Autonomy pipeline

Sensing
Physics-based models of camera, LIDAR, RADAR, GPS, etc.

Perception
Programs for object detection, lane tracking, scene understanding, etc.

Decisions and planning
Programs and multi-agent models of pedestrians, cars, etc.

Control
Dynamical models of engine, powertrain, steering, tires, etc.
Control

Dynamical models of engine, powertrain, steering, tires, etc.
Previously

• ODE language for control systems
  • Solutions, Lipschitz continuity, equilibria, steady state, transient

• Control design
  • Open loop vs closed loop
  • PID design
  • state-feedback

• Today
  • What are the requirements of a control system
    • Stability, Asymptotic stability
  • Lyapunov’s method for proving stability
  • Relationship to invariance
Example 1: Pendulum

Pendulum equation

\[ x_1 = \theta \quad x_2 = \dot{\theta} \]

\[ x_2 = \dot{x}_1 \]

\[ \dot{x}_2 = -\frac{g}{l} \sin(x_1) - \frac{k}{m} x_2 \]

\[
\begin{bmatrix}
\dot{x}_2 \\
\dot{x}_1
\end{bmatrix} = \begin{bmatrix}
-\frac{g}{l} \sin(x_1) - \frac{k}{m} x_2 \\
x_2
\end{bmatrix}
\]

\( k: \) friction coefficient

What is described?

-> Center of mass movement relative to the origin
A. M. Lyapunov (June 6 1857–November 3, 1918), Russian mathematician and physicist.

Defines stability of ordinary differential equations. In the theory of probability, he generalized the works of Chebyshev and Markov, and proved the Central Limit Theorem under more general conditions than his predecessors.
Lyapunov stability

Lyapunov stability: The system (1) is said to be **Lyapunov stable** (at the origin) if

\[ \forall \varepsilon > 0 \exists \delta_\varepsilon > 0 \text{ such that } |x_0| \leq \delta_\varepsilon \Rightarrow \forall t \geq 0, |\xi(x_0, t)| \leq \varepsilon. \]

How is this related to invariants and reachable states?
Asymptotically stability

The system (1) is said to be **Asymptotically stable (at the origin)** if it is Lyapunov stable and

\[ \exists \delta_2 > 0 \text{ such that } \forall |x_0| \leq \delta_2 \text{ as } t \to \infty, |\xi(x_0, t)| \to 0. \]

If the property holds for any \( \delta_2 \) then **Globally Asymptotically Stable**
Phase portrait of pendulum with friction
Butterfly*

\[
\begin{bmatrix}
\dot{x}_2 \\
\dot{x}_1
\end{bmatrix} = \begin{bmatrix}
2x_1 x_2 \\
x_1^2 - x_2^2
\end{bmatrix}
\]

All solutions converge to 0 but the equilibrium point (0,0) is not Lyapunov stable

*Not discussed in class
Verifying Stability for Linear Systems

Consider a linear system $\dot{x} = Ax$

**Theorem 1. (Stability of linear systems)**

1. It is asymptotically stable iff all the eigenvalues of $A$ have **strictly** negative real parts (*Hurwitz*).

2. It is Lyapunov stable iff all the eigenvalues of $A$ have real parts that are either zero or negative and the *Jordan blocks* corresponding to the eigenvalues with zero real parts are of size 1.
Jordan decomposition*

For every $n \times n$ matrix $A$, there exists a nonsingular $n \times n$ matrix $P$ such that

$$PAP^{-1} = J = \begin{bmatrix}
J_1 & 0 & 0 & \ldots & 0 \\
0 & J_2 & 0 & \ldots & 0 \\
0 & 0 & J_3 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & J_\ell
\end{bmatrix},$$

where each $J_i$ is an upper triangular matrix called a Jordan block.
\[
\begin{bmatrix}
\dot{x}_2 \\
\dot{x}_1 
\end{bmatrix} = \begin{bmatrix}
-1/4 & -2/5 \\
3 & -1/4 
\end{bmatrix} 
\]

\[
\lambda_1 = -0.25 - i1.10 \\
\lambda_2 = -0.25 + i1.10 
\]

\[\begin{bmatrix}
\dot{x}_2 \\
\dot{x}_1 
\end{bmatrix} = \begin{bmatrix}
1/4 & -2/5 \\
3 & -1/4 
\end{bmatrix} \]

\[
\lambda_1 = +i0.1066 \\
\lambda_2 = -i0.1066 
\]

\[
\begin{bmatrix}
\dot{x}_2 \\
\dot{x}_1 
\end{bmatrix} = \begin{bmatrix}
1/2 & -2/5 \\
3 & -1/2 
\end{bmatrix} \]

\[
\lambda_1 = 0.125 + i1.029 \\
\lambda_2 = -0.125 - i1.029 
\]

\[
\begin{bmatrix}
\dot{x}_2 \\
\dot{x}_1 
\end{bmatrix} = \begin{bmatrix}
-1/4 & -2/5 \\
3 & -1/2 
\end{bmatrix} \]

\[
\lambda_1 = -0.375 - i1.088 \\
\lambda_2 = -0.375 + i1.088 
\]
Nonlinear **hybrid** dynamics

\[
\frac{dx}{dt} = f(x,u) \quad \text{System dynamics}
\]

\[
x[t + 1] = f(x[t], u[t])
\]

\[
x = [v, s_x, s_y, \delta, \psi] \quad \text{State variables}
\]

\[
u = [a, v_\delta] \quad \text{Control inputs}
\]

**Physical plant**

**Decision and control software**

- Speed up
- Merge left
- Cruise
- Close to front car
- Speeding

- Merge right
- Slow down

State variables and control inputs are used to model the dynamics of a vehicle in a traffic scenario, where decisions are made based on the relative positions and speeds of surrounding vehicles.
Nonlinear hybrid dynamics
Interaction between computation and physics can lead to unexpected behaviors
Hybrid Instability: Switching between two stable linear models

Each of the modes of a walking robot are asymptotically stable.

Is it possible to switch between them to make the system unstable?
Yes! By switching between them the system becomes unstable.