ECE484 Principles of Safe Autonomy
Lecture 7

Modeling and Control

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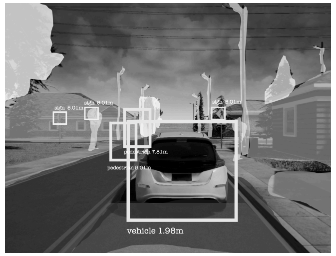
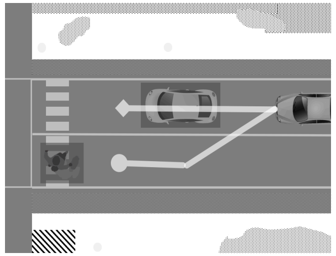
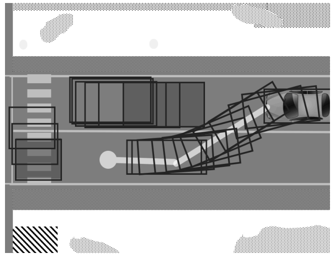
GEM platform

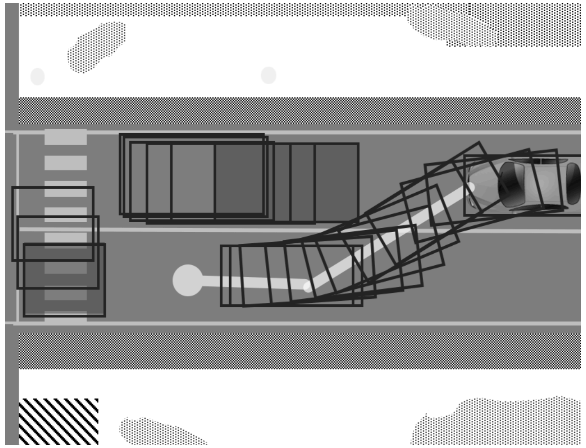
Autonomy pipeline







Sensing	Perception	Decisions and planning	Control
Physics-based models of camera, LIDAR, RADAR, GPS, etc.	Programs for object detection, lane tracking, scene understanding, etc.	Programs and multi-agent models of pedestrians, cars, etc.	Dynamical models of engine, powertrain, steering, tires, etc.



Control

Dynamical models of engine, powertrain, steering, tires, etc.

Previously

- ODE language for control systems
  - Solutions, Lipschitz continuity, equilibria, steady state, transient
- Control design
  - Open loop vs closed loop
  - PID design
  - state-feedback
- Today
  - What are the *requirements* of a control system
    - Stability, Asymptotic stability
  - Lyapunov's method for proving stability
  - Relationship to invariance

Example 1: Pendulum

Pendulum equation

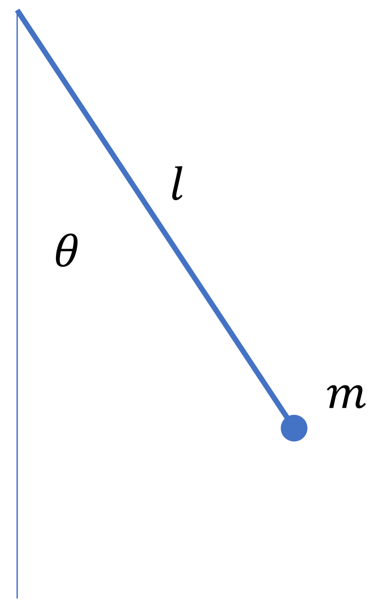
$x_1 = \theta \;\; x_2 = \dot{\theta}$

$x_2 = \dot{x}_1$

$\dot{x}_2 = -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2$

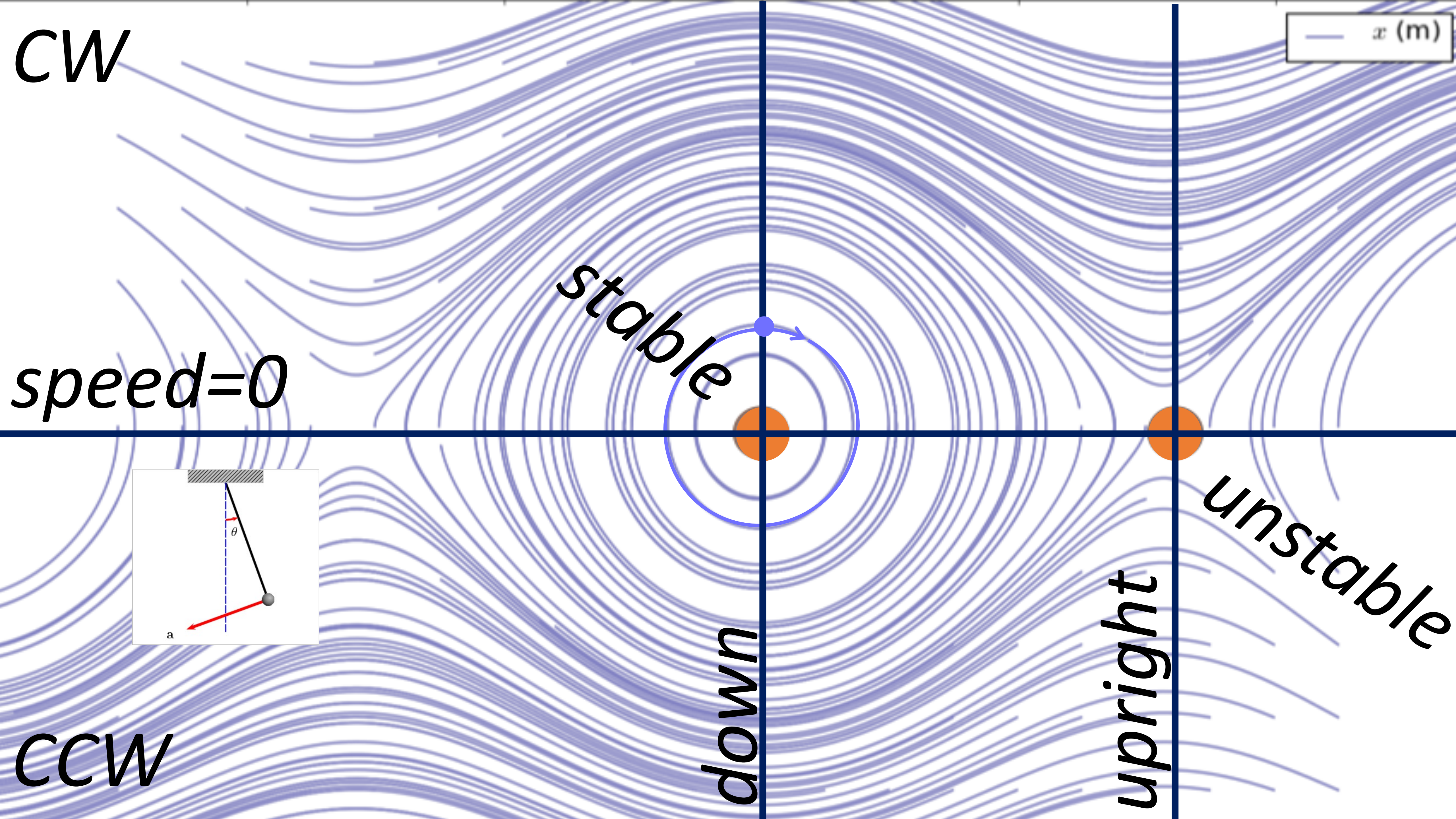
$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2 \\ x_2 \end{bmatrix}$

k: friction coefficient



What is described?

-> Center of mass movement relative to the origin



CW
speed=0
CCW
stable
unstable
down
upright
x (m)
θ
a

Aleksandr M. Lyapunov

A. M. Lyapunov (June 6 1857–November 3, 1918), Russian mathematician and physicist.

Defines stability of ordinary differential equations. In the theory of probability, he generalized the works of Chebyshev and Markov, and proved the Central Limit Theorem under more general conditions than his predecessors.

Lyapunov stability

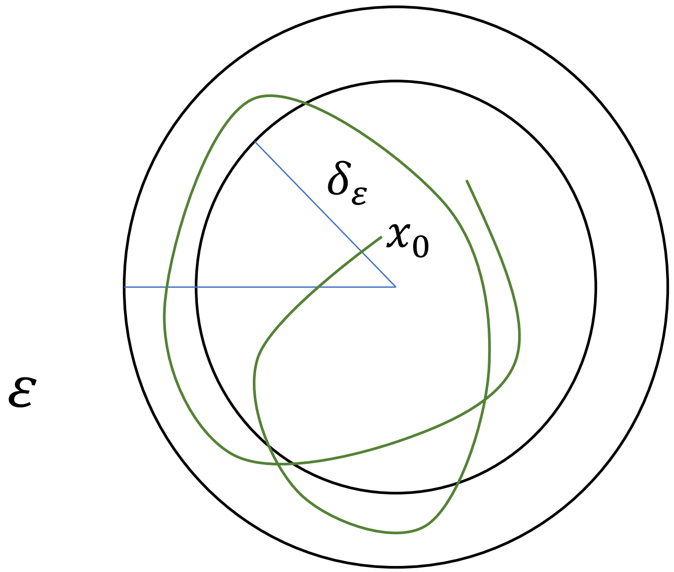
Lyapunov stability: The system (1) is said to be ***Lyapunov stable*** (at the origin) if

$\forall \varepsilon > 0 \; \exists \; \delta_\varepsilon > 0$ such that $|x_0| \leq \delta_\varepsilon \Rightarrow \forall \, t \geq 0, |\xi(x_0, t)| \leq \varepsilon$.

How is this related to

invariants and

reachable states ?

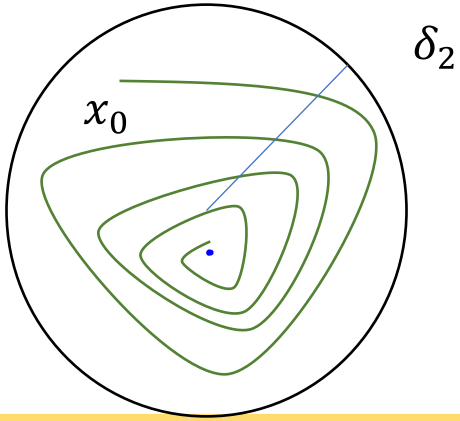


Asymptotically stability

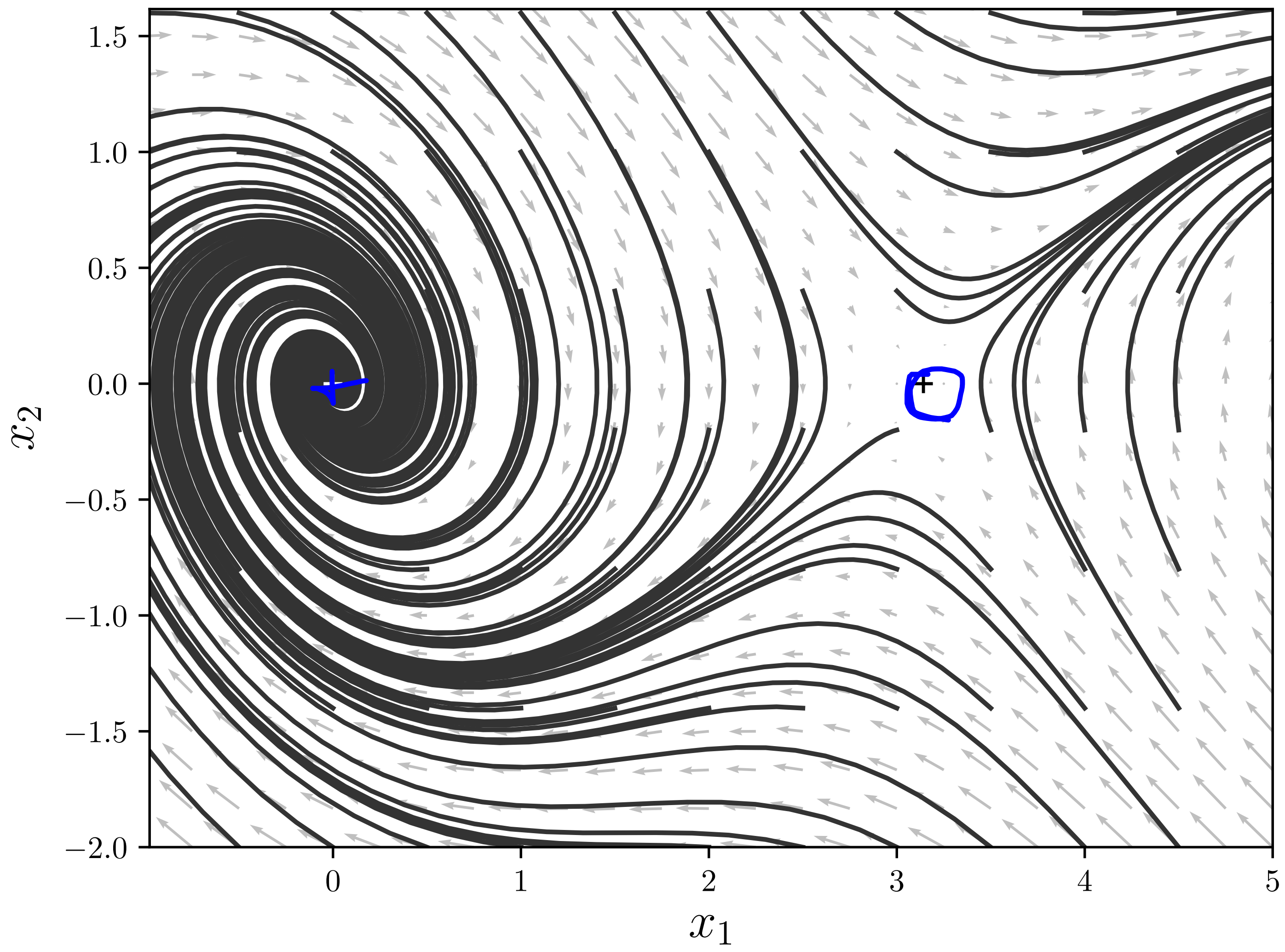
The system (1) is said to be ***Asymptotically stable*** *(at the origin)* if it is Lyapunov stable and

$\exists \delta_2 > 0$ such that $\forall |x_0| \leq \delta_2$ as t $\rightarrow \infty, |\xi(x_0, t)| \rightarrow \mathbf{0}$.

If the property holds for any δ_2 then **Globally Asymptotically Stable**



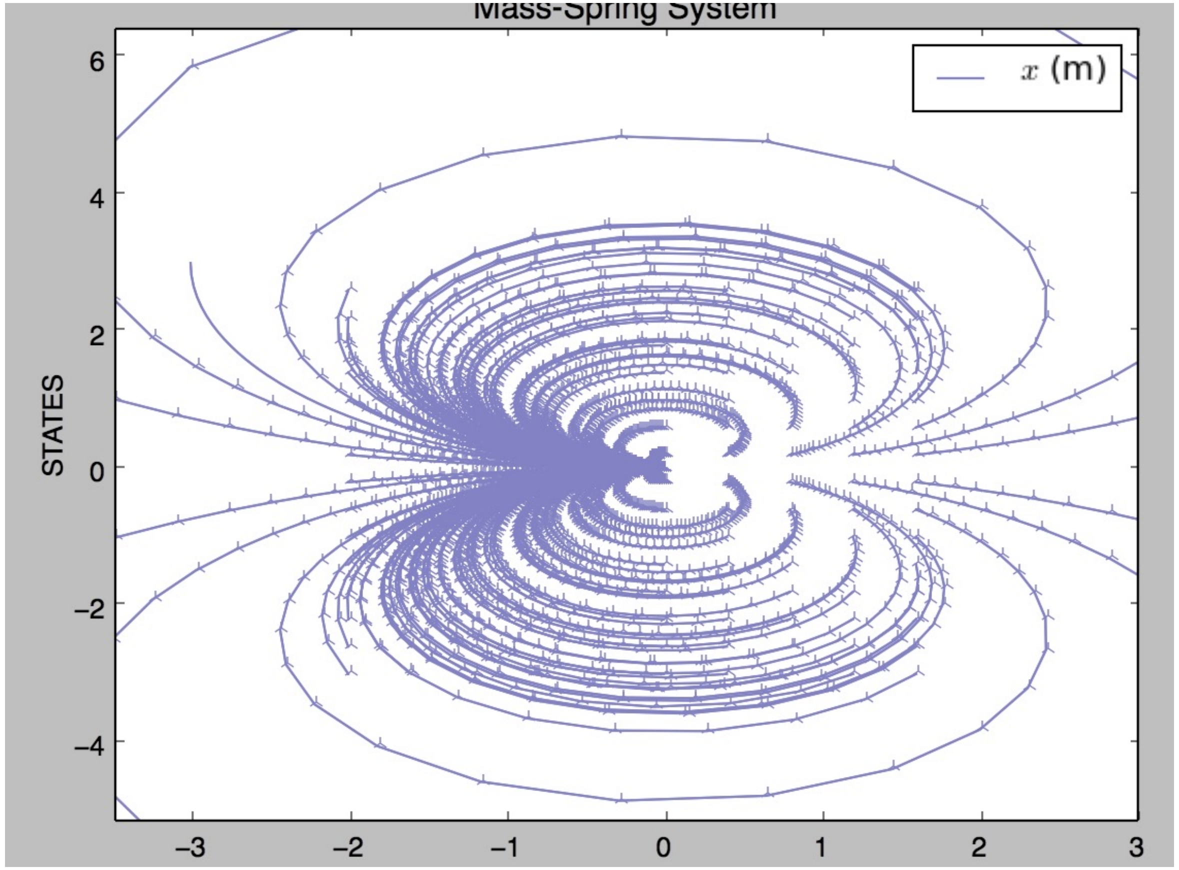
Phase portrait of pendulum with friction



Butterfly*

$$\begin{bmatrix} \dot{x_2} \\ \dot{x_1} \end{bmatrix} = \begin{bmatrix} 2x_1x_2 \\ x_1^2 - x_2^2 \end{bmatrix}$$

All solutions converge to 0 but the equilibrium point (0,0) is not Lyapunov stable



*Not discussed in class

Verifying Stability for Linear Systems

Consider a linear system $\dot{x} = Ax$

Theorem 1. (Stability of linear systems)

1. It is asymptotically stable iff all the eigenvalues of *A* have **strictly** negative real parts (*Hurwitz*).

2. It is Lyapunov stable iff all the eigenvalues of *A* have real parts that are either zero or negative and the *Jordan blocks* corresponding to the eigenvalues with zero real parts are of size 1.

Jordan decomposition*

For every *n x n* matrix *A*, there exists a nonsingular *n x n* matrix *P* such that

$$PAP^{-1} = J = \begin{bmatrix} J_1 & 0 & 0 & \dots & 0 \\ 0 & J_2 & 0 & \dots & 0 \\ 0 & 0 & J_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & J_\ell \end{bmatrix},$$

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \dots & 0 \\ 0 & 0 & \lambda_i & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_i \end{bmatrix}.$$

where each J_i is an upper triangular matrix called a *Jordan block*

Examples

$\begin{bmatrix}\dot{x_2}\\ \dot{x_1}\end{bmatrix} = \begin{bmatrix}-1/4 & -2/5\\ 3 & -1/4\end{bmatrix}$

λ_1=−0.25−i1.10å
λ_2=−0.25+i1.10

$\begin{bmatrix}\dot{x_2}\\ \dot{x_1}\end{bmatrix} = \begin{bmatrix}1/4 & -2/5\\ 3 & -1/4\end{bmatrix}$

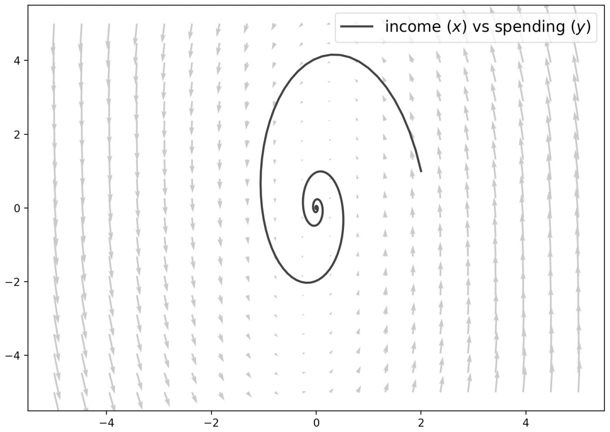
λ_1=+i0.1066
λ_2=-i0.1066

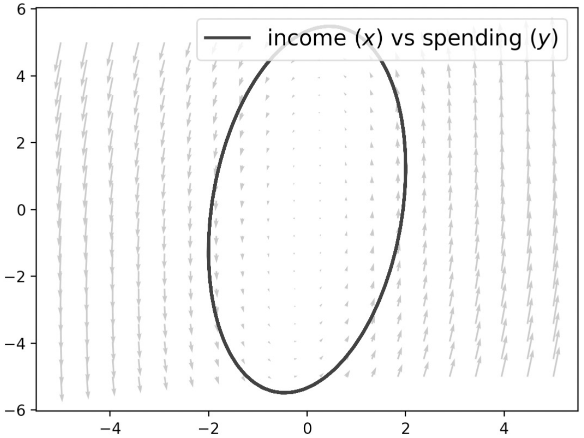
$\begin{bmatrix}\dot{x_2}\\ \dot{x_1}\end{bmatrix} = \begin{bmatrix}1/2 & -2/5\\ 3 & -1/4\end{bmatrix}$

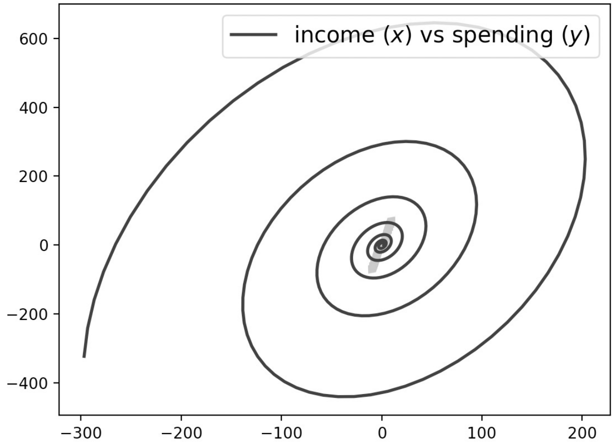
λ_1=0.125+i1.029
λ_2=-0.125-i1.029

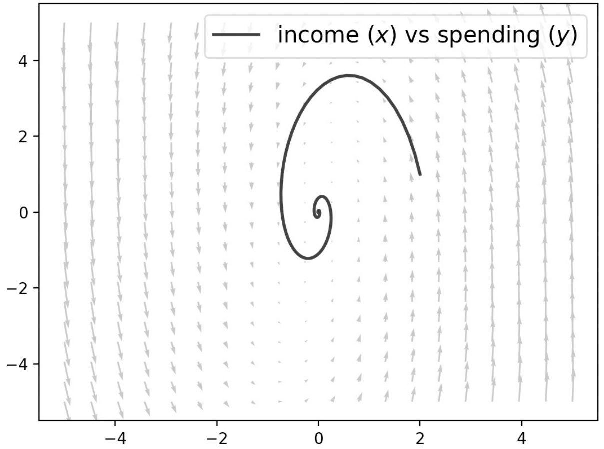
$\begin{bmatrix}\dot{x_2}\\ \dot{x_1}\end{bmatrix} = \begin{bmatrix}-1/4 & -2/5\\ 3 & -1/2\end{bmatrix}$

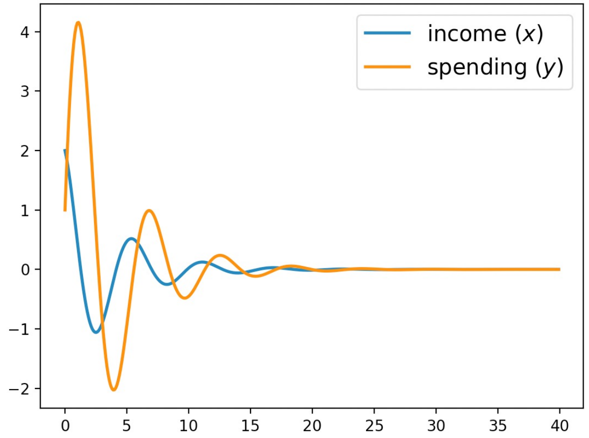
λ_1=−0.375−i1.088
λ_2=−0.375+i1.088

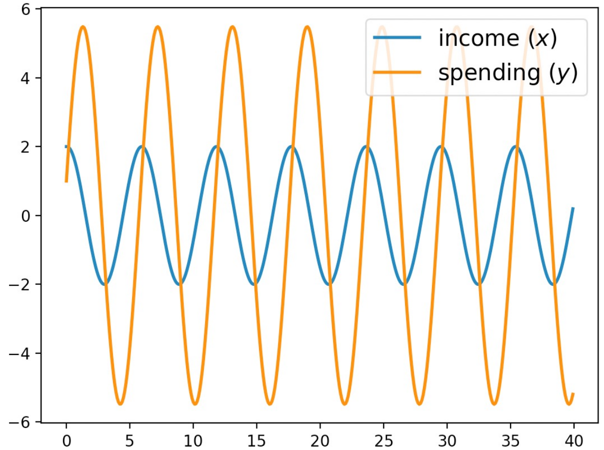


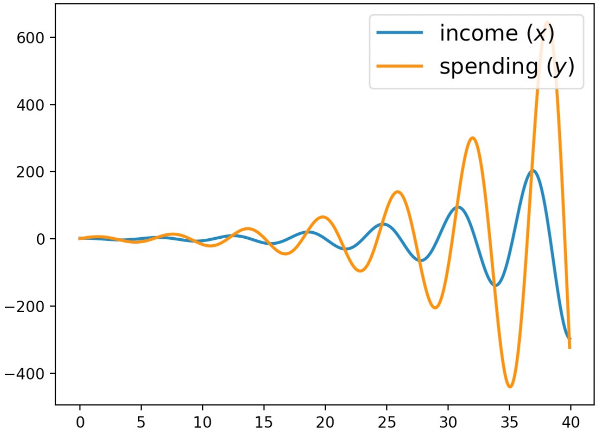


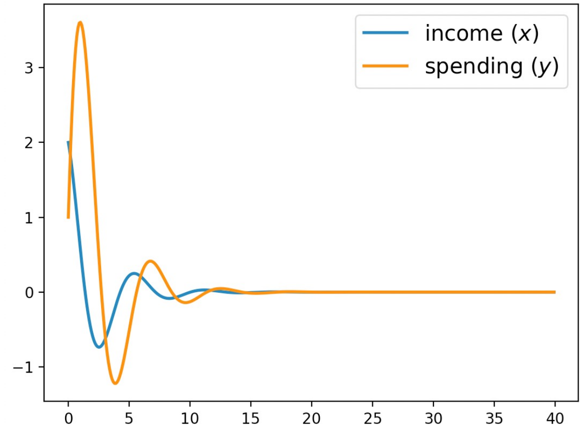




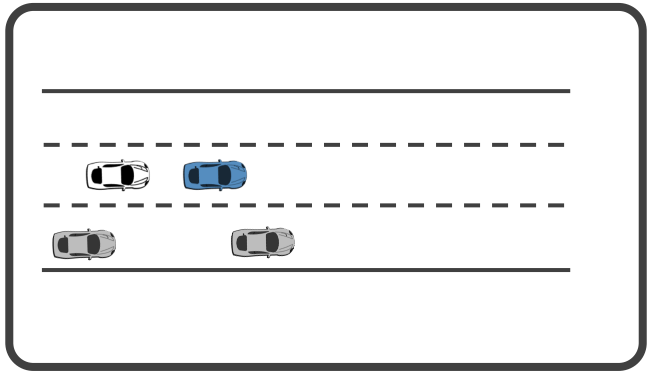








Nonlinear *hybrid* dynamics



Physical plant

$$\frac{dx}{dt} = f(x, u)$$

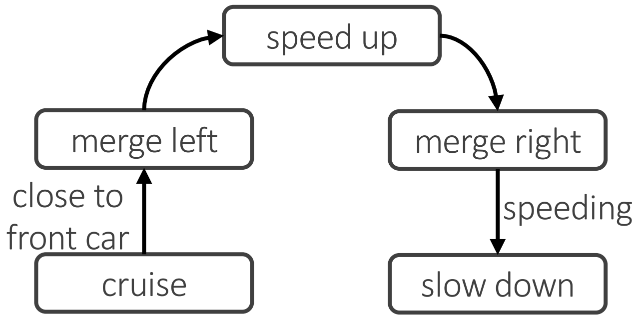
System dynamics

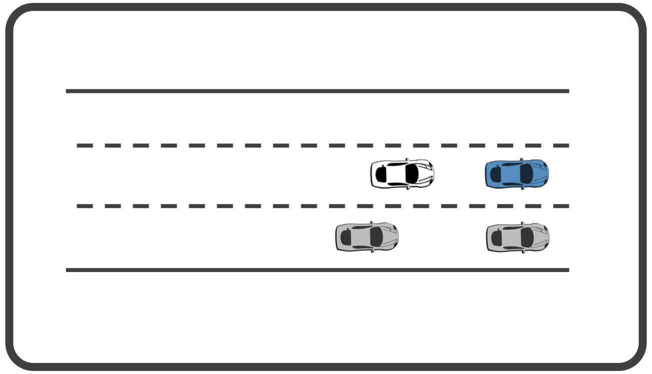
$$x[t+1] = f(x[t], u[t])$$

$x = [v, s_x, s_y, \delta, \psi]$ State variables

$u = [a, v_\delta]$ Control inputs

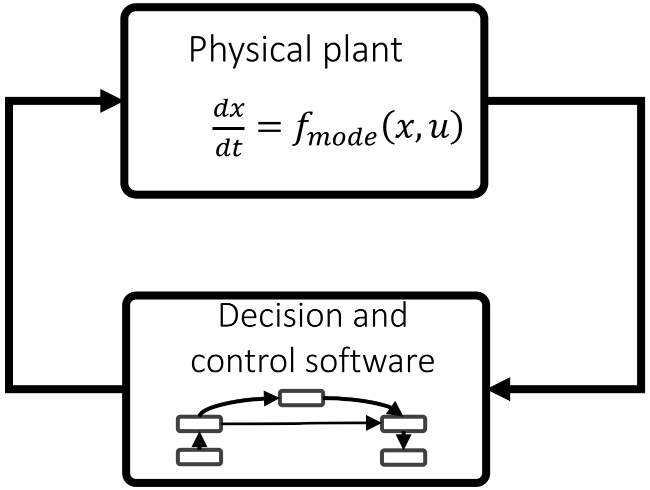
Decision and control software

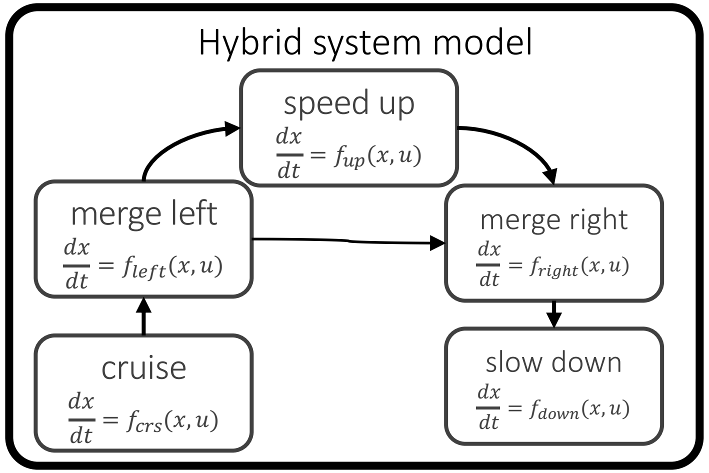




Nonlinear *hybrid* dynamics

Interaction between computation and physics can lead to unexpected behaviors

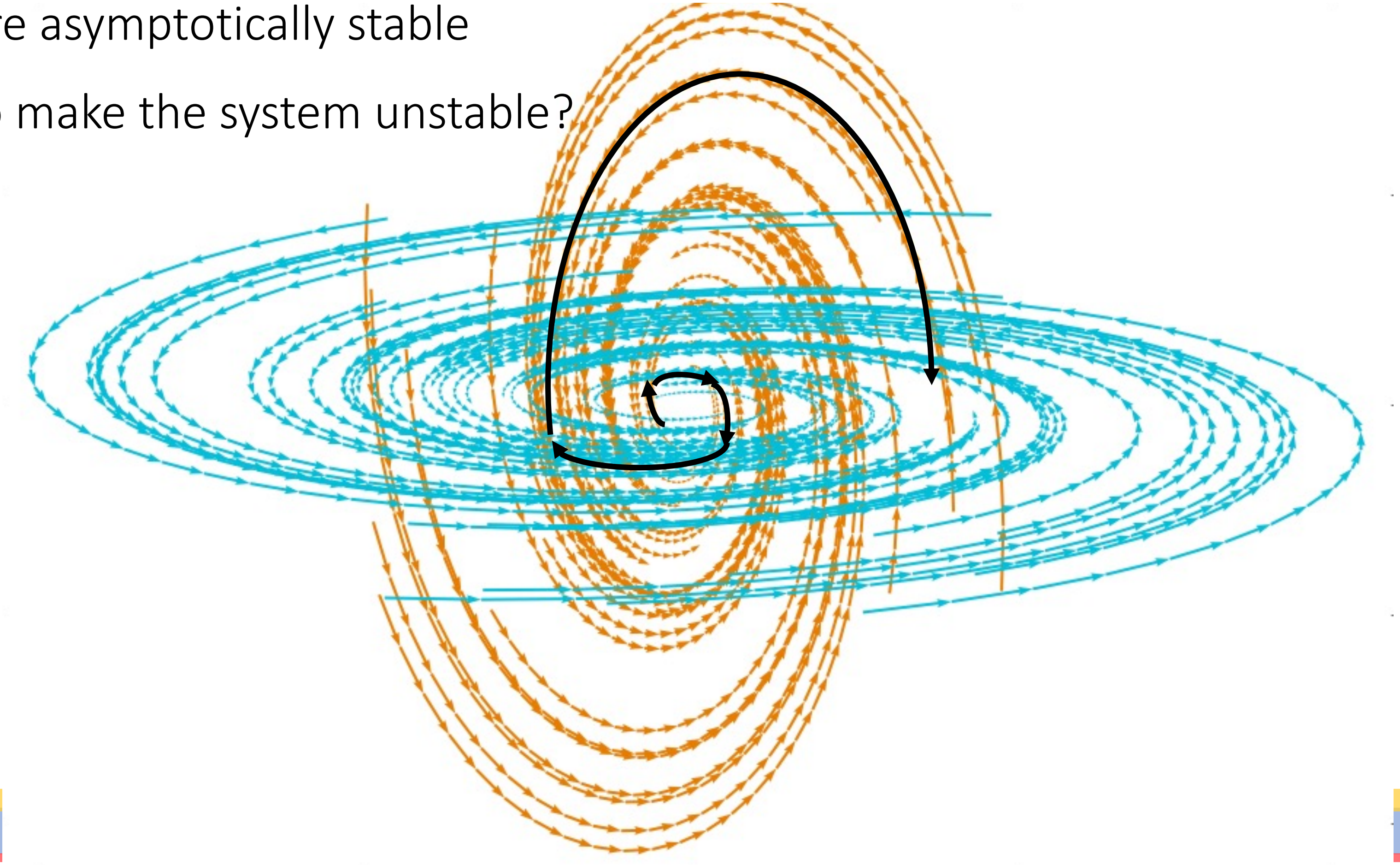


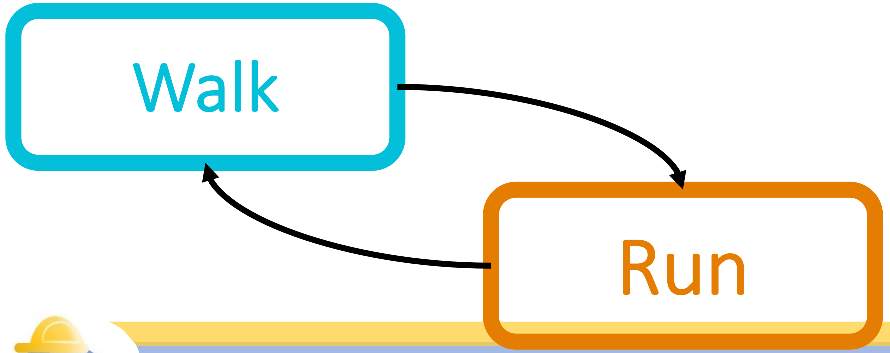


Hybrid Instability: Switching between two stable linear models

Each of the modes of a walking robot are asymptotically stable

Is it possible to switch between them to make the system unstable?





Yes! By switching between them the system becomes unstable