ECE/CS 498SM: Principles of Safe Autonomy

Dynamics, Control, and Stability
Lectures 16

April 1

Sayan Mitra

Outline

- Lyapunov stability and asymptotic stability
- Lyapunov functions
- Hybrid systems (gentle introduction)
  - Surprises with hybrid executions
- Stability of hybrid and switched systems
  - Common Lyapunov Functions
  - Multiple Lyapunov Function
  - Dwell-time Criteria

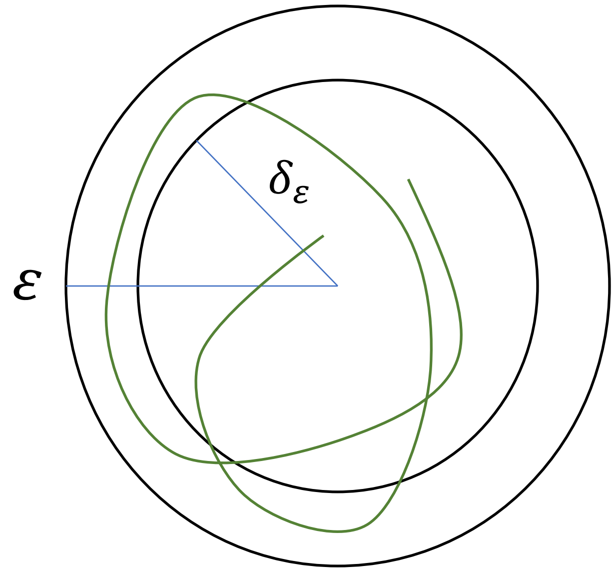
Lyapunov stability

Lyapunov stability: The system (1) is said to be ***Lyapunov stable*** (at the origin) if for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for every if $|\xi(0)| \leq \delta_\varepsilon$ then for all $t \geq 0, |\xi(t)| \leq \varepsilon$.

How is this related to

invariants and

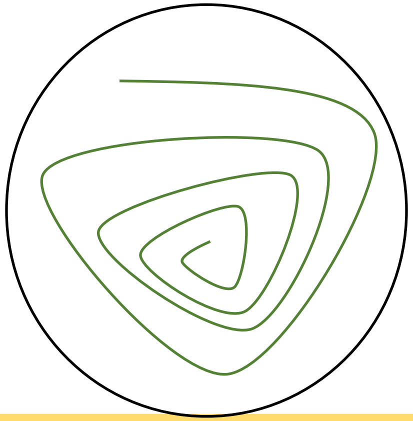
reachable states ?



Asymptotically stability

The system (1) is said to be ***Asymptotically stable*** *(at the origin)* if it is Lyapunov stable and there exists $\delta_2 > 0$ such that for every if $|\xi(0)| \leq \delta_2$ then $t \rightarrow \infty$, $|\xi(t)| \rightarrow \mathbf{0}$.

If the property holds for any δ_2 then **Globally Asymptotically Stable**



Example: Pendulum

Pendulum equation

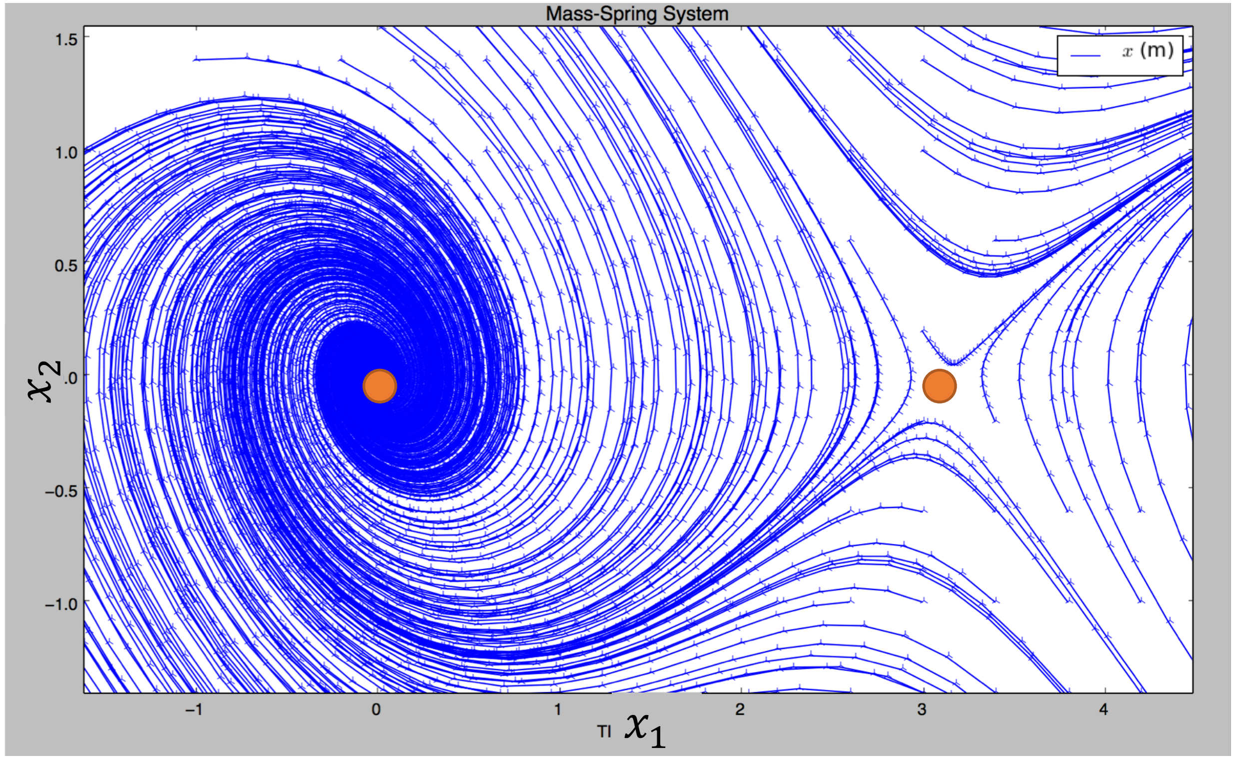
$x_1 = \theta \quad x_2 = \dot{\theta}$

$x_2 = \dot{x}_1$

$$\dot{x}_2 = -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2$$

$$\begin{bmatrix}\dot{x_2}\\ \dot{x_1}\end{bmatrix} = \begin{bmatrix}-\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2 \\ x_2\end{bmatrix}$$

Two equilibrium points: $(0,0), (\pi, 0)$



$x = (0, 0)$
asymptotically stable

$x = (\pi, 0)$
unstable

Example: Pendulum

Pendulum equation

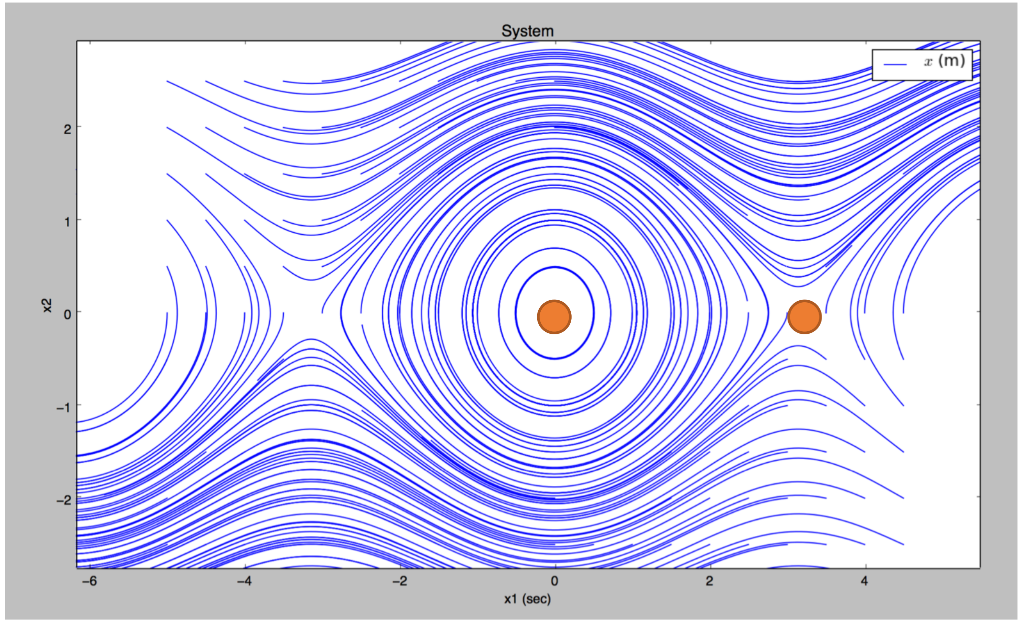
$x_1 = \theta \quad x_2 = \dot{\theta}$

$x_2 = \dot{x}_1$

$$\dot{x}_2 = -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2$$

$$\begin{bmatrix} \dot{x_2} \\ \dot{x_1} \end{bmatrix} = \begin{bmatrix} -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2 \\ x_2 \end{bmatrix}$$

$k = 0$ no friction



$x^* = (0, 0)$
stable but not asymptotically stable

$x^* = (\pi, 0)$
unstable

Van der pol oscillator

Van der pol oscillator

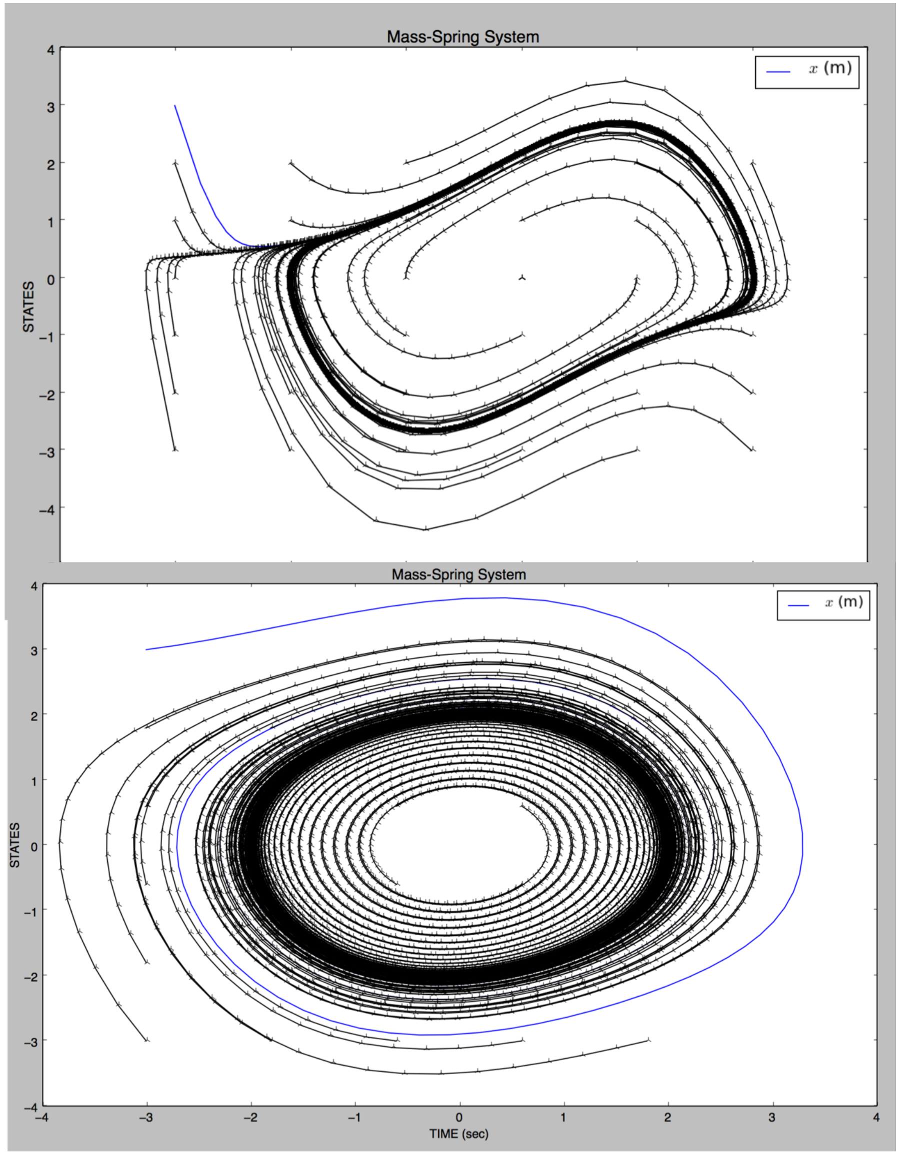
$$\frac{dx^2}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x = 0$$

$x_1 = x; x_2 = \dot{x}_1;$

coupling coefficient $\mu = 2 \;\; 0.1$

$$\begin{bmatrix} \dot{x_2} \\ \dot{x_1} \end{bmatrix} = \begin{bmatrix} \mu(1 - x_1^2)x_2 - \; x_1 \\ x_2 \end{bmatrix}$$

stable ?



Stability of solutions* (instead of points)

- For any $\xi \in \mathrm{PC}(\mathbb{R}^{\geq 0}, \mathbb{R}^n)$ define the s-norm $||\xi||_s = \sup_{t \in \mathbb{R}} ||\xi(t)||$

- A dynamical system can be seen as an operator that maps initial states to signals $T: \mathbb{R}^n \to PC(\mathbb{R}^{\geq 0}, \mathbb{R}^n)$

- Lyapunov stability required that this operator is continuous

- The solution ξ^* is ***Lyapunov stable*** if T is continuous as $\xi^*(0)$. $\mathrm{i.e.}$, for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for every $x_0 \in \mathbb{R}^n$ if $|\xi^*(0) - x_0| \leq \delta_\varepsilon$ then $\left|\left|T\left(\xi^*(t)\right) - T(x_0)\right|\right|_s \leq \varepsilon$.

*Not discussed in class

Verifying Stability for Linear Systems

Consider the linear system $\dot{x} = Ax$

Theorem.

1. It is asymptotically stable iff all the eigenvalues of A have **strictly** negative real parts (*Hurwitz*).

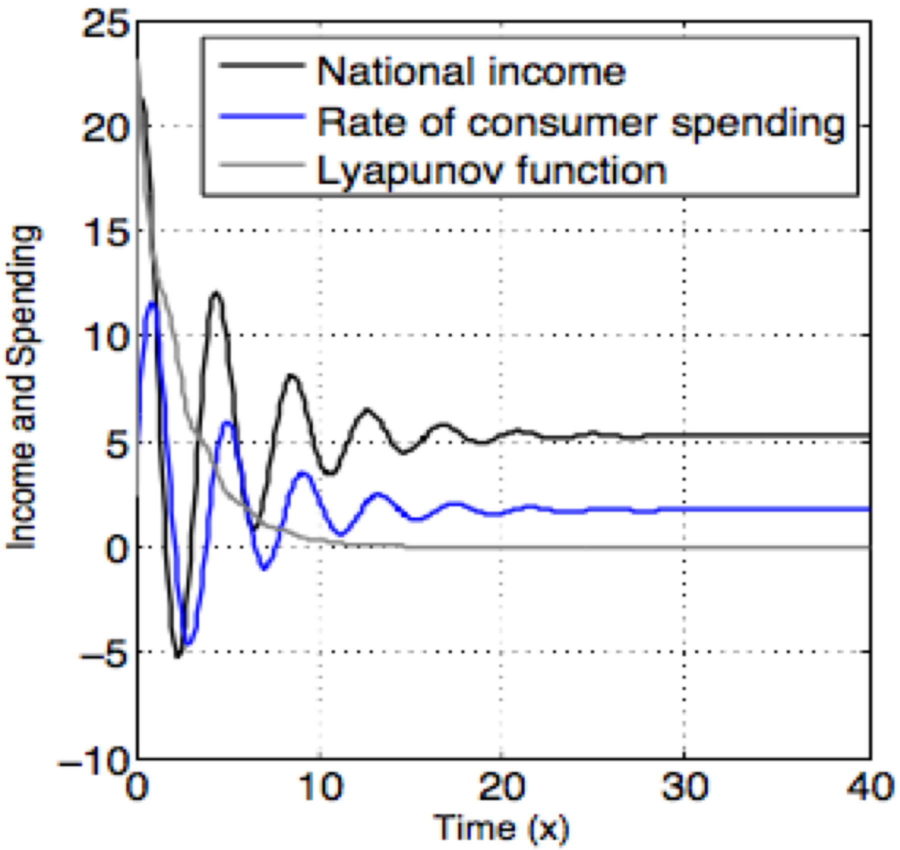
2. It is Lyapunov stable iff all the eigen values of A have real parts that are either zero or negative and the Jordan blocks corresponding to the eigenvalues with zero real parts are of size 1.

Example 1: Simple linear model of an economy

- x: national income y: rate of consumer spending
- g: rate of government expenditure
- $\dot{x} = x - \alpha y$
- $\dot{y} = \beta(x - y - g)$
- $g = g_0 + kx$ α, β, k are positive constants
- What is the equilibrium?
- $x^* = \frac{g_0 \alpha}{\alpha - 1 - k\alpha}$ $y^* = \frac{g_0 \alpha}{\alpha - 1 - k\alpha}$
- Dynamics:
- $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & -\alpha \\ \beta(1 - k) & -\beta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Example: Simple linear model of an economy

- $\alpha = 3, \beta = 1, k = 0$
- $\lambda_1, \lambda_1^* = (-.25 \pm i\ 1.714)$
- Negative real parts, therefore, asymptotically stable and the national income and consumer spending rate converge to $x = 1.764\ y = 5.294$



Lyapunov's method: Stability of nonlinear systems

- For any **positive definite** function of state $V: \mathbb{R}^n \to \mathbb{R}$
  - $V(x) \geq 0$ and $V(x) = 0$ iff $x = 0$
- Sub level sets of $L_p = \{x \in \mathbb{R}^n \mid V(x) \leq p\}$
- $V(\xi(t))$

V differentiable with continuous first derivative

- $\dot{V} = \ d\frac{V(\xi(t))}{dt} = ?$
- $\frac{\partial V}{\partial x} . \frac{d}{dt}(\xi(t)) = \frac{\partial V}{\partial x} . f(x)$ is also continuous
- V is radially unbounded if $||x|| \to \infty \Rightarrow V(x) \to \infty$

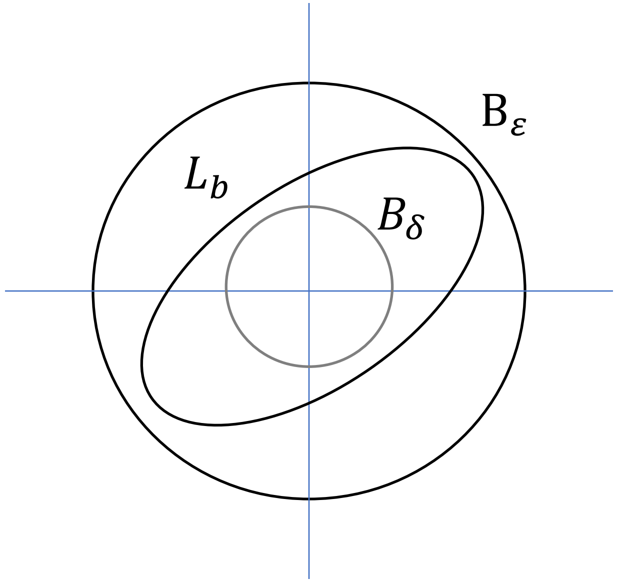
Verifying Stability

Theorem. (Lyapunov) Consider the system (1) with state space $\xi(t) \in \mathbb{R}^n$ and suppose there exists a positive definite, continuously differentiable function $V: \mathbb{R}^n \to \mathbb{R}$. The system is:

1. Lyapunov stable if $\dot{V}\big(\xi(t)\big) = \frac{\partial V}{\partial x} f(x) \leq 0$
2. Asymptotically stable if $\dot{V}\big(\xi(t)\big) < 0$
3. It is globally AS if *V* is also radially unbounded.

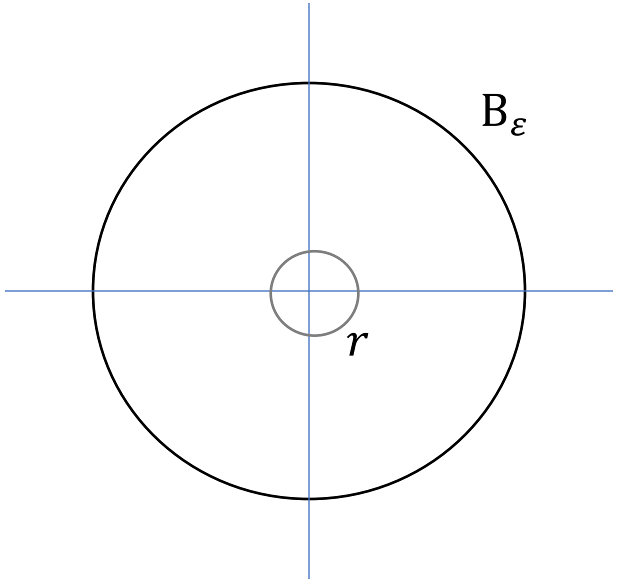
Proof sketch: Lyapunov stable if $\dot{V} \leq 0$

- Assume $\dot{V} \leq 0$
- Consider a ball B_ε around the origin of radius $\varepsilon > 0$.
- Pick a positive number $b < \min_{|x|=\varepsilon} V(x)$.
- Let δ be a radius of ball around origin which is inside $\mathrm{B}_\delta = \{x \mid V(x) \leq b\}$
- Since along all trajectories V is non-increasing, starting from B_δ each solution satisfies $V(\xi(t)) \leq b$ and therefore remains in B_ε



Proof sketch: Asymptotically stable if $\dot{V}(\xi(t)) < 0$

- Assume $\dot{V} < 0$
- Take arbitrary initial state $|\xi(0)| \leq \delta$, where this δ comes from some ε for Lyapunov stability
- Since $V(\xi(.)) > 0$ and decreasing along ξ it has a limit $c \geq 0$ at $t \to \infty$
- It suffices to show that this limit is actually 0
- Suppose not, c > 0 then the solution evolves in the compact set $S = \{x \mid r \leq |x| \leq \varepsilon\}$ for some sufficiently small r
- Let $d = \max_{x \in S} \dot{V}(x)$ [slowest rate]
- This number is well-defined and negative
- $\dot{V}(\xi(t)) \leq d$ for all t
- $V(t) \leq V(0) + dt$
- But then eventually $V(t) < c$



Example 2

- $\dot{x}_1 = -x_1 + g(x_2); \dot{x}_2 = -x_2 + h(x_1)$
- $|g(u)| \leq \frac{|u|}{2}, |h(u)| \leq \frac{|u|}{2}$
- Use $V = \frac{1}{2}(x_1^2 + x_2^2) \geq 0$
- $\dot{V} = x_1\, \dot{x}_1 + x_2\, \dot{x}_2$

$$= -x_1^2 - x_2^2 + x_1 g(x_2) + x_2 h(x_1)$$

$$\leq -x_1^2 - x_2^2 + \frac{1}{2}(|x_1 x_2| + |x_2 x_1|)$$

$$\leq -\frac{1}{2}(x_1^2 + x_2^2) = -V$$

$$(|x_1| - |x_2|)^2 \geq 0$$

$$x_1^2 + x_2^2 \geq 2|x_1 x_2|$$

$$|x_1 x_2| \leq \frac{1}{2}(x_1^2 + x_2^2)$$

We conclude global asymptotic stability (in fact global exponential stability) without knowing solutions

Proposition. Every sublevel set of V is an invariant

Proof. $V\big(\xi(t)\big) =$

$$= V\big(\xi(0)\big) + \int_0^t \dot{V}\big(\xi(\tau)\big)d\tau$$

$$\leq V(\xi(0))$$

An aside: Checking inductive invariants

- $\boldsymbol{A} = \langle X, Q_0, T \rangle$
  - X: set of variables
  - $Q_0 \subseteq val(X)$
  - $T \subseteq val(X) \times val(X)$ written as a program $x' \subseteq T(x)$
- How do we check that $I \subseteq val(X)$ is an inductive invariant?
  - $Q_0 \Rightarrow I(X)$
  - $I(X) \Rightarrow I(T(X))$
- Implies that $Reach_{\boldsymbol{A}}(Q_0) \subseteq I$ without computing the executions or reachable states of $\boldsymbol{A}$
- The key is to find such I

Finding Lyapunov Functions

- The key to using Lyapunov theory is to *find* a Lyapunov function and verify that it has the properties
- In general for nonlinear systems this is hard
- There are several approaches
  - Linear quadratic Lyapunov functions for linear systems
  - Decide the form/template of the function (e.g., quadratic), parameterized by some parameters
  - Try to find values of the parameters so that the conditions hold
  - NNs for learning Lyapunov functions from data [Billard`14]

Learning control Lyapunov function to ensure stability of dynamical system-based robot reaching motions, Khansari-Zadeh, Billard - Robotics and Autonomous Systems, 2014 - Elsevier

Linear autonomous systems

- $\dot{x} = Ax, A \in \mathbb{R}^{n\times n}$
- The Lyapunov equation: $A^T P + PA + Q = 0$

  where $P, Q \in \mathbb{R}^{n\times n}$ are symmetric

- Interpretation: $V(x) = x^T P x$ then

$$\dot{V}(x) = (Ax)^T P x + x^T P (Ax)$$

$$\left[\text{using } \frac{\partial u^T P v}{\partial t} = \frac{\partial u}{\partial t} P v + \frac{\partial v}{\partial t} P^T u\right]$$

$$= x^T (A^T P + PA) x = -x^T Q x$$

- If $x^T P x$ is the generalized energy then $-x^T Q x$ is the associated dissipation

Quadratic Lyapunov Functions

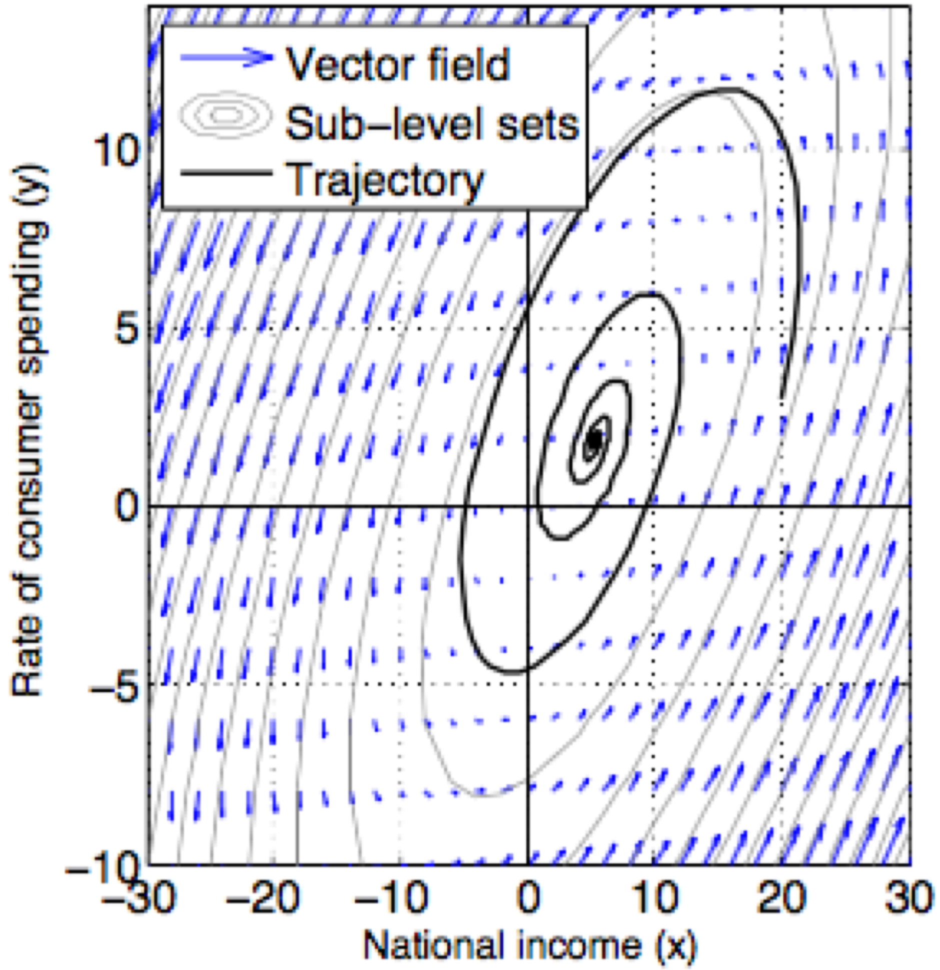
A **positive definite matrix** is a symmetric **matrix** with all **positive** eigenvalues.

- If $P > 0$ (positive definite)
- $V(x) = x^T P x = 0 \Leftrightarrow x = 0$
- The sub-level sets are ellipsoids
- If $Q > 0$ then the system is globally asymptotically stable

Same example

Lyapunov equations are solved as a set of $\frac{n(n+1)}{2}$ equations in $n(n+1)/2$ variables. Cost O(n^6)

Choose $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ solving Lyapunov equations we get $P = \begin{bmatrix} 2.59 & -2.29 \\ -2.29 & 4.92 \end{bmatrix}$ and we get the quadratic Lyapunov function $(x - x^*)P(x - x^*)^T$ an a sequence of invariants

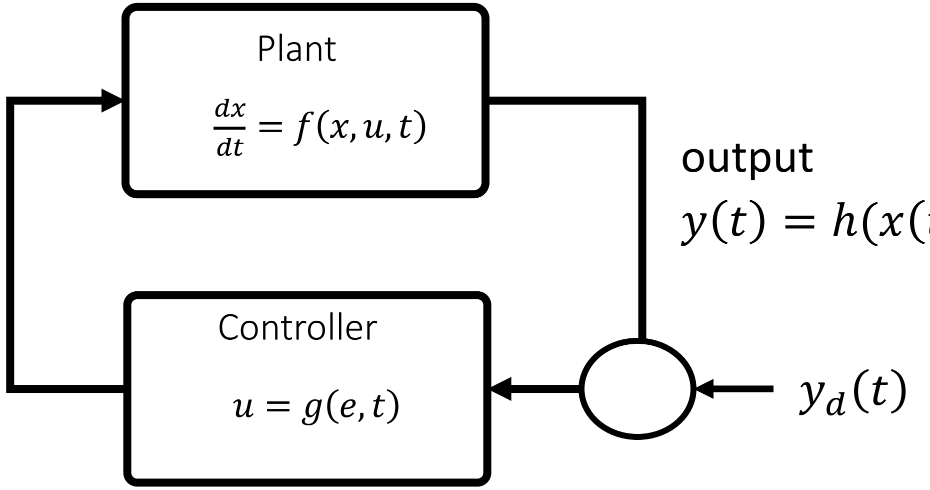


Converse Lyapunov

Converse Lyapunov theorems show that conditions of the previous theorem are also necessary. For example, if the system is asymptotically stable then there exists a positive definite, continuously differentiable function V, that satisfies the inequalities.

For example if the LTI system $\dot{x} = Ax$ is globally asymptotically stable then there is a quadratic Lyapunov function that proves it.

Plant and controller



$$\frac{dx}{dt} = f(x, u(t), t); \quad y(t) = h\big(x(t)\big);$$

$$e(t) = y(t) - y_d(t)$$

$$u(t) = g(e(t), t)$$

PID control

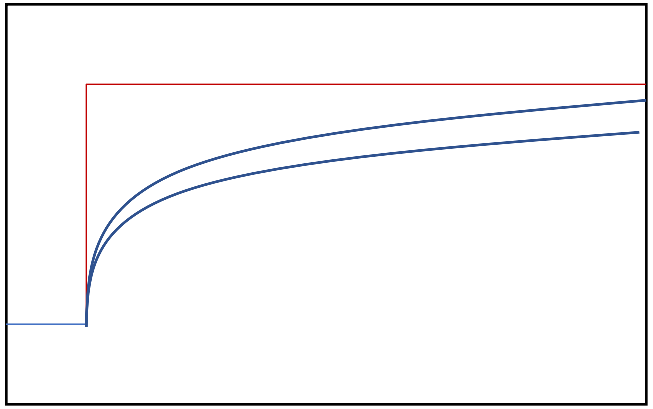
- 90% (or more) of control loops in industry are PID
- Simple control design model → simple controller
- The standard form of a PID controller:

$$u(t) = k_p e(t) + k_I \int_0^t e(\tau)\, d\tau + k_d \frac{de}{dt}$$

- where the error term $\mathrm{e}(t) = y(t) - y_d(t)$
- $y_d(t)$: desired output or setpoint value
- k_p, k_I, k_d: constant gains
- Many techniques for tuning these parameters: Ziegler-Nichols, relay method, Cohen-Coon method, etc.
- Analysis in frequency domain

P control

- Consider a simple integrator plant model
- $\dot{y}(t) = u(t) + d$
- $u(t) = -k_p(y(t) - y_d(t))$
- $\dot{y}(t) = -k_p\left(y(t) - y_d(t)\right) + d$
- $\dot{y}(t) = \ -k_p y(t) + \left(k_p y_d(t) + d\right)$



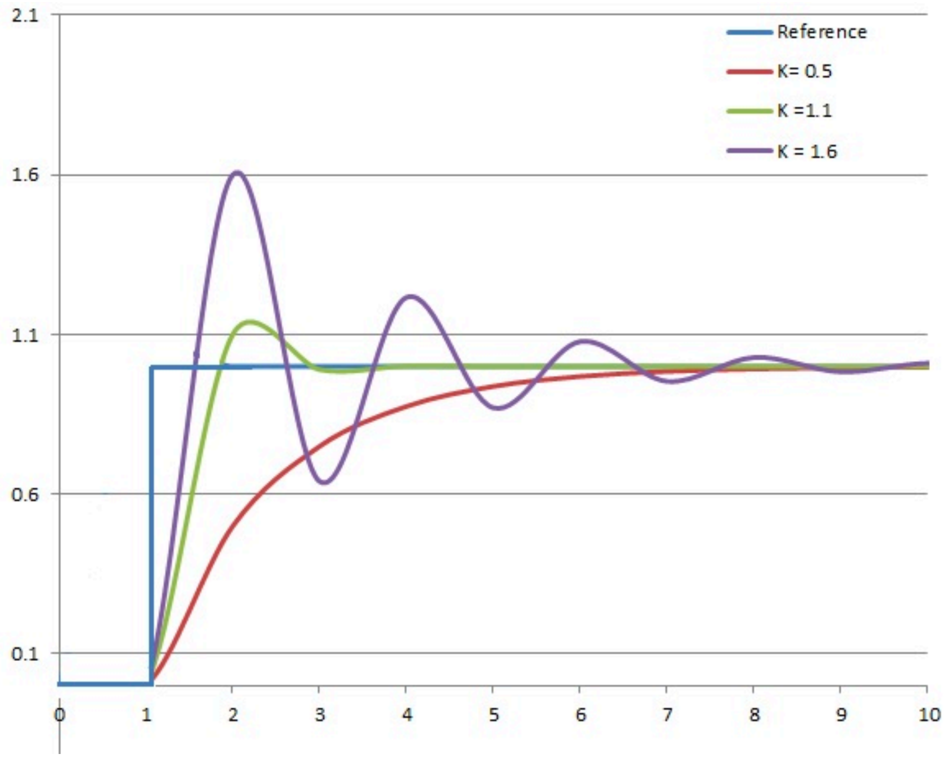
Steady state

- $0 = -k_p\left(y(t) - y_d(t)\right) + d$
- $y_{ss} = y_d - d/k_p$ steady state error

Transient

$$y(t) = y(0)e^{-\frac{t}{T}} + y_{ss}\left(1 - e^{-\frac{t}{T}}\right), T = 1/K_p$$

Choosing proportional gain k_p in PID

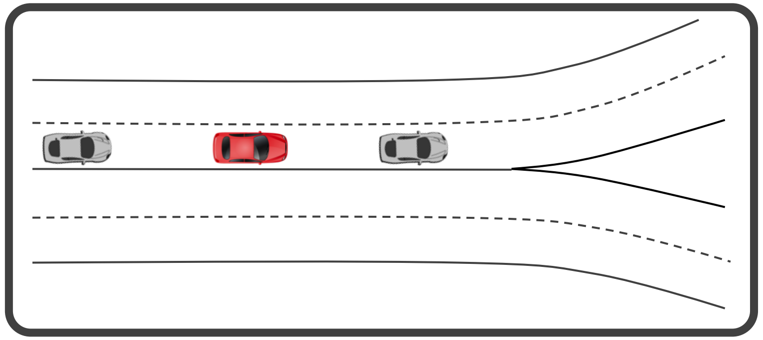


Response of y(t) to step change of $y_d(t)$ vs time, for three values of K_p (K_i and K_d held constant) Fig. from wikipedia

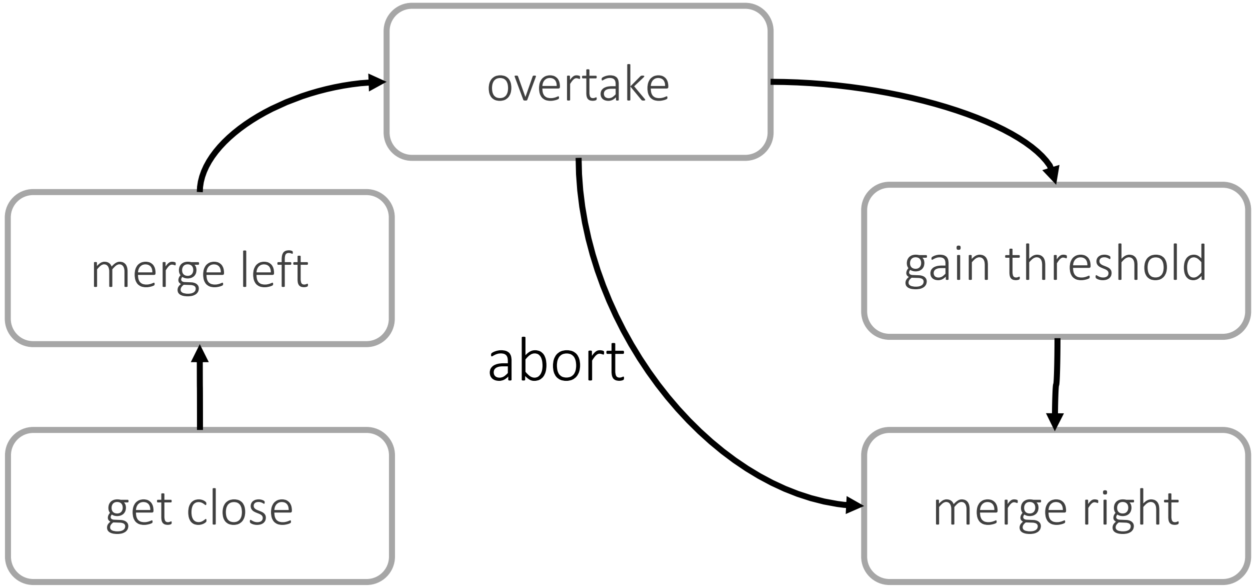
Summary (you should know)

- Definitions of solutions, stability, invariance, reach set
- Properties of solutions of linear systems
- Discrete abstractions
- Lyapunov's theorems and method for proving stability
- PID controller form, basic properties

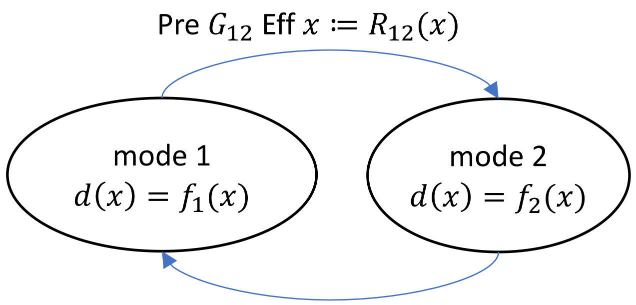
Hybrid system: Combining logic with dynamics



state machine

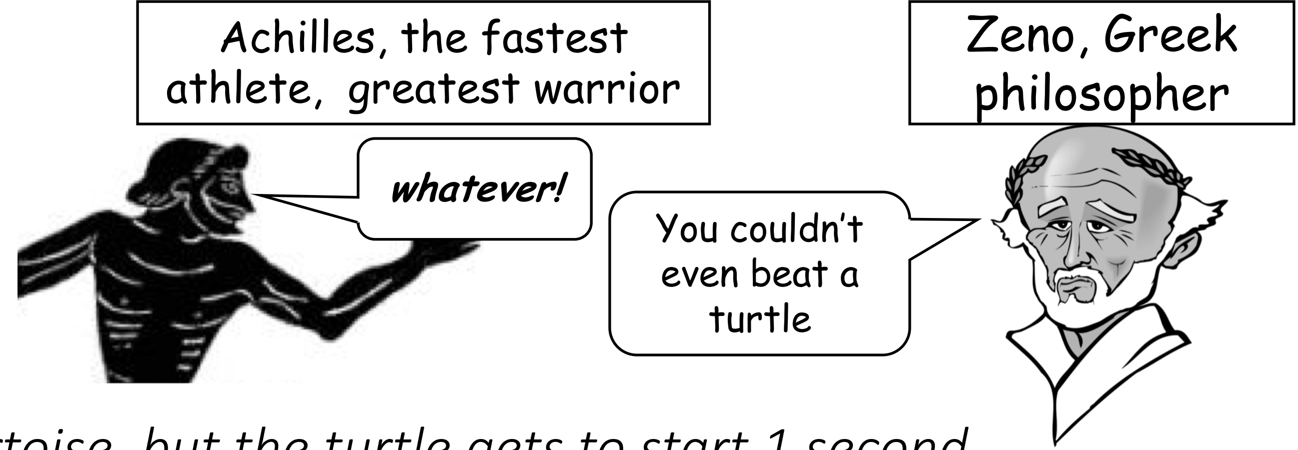


Gentle intro to hybrid systems

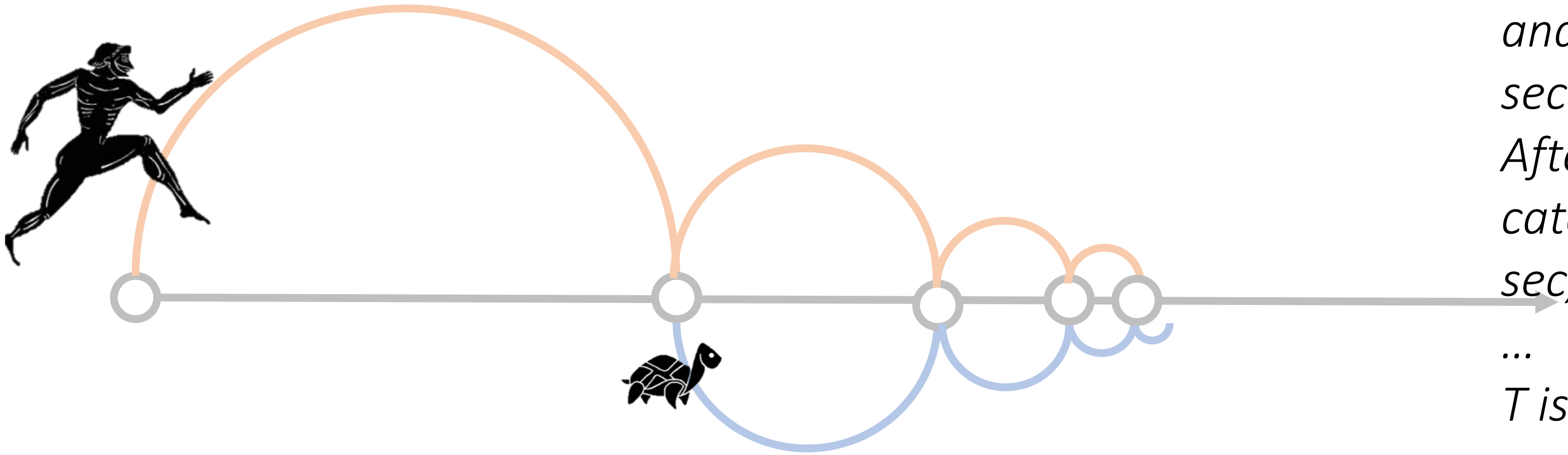


- Hybrid automaton: $\mathbf{A} = \langle V, A, D, \mathrm{T} \rangle$
  - $V = X \cup \{\ell\}$
  - X: continuous variables, e.g., temperature, position, orientation, speed
  - ℓ: mode, e.g., {on, off}, {cruising, braking, merging}
- Execution $\alpha = \tau_0 a_1 \tau_1 a_2 \ldots$
- Unexpected things can happen in hybrid executions

Zeno’s Paradox



Achilles runs 10 times faster than than the tortoise, but the turtle gets to start 1 second earlier. Can Achilles ever catch Turtle?



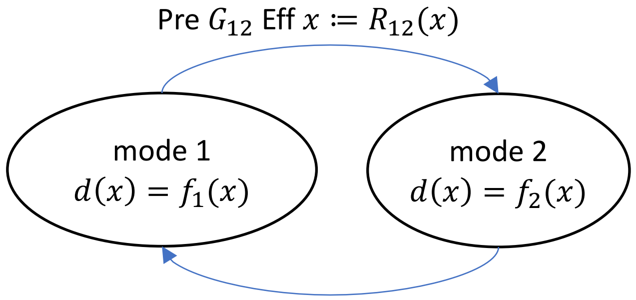
After 1/10th of a second, Achilles reaches where the Turtle (T) started, and T has a head start of 1/10th second.
After another 1/100th of a second, A catches up to where T was at t=1/10 sec, but T has a head start of 1/100th
…
T is always ahead …

Lesson: Mixing discrete time with continuous motion can be tricky!

Recall Stability

- Time invariant autonomous systems (closed systems, systems without inputs)
- $\dot{x}(t) = f\big(x(t)\big), x_0 \in \mathbb{R}^n, t_0 = 0$ –*(1)*
- $\xi(t)$ is the solution
- $|\xi(t)|$ norm
- $x^* \in \mathbb{R}^n$ is an **equilibrium point** if $f(x^*) = 0$.
- For analysis we will assume **0** to be an equilibrium point of (1) with out loss of generality

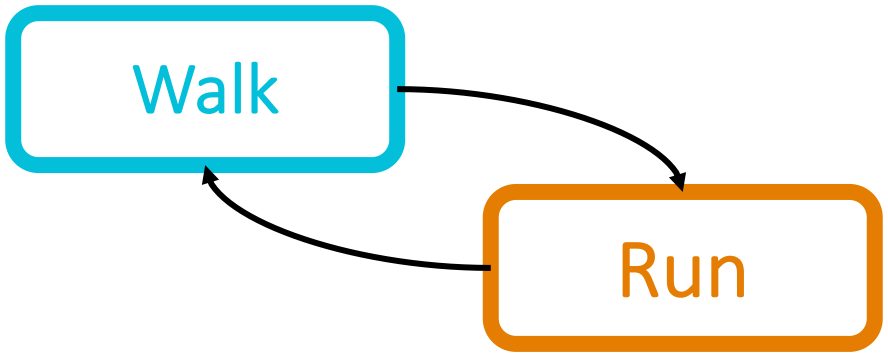
Gentle intro to hybrid systems

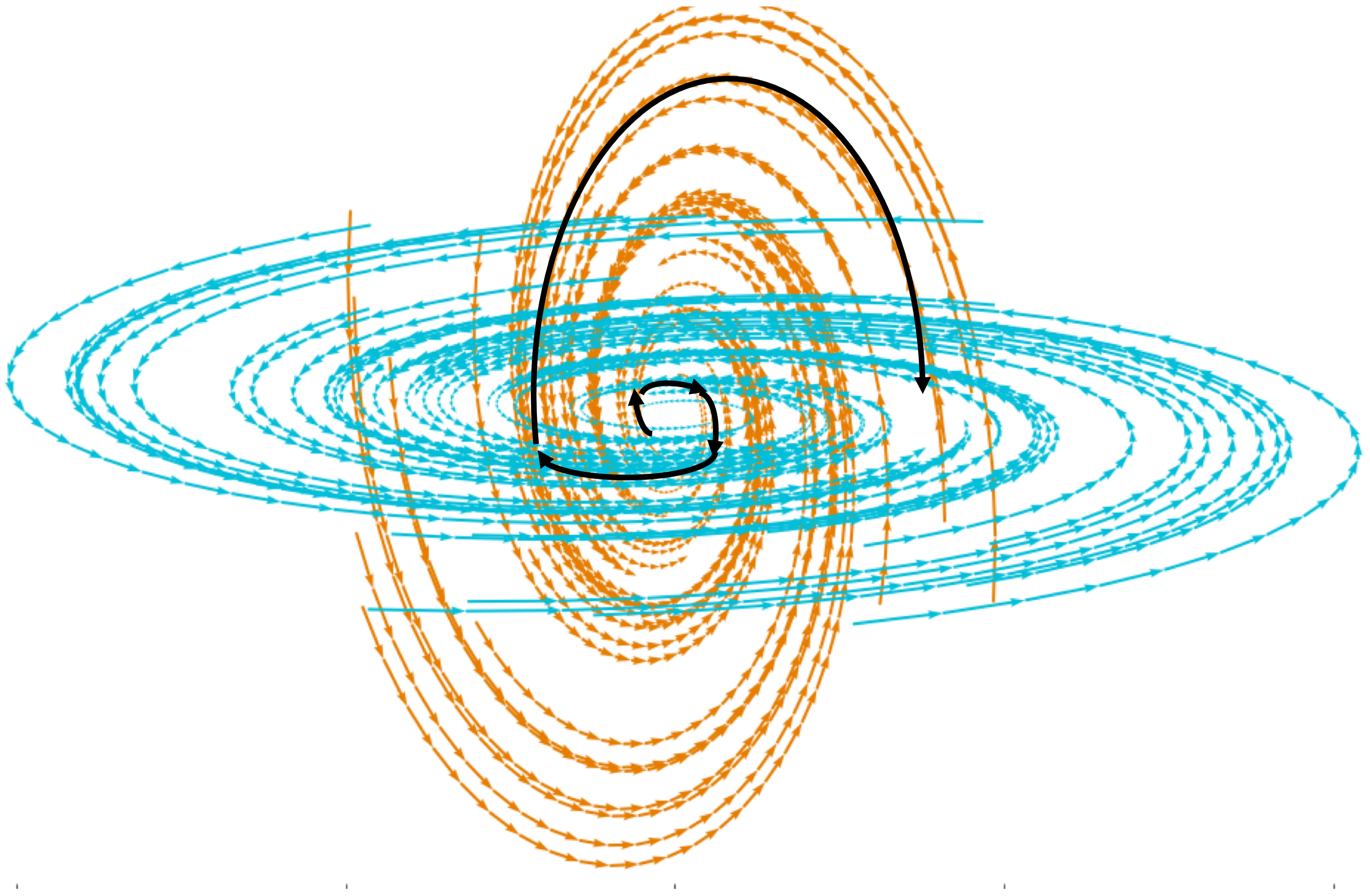


- Hybrid automaton: $\mathbf{A} = \langle V, A, D, \mathrm{T} \rangle$
  - $V = X \cup \{\ell\}$
  - X: continuous variables, e.g., temperature, position, orientation, speed
  - ℓ: mode, e.g., {on, off}, {cruising, braking, merging}
- Execution $\alpha = \tau_0 a_1 \tau_1 a_2 \ldots$
- Notation $\alpha(t)$: denotes the valuation $\beta.lstate$ where β is the longest prefix with $\beta.\mathrm{ltime} = t$
- $|\alpha(t)|$: norm of the continuous state X
- *A* is *Lyapunov stable* (at the origin) if for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for every if $|\alpha(0)| \leq \delta_\varepsilon$ then for all $\mathrm{t} \geq 0, |\alpha(t)| \leq \varepsilon.$
- *Asymptotically stable* if it is Lyapunov stable and there exists $\delta_2 > 0$ such that for every if $|\alpha(0)| \leq \delta_2$ then $\mathrm{t} \to \infty, |\alpha(t)| \to \mathbf{0}.$

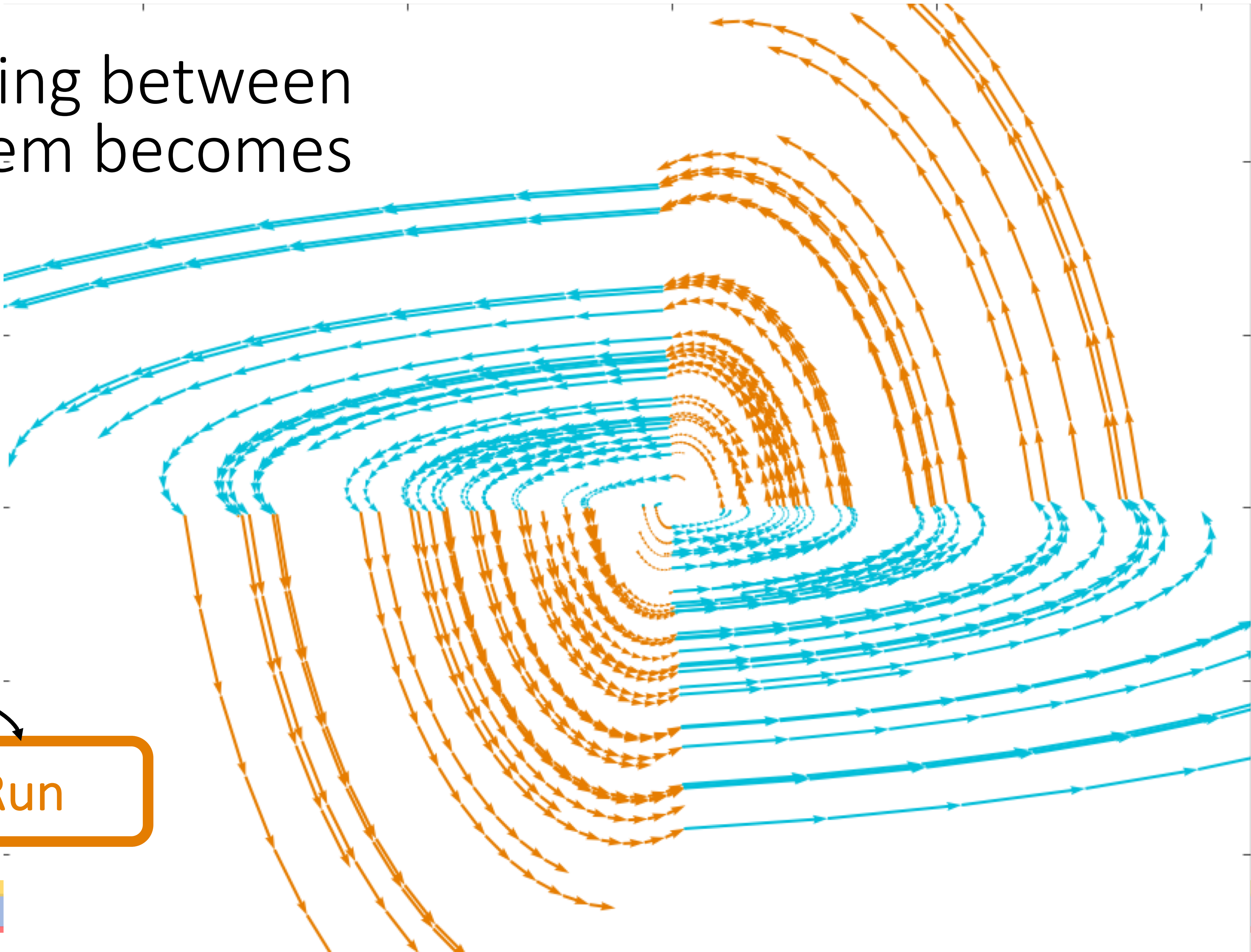
Each of the modes of a walking robot are asymptotically stable ☺

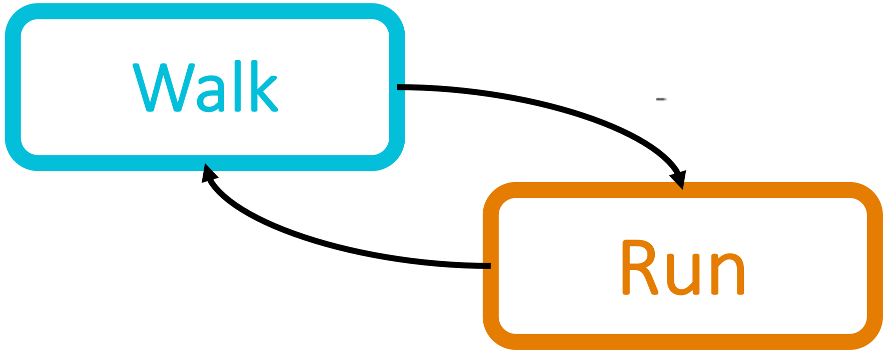
Is it possible to switch between them to make the system unstable ☹ ?

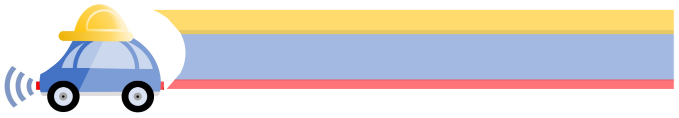




Yes! By switching between them the system becomes unstable







Common Lyapunov Function

- If there exists positive definite continuously differentiable function $V: \mathbb{R}^n \rightarrow \mathbb{R}$ and a positive definite function W: $\mathbb{R}^n \rightarrow \mathbb{R}$ such that for each mode i, $\frac{\partial V}{\partial t} f_i(x) < -W(x)$ for all $x \neq \mathbf{0}$ then V is called a common Lyapunov function for A.

- V is called a *common Lyapunov function*

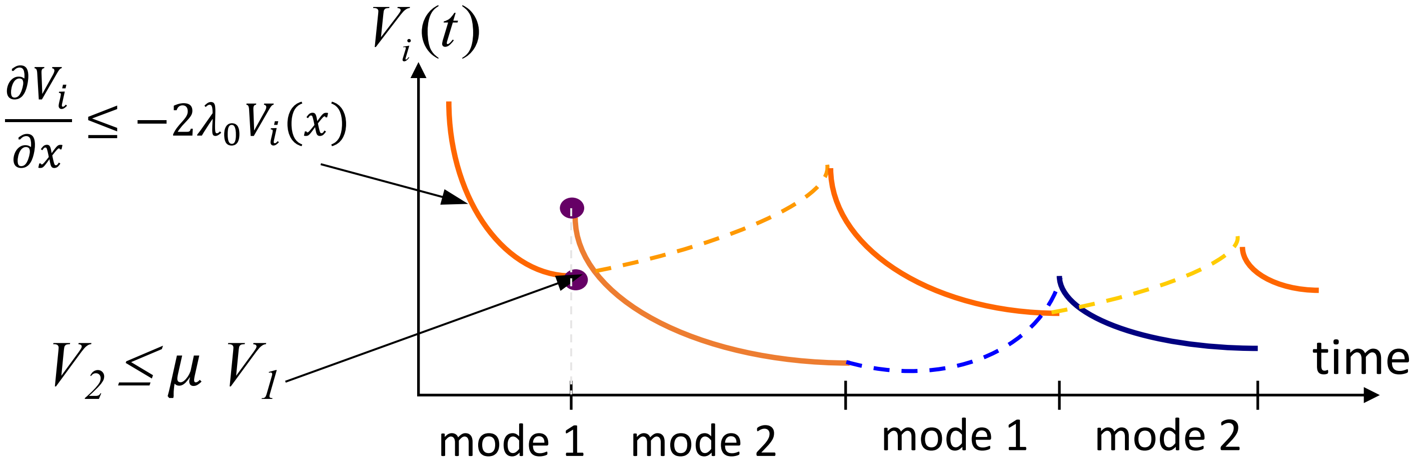
- **Theorem.** *A* is globally asymptotically stable if there exists a common Lyapunov function.

Multiple Lyapunov Functions

- In the absence of a common lyapunov function the stability verification has to rely of the discrete transitions.
- The following theorem gives such a stability in terms of *multiple Lyapunov function*.
- **Theorem** [Branicky] If there exists a family of positive definite continuously differentiable **Lyapunov** functions $V_i\colon \mathbb{R}^n \to \mathbb{R}$ and a positive definite function $\mathrm{W_i}\colon \mathbb{R}^n \to \mathbb{R}$ such that for any execution α and for any time t_1 t_2 $\alpha(t_1).\ell = \alpha(t_2).\ell = i$ and for all time $t \in (t_1, t_2), \alpha(t).\ell \neq i$
  - $V_i(\alpha(t_2).x) - V_i(\alpha(t_1).x) \leq -W_i(\alpha(t_1).x)$

Then the system is globally asymptotically stable.

Stability Under Slow Switching [Hespanha and Morse`99]



- Average Dwell Time (ADT) characterizes rate of mode switches
- Definition: H has ADT T if there exists a constant N_0 such that for every execution α,

$$N(\alpha) \leq N_0 + \text{duration}(\alpha)/T.$$

  $N(\alpha)$: number of mode switches in α

- **Theorem** [HM`99] H is asymptotically stable if its modes have a set of Lyapunov functions (μ, λ_0) and ADT(H) > log μ/λ_0 .

Remarks about ADT theorem assumptions

1. If f_i is globally asymptotically stable, then there exists a Lyapunov function V_i that satisfies $\frac{\partial V_i}{\partial x} \leq -2\lambda_i V_i(x)$ for appropriately chosen $\lambda_i > 0$
2. If the set of modes is finite, choose λ_0 independent of i
3. The other assumption restricts the maximum increase in the value of the current Lyapunov functions over any mode switch, by a factor of μ.
4. We will also assume that there exist strictly increasing functions β_1 and β_2 such that $\beta_1(|x|) \leq V_i(x) \leq \beta_2(|x|)$

Proof sketch

Suppose α is any execution of A.

Let $T = \alpha.ltime$ and $t_1, \ldots, t_{N(\alpha)}$ be instants of mode switches in α.

We will find an upper-bound on the value of $V_{\alpha(T).l}(\alpha(T).x)$

Define $W(t) = e^{2\lambda_0 t} V_{\alpha(t).l}(\alpha(t).x)$

W is non-increasing between mode switches $\left[\frac{\partial V_i}{\partial x} \le -2\lambda_0 V_i(x)\right]$

That is, $W(t_{i+1}^-) \le W(t_i)$

$W(t_{i+1}) \le \mu W(t_{i+1}^-) \le \mu W(t_i)$

Iterating this $N(\alpha)$ times: $W(T) \le \mu^{N(\alpha)} W(0)$

$$e^{2\lambda_0 T} V_{\alpha(T).l}(\alpha(T).x) \le \mu^{N(\alpha)} V_{\alpha(0).l}(\alpha(0).x)$$

$$V_{\alpha(T).l}(\alpha(T).x) \le \mu^{N(\alpha)} e^{-2\lambda_0 T} V_{\alpha(0).l}(\alpha(0).x) = e^{-2\lambda_0 T + N(\alpha)\log\mu} V_{\alpha(0).l}(\alpha(0).x)$$

If α has ADT τ_a then, recall, $N(\alpha) \le N_0 + T/\tau_a$ and $V_{\alpha(T).l}(\alpha(T).x) \le e^{-2\lambda_0 T + (N_0 + T/\tau_a)\log\mu} V_{\alpha(0).l}(\alpha(0).x) \le C\, e^{T(-2\lambda_0 + \log\mu/\tau_a)}$

If $\tau_a > \log\mu / 2\lambda_0$ then second term converges to 0 as $T \to \infty$ then from assumption 4 it follows that α converges to 0.

Summary and references

- Surprises with hybrid executions: Zeno, instability
- Common and Multiple Lyapunov Function Criteria
- Verification with Average Dwell Time