# Bayes Filter

- Derivation of Bayes Filter
- → Ref. Chapter 2 of Probabilistic Robotics by Thrun et al.
- → Slides March 11
- → Next: Particle filter algorithm

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$$P(x|y) = \frac{P(x,y)}{P(y)}$$ ← Conditional Prob

$$P(x,y) = P(x|y)\,P(y) \qquad P(y) > 0$$
$$= \underline{P(y|x)\,P(x)}$$

Posterior

Bayes Rule

$$\underline{P(x|y)} = \frac{P(y|x)\,P(x)}{\underline{P(y)}}$$

Prior

x: position

y: data measurement

inverse conditional Prob

Generative model

P(y) does not depend on x

$$\eta = 1/P(y)$$

$$\boxed{P(x|y) = \eta\, P(y|x)\, P(x)}$$

$$\underline{P(y)} = \sum_{x'} \underline{P(y|x')}\,\underline{P(x')}$$

Total prob

Discrete

$$= \int_{x'} p(y \mid x')\, p(x')\, dx'$$ Total prob Continuous

Bayes Rule $$p(x \mid y) = \frac{p(y \mid x)\, p(x)}{\sum_{x'} p(y \mid x')\, p(x')}$$

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Discrete Model

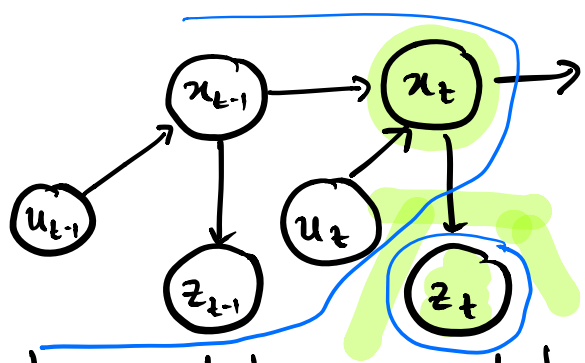
$x_{t_1:t_2} = x_{t_1}, x_{t_1+1}, \ldots x_{t_2}$

Sequence of actual robot states

$z_{t_1:t_2} = z_{t_1}, z_{t_1+1} \ldots$

Sequence of measurements

$u_{t_1:t_2}$ : Sequence of control inputs




State evolution model

$$p(x_t \mid x_{0:t-1}, z_{1:t-1}, u_{1:t-1})$$

State is complete Markovian model

$$= p(x_t \mid x_{t-1}, u_t)$$

$$= \underline{p(x' \mid x, u)}$$

## Measurement model

$$p(z_t \mid x_{0:t}, z_{1:t-1}, u_{1:t-1})$$

$$= p(z_t \mid x_t) \quad \text{if the state is complete}$$

$$= p(z \mid x)$$

measurements

inputs

belief $\underline{bel(x_t)} = \underline{p(x_t} \mid z_{1:t}, u_{1:t})$

$p(x_0)$ initial distribution

Posterior dist over state at time $t$ given all past measurements & control inputs

$$\left.\begin{aligned} \overline{bel}(x_t) &= p(\underline{x_t} \mid \underline{z_{1:t-1}}, \underline{u_{1:t}}) \\ bel(x_t) &= \eta \, p(z_t \mid x_t) \, \overline{bel}(x_t) \end{aligned}\right\}$$

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## Derivation of Bayes filter

$$bel(x_t) = \underline{p(x_t \mid z_{1:t}, u_{1:t})}$$

$$\rightarrow bel(x_{t-1}) = p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1})$$

inductive hypotesis

$$bel(x_0) = p(x_0)$$

Assumptions
1. $x_t$ is complete Markov model
2. $u_t$ is chosen at random

Bayes Rule $\quad P(x|y) = \dfrac{P(y|x)\,P(x)}{P(y)} \qquad P(y) > 0$

$$\Rightarrow P(x \mid y, z) = \frac{P(y \mid x, z)\, P(x \mid z)}{P(y \mid z)} \qquad P(y|z) > 0$$

$z_1, z_2, \ldots$

$$\mathrm{bel}(x_t) = P(x_t \mid z_{1:t}, u_{1:t})$$

$x = x_t \quad y = z_t \quad z = z_{1:t-1} \; u_{1:t}$

$$= \frac{P(z_t \mid x_t, z_{1:t-1}, u_{1:t})\, P(x_t \mid z_{1:t-1}, u_{1:t})}{P(z_t \mid z_{1:t-1}, u_{1:t})}$$

$$= \eta\, P(z_t \mid x_t, z_{1:t-1}, u_{1:t})\, P(x_t \mid z_{1:t-1}, u_{1:t})$$

Assume State is complete

$$= \eta\, P(z_t \mid x_t)\, \underbrace{P(x_t \mid z_{1:t-1}, u_{1:t})}_{\overline{\mathrm{bel}}(x_t)}$$

prediction

$$\boxed{\mathrm{bel}(x_t) = \eta\, P(z_t \mid x_t)\, \overline{\mathrm{bel}}(x_t)}$$

$$\overline{\mathrm{bel}}(x_t) = P(x_t \mid z_{1:t-1}, u_{1:t})$$

Use law of total probability

$$= \int_{x_{t-1}} P(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t})\, P(x_{t-1} \mid z_{1:t-1}, u_{1:t})\, dx_{t-1}$$

Markov property for state transitions

$$= \int_{x_{t-1}} P(x_t \mid x_{t-1}, u_t)\, P(x_{t-1} \mid z_{1:t-1}, u_{1:t})\, dx_{t-1}$$

$$= \int P(x_t \mid x_{t-1}, u_t)\, P(x_{t-1} \mid z_{1:t-1}, u_{1:t-1})\, dx_{t-1}$$

$\int_{x_{t-1}}$ $\underbrace{\text{assuming random inputs}}_{bel(x_{t-1})}$

$$\boxed{\overline{bel}(x_t) = \int_{x_{t-1}} p(x_t \mid x_{t-1}, u_t)\, bel(x_{t-1})\, d x_{t-1}}$$

$p(x_0) = bel(x_0)$

input

measurement

Algorithm Bayes_filter($bel(x_{t-1}), u_t, z_t$)

for all $x_t$ do:

Prediction update

1. $\overline{bel}(x_t) = \int p(x_t|u_t, x_{t-1}) bel(x_{t-1}) dx_{t-1}$
2. $bel(x_t) = \eta\, p(z_t|x_t)\, \overline{bel}(x_t)$

measurement update

end for

return $bel(x_t)$

Summary — Derived the Bayes filter Algo

→ Markov model / Complete state

→ Posterior over state $x_t$ conditioned on measurement $z_t$ control data $u_t$

How to implement? E.g. Histogram filter

Particle filter