Kalman Filtering
Motivation

- Suppose we have a simple robot that can move in 1D
  - The robot starts from some initial position $x_0$
  - We have a dynamics model of the robot but it’s not precise
  - We can periodically receive *noisy* measurements of the robot’s position
- Question: Can we combine our dynamics model with the noisy measurements to get a better prediction of the robot’s true position?
Kalman Filtering

- The Kalman filter combines our estimation of the current state (produced by the dynamics model) with noisy measurements to produce a better estimation.

Dynamics model:

\[ x_k = F_k x_{k-1} + B_k u_k + w_k \]

Where \( F \) is the state evolution model, \( B \) is the control input model, and \( w \) is the process noise (mean zero Gaussian distribution).
Kalman Filtering

Measurement model:

\[ z_k = H_k x_k + v_k \]

Where \( H \) is the measurement mapping (mapping state to observation) and \( v \) is the measurement noise (again, mean zero Gaussian distribution)

Our goal is to make **predictions** with the dynamics model, and then **update** the predictions using measurements
Prediction Step

We first predict the current state (given the last state) using the dynamics model:

$$\hat{x}_{k|k-1} = F_k x_{k-1|k-1} + B_k u_k$$

Note the process noise term is missing. The process noise is captured by the state covariance matrix given by:

$$\hat{P}_{k|k-1} = F_k P_{k-1|k-1} F_k^T + Q_k$$

Where $Q$ is the covariance of the process noise ($w$). At each prediction step, we generate both the state prediction and state covariance prediction.
Update Step

Whenever a new measurement is received, we compute the difference between the measurement and the predicted measurement:

\[ \tilde{y}_k = z_k - H_k \hat{x}_{k|k-1} \]

We then compute the Kalman gain (\( K_k \)) which is used to weight the update step:

\[ K_k = \hat{P}_{k|k-1} H_k^T (H_k \hat{P}_{k|k-1} H_k^T + R_k)^{-1} \]

We update the predicted state and covariance with the Kalman gain:

\[ x_{k|k} = \hat{x}_{k|k-1} + K_k \tilde{y}_k \]

\[ P_{k|k} = (I - K_k H_k) \hat{P}_{k|k-1} \]
Prior knowledge of state $\rightarrow \begin{align*}
    \mathbf{P}_{k-1|k-1} \\
    \hat{\mathbf{x}}_{k-1|k-1}
\end{align*}$

$\rightarrow$ Prediction step

$\rightarrow$ Based on e.g. physical model

$\rightarrow$ Next timestep

$\rightarrow k \leftarrow k + 1$

$\rightarrow \begin{align*}
    \mathbf{P}_{k|k} \\
    \hat{\mathbf{x}}_{k|k}
\end{align*}$

$\rightarrow$ Update step

$\rightarrow$ Compare prediction to measurements

$\rightarrow$ Measurements $\rightarrow y_k$

$\rightarrow$ Output estimate of state
1D Example

time = 1

prediction

measurement

correction

time = 2

prediction

measurement

correction
2D Example
Limitations

- Kalman filter is optimal (minimizes expected MSE) if:
  - Linear system dynamics
  - Gaussian process and measurement noise
- In practice the Kalman filter is fairly robust to non-Gaussian noise
- Modified versions of the Kalman filter exist to address non-linear systems
  - Extended Kalman filter (EKF)
  - Unscented Kalman filter