

ECE/CS 498SM: Principles of Safe Autonomy

Dynamical Systems and Control Lectures 14-15

March 25, 27

Sayan Mitra



Outline

- Dynamical systems
- Solutions
- Linear systems
- Connections to discrete time models
- Next time: Properties: stability, convergence, PID
- In 2 weeks: Verification



But, first

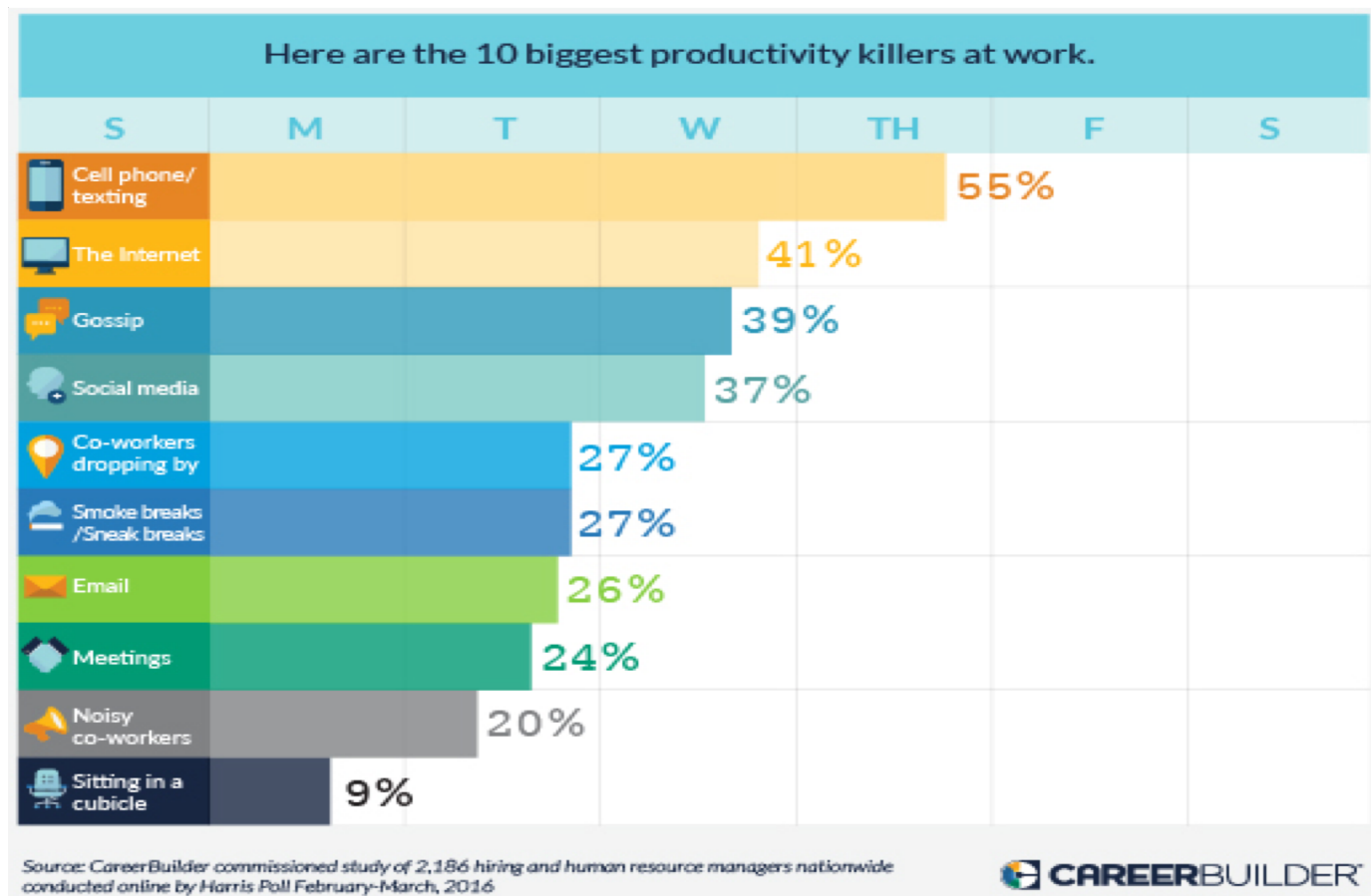
Please put away your laptops and phones

New tech policy for class: no tech

- Unless you are the scribe



The Biggest Productivity Killers



Observations (continued)

grade has contributed to a sense that one can receive a reward without putting in the effort.

Whatever the culprits are that contributed to this sense of entitlement, being aware of it and not demonstrating it will reward the job applicant. An attitude that cannot fail is “no one owes you anything.”

- **Control your on-the-job use of technology**

One of the best things a new employee can do is lock away their cell phone and disable access to the Internet when on the job. Abuse of technology has become common in the office. Other studies have found that persons feel lost without their cell phones. This research has discovered situations in which job applicants either have their cell phones on or have actually used their cell phones during an interview. The sad fact is some of these persons probably do not understand what is wrong with this.

It is time for employees to wean themselves from the addiction of constantly having to be in contact with others via technology. Twittering or checking Facebook are activities that do not belong on the job.

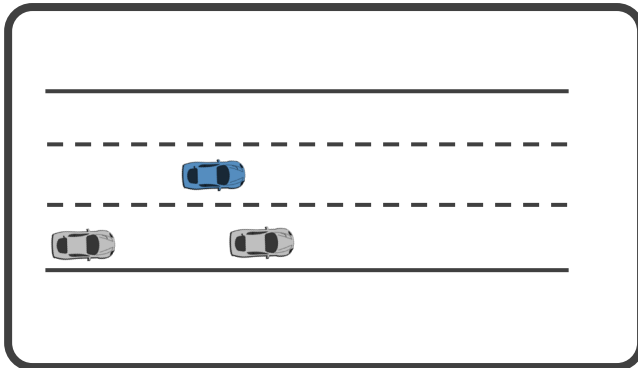
With job related activities, when you are sending an e-mail or text, ask yourself if a direct conversation with the person would not be more appropriate. The convenience of technology can prompt us to use it when other means of communication would be better or more effective.

An additional benefit of disciplining oneself in the use of technology is becoming more focused. Over a third of the respondents observed that there has been an increase in new employees who are unfocused. The number one cause of a lack of focus is identified by respondents as allowing technology to interrupt one's activities. Remember that the ability to multi-task effectively is a myth.

Nationwide random sample of 401 human resource professionals



Dynamical system models



Nonlinear dynamics

Generally, nonlinear ODEs do not have closed form solutions!

Dubin's car model

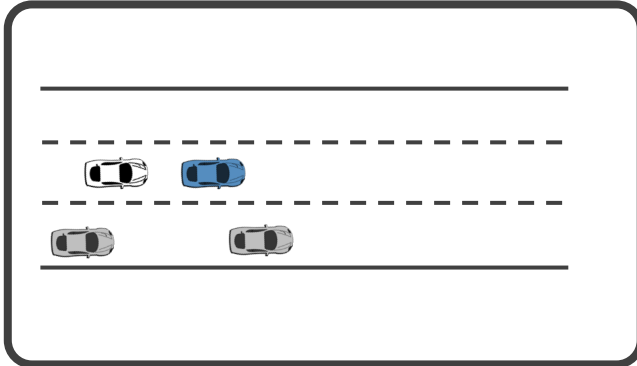
$\dot{v} = a$	Speed
$\frac{ds_x}{dt} = v \cos(\psi)$	Horizontal position
$\frac{ds_y}{dt} = v \sin(\psi)$	Vertical position
$\frac{d\delta}{dt} = v\delta$	Steering angle
$\frac{d\psi}{dt} = \frac{v}{l} \tan(\delta)$	Heading angle

Physical plant

$\frac{dx}{dt} = f(x, u)$	System dynamics
$x[t + 1] = f(x[t], u[t])$	
$x = [v, s_x, s_y, \delta, \psi]$	State variables
$u = [a, v_\delta]$	Control inputs



Nonlinear hybrid dynamics



Physical plant

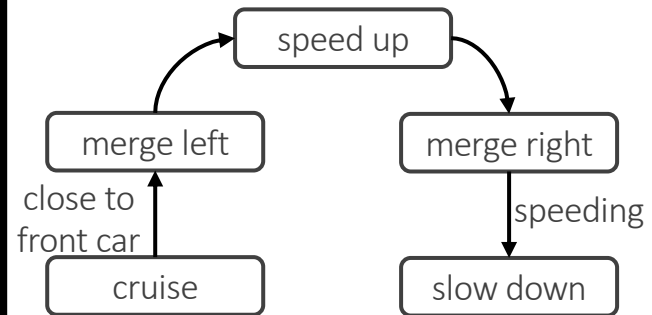
$$\frac{dx}{dt} = f(x, u) \quad \text{System dynamics}$$

$$x[t + 1] = f(x[t], u[t])$$

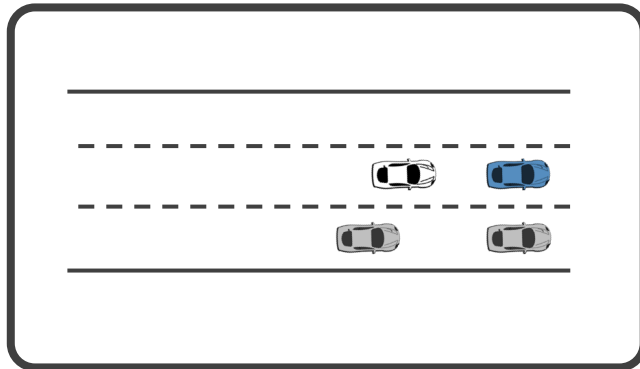
$$x = [v, s_x, s_y, \delta, \psi] \quad \text{State variables}$$

$$u = [a, v_\delta] \quad \text{Control inputs}$$

Decision and control software

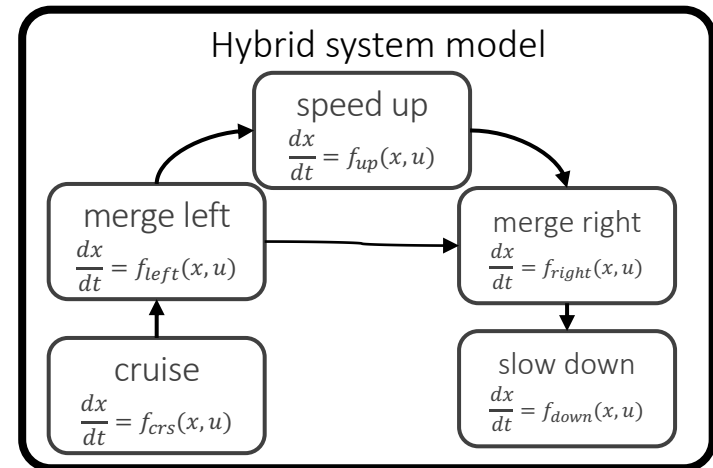
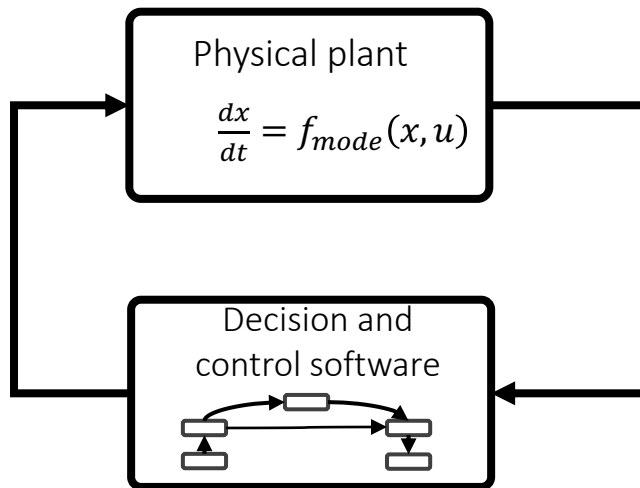


Cyberphysical system



Nonlinear hybrid dynamics

Interaction between computation and physics can lead to unexpected behaviors



Comparison of RL and Control approaches

RL setting

Unknown plant model

Unknown rewards

Typical setup: MDPs, generalized Markov Processes

Methods:

- Bellman equation
- Q-learning
- DQNN

Difficult to provide hard guarantees

Optimal control

Known plant models

Known cost/reward

Typical setup: ODE, hybrid systems, stochastic differential equations

Methods:

- Principle of optimality
- Lyapunov methods

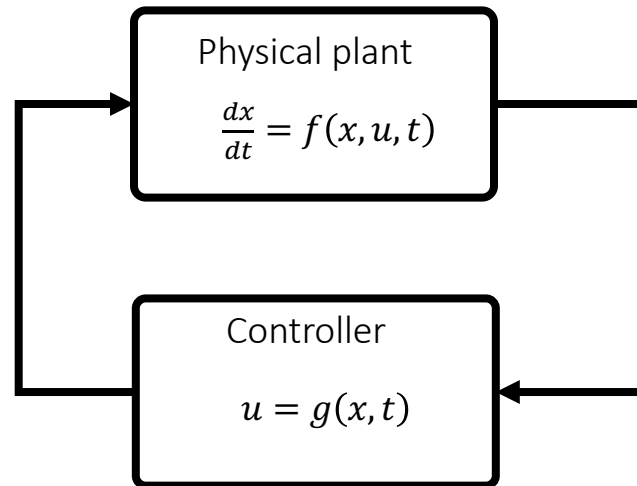
Hard guarantees

Reinforcement Learning Versus Model Predictive Control: A Comparison on a Power System Problem

[Damien Ernst](#) ; [Mevludin Glavic](#) ; [Florin Capitanescu](#) ; [Louis Wehenkel](#)



Simplified view of a plant and a controller



$$\frac{dx}{dt} = f(x, g(x), t)$$

$$\dot{x} = f(x, g(x), t)$$



Dynamical systems model

Describe behavior in terms of instantaneous laws

$$\frac{dx(t)}{dt} = f(x(t), u(t), t)$$

$$t \in \mathbb{R}, x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m$$

$f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}^n$ dynamic function

Dubin's car model

$\dot{v} = a$	Speed
$\frac{ds_x}{dt} = v \cos(\psi)$	Horizontal position
$\frac{ds_y}{dt} = v \sin(\psi)$	Vertical position
$\frac{d\delta}{dt} = v_\delta$	Steering angle
$\frac{d\psi}{dt} = \frac{v}{l} \tan(\delta)$	Heading angle



Linear system and solutions

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

For a given initial state $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$ and $u: \mathbb{R} \rightarrow \mathbb{R}^m$ a *solution* or a state trajectory is a function $\xi: \mathbb{R} \rightarrow \mathbb{R}^n$

What is a solution?

How to find one?

Does it always exist?

Is it unique?

How hard is it to compute a solution?



Notion of solution

- Fixing $u(t)$, we define $p(x(t), t) = f(x(t), u(t), t)$
- $p(., .)$ is piece-wise continuous in the second argument with a set of discontinuity points $D \subseteq \mathbb{R}$
- $\xi: \mathbb{R} \rightarrow \mathbb{R}^n$ is a solution passing through $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ iff
 1. $\xi(t_0) = x_0$ and
 2. $\frac{d}{dt}\xi(t) = p(\xi(t), t)$ for all $t \in \mathbb{R} \setminus D$



Is this enough to guarantee existence and uniqueness of solution

Consider $\dot{x}(t) = -\operatorname{sgn}(x(t)) = \begin{cases} -1 & x(t) \geq 0 \\ 1 & x(t) < 0 \end{cases}$

$x_0 = c, > 0 \quad t_0 = 0$

What is the solution?

Neither $\dot{x} = 1$ nor $\dot{x} = -1$ works at $x = 0$

Problem: f is discontinuous in x



Assume that $f()$ is continuous in x

Equivalently $p(x, t)$ is continuous in x

Consider $\dot{x}(t) = x^2$

Does it have a solution? What is the solution?

Check that $\xi(t) = \frac{c}{1-tc}$ is a solution from $x_0 = c, t_0 = 0$

But, as $t \rightarrow \frac{1}{c}$ the solution goes to $\rightarrow \infty$

So, no solution defined beyond $t = \frac{1}{c}$.

Problem: $p()$ grows too fast



Assume $f()$ changes slowly with respect to x

Lipschitz continuity.

Examples

Non-examples



Existence and uniqueness of solutions

$$\frac{dx(t)}{dt} = f(x(t), u(t), t)$$

Fixing $u(t)$, we define $p(x(t), t) = f(x(t), u(t), t)$

$p(.,.)$ is piece-wise continuous in the second argument with a set of discontinuity points $D \subseteq \mathbb{R}$

Theorem. If $p(.,.)$ is Lipschitz continuous in the first argument then the system has a unique solution.



Example: Pendulum

Pendulum equation

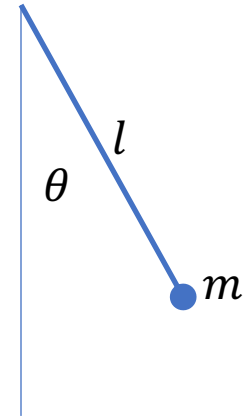
$$x_1 = \theta \quad x_2 = \dot{\theta}$$

$$x_2 = \dot{x}_1$$

$$\dot{x}_2 = -\frac{g}{l} \sin(x_1) - \frac{k}{m} x_2$$

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} -\frac{g}{l} \sin(x_1) - \frac{k}{m} x_2 \\ x_2 \end{bmatrix}$$

k : friction coefficient



Example: Pendulum

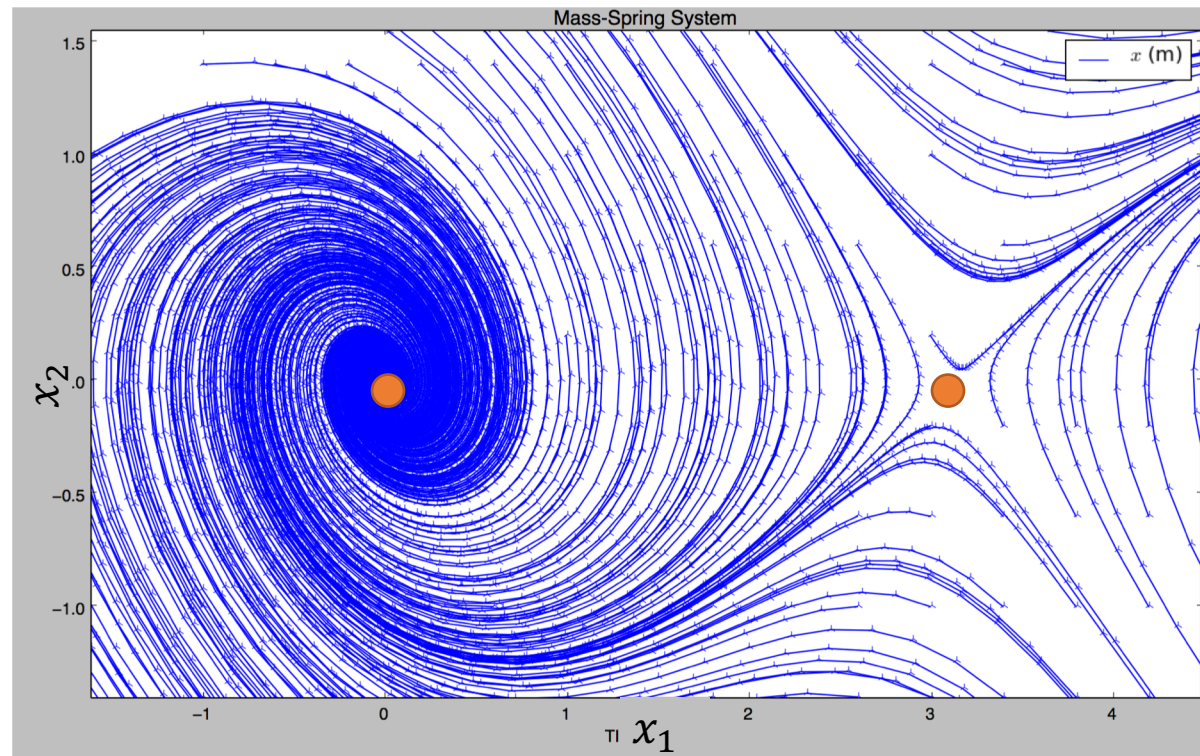
Pendulum equation

$$x_1 = \theta \quad x_2 = \dot{\theta}$$

$$x_2 = \dot{x}_1$$

$$\dot{x}_2 = -\frac{g}{l} \sin(x_1) - \frac{k}{m} x_2$$

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} -\frac{g}{l} \sin(x_1) - \frac{k}{m} x_2 \\ x_2 \end{bmatrix}$$



$x = (0, 0)$

asymptotically stable

$x = (\pi, 0)$

unstable



Special classes of systems

For general nonlinear dynamical systems, we may not have closed form expressions for the solution $\xi(t)$

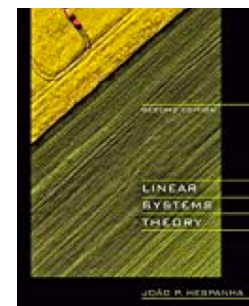
$$\dot{x}(t) = f(x(t), u(t), t)$$

Linear Time Varying Systems (LTV): $f(x(t), u(t), t)$ is a linear function of state and input

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \text{ --- state evolution}$$

$$y(t) = C(t)x(t) + D(t)u(t) \text{ --- output}$$

Linear Systems Theory
by [João P. Hespanha](#)



Properties of solutions of LTV systems

Theorem. Let $\xi(t, t_0, x_0, u): \mathbb{R} \rightarrow \mathbb{R}^n$ be the solution of LTV system with $D_x \subseteq \mathbb{R}$ points of discontinuity

1. (continuous w.r.t. time) For all t_0, x_0 , and $u(.) \in PC(\mathbb{R}, \mathbb{R}^n)$, $\xi(., t_0, x_0, u)$ is continuous and differentiable at all $t \in \mathbb{R} \setminus D_x$
2. (continuous w.r.t. initial state) For all t_0, t , and $u(.) \in PC(\mathbb{R}, \mathbb{R}^n)$, $\xi(t, t_0, ., u)$ is continuous
3. (Linearity w.r.t. initial states and inputs) For all $t, t_0, x_{01}, x_{02}, u_1, u_2, a_1, a_2$
$$\xi(t, t_0, a_1 x_{01} + a_2 x_{02}, a_1 u_1 + a_2 u_2) = a_1 \xi(t, t_0, x_{01}, u_1) + a_2 \xi(t, t_0, x_{02}, u_2)$$
4. (Linearity w.r.t. initial states and inputs) For all t, t_0, x_0, u ,
$$\xi(t, t_0, x_0, u) = \xi(t, t_0, x, \mathbf{0}) + \xi(t, t_0, 0, u)$$



Special Linear system and solutions

- Since $\xi(\cdot, t_0, x_0, u): \mathbb{R} \rightarrow \mathbb{R}^n$ is a linear function of the initial state and input,
- $\xi(t, t_0, x_0, u) = \xi(t, t_0, 0, u) + \xi(t, t_0, x_0, 0)$
- Let us focus on the linear function $\xi(\cdot, t_0, x_0, 0)$
- Define $\Phi(\cdot, t_0)x_0 = \xi(\cdot, t_0, x_0, 0)$
- This $\Phi(\cdot, t_0): \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ is called the state transition matrix



Linear time invariant system (LTI)

$$\dot{x}(t) = Ax(t) + Bu(t)$$

Matrix exponential:

$$\begin{aligned} e^{At} &= 1 + At + \frac{1}{2!} (At)^2 + \dots \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} (At)^k \end{aligned}$$

Theorem. $\xi(t, t_0, x_0, u) =$

$$x_0 e^{A(t-t_0)} + \int_{t_0}^t e^{A(t-\tau)} Bu(\tau) d\tau$$

SIAM REVIEW
Vol. 45, No. 1, pp. 3–000

© 2003 Society for Industrial and Applied Mathematics

Nineteen Dubious Ways to Compute the Exponential of a Matrix, Twenty-Five Years Later*

Cleve Moler[†]
Charles Van Loan[‡]

Abstract. In principle, the exponential of a matrix could be computed in many ways. Methods involving approximation theory, differential equations, the matrix eigenvalues, and the matrix characteristic polynomial have been proposed. In practice, consideration of computational stability and efficiency indicates that some of the methods are preferable to others, but that none are completely satisfactory. Most of this paper was originally published in 1978. An update, with a separate bibliography, describes a few recent developments.

Key words. matrix, exponential, roundoff error, truncation error, condition

AMS subject classifications. 15A15, 65F15, 65F30, 65L99

PII. S0036144502418010

1. Introduction. Mathematical models of many physical, biological, and economic processes involve systems of linear, constant coefficient ordinary differential equations

$$\dot{x}(t) = Ax(t).$$



Relating ODEs to discrete time models / discrete transition systems

- $x(t + 1) = f(x(t), u(t))$
- $x(t + 1) = f(x(t))$ **autonomous** system (with no inputs)
- Execution: $x_0, f(x_0), f^2(x_0), \dots$ takes the place of solutions
- System described as an automaton $\mathbf{A} = \langle Q, Q_0, T \rangle$
 - $Q = \mathbb{R}^n, Q_0 = \{x_0\}$
 - $T: \mathbb{R}^n \rightarrow \mathbb{R}^n; T(x) = f(x)$
 - This makes it a deterministic discrete time system
 - If we quantize $Q = \mathbb{Q}^n$: discrete time, discrete state system or automaton
 - If the state space is finite, then Deterministic Finite Automaton (DFA)

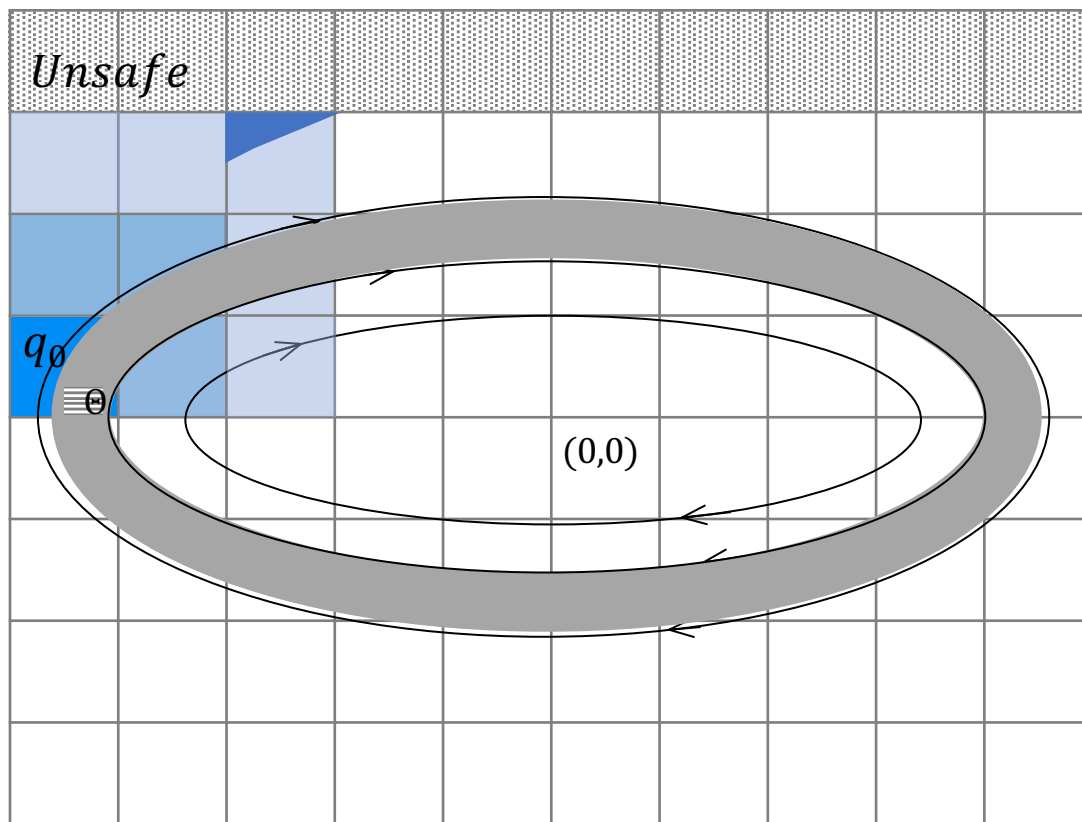


For systems with inputs: sampled-time model

- $\dot{x}(t) = f(x(t), u(t))$
- Assume: $u \in PC(\mathbb{R}, U)$ where $U \subseteq \mathbb{R}^m$ is a finite set
- $\xi(t, t_0, x_0, u)$
- Fix a sampling period $\delta > 0$
- $A_\delta = \langle Q, Q_0, U, T \rangle$
 - $Q = \mathbb{R}^n, Q_0 = \{x_0\}, Act = U,$
 - $T \subseteq \mathbb{R}^n \times U \times \mathbb{R}^n; (x, u, x') \in T \text{ iff } x' = \xi(\delta, 0, x, u)$
 - Nondeterministic transitions can capture dynamics conservatively
 - This leads to a Nondeterministic Automaton, NFA



Example: Descretization can lead to conservative analysis



Requirements for dynamical systems

What type of **requirements or properties** are we interested in?

- Invariance, safety
- State remains bounded
- Converges to target
- Bounded input gives bounded output (BIBO)



Invariance and safety

Given a system $\dot{x}(t) = f(x(t))$, and initial set $K \subseteq \mathbb{R}^n$ --- (1)

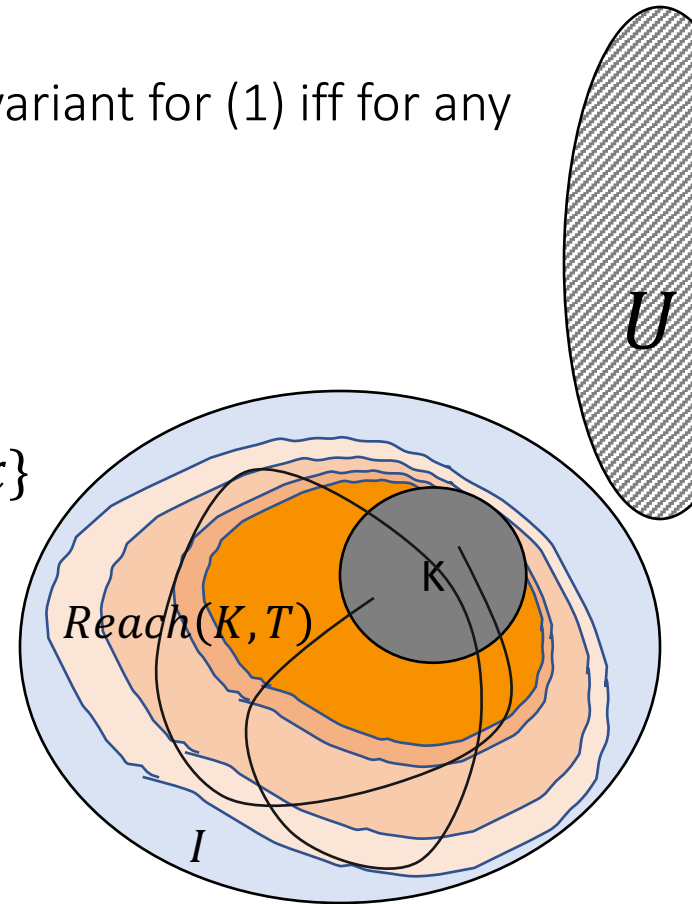
and a set of states $I \subseteq \mathbb{R}^n$ we say that I is an invariant for (1) iff for any initial $x_0 \in K$, and any time $t \geq 0$, $\xi(x_0, t) \in I$.

Related concept: Reachable states

$$Reach(K, T) = \{x | \exists x_0 \in K, t \leq T, \xi(x_0, t) = x\}$$

$$Reach(K) = \{x | \exists x_0 \in K, t > 0, \xi(x_0, t) = x\}$$

To check safety of a system relative to $U \subseteq \mathbb{R}^n$
 $Reach(K) \cap Unsafe = \emptyset$?



Aleksandr M. Lyapunov

Aleksandr M. Lyapunov (1857–1918), Russian mathematician and physicist.

His methods make it possible to define and prove stability of differential equations

Created the modern theory of the stability of dynamic systems.

Generalized the works of Chebyshev and Markov, and proved the Central Limit Theorem under more general conditions



Requirements for control systems: Stability

- We will focus on time invariant autonomous systems (closed systems, systems without inputs)
- $\dot{x}(t) = f(x(t)), x_0 \in \mathbb{R}^n, t_0 = 0 \text{ -(1)}$
- $\xi(t)$ is the solution
- $|\xi(t)|$ norm
- $x^* \in \mathbb{R}^n$ is an **equilibrium point** if $f(x^*) = 0$.
- For analysis we will assume **0** to be an equilibrium point of (1) without loss of generality



Example: Pendulum

Pendulum equation

$$x_1 = \theta \quad x_2 = \dot{\theta}$$

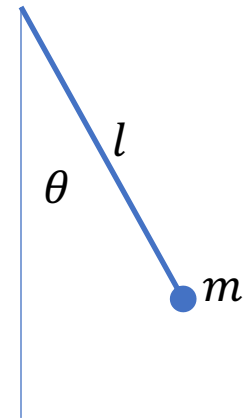
$$x_2 = \dot{x}_1$$

$$\dot{x}_2 = -\frac{g}{l} \sin(x_1) - \frac{k}{m} x_2$$

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} -\frac{g}{l} \sin(x_1) - \frac{k}{m} x_2 \\ x_2 \end{bmatrix}$$

k : friction coefficient

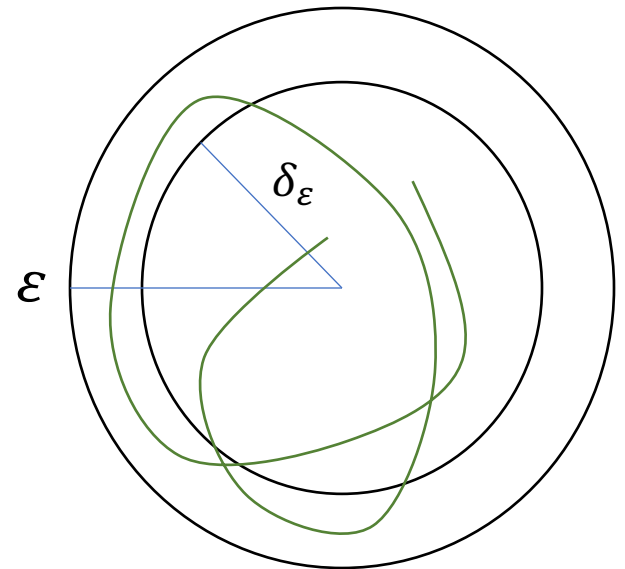
Two equilibrium points: $(0,0)$, $(\pi, 0)$



Lyapunov stability

Lyapunov stability: The system (1) is said to be ***Lyapunov stable*** (at the origin) if for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for every if $|\xi(0)| \leq \delta_\varepsilon$ then for all $t \geq 0$, $|\xi(t)| \leq \varepsilon$.

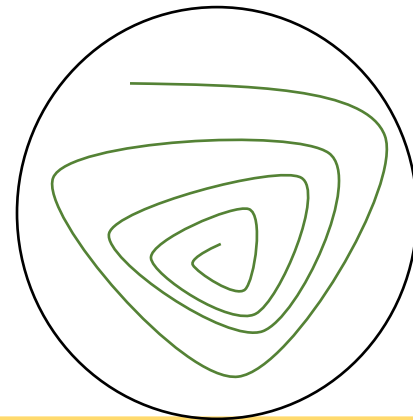
How is this related to
invariants and
reachable states ?



Asymptotically stability

The system (1) is said to be ***Asymptotically stable*** (at the origin) if it is Lyapunov stable and there exists $\delta_2 > 0$ such that for every if $|\xi(0)| \leq \delta_2$ then $t \rightarrow \infty, |\xi(t)| \rightarrow \mathbf{0}$.

If the property holds for any δ_2 then **Globally Asymptotically Stable**



Example: Pendulum

Pendulum equation

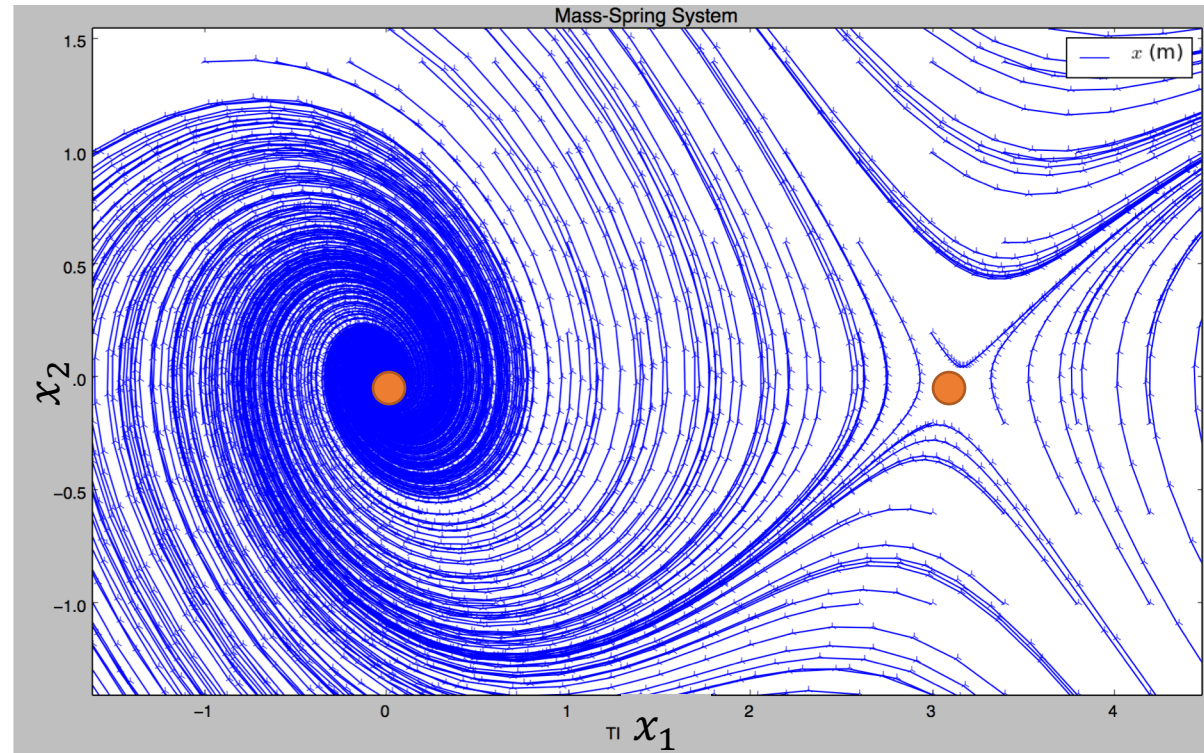
$$x_1 = \theta \quad x_2 = \dot{\theta}$$

$$x_2 = \dot{x}_1$$

$$\dot{x}_2 = -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2$$

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2 \\ x_2 \end{bmatrix}$$

Two equilibrium points: $(0,0)$, $(\pi, 0)$



$x = (0, 0)$
asymptotically stable

$x = (\pi, 0)$
unstable



Example: Pendulum

Pendulum equation

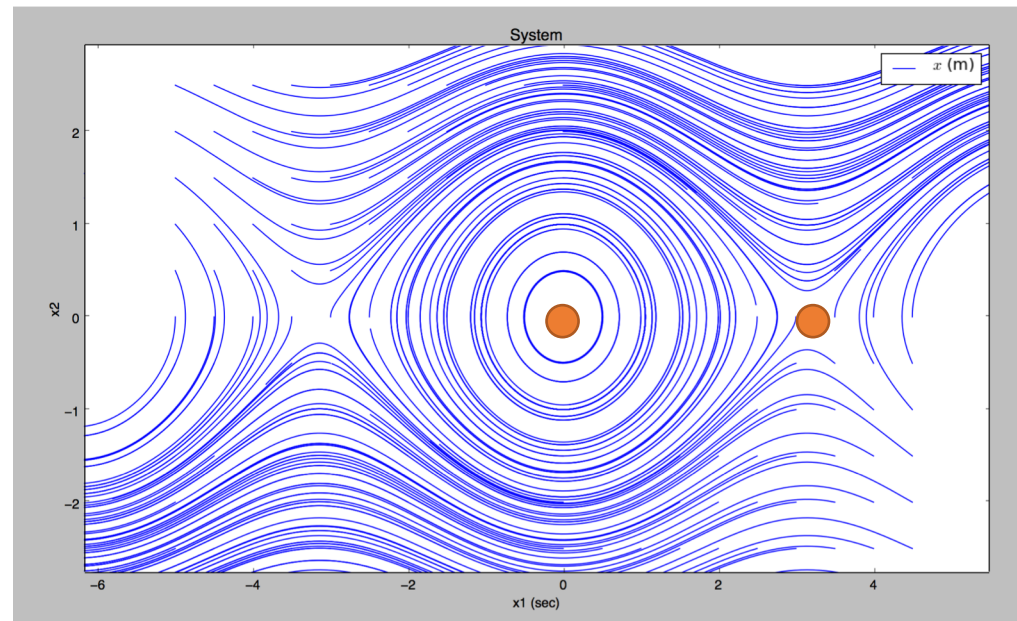
$$x_1 = \theta \quad x_2 = \dot{\theta}$$

$$x_2 = \dot{x}_1$$

$$\dot{x}_2 = -\frac{g}{l} \sin(x_1) - \frac{k}{m} x_2$$

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} -\frac{g}{l} \sin(x_1) - \frac{k}{m} x_2 \\ x_2 \end{bmatrix}$$

$k = 0$ no friction



$x^* = (0, 0)$
stable but not
asymptotically stable

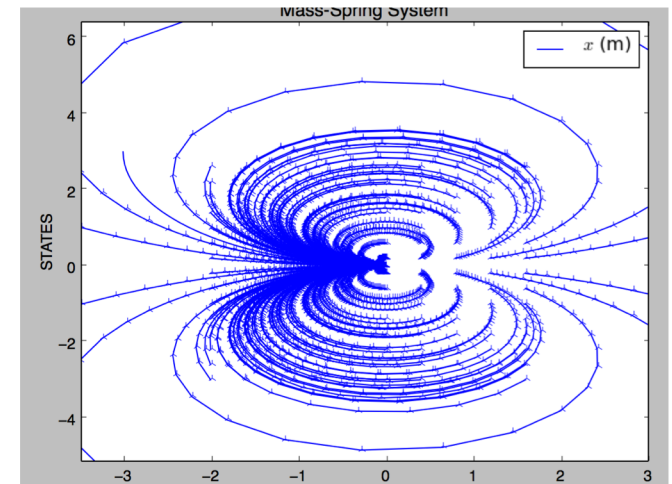
$x^* = (\pi, 0)$
unstable



Butterfly*

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} 2x_1x_2 \\ x_1^2 - x_2^2 \end{bmatrix}$$

All solutions converge to 0 but the equilibrium point (0,0) is not Lyapunov stable



*Not discussed in class

Van der pol oscillator

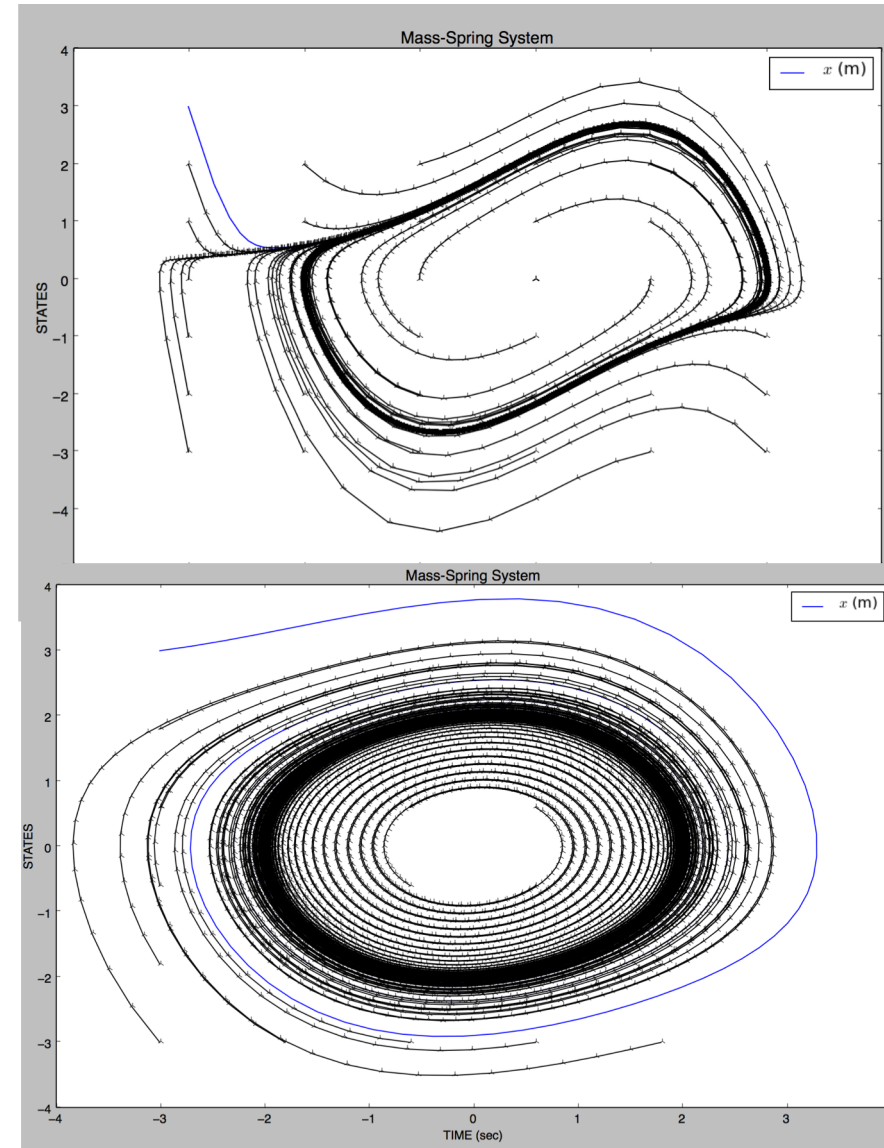
Van der pol oscillator

$$\frac{dx^2}{dt^2} - \mu(1 - x^2) \frac{dx}{dt} + x = 0$$

$x_1 = x; x_2 = \dot{x}_1;$
coupling coefficient $\mu = 2 \quad 0.1$

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} \mu(1 - x_1^2)x_2 - x_1 \\ x_2 \end{bmatrix}$$

stable ?



Stability of solutions* (instead of points)

- For any $\xi \in PC(\mathbb{R}^{\geq 0}, \mathbb{R}^n)$ define the s-norm $\|\xi\|_s = \sup_{t \in \mathbb{R}} \|\xi(t)\|$
- A dynamical system can be seen as an operator that maps initial states to signals $T: \mathbb{R}^n \rightarrow PC(\mathbb{R}^{\geq 0}, \mathbb{R}^n)$
- Lyapunov stability required that this operator is continuous
- The solution ξ^* is *Lyapunov stable* if T is continuous as $\xi^*(0)$. i. e., for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for every $x_0 \in \mathbb{R}^n$ if $|\xi^*(0) - x_0| \leq \delta_\varepsilon$ then $\|T(\xi^*(t)) - T(x_0)\|_s \leq \varepsilon$.



Verifying Stability for Linear Systems

Consider the linear system $\dot{x} = Ax$

Theorem.

1. It is asymptotically stable iff all the eigenvalues of A have **strictly** negative real parts (*Hurwitz*).
2. It is Lyapunov stable iff all the eigen values of A have real parts that are either zero or negative and the Jordan blocks corresponding to the eigenvalues with zero real parts are of size 1.



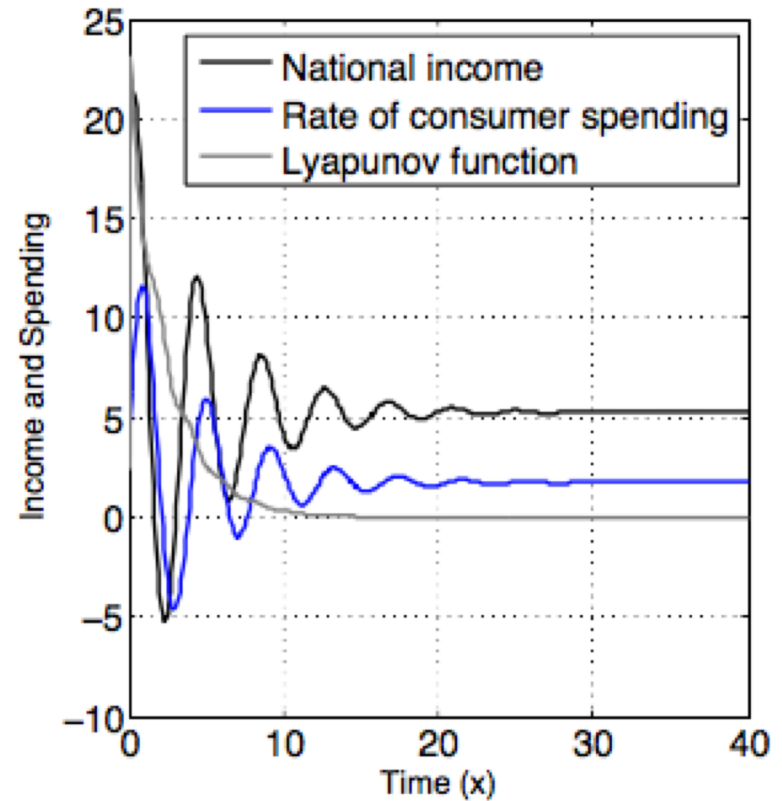
Example 1: Simple linear model of an economy

- x : national income y : rate of consumer spending; g : rate government expenditure
- $\dot{x} = x - \alpha y$
- $\dot{y} = \beta(x - y - g)$
- $g = g_0 + kx$ α, β, k are positive constants
- What is the equilibrium?
- $x^* = \frac{g_0\alpha}{\alpha-1-k\alpha} y^* = \frac{g_0\alpha}{\alpha-1-k\alpha}$
- Dynamics:
- $$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & -\alpha \\ \beta(1-k) & -\beta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Example: Simple linear model of an economy

- $\alpha = 3, \beta = 1, k = 0$
- $\lambda_1, \lambda_1^* = (-.25 \pm i 1.714)$
- Negative real parts, therefore, asymptotically stable and the national income and consumer spending rate converge to $x = 1.764$ $y = 5.294$



Stability of nonlinear systems

- For any **positive definite** function of state $V: \mathbb{R}^n \rightarrow \mathbb{R}$
 - $V(x) \geq 0$ and $V(x) = 0$ iff $x = 0$
- Sub level sets of $L_p = \{x \in \mathbb{R}^n \mid V(x) \leq p\}$
- $V(\xi(t))$

V differentiable with continuous first derivative

- $\dot{V} = d \frac{V(\xi(t))}{dt} = ?$
- $\frac{\partial V}{\partial x} \cdot \frac{d}{dt}(\xi(t)) = \frac{\partial V}{\partial x} \cdot f(x)$ is also continuous
- V is radially unbounded if $\|x\| \rightarrow \infty \Rightarrow V(x) \rightarrow \infty$



Verifying Stability

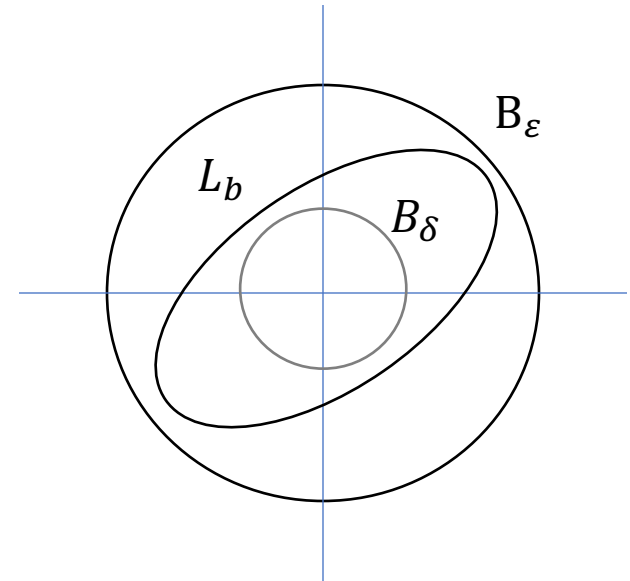
Theorem. (Lyapunov) Consider the system (1) with state space $\xi(t) \in \mathbb{R}^n$ and suppose there exists a positive definite, continuously differentiable function $V: \mathbb{R}^n \rightarrow \mathbb{R}$. The system is:

1. Lyapunov stable if $\dot{V}(\xi(t)) = \frac{\partial V}{\partial x} f(x) \leq 0$
2. Asymptotically stable if $\dot{V}(\xi(t)) < 0$
3. It is globally AS if V is also radially unbounded.



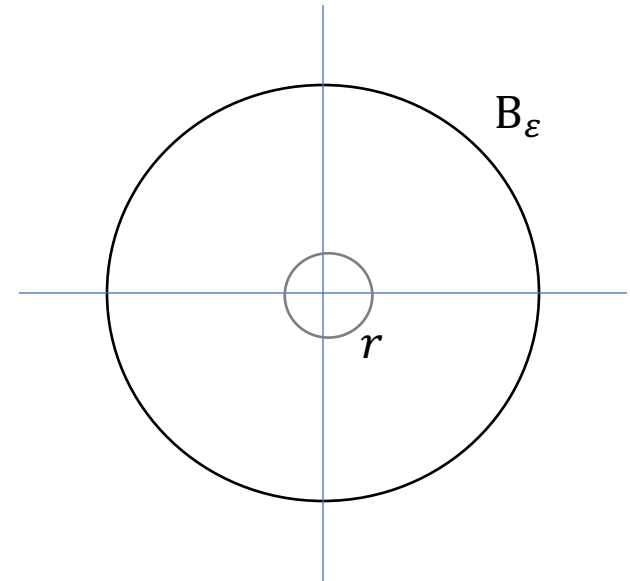
Proof sketch: Lyapunov stable if $\dot{V} \leq 0$

- Assume $\dot{V} \leq 0$
- Consider a ball B_ε around the origin of radius $\varepsilon > 0$.
- Pick a positive number $b < \min_{|x|=\varepsilon} V(x)$.
- Let δ be a radius of ball around origin which is inside $B_\delta = \{x \mid V(x) \leq b\}$
- Since along all trajectories V is non-increasing, starting from B_δ each solution satisfies $V(\xi(t)) \leq b$ and therefore remains in B_ε



Proof sketch: Asymptotically stable if $\dot{V}(\xi(t)) < 0$

- Assume $\dot{V} < 0$
- Take arbitrary $|\xi(0)| \leq \delta$, where this δ comes from some ε for Lyapunov stability
- Since $V(\xi(.)) > 0$ and decreasing along ξ it has a limit $c \geq 0$ at $t \rightarrow \infty$
- It suffices to show that this limit is actually 0
- Suppose not, $c > 0$ then the solution evolves in the compact set $S = \{x \mid r \leq |x| \leq \varepsilon\}$ for some sufficiently small r
- Let $d = \max_{x \in S} \dot{V}(x)$ [slowest rate]
- This number is well-defined and negative
- $\dot{V}(\xi(t)) \leq d$ for all t
- $V(t) \leq V(0) + dt$
- But then eventually $V(t) < c$



Example 2

- $\dot{x}_1 = -x_1 + g(x_2); \dot{x}_2 = -x_2 + h(x_1)$
- $|g(u)| \leq \frac{|u|}{2}, |h(u)| \leq \frac{|u|}{2}$
- Use $V = \frac{1}{2}(x_1^2 + x_2^2) \geq 0$
- $\dot{V} = x_1 \dot{x}_1 + x_2 \dot{x}_2$
$$= -x_1^2 - x_2^2 + x_1 g(x_2) + x_2 h(x_1)$$
$$\leq -x_1^2 - x_2^2 + \frac{1}{2}(|x_1 x_2| + |x_2 x_1|)$$
$$\leq -\frac{1}{2}(x_1^2 + x_2^2) = -V$$

$$(|x_1| - |x_2|)^2 \geq 0$$

$$x_1^2 + x_2^2 \geq 2|x_1 x_2|$$

$$|x_1 x_2| \leq \frac{1}{2}(x_1^2 + x_2^2)$$

We conclude global asymptotic stability (in fact global exponential stability)
without knowing solutions



Proposition. Every sublevel set of V is an invariant

Proof. $V(\xi(t)) =$

$$\begin{aligned} &= V(\xi(0)) + \int_0^t \dot{V}(\xi(\tau)) d\tau \\ &\leq V(\xi(0)) \end{aligned}$$



An aside: Checking inductive invariants

- $A = \langle X, Q_0, T \rangle$
 - X : set of variables
 - $Q_0 \subseteq \text{val}(X)$
 - $T \subseteq \text{val}(X) \times \text{val}(X)$ written as a program $x' \subseteq T(x)$
- How do we check that $I \subseteq \text{val}(X)$ is an inductive invariant?
 - $Q_0 \Rightarrow I(X)$
 - $I(X) \Rightarrow I(T(X))$
- Implies that $\text{Reach}_A(Q_0) \subseteq I$ without computing the executions or reachable states of A
- The key is to find such I



Finding Lyapunov Functions

- The key to using Lyapunov theory is to *find* a Lyapunov function and verify that it has the properties
- In general for nonlinear systems this is hard
- There are several approaches
 - Linear quadratic Lyapunov functions for linear systems
 - Decide the form/template of the function (e.g., quadratic), parameterized by some parameters
 - Try to find values of the parameters so that the conditions hold
 - NNs for learning Lyapunov functions from data [\[Billard`14\]](#)

Learning control Lyapunov function to ensure stability of dynamical system-based robot reaching motions, Khansari-Zadeh, Billard - Robotics and Autonomous Systems, 2014 - Elsevier



Linear autonomous systems

- $\dot{x} = Ax, A \in \mathbb{R}^{n \times n}$
- The Lyapunov equation: $A^T P + PA + Q = 0$
where $P, Q \in \mathbb{R}^{n \times n}$ are symmetric

- Interpretation: $V(x) = x^T P x$ then

$$\dot{V}(x) = (Ax)^T P x + x^T P (Ax)$$

$$[\text{using } \frac{\partial u^T P v}{\partial t} = \frac{\partial u}{\partial t} P v + \frac{\partial v}{\partial t} P^T u]$$

$$= x^T (A^T P + PA) x = -x^T Q x$$

- If $x^T P x$ is the generalized energy then $-x^T Q x$ is the associated dissipation



Quadratic Lyapunov Functions

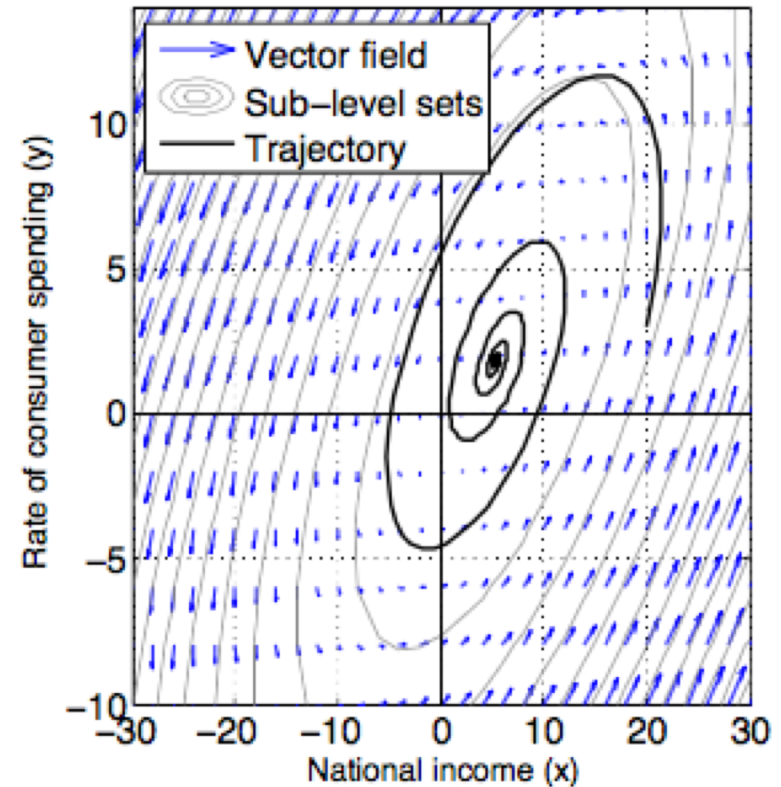
- If $P > 0$ (positive definite)
- $V(x) = x^T P x = 0 \Leftrightarrow x = 0$
- The sub-level sets are ellipsoids
- If $Q > 0$ then the system is globally asymptotically stable



Same example

Lyapunov equations are solved as a set of $\frac{n(n+1)}{2}$ equations in $n(n+1)/2$ variables. Cost $O(n^6)$

Choose $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ solving Lyapunov equations we get $P = \begin{bmatrix} 2.59 & -2.29 \\ -2.29 & 4.92 \end{bmatrix}$ and we get the quadratic Lyapunov function $(x - x^*)P(x - x^*)^T$ an a sequence of invariants



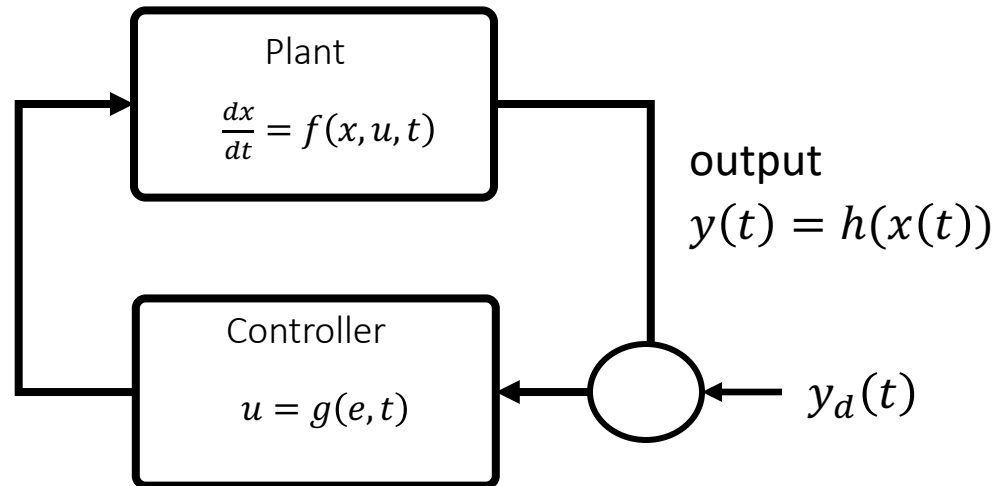
Converse Lyapunov

Converse Lyapunov theorems show that conditions of the previous theorem are also necessary. For example, if the system is asymptotically stable then there exists a positive definite, continuously differentiable function V , that satisfies the inequalities.

For example if the LTI system $\dot{x} = Ax$ is globally asymptotically stable then there is a quadratic Lyapunov function that proves it.



Plant and controller



$$\frac{dx}{dt} = f(x, u(t), t); \quad y(t) = h(x(t));$$
$$e(t) = y(t) - y_d(t)$$

$$u(t) = g(e(t), t)$$



PID control

- 90% (or more) of control loops in industry are PID
- Simple control design model → simple controller
- The standard form of a PID controller:

$$u(t) = k_p e(t) + k_I \int_0^t e(\tau) d\tau + k_d \frac{de}{dt}$$

- where the error term $e(t) = y(t) - y_d(t)$
- $y_d(t)$: desired output or setpoint value
- k_p, k_I, k_d : constant gains
- Many techniques for tuning these parameters: Ziegler-Nichols, relay method, Cohen-Coon method, etc.
- Analysis in frequency domain



P control

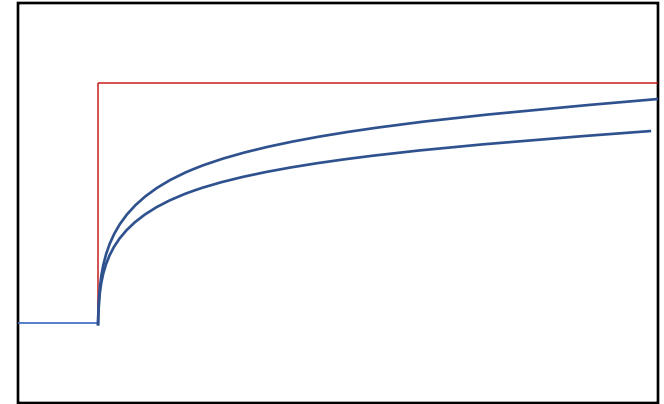
- Consider a simple integrator plant model
- $\dot{y}(t) = u(t) + d$
- $u(t) = -k_p(y(t) - y_d(t))$
- $\dot{y}(t) = -k_p(y(t) - y_d(t)) + d$
- $\dot{y}(t) = -k_p y(t) + (k_p y_d(t) + d)$

Steady state

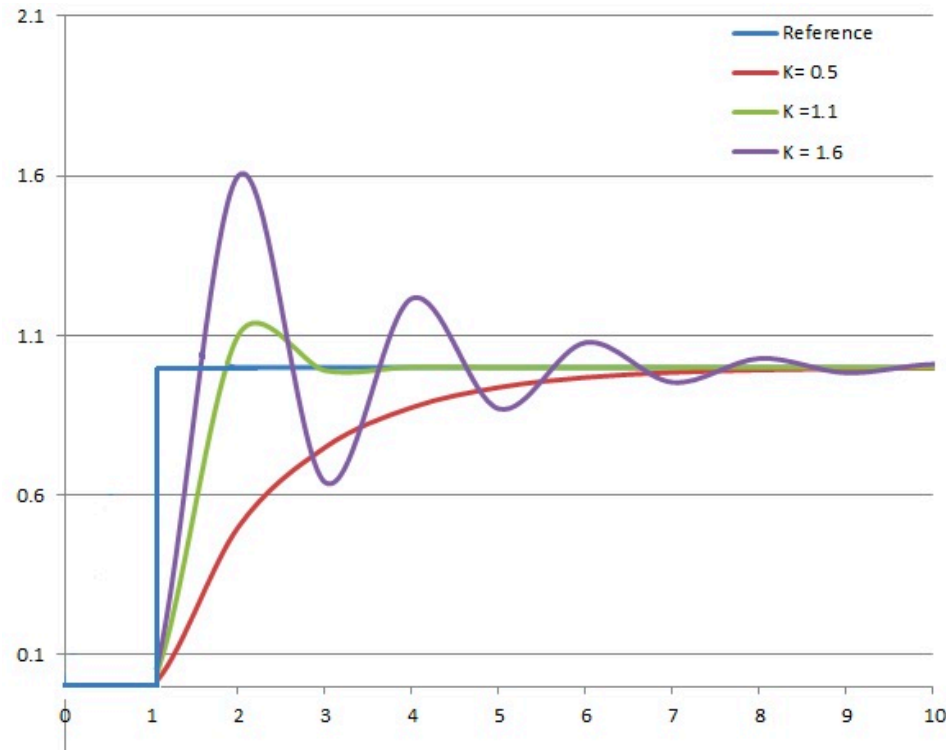
- $0 = -k_p(y(t) - y_d(t)) + d$
- $y_{ss} = y_d - d/k_p$ steady state error

Transient

$$y(t) = y(0)e^{-\frac{t}{T}} + y_{ss}\left(1 - e^{-\frac{t}{T}}\right), T = 1/K_p$$



Choosing proportional gain k_p in PID



Response of $y(t)$ to step change of $y_d(t)$ vs time, for three values of K_p (K_i and K_d held constant) Fig. from wikipedia



PI Control

- $c\dot{y}(t) = -y(t) + u(t) + d$ First order system
- $e(t) = y(t) - y_d(t)$
- $\dot{v} = e$, that is $v(t) = \int_0^t e(s)ds$
- $u = -K_I v - K_P e$
- $c\dot{y}(t) = -y(t) + -K_I v(t) - K_P e(t) + d$ Closed loop system
- $\dot{e}(t) = -\left(1 + \frac{K_P}{c}\right)e(t) + -\frac{K_I}{c}v(t) + \frac{1}{c}(d + (K_P - 1)y_d)$
- Steady state $\dot{v} = 0; e = 0; \dot{e} = 0$
- That is, $y_d(t) = y(t)$ no steady state error
- $\int e(s)ds = \frac{1}{K_I}(d + (K_P - 1)y_d)$ integral input



Summary (you should know)

- Definitions of solutions, stability, invariance, reach set
- Properties of solutions of linear systems
- Discrete abstractions
- Lyapunov's theorems and method for proving stability
- PID controller form, basic properties

