Lecture 11: Filtering & Localization

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ECE484: Principles of Safe Autonomy
Administrivia

• MP2 tutorial on Thursday
  ▪ Due 3/26

• Milestone report due Friday 3/19 by 5pm
  ▪ Submitted via Gradescope
  ▪ Rubric now online!
Thought Experiment

Navigating Intersections
Filtering and Localization Use Cases

HD Maps

Object Tracking

Image Credit: BMW China
Today’s Plan

• What is filtering, mapping, and localization?
  ▪ Probability review!
• Bayes Filters (discrete)
• Kalman Filters (continuous)
• Particle Filters
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Robot States and the Environment

- **State** represents the environment as well as the robot, for example:
  - location of walls or objects (environment or static)
  - pose of the robot (physical or dynamical)

- **Environment interaction** comes in the form of:
  - Sensor measurements
  - Control actions

- **Internal representation (or belief)** of the state of the world:
  - In general, the state (or the world) cannot be measured directly
  - Perception is the process by which the robot uses its sensors to obtain information about the state of the environment
Maps and Representations

- **Mapping** is one of the fundamental problems in (mobile) robotics.
- Maps allow robots to efficiently carry out their tasks and enable **localization**.
- Successful robot systems rely on maps for localization, navigation, path planning, activity planning, control, etc.
The General Problem of Mapping

- Sensor interpretation
  - How do we extract relevant information from raw sensor data?
  - How do we represent and integrate this information over time?
- Robot locations must be estimated
  - How can we identify that we are at a previously visited place?
  - This problem is the so-called data association problem

What does the environment look like?
Resulting Map Obtained with Ultrasound Sensors
The Localization Problem

• Determine the pose (state) of the robot relative to the given map of the environment.

• This is also known as position or state estimation problem.

• Given uncertainty in our measurements and ambiguity from locally symmetric environment, we need to recursively update our estimate or belief.
Probability review

• $P(A)$ denotes the probability that event A is true
  • $0 \leq P(A) \leq 1$, $P(\text{true}) = 1$, $P(\text{false}) = 0$
• $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Discrete Random Variables
• $X$ can take on a countable number of values in $\{x_1, x_2, \ldots, x_n\}$
• $P(X = x_i)$, or $P(x_i)$, is the probability that the random variable $X$ takes on value $x_i$
• $P(\cdot)$ is called probability mass function

Continuous Random Variables
• $X$ takes on values in the continuum
• $p(X = x)$, or $p(x)$, is a probability density function
• $P(x \in (a, b)) = \int_a^b p(x)dx$
Joint, Conditional, and Total Probability

• Joint Probability: $P(X = x \text{ and } Y = y) = P(x, y)$
  - If $x$ and $y$ are independent, then $P(x, y) = P(x) \cdot P(y)$

• Conditional Probability: $P(x|y)$
  - $P(x, y) = P(x|y)P(y) \Rightarrow P(x|y) = \frac{P(x,y)}{P(y)}$
  - If independent, then $P(x|y) = P(x)$

• Total Probability and Marginals
  
  Discrete case:
  - $\sum_x P(x) = 1$
  - $P(x) = \sum_y P(x, y) = \sum_y P(x|y)P(y)$
  
  Continuous case:
  - $\int_x p(x)dx = 1$
  - $p(x) = \int p(x, y)dy = \int p(x|y)p(y)dy$
Bayes’s Formula

Often write: $P(x|y) = \frac{P(y|x)P(x)}{P(y)}$

Recall: $P(x, y) = P(x|y)P(y) = P(y|x)P(x)$

Prior

$P(x|y)$ is diagnostic

$P(y|x)$ is causal

$P(y)$ is evidence

$P(x)$ is posterior

$P(y|x)$ is likelihood

$P(x)$ is prior

State to est given sensor/info

Normalization
**Robot’s Belief over States**

*Belief*: Robot’s knowledge about the state of the environment

\[ bel(x_t) = p(x_t|z_{1:t}, u_{1:t}) \]

**Posterior distribution** over state at time \( t \) given all past measurements and control

**Prediction**: \( \overline{bel}(x_t) = p(x_t|z_{1:t-1}, u_{1:t}) \)

Calculating \( bel(x_t) \) from \( \overline{bel}(x_t) \) is called **correction** or **measurement update**
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Notation and Definitions

• Discrete time model
  \[ x_{t_1:t_2} = x_{t_1}, x_{t_1+1}, x_{t_1+2}, ..., x_{t_2} \] sequence of states \( t_1 \) to \( t_2 \)

• Robot takes one measurement at a time
  \[ z_{t_1:t_2} = z_{t_1}, ..., z_{t_2} \] sequence of all measurements from \( t_1 \) to \( t_2 \)

• Control also exercised at discrete steps
  \[ u_{t_1:t_2} = u_{t_1}, u_{t_1+1}, u_{t_1+2}, ..., u_{t_2} \] sequence control inputs
State Evolution / Models

Evolution of the state and measurements are governed by probabilistic laws:

\[ p(x_t|x_{0:t-1}, z_{1:t-1}, u_{1:t}) \]

describes state evolution / motion model

If the state is *complete*, we can succinctly state:

\[ p(x_t|x_{0:t-1}, z_{1:t-1}, u_{1:t}) = p(x_t|x_{t-1}, u_t) \]

Measurement process given by:

\[ p(z_t|x_{0:t}, z_{1:t-1}, u_{0:t-1}) \]

Similarly, if measurement is complete:

\[ p(z_t|x_{0:t}, z_{1:t-1}, u_{1:t}) = p(z_t|x_t) \]
Discrete Bayes Filter Algorithm: Setup

- Evolution of the state is governed by probabilistic state transition:
  \[ p(x_t | x_{t-1}, u_t) \]
- Measurement process given by:
  \[ p(z_t | x_t) \]
Robot’s Belief over States

**Belief**: Robot’s knowledge about the state of the environment

\[ \text{bel}(x_t) = p(x_t|z_{1:t}, u_{1:t}) \]

**Posterior distribution** over state at time \( t \) given all past measurements and control

**Prediction**: \( \overline{\text{bel}}(x_t) = p(x_t|z_{1:t-1}, u_{1:t}) \)

Calculating \( \text{bel}(x_t) \) from \( \overline{\text{bel}}(x_t) \) is called **correction** or **measurement update**
Recursive Bayes Filter

Algorithm Bayes_Filter$(bel(x_{t-1}), u_t, z_t)$
for all $x_t$ do:

\[
\overline{bel}(x_t) = \int p(x_t|u_t, x_{t-1}) \text{bel}(x_{t-1}) \, dx_{t-1}
\]
\[
\text{bel}(x_t) = \eta \ p(z_t|x_t) \ \overline{\text{bel}}(x_t)
\]
end for
return $\text{bel}(x_t)$
Recursive Bayes Filter

Algorithm Bayes\_Filter(bel(x\_t-1), u\_t, z\_t)
for all x\_t do:
\[ bel(x\_t) = \sum_{x_{t-1}} p(x\_t | u\_t, x_{t-1}) bel(x_{t-1}) \]
\[ bel(x\_t) = \eta p(z\_t | x\_t) \overline{bel(x\_t)} \]
end for
return bel(x\_t)
Discrete Bayes Filter - Illustration

Time = 0

Time = 1

Time = 2
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Bayes Filter Reminder

\[ bel(x_t) = \eta p(z_t|x_t) \int p(x_t | u_t, x_{t-1}) bel(x_{t-1}) dx_{t-1} \]

→ Prediction

\[ \overline{bel}(x_t) = \int p(x_t | u_t, x_{t-1}) bel(x_{t-1}) dx_{t-1} \]

→ Correction

\[ bel(x_t) = \eta p(z_t|x_t) \overline{bel}(x_t) \]

What if we have a good model of our (continuous) system dynamics and we assume a Gaussian model for our uncertainty?

→ Kalman Filters!
Linear Systems with Gaussian Noise

Suppose we have a system that is governed by a linear difference equation:

\[ x_t = A_t x_{t-1} + B_t u_t + \epsilon_t \]

with measurement

\[ z_t = C_t x_t + \delta_t \]

Recall Multivariate Gaussians:

\[
p(x) \sim N(\mu, \Sigma) \equiv \frac{1}{(2\pi)^{d/2} |\Sigma|^{1/2}} e^{-1/2 (x-\mu)^T \Sigma^{-1} (x-\mu)}
\]

- \( A_t \) is an \( n \times n \) matrix that describes how the state evolves from \( t - 1 \) to \( t \) without controls or noise.
- \( B_t \) is an \( n \times m \) matrix that describes how the control \( u_t \) changes the state from \( t - 1 \) to \( t \).
- \( C_t \) is a \( k \times n \) matrix that describes how to map the state \( x_t \) to an observation \( z_t \).
- \( \epsilon_t \) and \( \delta_t \) are random variables representing the process and measurement noise, and are assumed to be independent and normally distributed with zero mean and covariance \( Q_t \) and \( R_t \), respectively.
What is a Kalman Filter?

Suppose we have a system that is governed by a linear difference equation:

\[ x_t = A_t x_{t-1} + B_t u_t + \epsilon_t \]

with measurement

\[ z_t = C_t x_t + \delta_t \]

• Tracks the estimated state of the system by the mean and variance of its state variables -- minimum mean-square error estimator

• Computes the Kalman gain, which is used to weight the impact of new measurements on the system state estimate against the predicted value from the process model

• Note that we no longer have discrete states or measurements!
Summary

• **Bayes filters** are a probabilistic tool for estimating the state of dynamic systems
  ▪ **They are everywhere!** Kalman filters, Particle filters, Hidden Markov models, Dynamic Bayesian networks, Partially Observable Markov Decision Processes (POMDPs), ...
  ▪ Bayes rule allows us to compute probabilities that are hard to assess otherwise
  ▪ Recursive Bayesian updating can efficiently combine evidence

• **Kalman filters** give you the optimal estimate for linear Gaussian systems
Fun Fact: Who is Bayes?

Bayes was an English **statistician**, philosopher, and minister who lived from 1701 to 1761, and is known for two works:

1. *Divine Benevolence, or an Attempt to Prove That the Principal End of the Divine Providence and Government is the Happiness of His Creatures* (1731)

2. *An Introduction to the Doctrine of Fluxions, and a Defence of the Mathematicians Against the Objections of the Author of The Analyst* (1736), in which he **defended the logical foundation of Isaac Newton's calculus** ("fluxions") against the criticism of George Berkeley, author of *The Analyst*

Bayes never published his most famous accomplishment **Bayes’ Theorem**. These notes were edited and published after his death by Richard Price.
Fun facts about Monte Carlo Methods

... a question which occurred to me in 1946 as I was convalescing from an illness and playing solitaires. The question was what are the chances that a Canfield solitaire laid out with 52 cards will come out successfully? After spending a lot of time trying to estimate them by pure combinatorial calculations, I wondered whether a more practical method than "abstract thinking" might not be to lay it out say one hundred times and simply observe and count the number of successful plays. This was already possible to envisage with the beginning of the new era of fast computers, and I immediately thought of problems of neutron diffusion and other questions of mathematical physics, and more generally how to change processes described by certain differential equations into an equivalent form interpretable as a succession of random operations. Later [in 1946], I described the idea to John von Neumann, and we began to plan actual calculations. - Stanislaw Ulam, ~1940s

Monte Carlo methods vary, but tend to follow this pattern:
1. Define a domain of possible inputs
2. Generate inputs randomly from a probability distribution over the domain
3. Perform a deterministic computation on the inputs
4. Aggregate the results

Monte Carlo methods were central to the simulations required for the Manhattan Project, and more recently has been extended to Sequential Monte Carlo in advanced signal processing and Bayesian inference. Also, the extension MC Tree Search enabled AlphaGo.
Who was Rudolf Kalman?

- Kálmán was one of the most influential people on control theory today and is most known for his co-invention of the Kalman filter (or Kalman-Bucy Filter).
- The filtering approach was initially met with vast skepticism, so much so that he was forced to do the first publication of his results in mechanical engineering, rather than in electrical engineering or systems engineering.
  - This worked out fine as some of the first use cases was with NASA on the Apollo spacecraft.
- Kalman filters are inside every robot, commercial airplanes, uses in seismic data processing, nuclear power plant instrumentation, and demographic models, as well as applications in econometrics.

Door example of Bayes Rule

Suppose a robot obtains measurement $z$. What is $P(\text{open}|z)$?

Given: $P(z|\text{open}) = 0.6$, $P(z|\text{closed}) = 0.3$

$P(\text{open}) = P(\text{closed}) = 0.5$

$$P(\text{open}|z) = \frac{P(z|\text{open})P(\text{open})}{P(z)}$$

$$= \frac{P(z|\text{open})P(\text{open})}{P(z|\text{open})P(\text{open}) + P(z|\text{closed})P(\text{closed})}$$

$$= \frac{0.6 \cdot 0.5}{0.6 \cdot 0.5 + 0.3 \cdot 0.5} = \frac{2}{3} = 0.67$$
Bayes Filters (1)

\[
\text{bel}(x_0 = \text{open}) = 0.5
\]

\[
P(z_t | x_t) \rightarrow \begin{array}{c}
\text{open or closed} \\
\text{sense - open (so)} \\
\text{sense - closed (sc)}
\end{array}
\]

\[
\text{bel}(x_t = \text{closed}) = 0.5
\]

\[
P(z_t = so | x_t = c)
\]

\[
\begin{array}{c|cc}
 & \text{open} & \text{closed} \\
\hline
so & 0.6 & 0.2 \\
sc & 0.4 & 0.8 \\
\end{array}
\]

\[
P(x_t | u_t, x_{t-1}) \rightarrow \begin{array}{c}
\text{nothing} \\
\text{push}
\end{array}
\]

\[
\begin{array}{c}
\text{nothing} \\
\text{push}
\end{array}
\]

\[
\begin{array}{c|cc}
 & \text{open} & \text{closed} \\
\hline
\text{nothing} & 0.7 & 0.3 \\
\text{push} & 0.2 & 0.8 \\
\end{array}
\]

\[
x_{t-1}
\]

\[
\begin{array}{c|cc}
 & \text{open} & \text{closed} \\
\hline
\text{open} & 0 & 0.1 \\
\text{closed} & 0.8 & 0.2 \\
\end{array}
\]

\[
x_t
\]

\[
\begin{array}{c|cc}
 & \text{open} & \text{closed} \\
\hline
\text{open} & 0 & 1 \\
\text{closed} & 1 & 0 \\
\end{array}
\]

\[
\begin{array}{c}
\text{nothing} \\
\text{push}
\end{array}
\]
Bayes Filters (2)

at \( t=1 \), do nothing : \( u_t = n \)

\[
\text{bel}(x_t) = \sum_{x_{t-1}} p(x_t | u_t, x_{t-1}) \text{bel}(x_{t-1})
\]

\[
= p(x_t | u_t=n, x_{t-1}=a) \text{bel}(x_{t-1}=a) +
\]

\[
p(x_t | u_t=n, x_{t-1}=c) \text{bel}(x_{t-1}=c)
\]

suppose that \( x_t = o \), now plug it in!

\[
\text{bel}(x_t=o) = p(x_t=o | u_t=n, x_{t-1}=o) \text{bel}(x_{t-1}=o) +
\]

\[
p(x_t=o | u_t=n, x_{t-1}=c) \text{bel}(x_{t-1}=c)
\]

\[
= 1 \cdot 0.5 + 0 \cdot 0.5 = 0.5
\]
Bayes Filters (3)

Suppose $x_i = c$

$$\text{bel}(x_i = c) = p(x_i = c | u_i = n, x_0 = o) \text{bel}(x_0 = o) + p(x_i = c | u_i = n, x_0 = c) \text{bel}(x_0 = c)$$

$$= 0.5 + 1.5 = 0.5$$

$$\text{bel}(x_i = o) = 0.5$$

$$\text{bel}(x_i = c) = 0.5$$
Bayes Filters (4) recall: \( \text{bel}(x_t) = \sum \text{p}(z_t | x_t) \overline{\text{bel}}(x_t) \)

suppose \( z_t = \text{sense-open} \) (so)

1) \( x_t = o \)
\[
\text{bel}(x_t = o) = \sum \text{p}(z_t = \text{so} | x_t = o) \overline{\text{bel}}(x_t = o)
\]
\[
= 2 \cdot 0.6 \cdot 5 = 2 \cdot 3 = 0.75
\]

2) \( x_t = c \)
\[
\text{bel}(x_t = c) = \sum \text{p}(z_t = \text{so} | x_t = c) \overline{\text{bel}}(x_t = c)
\]
\[
= 2 \cdot 0.2 \cdot 5 = 2 \cdot 1 = 0.25
\]

normalizer: \( \gamma = \frac{1}{0.75 + 0.25} = 2.5 \)
Bayes Filters (5)

Suppose we push door & get new measurement

\[ u_2 = \text{push (p)} \quad z_2 = \text{sense-open (so)} \]

\[
\bar{\text{bel}}(x_2 = o) = 1 \cdot 0.75 + 0.8 \cdot 0.25 = 0.95
\]

\[
\bar{\text{bel}}(x_2 = c) = 0 \cdot 0.75 + 0.2 \cdot 0.25 = 0.05
\]

\[
\text{bel}(x_2 = o) = \mathcal{Z} \cdot 0.95 \approx 0.983
\]

\[
\text{bel}(x_2 = c) = \mathcal{Z} \cdot 0.05 \approx 0.017
\]