Reference : Probabilistic Robotics by S. Thrun et al

Roadmap.

- Localization / estimation problem
- Review of Bayes rule
- One step estimation
- Multi-step estimation Bayes filter algorithm
  Histogram filter

Basic problem of localization / State estimation

State is evolving based on input and noisy dynamics

$$x_{t+1} = f(x_t, u_t) + \omega_t$$

x_t : State $\longrightarrow$ we cannot observe / measure this directly

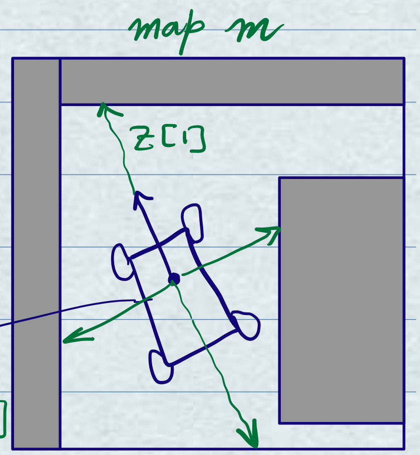
u_t : Control input $\rightarrow$ This is known

f, ω : model noise $\rightarrow$ we know model and distribution for ω_t

$$z_t = h(x_t)$$

z_t : measurement

g : measurement model



$\vec{x} = (x, y, \theta)$

$z_t = \langle z_t[1], \dots z_t[4] \rangle$

How to use measurements to improve estimation of x_t.

Review of Bayes Rule / Conditional probability

X : Random variable taking values $x_1, x_2 \dots x_k$

$P(X = x_i)$ written in short as $P(x_i)$

$P(X = x_i, Z = z_j)$ written as $P(x_i, y_j)$

$P(X = x \mid Z = z)$ Conditional probability of $X = x$ given

Def. $$P(x \mid z) = \frac{P(x, z)}{P(z)} \quad \text{— (1)}$$ Provided $P(z) > 0$

i.e. $$P(x, z) = P(x \mid z)\, P(z)$$

$$P(z \mid x) = \frac{P(x, z)}{P(x)}$$

$$P(x, z) = P(z \mid x)\, P(x) \quad \text{— (3)}$$

Replacing (3) in (1)

Bayes Rule

$$\boxed{P(x \mid z) = \frac{P(z \mid x)\, P(x)}{P(z)}}$$

Prior

Posterior

inverse Conditional Prob

$$\boxed{P(x \mid y, z) = \frac{P(y \mid x, z)\, P(x \mid z)}{P(y \mid z)}} \qquad (4)$$

Derive this (exercise)

$$P(x \mid y, z) = \frac{P(x\, y\, z)}{P(y, z)} \qquad P(z) > 0$$

$= P(y \mid x, z)$.

$$= \frac{P(y \mid x\, z)\, P(x, z)}{P(y\, z)}$$

$$= \frac{P(y \mid x, z)\, P(x \mid z)\, \cancel{P(z)}}{P(y \mid z)\, \cancel{P(z)}}$$

<u>Law of total probability</u>

$$\sum_{x_i} P(X = x_i) = \sum_{x_i} P(x_i) = 1$$

$$P(Y = y_j) = \sum_{x_i} P(Y = y_j \wedge X = x_i) = \sum_{x_i} P(y \mid x_i)\, P(x_i)$$

$$P(Y = y_j) = \int_{x_i} P(y_j \mid x_i)\, P(x_i)\, dx_i$$

in the continuous setting

In our estimation problem (one shot)

X : position

: measurement

$P(X=x)$: Prior knowledge about position

$P(X=x \mid Z=z)$: Posterior

$P(Z=z \mid X=x)$: Generative model for measurements (analogous to g)

$P(z)$: does not depend on X

- normalization constant $\eta = 1/pr$)

$$P(x \mid z) = \eta \, P(z \mid x) \, P(x)$$

Estimation over a sequence of steps

State evolves in discrete steps

$X_1 = x_1 \quad X_2 = x_2 \quad \cdots\cdots$ Sequence of RVs

$$P(X_{t+1} = x_{t+1} \mid X_t = x_t) \qquad P(x_{t+1} \mid x_t)$$

↑ State evolution model

In case there is no noise or input

$$P(X_{t+1} = f(x_t) \mid X_t = x_t) = \underline{1}$$

With inputs and measurements

$$P(X_t = x_t \mid X_0 = x_0, X_1 = x_1 \cdots X_t = x_{t-1}$$
$$U_1 = u_1 \quad U_2 = u_2 \cdots U_{t-1} = u_{t-1}$$
$$Z_1 = z_1 \cdots Z_{t-1} = z_{t-1})$$

We write in brief as

$$P(x_t \mid x_{0:t-1}, z_{1:t-1}, u_{1:t-1})$$

We assume Markovian State evolution model

$$P(x_t \mid x_{0:t-1}, z_{1:t-1}, u_{1:t-1}) = P(x_t \mid x_{t-1}, u_{t-1})$$

We also write this as

$$P(x' \mid x, u)$$

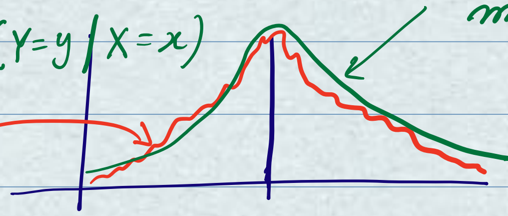
Measurement model

$P(z_t \mid x_{0:t}, z_{1:t-1}, u_{0:t-1})$

We assume $= P(z_t \mid x_t)$

state is complete $\uparrow$ measurement model

<u>Example</u> Simple range sensor



<u>Derivation of Bayes filter</u>

Def

Belief is defined as the posterior distribution over state (at time t) given all past measurements and control inputs.
Denoted by $bel(x_t)$

$$bel(x_t) = P(x_t \mid z_{1:t}, u_{1:t})$$

$$= P(\underbrace{x_t}_{x} \mid \underbrace{z_t}_{y}\ \underbrace{z_{1:t-1}\ u_{1:t}}_{z})$$

Using ④

$$= \frac{P(z_t \mid x_t\ z_{1:t-1}\ u_{1:t})\, P(x_t \mid z_{1:t-1}\ u_{1:t})}{P(z_t \mid z_{1:t-1}, u_{1:t})}$$

$$= \eta\, P(z_t \mid x_t\ z_{1:t-1}\ u_{1:t})\, P(x_t \mid z_{1:t-1}\ u_{1:t})$$

$$\boxed{bel(x_t) = \eta\, P(z_t \mid x_t)\, \overline{bel}(x_t)}$$ Correction based on measurement

because measurement model assumes state is complete

$\overline{bel}(x_t)$ belief about state at time t before using measurement info at t
but after using control at time t

$$\overline{bel}(x_t) = P(x_t \mid z_{1:t-1}, u_{1:t})$$

Using law of total Probability

$$\sum_{x_{t-1}} P(x_t \mid x_{t-1}\ z_{1:t-1}, u_{1:t})\, P(x_{t-1} \mid z_{1:t-1}\ u_{1:t})$$

Using Markov transition model

$$= \sum_{x_{t-1}} P(x_t \mid x_{t-1}, u_t)\, P(x_{t-1} \mid z_{1:t-1}, u_{1:t})$$

IF the control input u_t is chosen randomly (not using x_{t-1})

$$= \sum_{x_{t-1}} P(x_t \mid x_{t-1}, u_t)\, \underbrace{P(x_{t-1} \mid z_{1:t-1}, u_{1:t-1})}_{bel(x_{t-1})}$$

State transition model

Control input

$$\boxed{\overline{bel}(x_t) = \sum_{x_{t-1}} P(x_t \mid x_{t-1}\ u_t)\, bel(x_{t-1})}$$

prediction based on state transition model and control input u_t

In continuous form

$$\overline{bel}(x_t) = \int_{x_{t-1}} P(x_t \mid x_{t-1}\ u_t)\, bel(x_{t-1})\, dx_{t-1}$$

Algorithm Bayes_filter($bel(x_{t-1}), u_t, z_t$)

Prediction

for all x_t do:

$$\overline{bel}(x_t) = \int p(x_t | u_t, x_{t-1}) bel(x_{t-1}) dx_{t-1}$$

$$bel(x_t) = \eta \, p(z_t | x_t) \, \overline{bel}(x_t)$$

Correction

end for

return $bel(x_t)$