Filtering → iteratively updating beliefs
\[ \text{bel}(t) = P(x | z_{1:t}, u_{1:t}) \]

- Discrete distributions can be represented (and updated) as histograms
- What about continuous distributions

→ Particle filter

Represent \( \text{bel}(x_t) \) by a set of random state samples

Example

\[
P(x)
\]

Particle samples representing Gaussian

\[
P(y)
\]

\( \text{Gaussian} \)

Parametric representation \( \sigma, M \)

(Recall 24)
Advantages

Non-parametric representation

1. Can represent a much broader set of distributions

2. Can easily implement non-linear transformations of the distribution, e.g., operations needed for the predict & correct stages of the filter.

Def. The samples of the belief distribution are called particles.

We denote $M$ particles as

$$x_t^{[1]}, x_t^{[2]}, \ldots, x_t^{[M]} \in \mathbb{R}^n$$

Together $X_t = \{x_t^{[1]}, \ldots, x_t^{[M]}\}$

Each $x_t^{[m]}$ is a concrete state at time $t$ with $1 \leq m \leq M$. 
For example, for the rear wheel vehicle model:
\[ x_t^{[m]} = \langle \text{pos}_x, \text{pos}_y, \theta \rangle \]
\[ M \approx 100 \]

In representing \( \text{bel}(x_t) \) with \( X_t \), ideally \( x_t^{[m]} \) should be included in \( X_t \) with probability proportional to \( \text{bel}(x_t) = P(x_{t+1} \ldots) \)

This will hold asymptotically as \( M \to \infty \)
Basic Particle Filtering algorithm

- Same structure as Bayes filter
  - input: bet \( (x_{t+1}) \leftrightarrow X_{t-1} \)
  - control: \( u_t \)
  - measurement: \( z_t \)
  - output: bet \( (x_t) \leftrightarrow X_t \)
  - prediction from bet \( (x_{t-1}) \) using \( u_t \)
  - correction from \( \text{bel}(x_t) \) using \( z_t \)

\[ X_t = x_t^{[1]}, x_t^{[2]}, \ldots, x_t^{[M]} \] particles

Algorithm Particle_filter\((X_{t-1}, u_t, z_t):\)

\[ \hat{X}_{t-1} = X_t = \emptyset \]

for all \( m \) in \([M]\) do:

- importance sample \( x_t^{[m]} \sim \text{p} (x_t | u_t, x_t^{[m]}) \)

- factor \( w_t^{[m]} = \text{p} \left( z_t | x_t^{[m]} \right) \)

- \( \text{bel} \)

\[ \bar{X}_t = \bar{X}_t + \left( x_t^{[m]}, w_t^{[m]} \right) \]

end for

for all \( m \) in \([M]\) do:

- draw \( i \) with probability \( \propto w_t^{[i]} \)

- add \( x_t^{[i]} \) to \( X_t \)

end for

return \( X_t \)
Resampling step

“Survival of the fittest” based on measurement

We want to sample particles from \( \text{bel}(x_t) \)

But we only have samples from \( \text{bel}(x_t) \)

More generally we want to sample from a distribution \( f \)

but we can only sample from another distribution \( g \)

How?

\[
E_f [I(x \in A)] = \int f(x) I(x \in A) \, dx
= \int \frac{f(x)}{g(x)} \cdot g(x) I(x \in A) \, dx
= E_g [\omega(x) I(x \in A)]
\]

Provided \( f(x) > 0 \Rightarrow g(x) > 0 \)
Samples from $f_{\text{bel}(x_t)}$

Samples from $g_{\frac{f(x)}{g(x)}}$

Samples of $f$ obtained by attaching a weight $\frac{f(x)}{g(x)}$ to each sample of $g$.