(a) Is this system stable?

eigenvalues of A

$\det(\lambda I - A)$

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\det \begin{bmatrix} \lambda - 1 & -1 \\ -1 & \lambda - 2 \end{bmatrix} = 0$$

* Necessary & sufficient cond. for stability & asymptotic stability

$$(\lambda - 1)(\lambda - 2) - 1 = 0$$

$$\lambda^2 - 3\lambda + 1 = 0$$

$$\text{roots} = \frac{-(-3) \pm \sqrt{9 - 4}}{2}$$

Unstable

$$= \frac{3 \pm \sqrt{5}}{2}$$

both roots are positive

(b) $\rightarrow$ $\underset{\uparrow}{u}$ Control

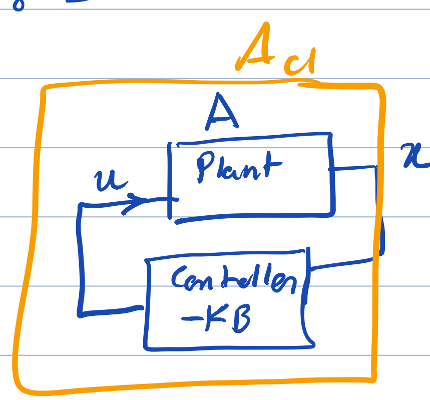
$$u = -\begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -k_1 x_1 - k_2 x_2$$

$$A_{cl} = A - BK \qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ 0 & 0 \end{bmatrix}$$

$$A_{cl} = \begin{bmatrix} 1 - k_1 & 1 - k_2 \\ 1 & 2 \end{bmatrix}$$



(c) $\det(\lambda I - A_{cl}) = 0$

$$= \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1-k_1 & 1-k_2 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda - 1 + k_1 & -1 + k_2 \\ -1 & \lambda - 2 \end{bmatrix}$$

$$= (\lambda - 1 + k_1)(\lambda - 2) + (k_2 - 1)$$

$$= \lambda^2 + \lambda(-2 + k_1 - 1) + (k_2 - 1) + (-2)(k_1 - 1)$$

$$= \lambda^2 + \lambda(k_1 - 3) + k_2 - 2k_1 + 1 \quad \text{— } 1$$

roots $\dfrac{-(k_1-3) \pm \sqrt{(k_1-3)^2 - 4(k_2 - 2k_1 + 1)}}{2}$

Yes we can

roots of characteristic equation

(d) $\lambda_1 = -5 \qquad \lambda_2 = -6$

How to choose k_1 and k_2? — 2

$(\lambda+5)(\lambda+6) = 0 \qquad \lambda^2 + 11\lambda + 30 = 0$

$\lambda^2 -$ Compare the coeffs of 1 & 2

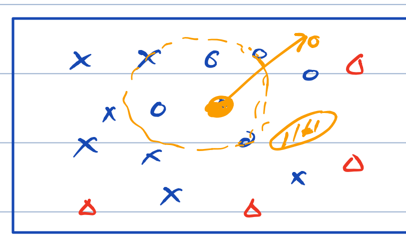
$k_1 - 3 = 11 \qquad k_1 = 14$

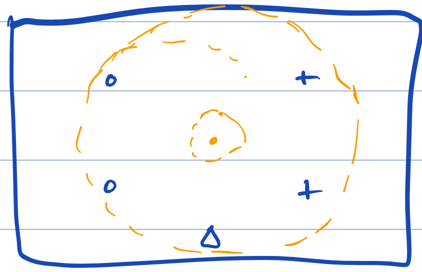
$k_2 - 2k_1 + 1 = 30$

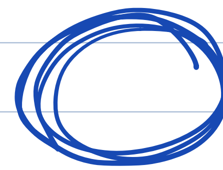
$k_2 = 57.$

Problem

(a) $k = \boxed{5}$ NN







Problem 4

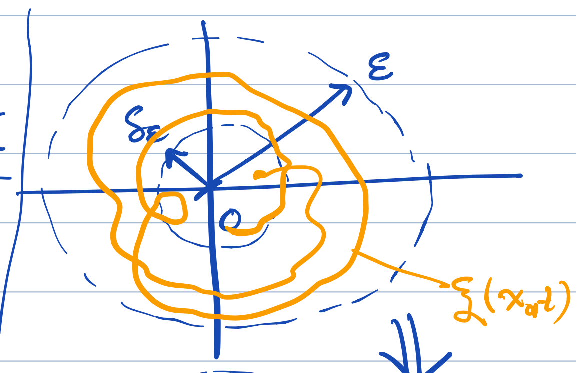
$\dot{x} = f(x)$ $\boxed{\Theta \subseteq \mathbb{R}^n}$

Lyapunov stable : $\forall \varepsilon > 0\ \exists \delta_\varepsilon > 0$ s.t $\boxed{\forall x_0}$
$|\xi(x_0, t)| \leq \delta_\varepsilon \Rightarrow \forall t \geq 0\ |\xi(x_0, t)| \leq \varepsilon$

Invariant

$I \subseteq \mathbb{R}^n$

$\forall x_0 \in \Theta\ \forall t\ \xi(x_0, t) \in I$



Fix an $\varepsilon > 0$ in def LS.

Such that $B_\varepsilon \subseteq I$

From LS we know $\exists \delta_\varepsilon > 0$

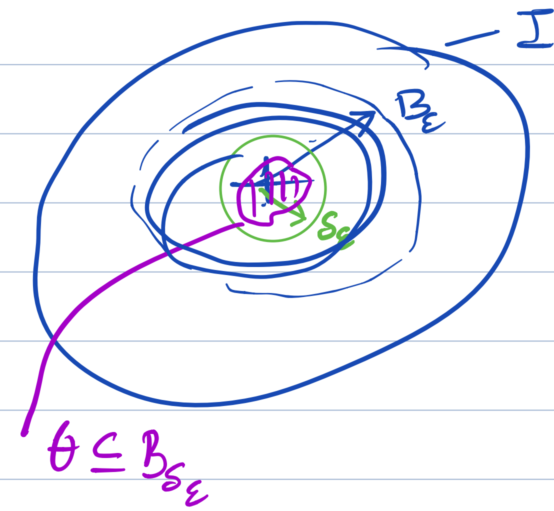
Such that $\boxed{\text{if we start from}}$

B_{δ_ε} the System $\forall t\ \xi(x_0, t) \in B_\varepsilon$

(1) I must contain B_ε for some $\varepsilon > 0$

(2) $\Theta \subseteq B_{\delta_\varepsilon}$ the corresp. to ε

$\Rightarrow$ I must be an invariant. I must contain B_ε for some $\varepsilon > 0$



Problem 5

Lipschitz Continuity $\quad f: \mathbb{R}^n \to \mathbb{R}^n$

$\exists L \; \forall x_1 x_2 \, n \cdots$

$\forall x_1, x_2 \in \mathbb{R} \quad \| f(x_1) - f(x_2) \| \leq L \, \| x_1 - x_2 \|$

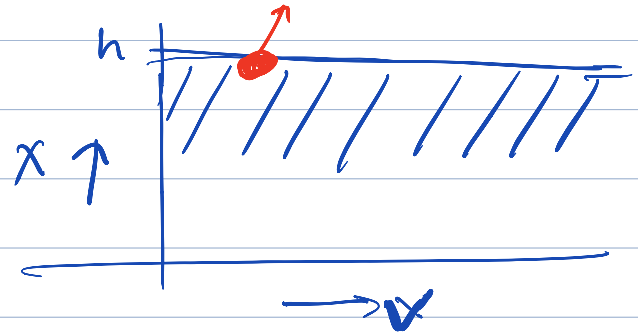
$\sin(x)$

$$\| \sin(x_1) - \sin(x_2) \|$$

$$= \left\| 2 \cos\left(\frac{x_1 + x_2}{2}\right) \sin\left(\frac{x_1 - x_2}{2}\right) \right\| \quad [\text{standard trig equality}]$$

$$\leq 2 \left\| \sin\left(\frac{x_1 - x_2}{2}\right) \right\| \quad [\forall x \ \cos(x) \leq 1]$$

$$\leq 2 \left\| \frac{x - x_2}{2} \right\| \leq \| x_1 - x_2 \| \quad [\forall x \ |\sin(x)| \leq |x|]$$



Problem 6 inductive invariants.

(a) Inv1. $\forall t \quad x(t) \leq h$

How to prove invariant I inductively

(i) Check base case

$\forall x(0), \quad x(0)$ satisfies I

(ii) Check inductive step

$\forall \; x(t) \rightarrow x(t+1)$

if $x(t)$ satisfies I then

$x(t+1)$ also satisfies I

I : $\quad x \leq h$

Base $\quad x(0) = h \leq h$ ✓

Inductive step

$$x(t+1) = \boxed{x(t)} + v(t).d - \tfrac{1}{2}gd^2 \quad \text{[Pline3]}$$

$$\leq h + v(t).d - \tfrac{1}{2}gd^2$$

$x(t+1) \leq h$

we cannot conclude

$x(t+1) \leq h$

(b) $I_2 = v^2(t) - 2g(h - x(t)) = 0$ ✓

Base

Inductive step

Base $v^2(0) - 2g(h - x(0))$

$0 - 2g(h - h) = 0$ ✓

Inductive step. Assume $\boxed{v^2(t) - 2g(h - x(t)) = 0}$

We have to show I_2 holds at $t+1$

<u>Case 1</u> $(x(t+1) \geq 0)$

$v(t+1) = v(t) - g.d$ [PL 3]

$v^2(t+1) = v^2(t) + g^2.d^2 - 2v(t)g.d$

$x(t+1) = x(t) + v(t).d - \frac{1}{2}gd^2$ [PL4]

$v^2(t+1) - 2g(h - x(t+1)) \Leftarrow I_2$

$= v^2(t) + g^2d^2 - 2v(t)gd$

$- 2gh + 2g[x(t) + v(t).d - \frac{1}{2}gd^2]$

$= v^2(t) - 2g(h - x(t))$

$= 0$ [by induction hypothesis]

<u>Case 2</u> $x(t+1) = 0$

$v^2(t+1) = v^2(t) + 2g\,x(t)$

$I_2 = 0$

Problem 7

(a) — (2) (b) — (1) (c) — (3)