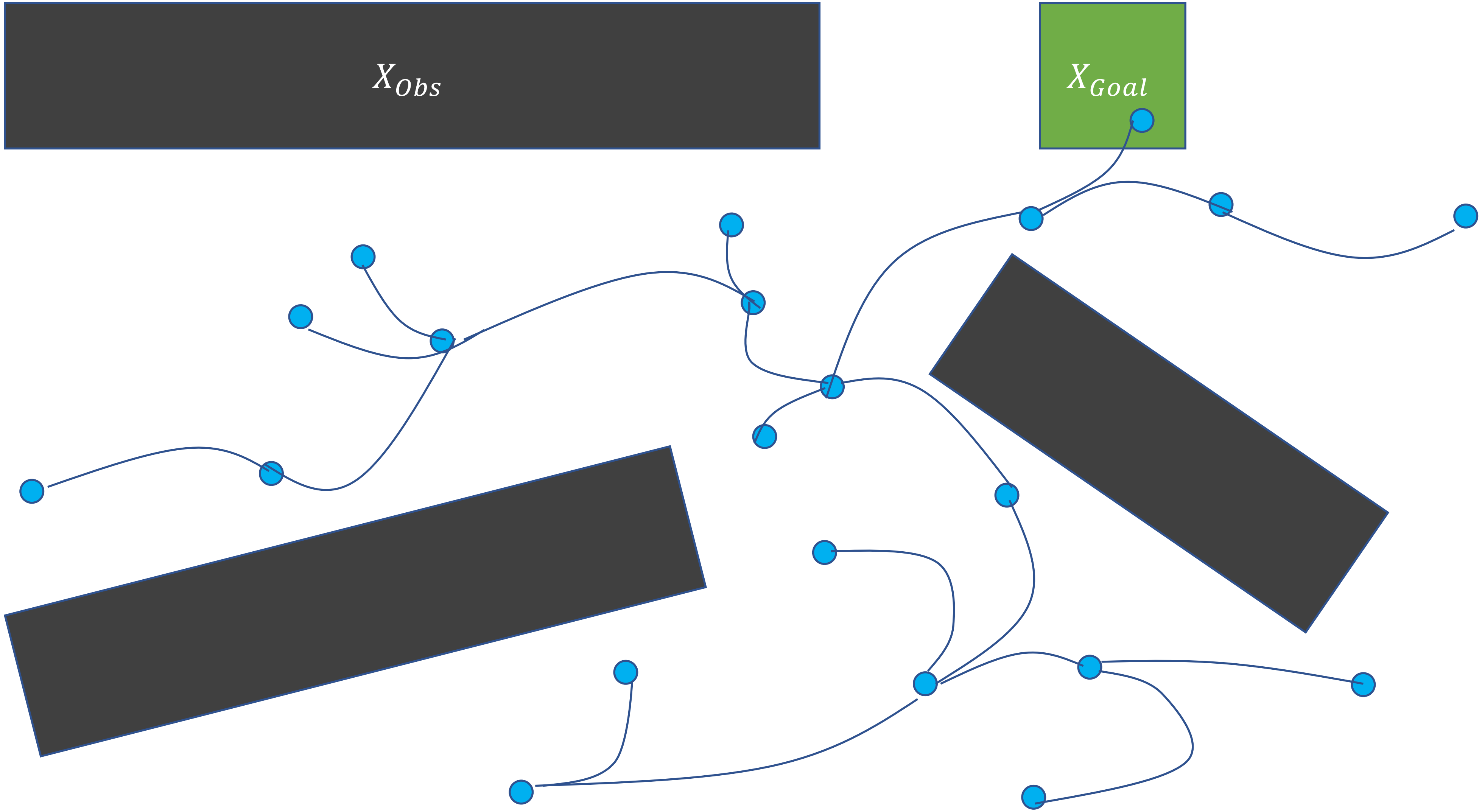
# RRT

LaValle, Steven M.; Kuffner Jr., James J. (2001). "Randomized Kinodynamic Planning" (PDF). *The International Journal of Robotics Research (IJRR)*. **20** (5): 378–400. doi:10.1177/02783640122067453. S2CID 40479452.

# RRT

$V \leftarrow \{x_{init}\}$; $E \leftarrow \emptyset$
**for** $i = 1, \ldots, N$ **do**
  $x_{rand} \leftarrow \text{SampleFree}_i$
  $x_{nearest} \leftarrow \text{Nearest}(G = (V, E), x_{rand})$
  $x_{new} \leftarrow \text{Steer}(x_{nearest}, x_{rand})$
  if $\text{ObtacleFree}(x_{nearest}, x_{new})$ **then**
    $V \leftarrow V \cup \{x_{new}\}$
    $E \leftarrow E \cup \{(x_{nearest}, x_{new})\}$
**return** $G = (V, E)$


$X_{Obs}$
$X_{Goal}$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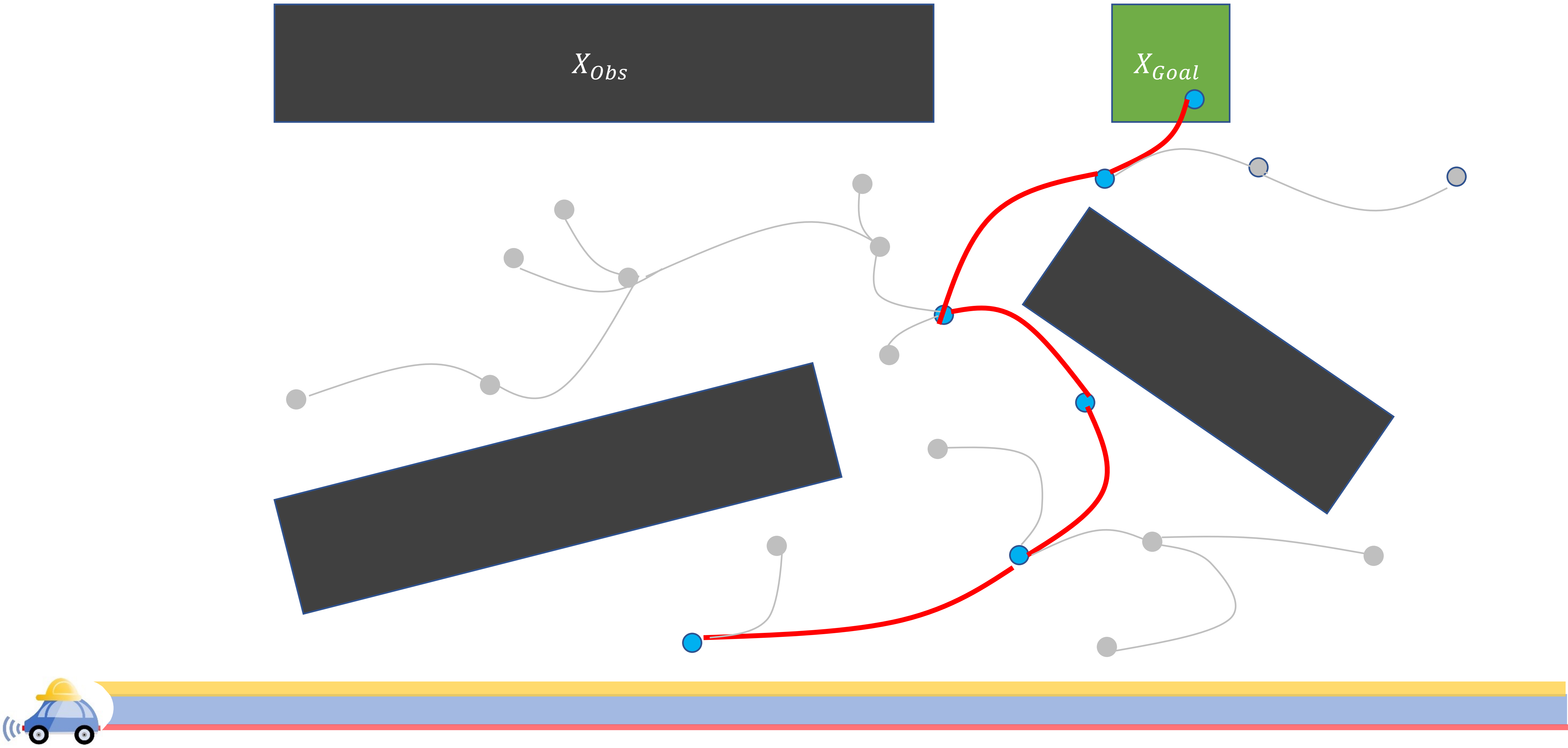
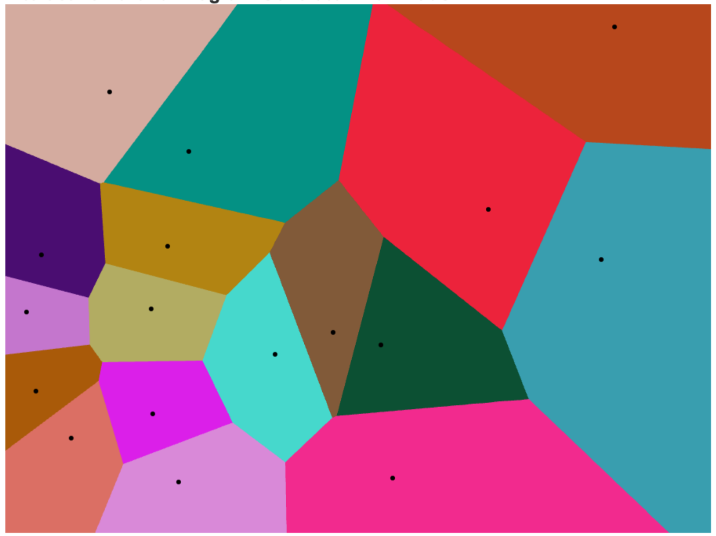

$X_{Obs}$
$X_{Goal}$

# Voronoi bias



Given n points in d dimensions, the ***Voronoi diagram*** of the points is a partition of $R^d$ into regions, one region per point, such that all points in the interior of each region lie closer to that regions site than to any other site.

Try it: [http://alexbeutel.com/webgl/voronoi.html](http://alexbeutel.com/webgl/voronoi.html)

***Voronoi bias.*** Vertices of the RRT that are more “isolated” (e.g., in unexplored areas, or at the boundary of the explored area) have larger Voronoi regions—and are more likely to be selected for extension.

# RRT in action [Frazzoli]

- Talos, the MIT entry to the 2007 DARPA Urban Challenge, relied on an“RRT-like” algorithm for real-time motion planning and control.
- The devil is in the details: provisions needed for, e.g.,
  - Real-time, on-line planning for a safety-critical vehicle with substantial momentum.
  - Uncertain, dynamic environment with limited/faulty sensors.
- Main innovations [Kuwata, et al. ’09]
  - Closed-loop planning: plan reference trajectories for a closed-loop model of the vehicle under a stabilizing feedback
  - Safety invariance: Always maintain the ability to stop safely within the sensing region.
  - Lazy evaluation: the actual trajectory may deviate from the planned one,need to efficiently re-check the tree for feasibility.
- The RRT-based P+C system performed flawlessly throughout the race.
- https://journals.sagepub.com/doi/abs/10.1177/0278364911406761

# Limitations

The MIT DARPA Urban Challenge code, as well as other incremental sampling methods, suffer from the following limitations:

- No characterization of the quality (e.g., "cost") of the trajectories returned by the algorithm.
- Keep running the RRT even after the first solution has been obtained, for as long as possible (given the real-time constraints), hoping to find a better path than that already available.
- No systematic method for imposing temporal/logical constraints, such as, e.g., the rules of the road, complicated mission objectives, ethical/deontic code.
- In the DARPA Urban Challenge, all logics for, e.g., intersection handling, had to be hand-coded, at a huge cost in terms of debugging effort/reliability of the code.

# RRTs and Asymptotic Optimality

- RRTs are great at finding feasible trajectories quickly, however, RRTs are apparently terrible at finding good trajectories. Why?
- Let $Y^{RRT}_n$ be the cost of the best path in the RRT at the end of iteration n.
- It is easy to show that $Y^{RRT}_n$ converges (to a random variable), $\lim_{n\to\infty} Y_n^{RRT} = Y_\infty^{RRT}$.
- The random variable $Y_\infty^{RRT}$ is sampled from a distribution with zero mass at the optimum

**Theorem [Karaman & Frizzoli`10]** (Almost sure ***sub-optimality*** of RRTs) If the set of sampled optimal paths has measure zero, the sampling distribution is absolutely continuous with positive density in $X_{free}$, and d ≥ 2, then the best path in the RRT converges to a sub-optimal solution almost surely, i.e.,

$$\Pr[Y_\infty^{RRT} > c^*] = 1.$$

# Why is RRT not asymptotically optimal?

- Root node has infinitely many subtrees that extend at least a distance $\epsilon$ away from $x_{init}$.
- The RRT algorithm "traps" itself by disallowing new better paths to emerge. Why?
- Heuristics such as running the RRT multiple times, running multiple trees concurrently etc., work better than the standard RRT, but also result in almost-sure sub-optimality.
- A careful rethinking of the RRT algorithm is required for (asymptotic) optimality.

# Rapidly Exploring Random Graphs (possibly cyclic)

$V \leftarrow \{x_{init}\}$; $E \leftarrow \emptyset$;

**for** i = 1, . . . , N **do**

  $x_{rand} \leftarrow$ SampleFreei;

  $x_{nearest} \leftarrow$ Nearest($G = (V, E), x_{rand}$);

  $x_{new} \leftarrow$ Steer($x_{nearest}, x_{rand}$) ;

  **if** ObtacleFree($x_{nearest}, x_{new}$) **then**

    $X_{near} \leftarrow$ Near($G = (V, E), x_{new}, \min\{\gamma_{RRG}(\log(\text{card } V)/ \text{card } V)^{1/d}, \eta\}$) ;

    $V \leftarrow V \cup \{x_{new}\}$; $E \leftarrow E \cup \{(x_{nearest}, x_{new}),(x_{new}, x_{nearest})\}$ ;

    **foreach** $x_{near} \in X_{near}$ **do**

      if CollisionFree($x_{near}, x_{new}$) **then** $E \leftarrow E \cup \{(x_{near}, x_{new}),(x_{new}, x_{near})\}$

**return** $G = (V, E)$;

At each iteration, the RRG tries to connect the new sample $x_{new}$ to all vertices in a ball of radius $r_n$ centered at it. (Or just default to the nearest one if such b all is empty.)

# Theorems [Proofs not required for exam]

- **Probabilistic completeness.** Since $V_n^{RRG} = V_n^{RRT}$, for all n RRG has the same completeness properties as RRT, i.e.,

$$Pr\left[V_n^{RRG} \cap X_{goal} = \emptyset\right] = O\left(e^{-bn}\right).$$

- **Asymptotic optimality.** If the **Near** procedure returns all nodes in V within a ball of volume $Vol = \frac{\gamma \log n}{n}, \gamma > 2^d \left(1 + \frac{1}{d}\right)$, under some additional technical assumptions (e.g., on the sampling distribution, on the $\epsilon$ clearance of the optimal path, and on the continuity of the cost function), the best path in the RRG converges to an optimal solution almost surely, i.e.,

$$\Pr[Y_\infty^{RRG} = c^*] = 1.$$

# Final thoughts on RRG

- What is the additional computational load?
  - O(log n) extra calls to ObstacleFree compared to RRT
- Key idea in RRG/RRT*:
  - Combine optimality and computational efficiency, it is necessary to attempt connection to Θ(log N) nodes at each iteration.
  - Reduce volume of the "connection ball" as log(N)/N;
  - Increase the number of connections as log(N).
- These principles can be used to obtain "optimal" versions of PRM, etc.

# Summary and future directions

- State-of-the-art algorithms such as RRT converge to a NON-optimal solution almost-surely
- new algorithms (RRG and the RRT$*$ ), which almost-surely converge to optimal solutions while incurring no significant cost overhead
- Bibliographical reference: S. Karaman and E. Frazzoli. Sampling-based algorithms for optimal motion planning. Int. Journal of Robotics Research, 2011. TAlso available at http://arxiv.org/abs/1105.1186.
- research directions:
  - Optimal motion planning with temporal/logic constraints
  - Anytime solution of differential games
  - Stochastic optimal motion planning (process + sensor noise)
  - Multi-agent problems.

| Algorithm | Prob. Completeness | Asymptotic Optimality | Complexity |
|---|---|---|---|
| sPRM | Yes | Yes | O(N) |
| k-nearest sPRM | No | No | O(log N) |
| RRT | Yes | No | O(log N) |
| PRM* | Yes | Yes | O(log N) |
| k-nearest PRM* | Yes | Yes | O(log N) |
| RRG | Yes | Yes | O(log N) |
| k-nearest RRG | Yes | Yes | O(log N) |
| RRT* | Yes | Yes | O(log N) |
| k-nearest RRT* | Yes | Yes | O(log N) |