Check if the following is valid.

$$\alpha = (\neg x_2 \wedge x_5 \wedge x_4) \vee (\neg x_4 \wedge x_3) \vee (\neg x_3 \wedge x_1) \vee (\neg x_2 \wedge x_1) \vee x_3$$ DNF

$\rightarrow$ α is valid iff $\forall \bar{x}$ $\alpha(\bar{x}) = \text{True}$

α_1 is satisfiable iff $\exists \bar{x}$ $\alpha_1(\bar{x}) = \text{True}$

(i) What is the satisfiability problem we need to solve?

$\alpha_1 = \neg\alpha$

Thm

$\boxed{\text{if } \neg\alpha \text{ is not satisfiable then } \alpha \text{ must be valid}}$

$\alpha_1 =$

$$(x_2 \vee \neg x_5 \vee \neg x_4) \wedge (x_4 \vee \neg x_3) \wedge (x_3 \vee \neg x_1) \wedge (x_2 \vee \neg x_1) \wedge \neg x_3$$ CNF

(ii) We are going to use DP(LL) for solving α_1. Which rule is applicaple to variable x_1? And what is the result of applying that rule?

Ans Pure literal as only $\neg x_1$ appears

$x_1 = \text{False} \quad \neg x_1 = T$

$\alpha_2 = (x_2 \vee \neg x_5 \vee \neg x_4) \wedge$
$(x_4 \vee \neg x_3) \wedge \underline{\neg x_3}$

(ii) Which is applicable to x_3 ?

$x_3 = F$

$\alpha_3 = (x_2 \vee \neg x_5 \vee \neg x_4)$

(iii) What can we conclude about satisfiability of α_3.

Ans. α_3 is SAT $\therefore$ α was not valid.

Problem 1

Measurement model $P(z|x)$

Prior distribution $p(X=x_i) = \frac{1}{3} \; \forall i \in \{1,2,3\}$

Actual measurement is $z=1$.

We have to calculate the posterior

$$P(X=x_i \mid z=1) = \frac{P(z=1 \mid x=x_i)\, P(X=x_i)}{P(z=1)}$$

$$P(X=x_1 \mid z=1) = \frac{0.8 \cdot \frac{1}{3}}{P(z=1)} = \frac{0.8 \times \frac{1}{3}}{\sum_{i=1}^{3} P(z=1 \mid x=x_i)\, P(x=x_i)}$$

$$= \frac{0.8 \times \frac{1}{3}}{0.8 \cdot \frac{1}{3} + 0.4 \times \frac{1}{3} + 0.1 \times \frac{1}{3}} = \frac{8}{13}$$

$$P(X=x_2 \mid z=1) = \frac{0.4 \times \frac{1}{3}}{\ldots} = 4/13$$

$$P(X=x_3 \mid z=1) = \qquad 1/13$$

Problem 2   k-means clustering for N vectors in $\mathbb{R}^n$

(a) input : $k \in \mathbb{N}$, $\{x_1, \dots, x_N \in \mathbb{R}^n\}$
output : $z_1, \dots, z_k \in \mathbb{R}^n$

Randomly initialize z_i, z_i' in $\mathbb{R}^n$

Repeat {

$z_m = z_m' \quad \forall m \in \{1, \dots k\}$

For all $j \in \{1, \dots N\}$

$c_j = \arg\min_{i \in \{1,\dots k\}} \|x_j - z_i\|$ // assigns j to cluster i

start here

For each $m \in \{1, \dots k\}$

$G_m = \{j \in \{1, \dots N\} \mid c_j = m\}$ // Point in cluster m

$z_m' = \frac{1}{|G_m|} \sum_{j \in G_m} x_j$

} until $z_m' = z_m \quad \forall m \in \{1, \dots k\}$

(b) Claim 1 $z_m' = \alpha z_m \quad \forall m$

$$z_m' = \frac{1}{|G_m|} \sum x_i' = \frac{1}{|G_m|} \sum \alpha x_i$$

$$= \frac{\alpha}{|G_m|} \sum x_i = \alpha z_m$$

Claim 2 After the z_m's are updated c_i's G_m's do not change.

Proof Fix any x_i and any two z_1, z_2

$$|x_i - z_1| < |x_i - z_2| \Rightarrow$$
$$|x_i' - z_1'| < |x_i' - z_2'|$$

$$\Rightarrow \quad c_i' = c_i$$

$$\Rightarrow \quad G_m' = G_m$$

RRT

$V \leftarrow \{x_{init}\}$; $E \leftarrow \emptyset$

for $i = 1, \ldots, N$ **do**

  $x_{rand} \leftarrow \text{SampleFree}_i$

  $x_{nearest} \leftarrow \text{Nearest}(G = (V, E), x_{rand})$ // Find node in G that is closest to x_{rand}

  $x_{new} \leftarrow \text{Steer}(x_{nearest}, x_{rand})$ // Use local controller to steer $x_{nearest}$ to x_{rand}

  if ObtacleFree($x_{nearest}$, x_{new}) **then**

    $V \leftarrow V \cup \{x_{new}\}$

    $E \leftarrow E \cup \{(x_{nearest}, x_{new})\}$

return $G = (V, E)$