Search and Planning

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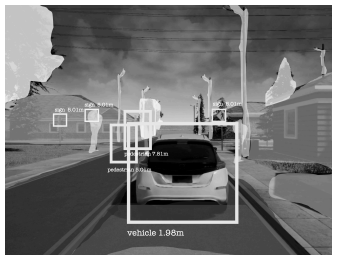
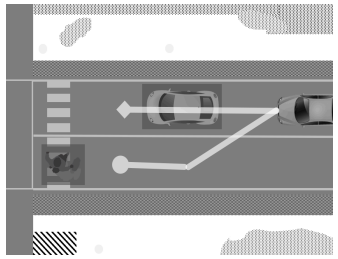
Based on some lectures by Emilio Frazzoli

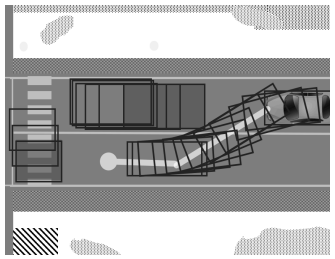
March 24

GEM platform

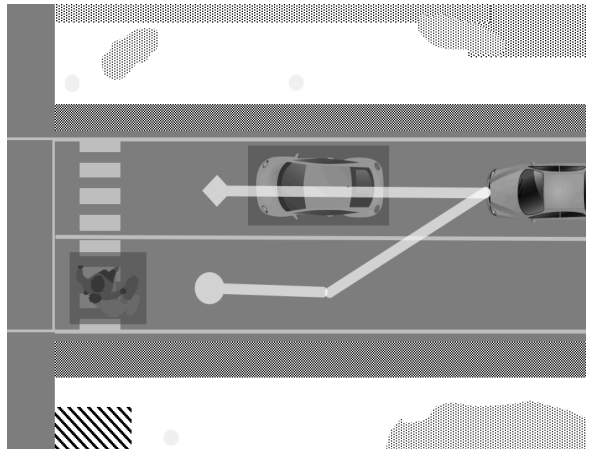
Autonomy pipeline







Sensing	Perception	Decisions and planning	Control
Physics-based models of camera, LIDAR, RADAR, GPS, etc.	Programs for object detection, lane tracking, scene understanding, etc.	Programs and multi-agent models of pedestrians, cars, etc.	Dynamical models of engine, powertrain, steering, tires, etc.

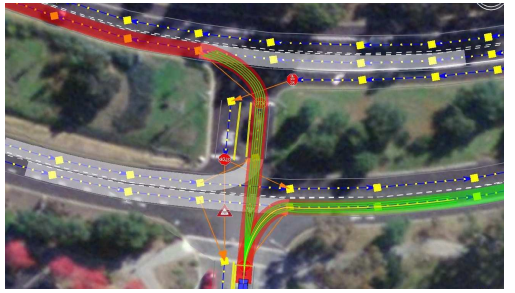


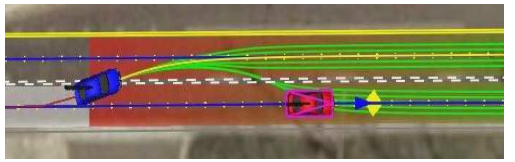
Decisions and planning

Programs and multi-agent models of pedestrians, cars, etc.

Search in different levels of navigation, planning, decision making, and control

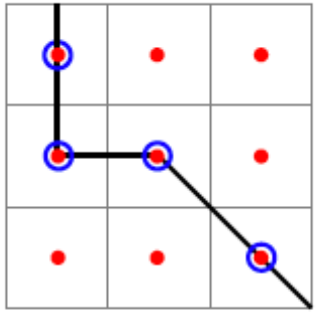
- Global path planner --- invoked at each new *checkpoint*
  - finds paths from every point in the map to next checkpoint
  - dynamic programming (Howard 1960)
- Road navigation
  - For each path, the planner rolls out several discrete trajectories that are parallel to the smoothed center of the lane
- Freeform navigation (parking lots)
  - Generate arbitrary trajectories (irrespective of road structure) using modified A*



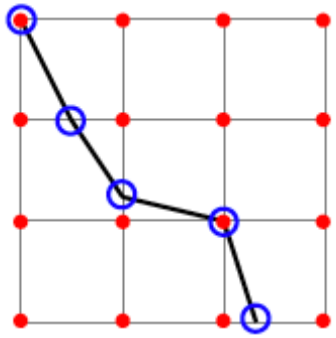


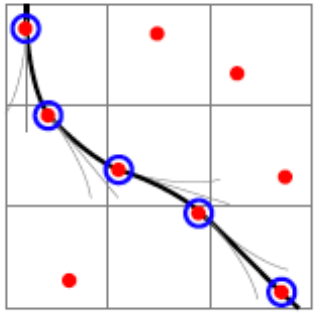
Junior: The Stanford Entry in the Urban Challenge, Thrun et al., 2008

A search-based strategy for planning



- Represent vehicle state in a *uniform* discrete grid
  - 4D grid: $x, y, \theta\ (heading), dir$ (fwd,rev)
- A path (a) over this discrete grid is a start for a plan
- But, the discrete path (a) may not be executable by the vehicle dynamics
- *Hybrid A** solves this problem by shifting the points that represent the discrete cells
  - More on this in the next lecture





Outline

- Informed search
- Optimal search
- Dynamic programming

Starting from uninformed graph search

- Search for collision free trajectories can be converted to graph search
- Hence, we can solve such problems using the graph search algorithms like (uninformed) Breadth-First Search and Depth-First Search
- However, roadmaps are not just “generic” graphs
  - Some paths are much more preferable with respect to others (e.g., shorter, faster, less costly in terms of fuel/tolls/fees, more stealthy, etc.).
  - Distances have a physical meaning
  - Good guesses for distances can be made, even without knowing optimal paths.
- Can we *utilize this information* to find efficient paths, efficiently?

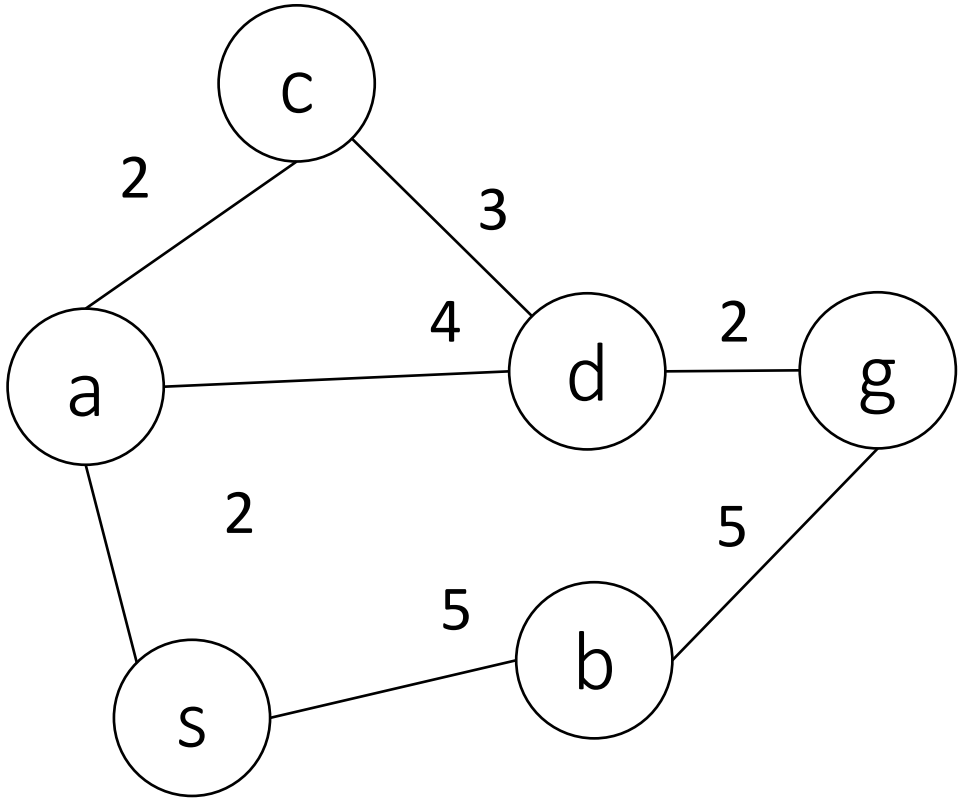
Shortest path problems

- Input: $\langle V, E, w, start, goal \rangle$
  - V: (finite) set of vertices
  - $E \subseteq V \times V$: (finite) set of edges
  - $w : E \to \mathbb{R}_{>0}$: a function that associates to each edge e to a strictly positive weight $w(e)$ (cost, length, time, fuel, prob. of detection)
  - $start, goal \in V$: respectively, start and end vertices.
- Output: $\langle P \rangle$
  - P is a path (seq of vertices)
  - The weight of a path is the sum of the weights of its edges
    - Ultimately, we'd want a path starting in start and ending in goal, such that its weight $w(P)$ is minimal among all such paths
  - The graph may be unknown, partially known, or known

Example: Find the minimal path from s to g:

a simple path P:

w(P):



Search Performance Metrics

- Soundness: when a solution is returned, is it guaranteed to be correct
- Completeness: – the algorithm guaranteed to find a solution when one exists
- Optimality: How close is the found solution to the best solution
- Space complexity: memory needed
- Time complexity: running time; can it be used for online planning?

Uniform cost search (Uninformed search)

$Q \leftarrow \langle start \rangle$ *// initialize a queue of paths with start*

while $Q \neq \emptyset$:

  from Q pick $(and\ remove)\ the\ path\ P\ with\ lowest\ cost,\ say\ \ g = w(P)$

  if $head(P) = \ goal$ then return P ; *// Reached the goal*

  foreach $vertex\ v\ such\ that\ (head(P), v) \in \ E$, do *// for all neighbors*

    add $\langle v, P \rangle\ to\ Q$; *// Add expanded paths*

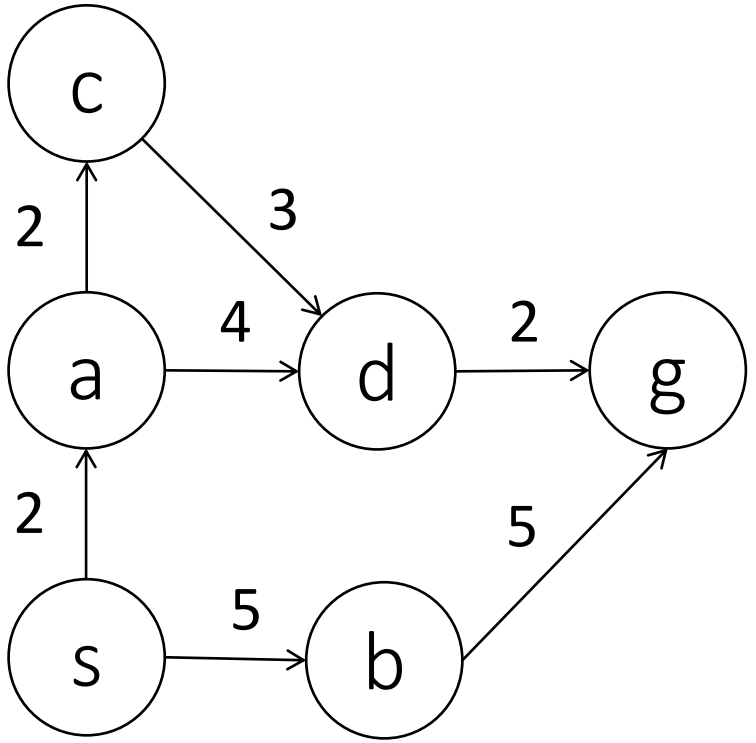
return $FAILURE$; *// nothing left to consider*

Note no visited list; Use no information obtained from the environment

Example of Uniform-Cost Search

Q:

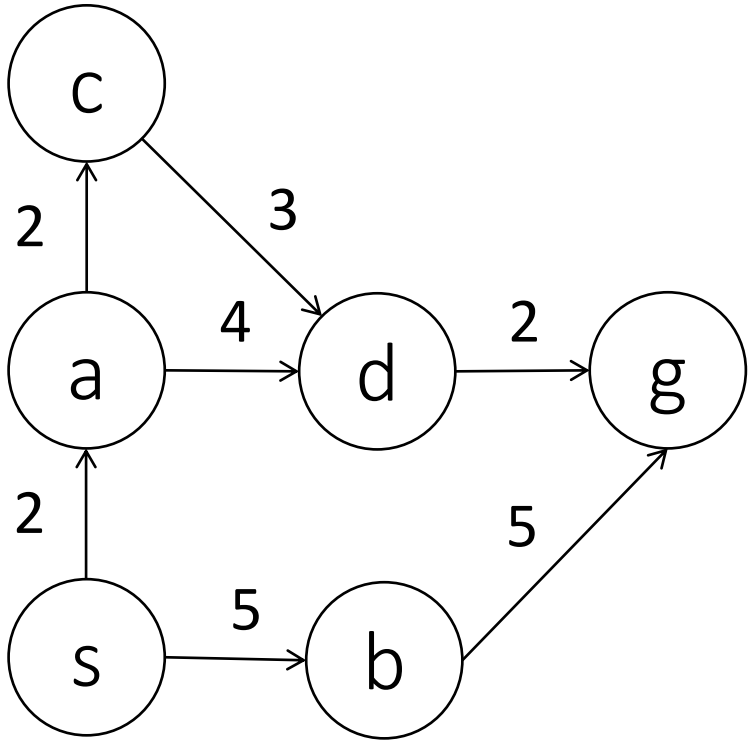
Path	Cost
$\langle s \rangle$	0



Example of Uniform-Cost Search

Q:

Path	Cost
$\langle a, s \rangle$	2
$\langle b, s \rangle$	5



Example of Uniform-Cost Search

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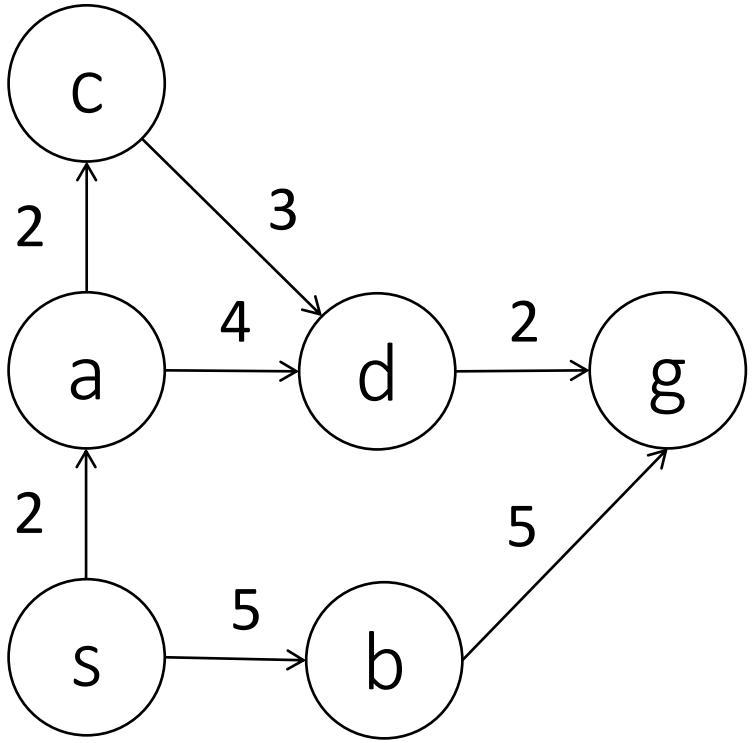
Path	Cost



Example of Uniform-Cost Search

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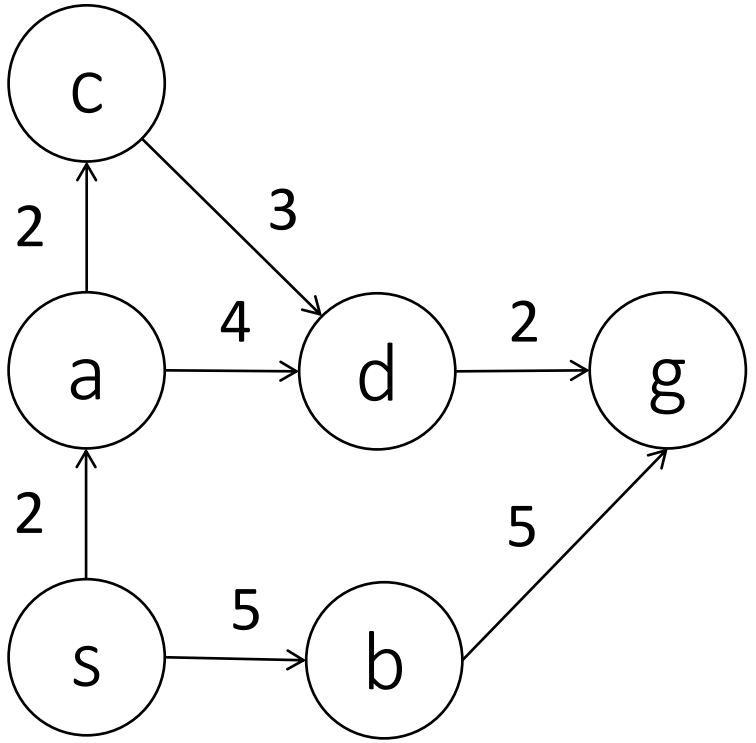
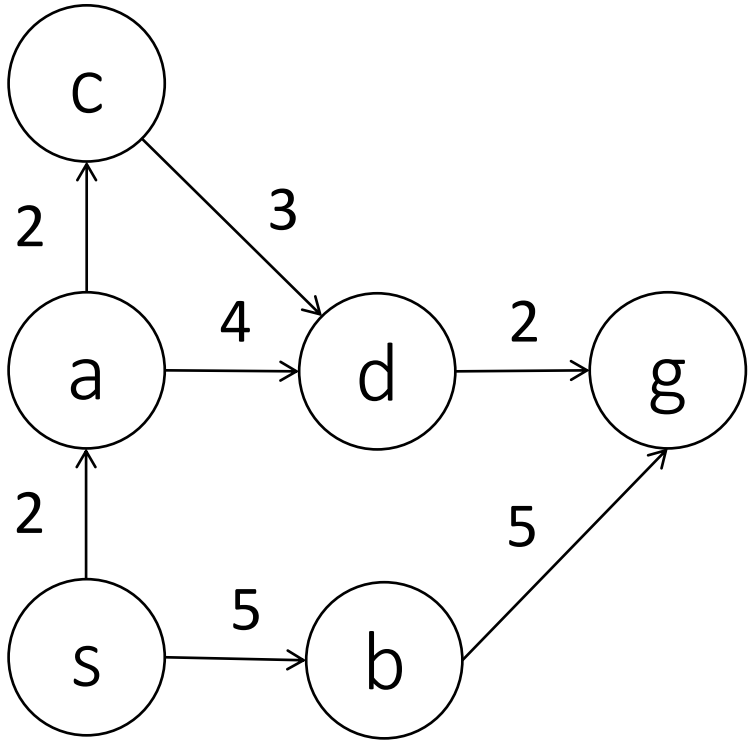
Path	Cost



Example of Uniform-Cost Search

Q:

Path	Cost



Remarks on Uniform Cost Search

- UCS is an extension of BFS to the weighted-graph case (UCS = BFS if all edges have the same cost)
- UCS is *complete* and *optimal* (assuming costs bounded away from zero)
- UCS is guided by path cost rather than path depth, so it may get in trouble if some edge costs are very small
- Worst-case time and space complexity $O(b^{W^*/\epsilon})$, where W^* is the optimal cost, and ϵ is such that all edge weights are no smaller than
  - b is the max number of branches out of each node

Greedy or Best-First Search

- UCS explores paths in all directions, with no bias towards the goal state
- What if we try to get "closer" to the goal?
- We need a measure of distance to the goal. It would be ideal to use the length of the shortest path... but this is exactly what we are trying to compute!
- We can estimate the distance to the goal through a "heuristic function," $h : V \rightarrow \mathbb{R}_{\geq 0}$. E.g., the Euclidean distance to the goal (as the crow flies)
- A reasonable strategy is to always try to move in such a way to minimize the estimated distance to the goal: this is the basic idea of the greedy (best-first) search

Greedy/Best-first search

$Q \leftarrow \langle start \rangle$ *// initialize queue with start*

while $Q \neq \emptyset$:

  from Q pick $(and\ remove)\ the\ path\ P\ with\ lowest$ *heuristic cost* $h(head(P))$

  if $head(P) = goal$ then return P *// Reached the goal*

  foreach $vertex\ v\ such\ that\ (head(P), v) \in E$, do *// for all neighbors*

    add $\langle v, P \rangle\ to\ Q$; *// Add expanded paths*

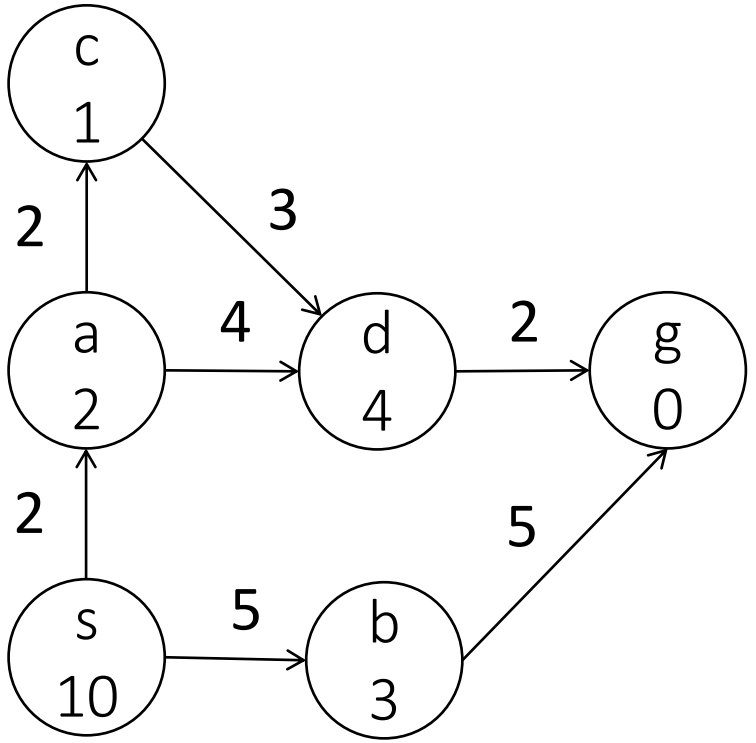
return $FAILURE$; *// nothing left to consider*

Note no visited list

Example of Greedy search

Q:

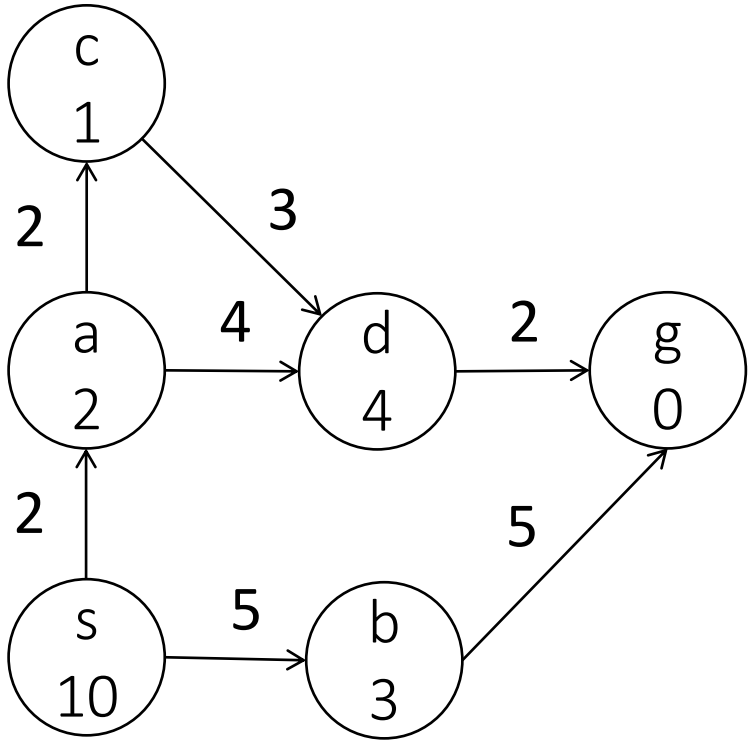
Path	Cost	h
$\langle s \rangle$	0	10



Example of Greedy search

Q:

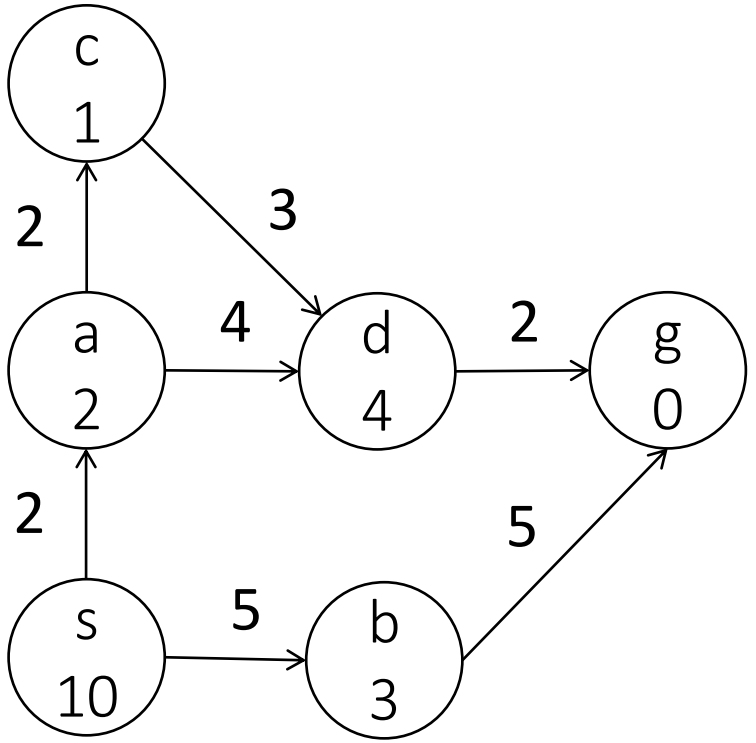
Path	Cost	h
$\langle a, s \rangle$	2	2
$\langle b, s \rangle$	5	3



Example of Greedy search

Q:

Path	Cost	h
$\langle s \rangle$	0	10



Remarks on greedy/best-first search

- Greedy (Best-First) search is similar to Depth-First Search
  - keeps exploring until it has to back up due to a dead end
- Not complete and not optimal, but is often fast and efficient, depending on the heuristic function h
- Worst-case time and space complexity $O(b^m)$

A search

- The problems
  - UCS is optimal, but may wander around a lot before finding the goal.
  - Greedy search is not optimal, but can be efficient, as it is heavily biased towards moving towards the goal. The non-optimality comes from neglecting “the past.”

- The idea
  - Keep track both of the cost of the partial path to get to a vertex, say g(v), and of the heuristic function estimating the cost to reach the goal from a vertex, h(v).
  - In other words, choose as a “ranking” function the sum of the two costs:

$$f(v) = g(v) + h(v)$$

  - g(v) cost-to-come (from the start to v)
  - h(v): cost-to-go estimate (from v to the goal)
  - f (v): estimated cost of the path (from the start to v and then to the goal).

A search

open set and closed set

$Q \leftarrow \langle start \rangle$ // *initialize queue with start*

while $Q \neq \emptyset$:

pick $(and\ remove)\ \ path\ P\ with\ lowest\ estimated\ cost\ f(P) = g(P) + h\big(head(P)\big)\ from\ Q$

if $head(P) = \ goal$ then return P // *Reached the goal*

foreach $vertex\ v\ such\ that\ (head(P), v) \in \ E$, do // *for all neighbors*

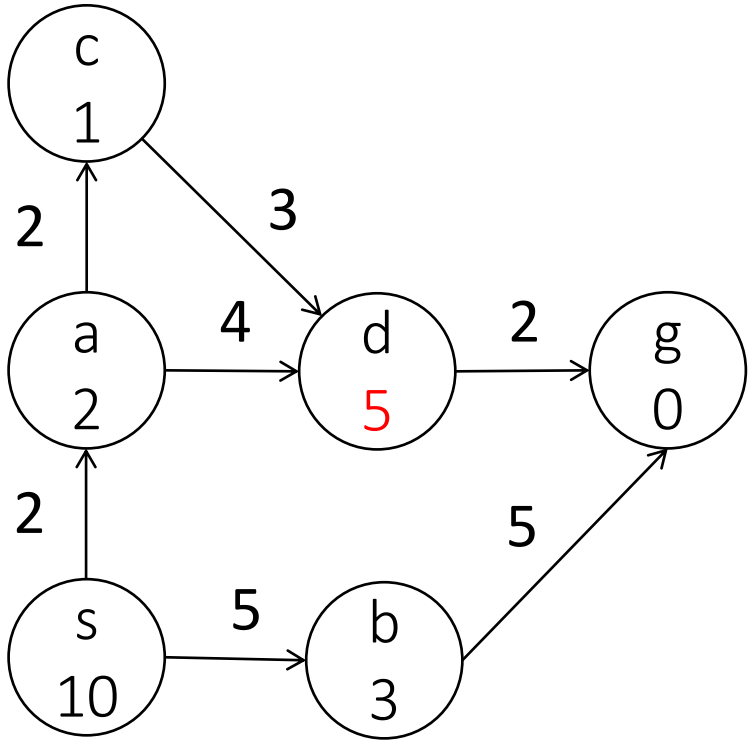
add $\langle v, P \rangle\ to\ Q$; // *Add expanded paths*

return $FAILURE$; // *nothing left to consider*

Example of A search

Q:

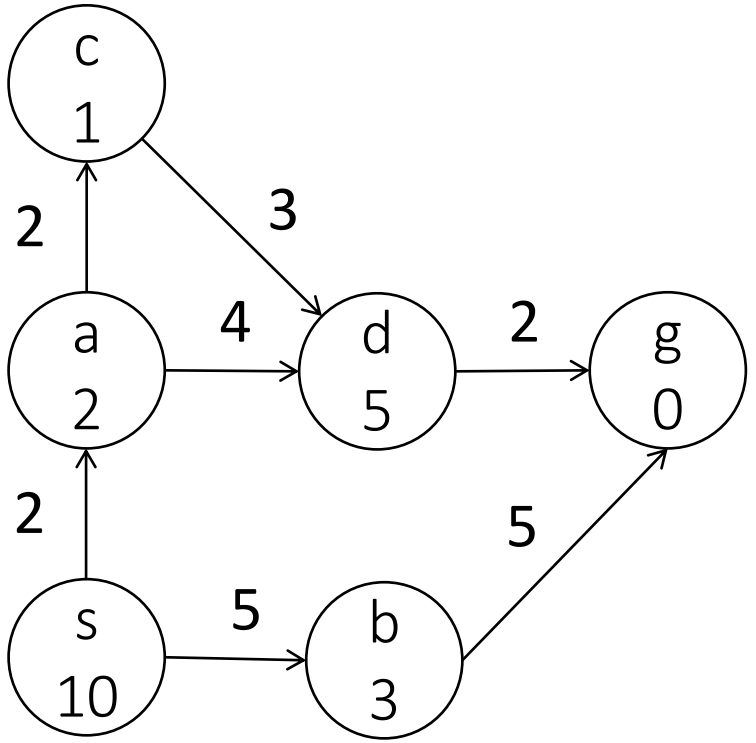
Path	g	h	f
$\langle s \rangle$	0	10	10



Example of A search

Q:

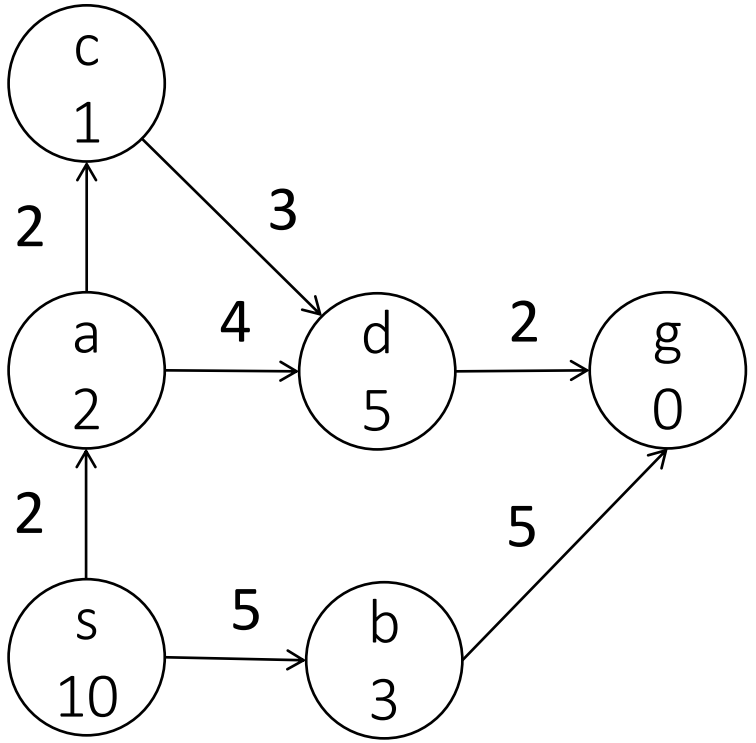
Path	g	h	f
$\langle a, s \rangle$	2	2	4
$\langle b, s \rangle$	5	3	8



Example of A search

Q:

Path	g	h	f
$\langle a, s \rangle$	2	2	4
$\langle b, s \rangle$	5	3	8



Remarks on A search

- A search is similar to UCS, with a bias induced by the heuristic h
- If h = 0, A = UCS.
- The A search is complete, but is *not optimal*
  - What is wrong? (Recall that if h = 0 then A = UCS, and hence optimal...)

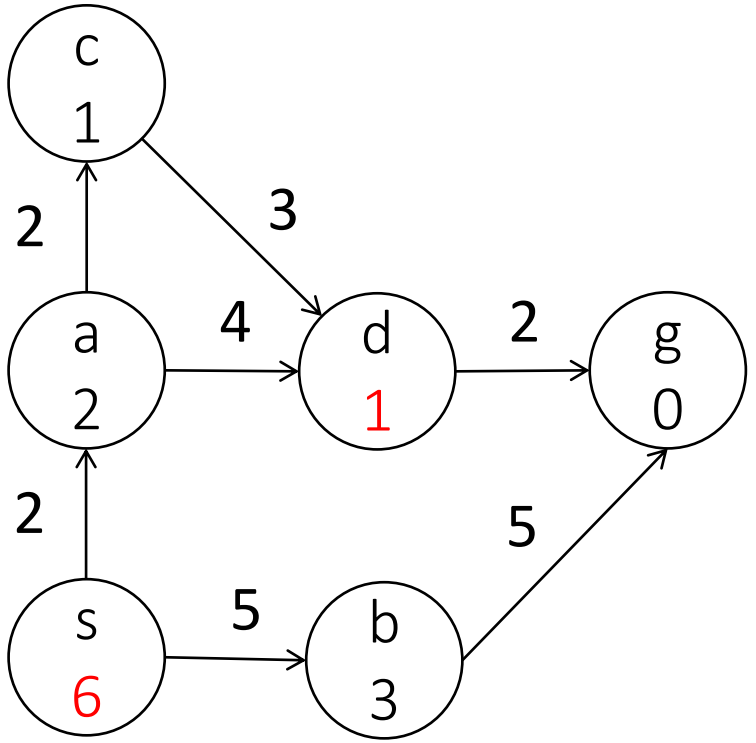
A $*$ Search

- Choose an admissible heuristic, i.e., such $that\ h(v)\ \leq h^*(v)$
  - $h^*(v)$ is the "optimal" heuristic---perfect cost to go
  - To be admissible $h(v)$ should be at most $h^*(v)$
  - A search with an admissible heuristic is called A* --- guaranteed to find optimal path

Example of A* search

Q:

Path	g	h	f
$\langle s \rangle$	0	6	6



Proof of optimality of A*

- Let w* be the cost of the optimal path
- Suppose for the sake of contradiction, that A* returns P with w(P) > w*
- Find the first unexpanded node on the optimal path P*; call it n
- f(n) > w(P), otherwise n would have been expanded
- f(n) = g(n) + h(n)

  = g*(n) + h(n) [since n is on the optimal path]

  <= g*(n) + h*(n) [since h is admissible]

  = f*(n) = w* [by def. of f, and since w* is the cost of the optimal path]
- Hence w* >= f(n) = w(P), which is a contradiction

Admissible heuristics

- How to find an admissible heuristic? i.e., a heuristic that never overestimates the cost-to-go.
- Examples of admissible heuristics
  - $h(v) = 0$: this always works! However, it is not very useful, A$*$ = UCS
  - $h(v) = distance(v, g)$ when the vertices of the graphs are physical locations
  - $h(v) = \left|\left|v - g\right|\right|_p$, when the vertices of the graph are points in a normed vector space
- A general method
  - Choose h as the optimal cost-to-go function for a relaxed problem, that is easy to compute
  - Relaxed problem: ignore some of the constraints in the original problem

Admissible heuristics for the 8-puzzle

Initial state:

1		5
2	6	3
7	4	8

Goal state:

1	2	3
4	5	6
7	8	

Which of the following are admissible heuristics?

- h = 0 YES, always good
- h = 1 NO, not valid in goal state
- h = number of tiles in the wrong positon YES, “teleport” each tile to the goal in one move
- h = sum of (Manhattan) distance between tiles and their goal position. YES, move each tile to the goal ignoring other tiles.

A partial order of heuristic functions

- Some heuristics are better than others
  - h = 0 is an admissible heuristic, but is not very useful
  - h = h^* is also an admissible heuristic, and it the “best” possible one (it give us the optimal path directly, no searches/backtracking)
- Partial order
  - We say that h_1 dominates h_2 if $h_1(v) \geq h_2(v)$ for all vertices v.
  - h^* dominates all admissible heuristics, and 0 is dominated by all admissible heuristics
- Choosing the right heuristic
  - In general, we want a heuristic that is as close to h $*$ as possible.
  - However, such a heuristic may be too complicated to compute. There is a tradeoff between complexity of computing h and the complexity of the search

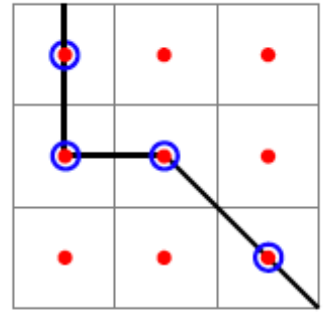
Consistent heuristics

- An additional useful property for A∗ heuristics is called consistency
  - A heuristic $h: X \to \mathbb{R}_{\geq 0}$ is said consistent if $h(u) \leq w(e = (u,v)) + h(v), \forall (u,v) \in E$
  - In other words, a consistent heuristics satisfies a triangle inequality
- If h is a consistent heuristics, then $f = g + h$ is non-decreasing along paths: $f(v) = g(v) + h(v) = g(u) + w(u,v) + h(v) \geq f(u)$
- Hence, the values of f on the sequence of nodes expanded by A∗ is non-decreasing: the first path found to a node is also the optimal path ⇒ no need to compare costs!

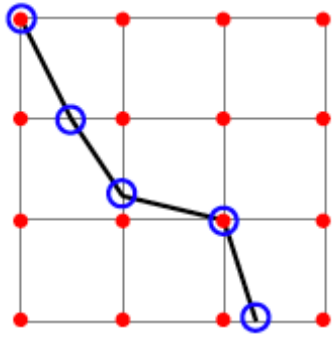
Hybrid A*

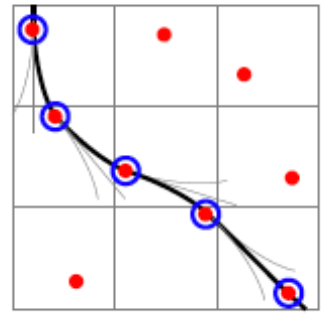
open cells: cells that are accessible from root and closed cells

(a)



- Represent vehicle state in a ~~*uniform*~~ discrete grid
  - 4D grid: $x, y, \theta\ (heading), dir$ (fwd,rev)
- If the current coordinate is $\langle x, y, \theta \rangle$ and those coordinates lie in cell c_i then the *representative continuous state* for cell c_i will be $x_i = x, y_i = y, \theta_i = \theta$
- After applying control input u to vehicle, suppose the predicted state is x', y', θ'
  - $x', y', \theta' = f(x, y, \theta, u); \dot{x} = \cdots$
  - representative for $c_j = x', y', \theta'$
  - This defines a transition from c_i to c_j
- More details in the next lecture





Summary

- A* algorithm combines cost-to-come g(v) and a heuristic function h(v) for cost-to-go to find shortest path
  - informed search
- heuristic function must be *admissible* $h(v) \leq h^*(v)$
  - Are all $h(v)$ values needed ?
  - What if h is not admissible
  - How to find heuristics

Dynamic programming/Dijkstra

- The optimality principle
  - Let P = (s, . . . , v, . . . g) be an optimal path (from s to g).
  - Then, for any v ∈ P, the sub-path S = (v, . . . , g) is itself an optimal path (from v to g)
- Using the optimality principle
  - Essentially, optimal paths are made of optimal paths. Hence, we can construct long complex optimal paths by putting together short optimal paths, which can be easily computed. Fundamental formula in dynamic programming: $h*(u) = \min_{(u,v)\in E} [w((u, v)) + h*(v)]$. Typically, it is convenient to build optimal paths working backwards from the goal.