Bayes Filter

- Derivation of Bayes Filter
  → Ref. Chapter 2 of Probabilistic Robotics by Thrun
  → Slides March 11 et al.
  → Next: Particle filter algorithm

\[
P(x|y) = \frac{P(x, y)}{P(y)} \leftarrow \text{Conditional Prob}
\]

\[
P(x, y) = P(x|y) \cdot P(y) \quad P(y) > 0
\]

\[
= P(y|x) \cdot P(x)
\]

\[
P(x|y) = \frac{P(y|x) \cdot P(x)}{P(y)} \leftarrow \text{Bayes Rule}
\]

 Prior

\[
\eta = \frac{1}{P(y)}
\]

\[
P(x|y) = \eta \cdot P(y|x) \cdot P(x)
\]

\[
P(y) = \sum_{x'} P(y|x') \cdot P(x') \quad \text{Total prob Discrete}
\]

x: position

y: data measurement

\[P(y)\] does not depend on \(x\)
\[
= \int \ p(y|x')\ p(x')\ dx'\quad \text{Total prob continuous}
\]

Bayes Rule \[ p(x|y) = \frac{p(y|x)\ p(x)}{\sum_{x'} p(y|x')\ p(x')} \]

**Discrete Model**

\[ x_{t_1:t_2} = x_{t_1}, x_{t_1+1}, \ldots, x_{t_2} \]
Sequence of actual robot states

\[ z_{t_1:t_2} = z_{t_1}, z_{t_1+1}, \ldots \]
Sequence of measurements

\[ u_{t_1:t_2} \] : Sequence of control inputs

**State evolution model**

\[ p(x_t | x_{0:t-1}, z_{1:t-1}, u_{1:t-1}) \]

State is complete Markovian model

\[
= p(x_t | x_{t-1}, u_t) \\
= p(x' | x_t, u_t)
\]
**Measurement model**

\[
P(Z_t \mid x_{0:t}, z_{1:t-1}, u_{1:t-1}) = \frac{p(Z_t \mid v_t)}{p(z_{1:t})}
\]

if the state is complete
\[
p(Z_t \mid v_t)
\]

measurements

\[
bel(x_t) = \frac{p(x_t \mid z_{1:t}, u_{1:t})}{p(x_0)} \quad \text{initial distribution}
\]

Posterior distribution over state at time \( t \)
given all past measurements & control inputs

\[
\bar{bel}(x_t) = \mathbb{E}_{p(z_{1:t-1}, u_{1:t})} \left[ bel(x_t) \right]
\]

\[
bel(x_t) = \mathbb{E}_{p(z_{t+1} \mid x_t)} \bar{bel}(x_t)
\]

Derivation of Bayes filter

\[
bel(x_t) = p(x_t \mid z_{1:t}, u_{1:t})
\]

\[
\Rightarrow bel(x_{t-1}) = p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1})
\]

inductive hypothesis

\[
bel(x_0) = p(x_0)
\]

Assumptions:

1. \( x_t \) is complete Markov model
2. \( u_t \) is chosen at random
Bayes Rule \( P(x | y) = \frac{P(y | x) P(x)}{P(y)} \), \( P(y) > 0 \)

\[
\begin{align*}
\Rightarrow P(x | y, z_t) &= \frac{P(y | x, z_t) P(x | z_t) P(b_t)}{P(y | z_t)} \\
&= \frac{P(z_t | x_t, 2_{1:t-1}, u_{1:t}) P(x_t | z_t, 2_{1:t-1}, u_{1:t})}{P(z_t | 2_{1:t-1}, u_{1:t})} \\
&= \eta \frac{P(z_t | x_t, 2_{1:t-1}, u_{1:t}) P(x_t | z_t, 2_{1:t-1}, u_{1:t})}{P(z_t | 2_{1:t-1}, u_{1:t})} \\
&= \eta \frac{P(x_t | z_t, 2_{1:t-1}, u_{1:t})}{P(z_t | 2_{1:t-1}, u_{1:t})} \\
&= \eta \frac{P(z_t | x_t, 2_{1:t-1}, u_{1:t})}{P(z_t | 2_{1:t-1}, u_{1:t})} \\
\end{align*}
\]

Assume state is complete

\[
\text{bel}(x_t) = \eta \frac{P(z_t | x_t, 2_{1:t-1}, u_{1:t})}{P(z_t | 2_{1:t-1}, u_{1:t})} \\
\overline{\text{bel}}(x_t) = p(x_t | 2_{1:t-1}, u_{1:t}) \\
\text{Use law of total probability}
\]

\[
\begin{align*}
\overline{\text{bel}}(x_t) &= \int p(x_t | x_{t-1}, 2_{1:t-1}, u_{1:t}) \ d x_{t-1} \\
&= \int p(x_t | x_{t-1}, u_t) \ d x_{t-1} \\
&= \int p(x_t | x_{t-1}, u_t) \ d x_{t-1} \\
&= \int p(x_t | x_{t-1}, u_t) \ P(x_{t-1} | 2_{1:t-1}, u_{1:t}) d x_{t-1}
\end{align*}
\]

Markov property for state transitions

\[
\begin{align*}
&= \int p(x_t | x_{t-1}, u_t) \ P(x_{t-1} | 2_{1:t-1}, u_{1:t}) d x_{t-1} \\
&= \left[ p(x_t | x_{t-1}, u_t) \ P(x_{t-1} | 2_{1:t-1}, u_{1:t}) \right] d x_{t-1}
\end{align*}
\]
\[ \bar{\text{bel}}(x_t) = \int p(x_t|u_t, x_{t-1}) \text{bel}(x_{t-1}) \, dx_{t-1} \]

\[ \text{bel}(x_0) = \text{bel}(x_0) \]

Algorithm Bayes_filter(\text{bel}(x_{t-1}), u_t, z_t)

for all \( x_t \) do:

1. \( \overline{\text{bel}}(x_t) = \int p(x_t|u_t, x_{t-1}) \text{bel}(x_{t-1}) \, dx_{t-1} \)
2. \( \text{bel}(x_t) = \eta p(z_t|x_t) \text{bel}(x_t) \)  

end for

return \( \text{bel}(x_t) \)

Summary — Derived the Bayes filter Algo

⇒ Markov model / Complete state
⇒ Posterior over state \( x_t \)
⇒ Conditioned on measurement \( z_t \)
⇒ Control data, \( u_t \)

How to implement? E.g. Histogram filter
⇒ Particle filter