ANALYTIC AND HOLOMORPHIC STRUCTURES IN LIE GROUPOIDS, ALGEBROIDS, AND POISSON GEOMETRY

BY

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DISSERTATION

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Abstract

This thesis explores analytic and holomorphic structures in Lie groupoids, Lie algebroids, and Poisson geometry. The first chapter provides background material on Lie groupoids, Lie algebroids, and Poisson structures. In the second chapter, we prove that any analytic s-proper Lie groupoid is analytically linearizable. Our approach relies on constructing an analytic Haar density and an analytic 2-metric on groupoids. The third chapter introduces the notion of complexification for analytic Lie algebroids. We establish that when the anchor map is injective or surjective, the integrability of the original algebroid implies the integrability of its complexification. Moreover, if the analytic algebroid is of s-proper type, its complexification is locally integrable. In the fourth chapter, we develop a computer program for computing the holomorphic Poisson cohomology of projective spaces.

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Introduction

Lie groupoids and Lie algebroids are geometric structures that generalize Lie groups and Lie algebras, respectively. While Lie groups capture global symmetries and Lie algebras describe infinitesimal ones, Lie groupoids and algebroids provide a framework for describing symmetries and constraints that vary from point to point on a manifold. These objects can be defined not only in the smooth category, but also in the real analytic and holomorphic settings. Much of the foundational theory has been developed in the smooth case. In this thesis, we aim to extend key results and techniques from the smooth category to the analytic category.

The first topic of the thesis is linearization of proper groupoid. It is known that proper Lie groupoids are linearizable around their orbits, originally conjectured by Weinstein [1], [2] and later proved in full generality in [3], [4] (see also [5], [6]). A special case is the classical Bochner linearization theorem, which states that every compact group action can be linearized around a fixed point. To prove this theorem, one can firstly find a Riemannian metric invariant under the group action and use the exponential map to achieve the linearization. This approach was generalized to proper Lie groupoids through the introduction of 2-metrics in [6], which enable linearization via the exponential map of the 2-metric.

These results, however, are all formulated in the smooth setting. In Chapter 2, we investigate whether the linearization theorem holds in the analytic category. We show that an analytic s-proper Lie groupoid admits an analytic 2-metric, constructed using an analytic averaging process based on the existence of an analytic Haar measure. As a consequence, we obtain an analytic version of the linearization theorem:

Theorem 1. *Every s-proper analytic groupoid admits an analytic invariant linearization.*

The second topic is to study the complexification of an analytic Lie algebroids. Given a real Lie algebra $\mathfrak{g}$, we can complexify the underlying vector space and extend the Lie algebra structure linearly to get the complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$. We say that $\mathfrak{g}$ is a real form of $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}_{\mathbb{C}}$ is a complexification of $\mathfrak{g}_{\mathbb{C}}$.

One has analogues of the concepts of real form and complexification for Lie algebroids. Given an analytic manifold M, by [7], we can always embed M into a holomorphic manifold $M_{\mathbb{C}}$, such that on local charts it looks like the standard embedding of $\mathbb{R}^n$ in $\mathbb{C}^n$. Given a real analytic Lie algebroid $A \to M$ one can extend it to be a holomorphic Lie algebroid $A_{\mathbb{C}} \to M_{\mathbb{C}}$. In this case, we say that A is a real form of $A_{\mathbb{C}}$ and $A_{\mathbb{C}}$ is a complexification of A, which was original proposed by Weinstein in [8].

Let G be a Lie group. Recall that its Lie algebra $\mathfrak{g}$ consists of all left invariant vector fields on G. Similarly, given a Lie groupoid $\mathcal{G}$, we can associate to $\mathcal{G}$ a Lie algebroid $A = Lie(\mathcal{G})$, such that sections of A correspond left invariant vector fields in $\mathcal{G}$. Lie's third theorem states that every Lie algebra $\mathfrak{g}$ is the Lie algebra of some Lie group G. Lie's third theorem fails, in general, for Lie algebroids, and Crainic and Fernandes [9] found the necessary and sufficient conditions for integrability of a real Lie algebroid. Later [10] showed that a holomorphic Lie algebroid is integrable if and only if its underlying real Lie algebroid is integrable. Hence the Cranic-Fernandes criterion applies also in the holomorphic context. Since a subalgebroid of an integrable Lie algebroid is integrable, if $A_{\mathbb{C}}$ is integrable it follows that A must be integrable. We want to explore whether, conversely, the integrability of A implies the integrability of its complexification $A_{\mathbb{C}}$ near M.

In Chapter 3, we will show that:

Theorem 2. *When the anchor map is "almost" injective or surjective everywhere, the integrability of A implies the integrability of $A_{\mathbb{C}}$ near M.*

We will also look at the case when the real analytic algebroid A is of s-proper type, meaning that it can be integrated to an analytic s-proper groupoid. Since every s-proper analytic groupoid is analytically linearizable, by complexifying the linear model, we obtain a local integration of $A_{\mathbb{C}}$ near A. This leads to the following theorem:

Theorem 3. *If an analytic groupoid A is of s-proper type, then its complexification $A_{\mathbb{C}}$ is locally integrable near A.*

While integrability implies the local integrability, the converse is not true. There exist locally integrable Lie algebroids that are not globally integrable, as well as Lie algebroids that are not even locally integrable (cf. [9]). However, we can not establish the global integrability in this context.

The final topic of this thesis concerns the Poisson cohomology of projective spaces. Poisson cohomology groups are invariants associated with Poisson manifolds and can be defined in both the smooth and holomorphic categories. In the smooth setting, these groups may be infinite-dimensional even when the underlying manifold is compact. However, in the holomorphic setting, Poisson cohomology groups are always finite-dimensional for compact manifolds (cf. [10]).

In projective space $\mathbb{P}^n$, a Poisson structure can be described via a homogeneous quadratic bivector field on $\mathbb{C}^{n+1}$, and multivector fields on $\mathbb{P}^n$ correspond to homogeneous multivector fields on $\mathbb{C}^{n+1}$. This correspondence allows for an explicit, linear-algebraic description of the Poisson differential. Based on this observation, we develop a computer program that computes holomorphic Poisson cohomology for projective spaces in arbitrary dimensions.

As an application, we examine the case of $\mathbb{P}^3$, whose holomorphic Poisson structures have been classified in [11], [12]. We compute the Poisson cohomology for some Poisson structures on $\mathbb{P}^3$, illustrating the effectiveness of our algorithm.

Chapter 1

Background

This chapter provides an overview of Lie algebroids, Lie groupoids and Poisson geometry. For further details, the reader is referred to the textbook [13]. Readers already familiar with the material may wish to proceed directly to Chapter 2. In this thesis, when we say analytic, we always mean real analytic. We will use holomorphic for complex analytic.

1.1 Lie Algebroids and Lie Groupoids

Definition 1.1.1. A **Lie algebroid** over a manifold M is a triple $(A, \rho, [\cdot,\cdot])$ where:

- $A \to M$ is a vector bundle;
- the anchor map $\rho : A \to TM$ is a bundle map;
- $[\cdot,\cdot]$ is a Lie bracket on the $\mathcal{A}$, sheaf of smooth sections on A,

such that:

$$[X, fY] = \rho(X)(f)Y + f[X,Y] \qquad \text{(Leibniz rule)}$$

for any $X, Y \in \mathcal{A}(U)$, $f \in C^\infty(U)$.

Remark. We can define the notion of an algebroid in different categories. We say that A is an **analytic/holomorphic algebroid** by requiring A to be a analytic/holomorphic bundle, the anchor map ρ to be an analytic/holomorphic bundle map and $\mathcal{A}$ to be the sheaf of analytic/holomorphic sections.

Lie algebroids are generalization of Lie algebras. Let M be a point, $\mathfrak{g}$ be a Lie algebra, and ρ be the trivial map, then $\mathfrak{g}$ becomes a Lie algebroid.

Example 1.1.2. *For any smooth manifold, TM is a Lie algebroid with identity anchor map and usual Lie brackets for vector fields.*

Example 1.1.3. *Let $A \to M$ be a bundle of Lie algebra, where each fiber A_x comes with a Lie algebra structure varying smoothly. Then A is a Lie algebroid with zero anchor map.*

Example 1.1.4. *Let $a : \mathfrak{g} \to \mathfrak{X}(M)$ be a Lie algebra homomorphism. The* ***action algebroid*** *$\mathfrak{g} \ltimes M$ is defined as follows. The vector bundle A is the trivial vector bundle $M \times \mathfrak{g}$, the anchor map $\rho : M \times \mathfrak{g} \to TM$ is:*

$$\rho(x, v) = a(v)_x.$$

Since the section of $M \times \mathfrak{g}$ can be viewed as smooth $\mathfrak{g}$-valued functions on M, the bracket can be defined by:

$$[f,g]_x = [f(x), g(x)]_{\mathfrak{g}} + a(f(x))_x(g) - a(g(x))_x(f)$$

Definition 1.1.5. Let A be a Lie algebroid over M. At each point $x \in M$, the kernel of the anchor map $\mathfrak{g}_x := Ker(\rho)_x$ is a Lie algebra with the Lie bracket defined by:

$$[v,w]_{\mathfrak{g}_x} = [\tilde{v}, \tilde{w}]_A(x)$$

where $\tilde{v}, \tilde{w}$ are local sections of A with $\tilde{v}(x) = v$ and $\tilde{w}(x) = w$. $\mathfrak{g}_x$ is called the **isotropy Lie algebra** at x.

One can show by Leibniz rule that the above definition does not depend on the choice of the local extension $\tilde{v}$ and $\tilde{w}$.

We say that a Lie algebroid is called **regular** if the anchor map is of constant rank. When the anchor map ρ is regular, since ρ preserves Lie brackets, $Im(\rho) \subset TM$ gives an involutive distribution on TM, hence define a foliation $\mathcal{F}_A$ with

$$T\mathcal{F}_A = Im(\rho) \subset TM.$$

The leaf of of this foliation is called the **orbit** of A.

In the case when A is not regular, $Im(\rho) \subset TM$ is a singular distribution, which gives a singular foliation on M (cf. [14]). We also call the leaf this this foliation the **orbit** of A. Points in the same leaf has isomorphic isotropy Lie algebras. We say that a Lie algebra is **transitive** if the anchor map is surjective.

Definition 1.1.6. Let

$$(A \to M,\ [\cdot,\cdot]_A,\ \rho_A) \quad \text{and} \quad (B \to N,\ [\cdot,\cdot]_B,\ \rho_B)$$

be two Lie algebroids. A **morphism of Lie algebroids** F is a vector-bundle map:

$$\begin{array}{ccc} A & \xrightarrow{F} & B \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array}$$

such that:

(i) *Anchor compatibility:*

$$\rho_B \circ F \ = \ Tf \circ \rho_A$$

(ii) *Bracket compatibility:* For all section $\alpha \in \mathcal{A}(U)$, we write

$$F(\alpha) = F \circ \alpha, \quad f^*(\alpha) = \alpha \circ f.$$

Given a section $\alpha, \beta \in \mathcal{A}(U)$, we can write their image of F as finite linear combinations:

$$F(\alpha) = \sum_i c_i\, f^*(\alpha_i), \quad F(\beta) = \sum_i c_i'\, f^*(\beta_i),$$

where $c_i, c_i' \in C^\infty(U)$ and $\alpha_i, \beta_i \in \mathcal{B}(U)$. We require the following compatibility for Lie bracket:

$$F\big([\alpha,\beta]_A\big) = \sum_{i,j} c_i\, c_j'\, f^*\big([\alpha_i, \beta_j]_B\big) \ + \ \sum_j \mathcal{L}_{\rho(\alpha)}\big(c_j'\big)\, f^*(\beta_j) \ - \ \sum_i \mathcal{L}_{\rho(\beta)}\big(c_i\big)\, f^*(\alpha_i).$$

Definition 1.1.7. A **groupoid** is a category where every morphism is invertible. We denote the space of arrows by $\mathcal{G}$, the space of objects by M and the groupoid by $\mathcal{G} \rightrightarrows M$.

A groupoid consists of the following structure maps:

- source/target map $s, t : \mathcal{G} \to M$
- multiplication $m : \mathcal{G}^{(2)} := \mathcal{G}_s \times_t \mathcal{G} \to \mathcal{G}$
- unit map $u : M \to \mathcal{G}$, $u(x) = 1_x$
- inverse map $\iota : \mathcal{G} \to G$, $\iota(g) = g^{-1}$

We denote an arrow $g \in \mathcal{G}$ with $s(x) = y$, $t(g) = y$ by $g : y \leftarrow x$.

$\mathcal{G}$ is a **Lie groupoid** if M and $\mathcal{G}$ are manifolds, all structure maps are smooth, and s, t are submersions. We say that the Lie groupoid is **analytic/holomorphic** if M and $\mathcal{G}$ are analytic/complex manifold and all structure maps are analytic/holomorphic.

Let $\mathcal{G}_x := s^{-1}(x) \cap t^{-1}(x)$. Since every two arrows in $\mathcal{G}_x$ can be multiplied, $\mathcal{G}_x$ is a Lie group, called the **isotropy Lie group** at x. $\mathcal{G}$ defines an equivalence relation $\sim_{\mathcal{G}}$ on M: $x \sim_{\mathcal{G}} y$ if there is an arrow $g \in \mathcal{G}$ such that $s(x) = x$ and $t(x) = g$. The equivalence class of x is called the **orbit** $\mathcal{O}_x$ through x. Points in the same orbits share some common properties, they have:

- isomorphic source fiber $s^{-1}(x)$ and target fiber $t^{-1}(x)$,
- isomorphic isotropy groups.

Proposition 1.1.8 ([15])**.** *We have the following properties for a Lie groupoid:*

1. *$u : M \to \mathcal{G}$ is an embedding,*
2. *every orbit is an immersed submanifold,*
3. *$t : s^{-1}(x) \to \mathcal{O}_x$ is a principal $\mathcal{G}_x$ bundle for all $x \in M$.*

Lie groupoids generalize Lie groups. Every Lie group is a Lie groupoid over a point.

Example 1.1.9 (Pair groupoids)**.** *Let M be a smooth manifold. $M \times M$ is a Lie groupoid with $s(y, x) = x$, $t(y, x) = y$ and $(z, y) \cdot (y, x) = (z, x)$.*

Definition 1.1.10. Let $\mathcal{G}$ be a groupoid over M. A $\mathcal{G}$ action on a manifold E with moment map $\mu : E \to M$ is defined by a map:

$$\mathcal{G} \times_M E := \{(g, e) | s(g) = \mu(e)\} \to E, \quad (g, e) \mapsto ge$$

satisfying:

1. $\mu(ge) = t(g)$,
2. $g(he) = (gh)e$, where every both sides are defined,
3. $1_{\mu(e)} e = e$ for all $e \in E$.

Given a $\mathcal{G}$ action on E. We say that e and e' in E are in the same orbit if there exists $g \in G$, such that $e' = ge$.

Example 1.1.11. *Let G be a Lie group acting on a manifold M. The manifold $G \times M$ has a groupoid structure with $s(g,x) = x$, $t(g,x) = g \cdot x$ and $(h, g \cdot x) \cdot (g,x) = (hg, x)$. We denote the groupoid by $G \ltimes M$.*

Example 1.1.12. *Given a $\mathcal{G}$ action on E with moment map $\mu : E \to M$. We can define the* ***action groupoid*** *$\mathcal{G} \times \ltimes E$, with space of arrows is $\mathcal{G} \times_M E$. The structure maps are:*

$$s(g,e) = e, t(g,e) = ge, (h, ge) \cdot (g,e) = (hg, e)$$

Example 1.1.13 (Gauge groupoids)**.** *Let P be a principal G bundle. The gauge groupoid $P \otimes_G P$ is defined as follows: the space of arrows is the quotient space $(P \times P)/G$, where G acts on $P \times P$ by*

$$g \cdot (q,p) = (qg^{-1}, pg^{-1}),$$

the structure maps are: $s([q,p]) = p, t([q,p]) = q, [r,q] \cdot [q,p] = [r,p]$.

We say that a groupoid is **transitive** if it consists of a single orbit. Gauge groupoids are examples of transitive groupoids. In fact, every transitive groupoid can be described as a gauge groupoid.

Let $\mathcal{G}$ be a transitive groupoid. Pick a point $x \in M$. Let $G_x = s^{-1}(x) \cap t^{-1}(x)$ be the isotropy group. According to Proposition 1.1.8, $P_x = s^{-1}(x)$ is a Principal G_x-bundle. Then $\mathcal{G}$ is isomorphic to $P \otimes_{G_x} P$.

Let $\mathcal{G}$ be a Lie groupoid. Let $A := Kerds|_M$ be the kernel of ds restricting on M. A becomes a Lie algebroid with anchor map $\rho = dt|_A : Kerds|_M \to TM$. Let $g : y \leftarrow x \in G$. Let $R_g : s^{-1}(x) \to s^{-1}(y)$ be the right multiplication by g. Given a section α on A, we can extend it to obtain a section $\tilde{\alpha}$ on $Kerds \subset T\mathcal{G}$ by

$$\tilde{\alpha}(g) = dR_g(\alpha(1_{t(g)})).$$

Then bracket of A is defined by:

$$[\alpha, \beta]_A(x) = [\tilde{\alpha}, \tilde{\beta}](1_x),$$

where the left hand side is just the usual Lie bracket of vector fields. We call A **the Lie algebroid of the groupoid** $\mathcal{G}$, denoted by $Lie(\mathcal{G})$. If we assume that the groupoid $\mathcal{G}$ is analytic/holomorphic, then $Lie(\mathcal{G})$ is an analytic/holomorphic Lie algebroid.

Let A be the Lie algebroid of a Lie groupoid $\mathcal{G}$. The orbit of A is the connected component of the orbit of $\mathcal{G}$. If $\mathcal{G}$ is source connected, then the orbit of A coincides the orbit of $\mathcal{G}$.

Definition 1.1.14. Given two Lie groupoids $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$, a **morphism of groupoids** is a pair of smooth maps $(\mathcal{F}, f)$:

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{F} & \mathcal{H}, \\ \downarrow\downarrow & & \downarrow\downarrow \\ M & \xrightarrow{f} & N \end{array}$$

which is compatible with all structure maps. We say that $(\mathcal{F}, f)$ is an **isomorphism of groupoids** if it is invertible.

By compatibility of $(\mathcal{F}, f)$, we mean that the following conditions hold:

1. $\mathcal{F}(g)$ has source $f \circ s(g)$ and target $f \circ t(g)$,

2. For any composable arrows $g, h \in G$, $F(gh) = F(g)F(h)$,

3. $F(1_x) = 1_{f(x)}$ for any $x \in M$.

4. $F(g^{-1}) = (F(g))^{-1}$.

Given a morphism $\mathcal{F} : \mathcal{G} \to \mathcal{H}$ between two Lie groupoids, the differential of $\mathcal{F}$ induces a morphism $Lie(\mathcal{F}) : Lie(\mathcal{G}) \to Lie(\mathcal{H})$ between their corresponding Lie algebroids.

1.2 Integrability of Algebroids

1.2.1 Source Simply Connected Integration

Given a Lie algebra $\mathfrak{g}$, there is a unique simply connected Lie group integrating G, which is known as Lie's third theorem. Lie's third theorem fails, in general, for Lie algebroids.

Definition 1.2.1. Let A be a (holomorphic/analytic) Lie algebroid. We say that A is **(holomorphically/analytically) integrable** if there is a (holomorphic/analytic) Lie groupoid $\mathcal{G}$ with $\mathrm{Lie}(\mathcal{G}) = A$. In that case, we say that $\mathcal{G}$ is an **integration** of A.

Crainic and Fernandes [9] found the necessary and sufficient conditions for integrability of a real Lie algebroid. Later [16] showed that a holomorphic Lie algebroid is integrable if and only if its underlying real Lie algebroid is integrable. Hence the Crainic-Fernandes criterion applies also in the holomorphic context.

There may be multiple Lie groupoids integrating the same Lie algebroid A. A groupoid is said to be **source connected** if its source fibers are connected. It is called **source simply connected** if the source fibers $s^{-1}(x)$ are simply connected. The source simply connected groupoid plays a central role: if A is integrable, there exists a unique source simply connected Lie groupoid integrating A.

Proposition 1.2.2 ([15])**.** *Let $\mathcal{G}$ be a source connected Lie groupoid. There exist a groupoid morphism $\pi : \tilde{\mathcal{G}} \to \mathcal{G}$ covering the identity, where:*

- *$\tilde{\mathcal{G}}$ is source simply connected,*
- *π is a local diffeomorphism,*
- *$\mathcal{G}$ and $\tilde{\mathcal{G}}$ has the same Lie algebroid.*

A morphism of Lie algebroids can be integrated to obtain a morphism from the corresponding simply connected Lie groupoid. We get the following algebroid version of Lie's II theorem.

Theorem 1.2.3. *[[9], [17], [18]] Let $F : A \to B$ be a morphism between integrable Lie algebroids. Let $\mathcal{G}$ and $\mathcal{H}$ be integrations of A and B, respectively. Assume that $\mathcal{G}$ is simply connected, then F can be integrated to a Lie groupoid morphism $\mathcal{F} : \mathcal{G} \to \mathcal{H}$.*

Proposition 1.2.2 and Theorem 1.2.3 show that: once a Lie algebroid A is integrable, its source simply connected integration exists and is unique up to isomorphism. The source simply connected integration can be constructed as **Weinstein groupoid** of A, denoted by $\mathcal{G}(A)$ (see [9] for the detail). Without assuming the integrability of A, one can define $\mathcal{G}(A)$ as a topological groupoid (without smooth structure). It turns out that the Weinstein groupoid is the canonical model for measuring the integrability:

- A is integrable iff $\mathcal{G}(A)$ is a Lie groupoid ([9]),

- A is holomorphically integrable iff $\mathcal{G}(A)$ is a holomorphic groupoid ([16]),
- A is analytically integrable iff $\mathcal{G}(A)$ is an analytic groupoid.

The last argument can be proved as follows:

Proposition 1.2.4. *If A be an analytically integrable algebroid, then its source simply connected integration is analytic.*

Proof. Let $\mathcal{G}$ be an analytic groupoid integrating A. On each fiber $s^{-1}(x)$, we take the connected component containing the identity 1_x. Union all those connected components gives us a subgroupoid whose source fibers are connected. Hence we may assume that the $\mathcal{G}$ is source connected. Let $\tilde{\mathcal{G}}$ be the source simply connected integration with the groupoid morphism:

$$\begin{array}{ccc} \widetilde{\mathcal{G}} & \xrightarrow{\pi} & \mathcal{G} \\ & \tilde{s} \searrow\searrow \tilde{t} & t \downarrow \downarrow s \\ & & M, \end{array}$$

where π is a local diffeomorphism.

Let (U_α, ϕ_α) be an analtyic atlas for $\mathcal{G}$ with $\phi_\alpha : U_\alpha \to \mathbb{R}^n$. For any point $p \in \tilde{\mathcal{G}}$, there is an open neighborhood $V \subset \tilde{\mathcal{G}}$ such that $\pi|_V : V \to \pi(V)$ is a diffeomorphism. Possibly by shrinking the V, we may assume that $\pi(V)$ is contained in some chart U_α. Then $(V, \phi \circ \pi)$ is a coordinate chart on $\tilde{\mathcal{G}}$ with

$$\phi_\alpha \circ \pi|_V : V \to \mathbb{R}^n.$$

Combining such coordinate chart gives us an atlas on $\tilde{\mathcal{G}}$, whose transition functions:

$$(\phi_\beta \circ \pi|_W) \circ (\phi_\alpha \circ \pi|_V)^{-1} = \phi_\beta \circ \pi|_W \circ (\pi|_V)^{-1} \circ \phi_\alpha^{-1} = \phi_\beta \circ \phi_\alpha^{-1}$$

are analytic. With such analytic structure on $\tilde{\mathcal{G}}$, π is just the identity map on the local coordinates. Since π is a groupoid morphism, on local coordinates, the structure maps of $\tilde{\mathcal{G}}$ has the same representations as in $\mathcal{G}$, hence are analytic. □

1.2.2 Cranic-Fernandes Criterion

In this subsection, we will describe the construction of Weinstein groupoid and the obstruction for the integrability. Given a Lie algebroid A, let $p : A \to M$ be the projection onto the base manifold. We say that a C^1 curve $a : I \to A$ is an **A-path** if

$$\rho(a(t)) = \frac{d}{dt}\gamma(t)$$

where $\gamma(t) = p \circ a(t)$ is the base curve. Let $P(A)$ denote the space of all A-paths, which is a Banach manifold with the topology of uniform convergence.

Definition 1.2.5. We say that two A-path $a_0, a_1 : I \to A$ are **A-path homotopic** if there exists a Lie algebroid morphism

$$\Phi : T(I \times I) \to A, \quad \Phi = \Phi_t dt + \Phi_\epsilon d\epsilon$$

such that

$$\Phi_t(t,0) = a_0, \quad \Phi_t(t,1) = a_1, \quad \Phi_\epsilon(0,\epsilon) = \Phi_\epsilon(1,\epsilon) = 0$$

In this case, we say that Φ is an **A-path homotopy** between a_0 and a_1.

The A-homotopy defines an equivalence relationship on the space of A-path, $P(A)$. The Weinstein groupoid $\mathcal{G}(A) = P(A)/\sim$, where $s(a) = a(0)$, $t(a) = a(1)$ and the multiplication is induced by concatenating two paths (see [9] for details). With the quotient topology, $\mathcal{G}(A)$ becomes a topological groupoid. $\mathcal{G}(A)$ is smooth if and only if A is integrable.

Example 1.2.6. *Let $\mathfrak{g}_x$ be the isotropy Lie algebra of at $x \in M$, which can be viewed as a Lie algebroid over a point. The simply connected Lie group integrating $\mathfrak{g}_x$ is $\mathcal{G}(\mathfrak{g}_x)$, which can also be viewed as a Lie groupoid over a point.*

Let $F : A \to B$ be a morphism between two Lie algebroids. F induces a morphism $\mathcal{G}(F) : \mathcal{G}(A) \to \mathcal{G}(B)$ between the corresponding Weinstein groupoids. For any A-path $a : I \to A$, $\mathcal{G}(F)([a]) = [F \circ a]$. If the both A and B are integrable, then $\mathcal{G}(F)$ is a smooth map.

For any $x \in M$, that the inclusion $i : \mathfrak{g}_x \to A$ is a morphism of Lie algebroids. Hence it induces a morphism between topological groupoids $\mathcal{G}(i) : \mathcal{G}(\mathfrak{g}_x) \to \mathcal{G}(A)$. Let $\mathcal{G}(A)_x = s^{-1}(x) \cap t^{-1}(x)$ be the isotropy group and $\mathcal{G}(A)^0_x$ be its connected component of containing the identity. We have the image of $\mathcal{G}(i)$ is in $\mathcal{G}(A)^0_x$. Suppose that A is integrable, then $\mathcal{G}(i) : \mathcal{G}(\mathfrak{g}_x) \to \mathcal{G}(A)^0_x$ becomes a morphism of Lie groups. From the classical Lie theory we know that $\mathcal{G}(i)$ gives a covering map and $\mathcal{G}(A)^0_x \cong \mathcal{G}(\mathfrak{g}_x)/ker(\mathcal{G}(i))$. Therefore, $ker(\mathcal{G}(i))$ must be discrete when A is integrable. The kernel of $\mathcal{G}(i)$ is called the **monodromy group** of A, denoted by $\tilde{\mathcal{N}}_x(A)$. More explicitly,

$$\tilde{\mathcal{N}}_x(A) := \{[a] \in \mathcal{G}(\mathfrak{g}_x) |\ a \sim 0_x \text{ as } A\text{-paths } \}$$

The usual exponential map $exp : \mathfrak{g}_x \to G(\mathfrak{g}_x)$ between the Lie algebra and the corresponding Lie group is given by mapping $v \in \mathfrak{g}_x$ to its $\mathfrak{g}_x$-homotopy class, i.e., $exp(v) = [v]$. Define

$$\mathcal{N}_x(A) := exp^{-1}(\tilde{\mathcal{N}}_x(A)) = \{v \in g_x |\ v \sim 0_x \text{ as } A\text{-paths}\}$$

It turns out that discreteness of $\mathcal{N}_x(A)$ and $\tilde{N}_x(A)$ are equivalent to each other.

Proposition 1.2.7 ([9])**.** *For any Lie algebroid A and any $x \in M$, the following are equivalent:*

1. *$N_x(A)$ is discrete,*
2. *$\tilde{N}_x(A)$ is discrete,*
3. *$N_x(A)$ is closed,*
4. *$\tilde{N}_x(A)$ is closed.*

For the points in the same orbit, their monodromy groups coincide.

Proposition 1.2.8 ([15])**.** *If $x, y \in M$ are in the same orbit of A, then there are canonical isomorphisms:*

$$N_x(A) \cong N_y(A), \quad \tilde{N}_x(A) \cong \tilde{N}_y(A)$$

Combining all the monodromy groups at each point, we get:

$$\mathcal{N}(A) := \cup_x \mathcal{N}_x(A) \subset A$$

The property of $\mathcal{N}(A)$ gives a necessary and sufficient condition for measuring the integrability of A, which is called Cranic-Fernandes criterion.

Theorem 1.2.9 ([9]). *A is integrable if and only if there is an open neighborhood $U \subset A$ containing the zero section 0_M such that $U \cap N(A) = 0_M$.*

The condition for $N(A)$ in the above theorem is called the **uniform discreteness** of $N(A)$. Notice that if $N(A)$ is uniformly discrete, then $N_x(A)$ is discrete for all $x \in M$, which implies each $\tilde{N}_x(A)$ is also discrete. Later we will see that the converse also holds when the algebroid is transitive.

According to Proposition 1.2.8, for a transitive Lie algebroid, to check the integrability, we just need to check discreteness of the monodromy group at one point.

Corollary 1.2.10. *[9] Let A be a transitive Lie algebroid. A is integrable if and only if one of the following holds for some point $x \in M$:*

1. *$N_x(A)$ is discrete,*
2. *$\tilde{N}_x(A)$ is discrete,*
3. *$N_x(A)$ is closed,*
4. *$\tilde{N}_x(A)$ is closed.*

Let $\mathcal{O} \subset M$ be an orbit passing through x. Let $[\gamma] \in \pi_2(\mathcal{O}, x)$ be represented by a smooth map $\gamma : I \times I \to \mathcal{O}$ with the $\gamma(\partial(I \times I)) = x$. We choose an A-path homotopy:

$$\Phi : T(I \times I) \to A|_{\mathcal{O}}, \quad \Phi = \Phi_t dt + \Phi_\epsilon d\epsilon$$
$$\Phi_t(t, 0) = 0_x, \quad \Phi_\epsilon(0, \epsilon) = \Phi_\epsilon(1, \epsilon) = 0_x,$$

such that Φ covers γ in the commutative diagram:

$$\begin{array}{ccc} T(I \times I) & \xrightarrow{\Phi} & A|_{\mathcal{O}} \\ \downarrow & & \downarrow p \\ I \times I & \xrightarrow{\gamma} & \mathcal{O}. \end{array}$$

Write $a(t) = \Phi_t(t, 1)$. Since $\rho \circ \Phi = d\gamma$ and γ is constant on the boundary of $I \times I$, we have $a(t) \in \mathfrak{g}_x$ for all $t \in [0, 1]$. Hence a represents $[a] \in \mathcal{G}(\mathfrak{g}_x)$. It has been shown that the equivalence class $[a]$ does not the choice of the A-path homotopy Φ. It only depends on the homotopy class $[\gamma]$. Hence we obtain a morphism of groups:

$$\partial_x : \pi_2(\mathcal{O}, x) \to \mathcal{G}(\mathfrak{g}_x), \quad [\gamma] \mapsto [a] = \partial_x[\gamma].$$

Moreover, we have

$$Im(\partial_x) = \tilde{\mathcal{N}}_x(A),$$

which can be seen from the following proposition.

Proposition 1.2.11. *[9] For any Lie algebroid A, there is a short exact sequence of groups:*

$$\pi_2(\mathcal{O}, x) \xrightarrow{\partial_x} \mathcal{G}(\mathfrak{g}_x) \xrightarrow{\mathcal{G}(i)} \mathcal{G}(A)_x \xrightarrow{p_x} \pi_1(\mathcal{O}, x) \to 1,$$

where $p_x([a]) = [p \circ a]$.

If a Lie algebroid has zero anchor map, it is a bundle of Lie algebras. Since the orbits are discrete points, we have that the monodromy group $\tilde{N}_x(A)$ is trivial. Hence, the algebroid is integrable.

Corollary 1.2.12 ([19])**.** *A bundle of Lie algebras can be integrated to a bundle of Lie groups.*

1.2.3 Holomorphic Lie Algebroid

In this section, we will see that the Cranic-Fernandes criterion also works on the holomorphic category.

Let $(A, \rho, [\cdot,\cdot])$ be a holomorphic algebroid and let J be the complex structure on A. Forgetting about the complex structure on A, we can view A as a real vector bundle, and the anchor map $\rho : A \to TM$ as a real vector bundle map. Let $\mathcal{A}$ be the sheaf of holomorphic sections on A and $\mathcal{A}_\infty$ be the sheaf of smooth sections on A. By choosing local trivialization, any smooth section $s \in \mathcal{A}_\infty(U)$ can be written as linear combination:

$$s = \sum_{k=1}^{m} (f_k a_k + g_k J(a_k))$$

where $f_i, g_i \in C^\infty(U)$ are smooth functions on U and $a_k \in \mathcal{A}(U)$ is a holomorphic section. We can define a bracket $[\cdot,\cdot]_R$ in $\mathcal{A}_\infty(U)$ by extending the bracket on $\mathcal{A}(U)$ using Leibniz rule and $\mathbb{C}$-linearity. Explicitly, the bracket of $s = \sum_{k=1}^{m} (f_k a_k + g_k J(a_k))$ and $s' = \sum_{k=1}^{m} (f'_k a_k + g'_k J(a_k))$ is given by:

$$\begin{aligned}
[s, s']_R = &\sum_{k,s} \left(f_k\, f'_s\, [a_k, a_s] \;+\; f_k\, f'_s\, [a_k, a_s] \;+\; f_k\, f'_s\, [a_k, a_s] \;+\; f_k\, f'_s\, [a_k, a_s]\right) \\
&+ \sum_{s} \left(\mathcal{L}_{\rho(s)}\big(f'_s\big)\, a_s \;+\; \mathcal{L}_{\rho(s)}\big(g'_s\big)\, J(a_s)\right) \\
&- \sum_{k} \left(\mathcal{L}_{\rho(s')}\big(f_k\big)\, a_k \;+\; \mathcal{L}_{\rho(s')}\big(g_k\big)\, J(a_k)\right)
\end{aligned}$$

Let A_R be the underlying real vector bundle of A and $\rho_R = \rho$. It has been shown that the above bracket defines a real Lie algebroid $(A_R, \rho_R, [\cdot,\cdot]_R)$, which is called the **underlying real Lie algebroid** of the holomorphic Lie algebroid A.

Theorem 1.2.13 ([10])**.** *Let $(A, \rho, [\cdot,\cdot])$ be a holomorphic Lie algebroid. Then there is a unique real Lie algebroid structure on the real vector bundle A with the same anchor map ρ such that the inclusion of the sheaf $\mathcal{A} \to \mathcal{A}_\infty$ is a morphism of sheaf of Lie algebras.*

The integrability of A is equivalent to the integrability of its underlying real Lie algebroid A_R.

Theorem 1.2.14 ([16])**.** *Let A be a holomorphic Lie algebroid. A is holomorphically integrable if and only if the underlying real Lie algebroid A_R is integrable as a real Lie algebroid.*

Moreover, Lie's II theorem holds for holomorphic algebroids.

Theorem 1.2.15 ([16]). *Let $\mathcal{G}$ and $\mathcal{F}$ be holomorphic Lie groupoids. Let $\mathcal{F} : \mathcal{G} \to \mathcal{H}$ be a real Lie groupoids morphism between the underlying real Lie groupoids. Assume that $Lie(\mathcal{F}) : Lie(\mathcal{G}) \to Lie(\mathcal{H})$ is a morphism of holomorphic algebroids. Then F is a morphism of holomorphic Lie groupoids.*

Corollary 1.2.16. *Let $F : A \to B$ be a morphism between integrable holomorphic Lie algebroids. Let $\mathcal{G}$ and $\mathcal{H}$ be integrations of A and B, respectively. Assume that $\mathcal{G}$ is source simply connected, then F can be integrated to a holomorphic Lie groupoid morphism $\mathcal{F} : \mathcal{G} \to \mathcal{H}$.*

Proof. According to Theorem 1.2.3, F can be integrated to a morphism $\mathcal{F} : \mathcal{G} \to \mathcal{H}$ of real Lie algebroid. Since $F = Lie(\mathcal{F}) : A \to B$ is holomorphic, $\mathcal{F}$ is actually a morphism between holomorphic Lie groupoids. □

1.3 Poisson Manifolds

Let M be a smooth manifold and denote by C^∞_M sheaf of smooth function of M.

Definition 1.3.1. A **Poisson bracket** on M is a sheaf morphism

$$\{\cdot,\cdot\} \;:\; C^\infty_M \times C^\infty_M \;\longrightarrow\; C^\infty_M$$

such that for all $f, g, h \in C^\infty_M(U)$:

1. **Skew-symmetry:** $\{f, g\} = -\{g, f\}$.
2. **Leibniz rule:** $\{f, gh\} = \{f, g\}\, h + g\,\{f, h\}$.
3. **Jacobi identity:** $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$.

In local chart, by Leibniz rule, the Poisson bracket $f, g \in C^\infty_M(U)$ can be written as:

$$\{f, g\} = \sum_{i,j} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}$$

Locally, a Poisson structure can be expressed as:

$$\pi = \sum_{i<j} \pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}.$$

Hence the above definition is equivalent to specifying a bivector field:

$$\pi \in \wedge^2 TM$$

satisfying the Schouten bracket $[\pi, \pi] = 0$ (cf. [13, Chapter 2]). The Poisson bracket can be recovered form the bivector by

$$\{f, g\} = \pi(df, dg).$$

Remark. The above definitions extend to the categories of real-analytic and complex-analytic manifolds. If M is a real-analytic manifold and the bracket takes analytic functions to analytic functions, then π defines an **analytic Poisson structure**. If X is a complex manifold, replacing $C^\infty(M)$ by the sheaf $\mathcal{O}_X$ of holomorphic functions, and TM by the holomorphic tangent bundle, the same axioms yield a **holomorphic Poisson structure** on X.

Let (M, π) be a Poisson manifold. π induces a bundle map

$$\pi^{\#} : T^*M \to TM \quad \alpha \mapsto i_\alpha \pi,$$

and a Lie bracket on the space of differential one forms:

$$[\alpha, \beta]_\pi := \mathcal{L}_{\pi^\sharp \alpha}(\beta) - \mathcal{L}_{\pi^\sharp \beta}(\alpha) - d\big(\pi(\alpha, \beta)\big)\,.$$

Then the triple $(T^*M, \pi^{\#}, [\cdot, \cdot]_\pi)$ define a Lie algebroid over M.

Example 1.3.2. *Any smooth manifold has a zero Poisson structure $\pi \equiv 0$.*

Example 1.3.3. *Let $\mathfrak{g}$ be a Lie algebra. For any $\xi \in \mathfrak{g}^*$, the tangent bundle $T_\xi \mathfrak{g}^*$ can be identified with $\mathfrak{g}$. There is a Poisson structure π on $\mathfrak{g}^*$ called the **Kostant-Kirilov-Souriau (KKS)** Poisson structure, which is defined by:*

$$\pi_\xi(df, dg) :=< [d_\xi f, d_\xi g], \xi >$$

for any holomorphic functions g, h on $\mathfrak{g}^*$.

More explicitly, if $\mathfrak{g}$ has a basis $\{e_1, \cdots, e_n\}$ with the bracket

$$[e_i, e_j] = \sum_{k=1}^{n} c_{ij}^k e_k.$$

Then the KKS Poisson structure on $\mathfrak{g}^* \cong \mathbb{C}^n$ can be written as:

$$\pi = \sum_{1 \le i < j \le n} c_{ij}^k z_k \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}.$$

*Such Poisson structures are also called **linear Poisson structure**.*

Definition 1.3.4. For a smooth Poisson manifold (M, π). The **Poisson differential** is the linear map $d_\pi : \Gamma(\wedge^k TM) \to \Gamma(\wedge^{k+1} TM)$ defined by:

$$\begin{aligned}(d_\pi \vartheta)(\alpha_0, \ldots, \alpha_k) = &\sum_{i=0}^{k} (-1)^i \, \mathcal{L}_{\pi^\sharp(\alpha_i)}\big(\vartheta(\alpha_0, \ldots, \widehat{\alpha_i}, \ldots, \alpha_k)\big) \\ &+ \sum_{0 \le i < j \le k} (-1)^{i+j} \, \vartheta\big([\alpha_i, \alpha_j]_\pi, \, \alpha_0, \ldots, \widehat{\alpha_i}, \ldots, \widehat{\alpha_j}, \ldots, \alpha_k\big).\end{aligned}$$

The Poisson differential gives a cochain complex:

$$0 \to C^\infty(M) \xrightarrow{d_\pi} TM \xrightarrow{d_\pi} \cdots \xrightarrow{d_\pi} \wedge^k TM \xrightarrow{d_\pi} \cdots \xrightarrow{d_\pi} \wedge^n TM \to 0$$

Definition 1.3.5. The **Poisson cohomology** of a Poisson manifold (M, π) is the cohomology of the cochain complex $(\Gamma(\wedge^k TM), \, d_\pi)$:

$$H_\pi^k(M) := \frac{\ker\big(d_\pi : \Gamma(\wedge^k TM) \to \wedge^{k+1} TM\big)}{\mathrm{Im}\big(d_\pi : \Gamma(\wedge^{k-1} TM) \to \Gamma(\wedge^k TM)\big)}.$$

Remark. For a holomorphic Poisson manifold, the Poisson differential is defined on the sheaf of multi-vector fields, and the Poisson cohomology is the sheaf cohomology of the cochain complex.

Example 1.3.6. *Let M be a smooth manifold and $\pi \equiv 0$ be the zero Poisson structure on M. Then the Poisson cohomology is $H^k_\pi(M) = \Gamma(\wedge^k TM)$, which is always an infinitely dimensional vector space.*

Example 1.3.7. *Let $\mathfrak{g}$ be a Lie algebra. Then $\mathfrak{g}^*$ is a Poisson manifold with linear Poisson structure π. The Poisson cohomology of $(\mathfrak{g}^*, \pi)$ can be expressed as the Lie algebra cohomology with coefficients:*

$$H^k_\pi(\mathfrak{g}^*) \cong H^*(\mathfrak{g}, C^\infty(\mathfrak{g}^*)),$$

Here we view $C^\infty(\mathfrak{g}^*)$ as an infinitely dimensional representation induced by Poisson structure:

$$\rho : \mathfrak{g} \to \mathfrak{gl}(C^\infty(\mathfrak{g}^*)) \ \rho(v)(f) = \pi(dg_v, df),$$

where any $v \in \mathfrak{g}$ can be viewed as a linear function $g_v : \alpha \mapsto < \alpha, v >$ on $\mathfrak{g}^*$.

In the smooth category, Poisson cohomology groups may have infinitely many dimensions and are difficult to compute in most cases. However, Poisson cohomology groups for holomorphic Poisson manifold is always of finite dimension ([10]).

Chapter 2

Linearization of Groupoids

2.1 Linear Model of Groupoids

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. We say that a submanifold $S \subset M$ is a saturated submanifold if S is a union of orbits. Denote $\mathcal{G}_S = s^{-1}(S)$. Then we have the following groupoid:

$$\mathcal{N}_{\mathcal{G}_S} := T\mathcal{G}/T\mathcal{G}_S \rightrightarrows \mathcal{N}_S := TM/TS,$$

whose structure maps are induced by the differential of structure maps of $\mathcal{G}$. We call the groupoid $\mathcal{N}_{\mathcal{G}_S} \rightrightarrows \mathcal{N}_S$ the **linear model** of $\mathcal{G}$ at S.

There are various ways of describing the groupoid structure of the linear model. The normal bundle $\mathcal{N}_{\mathcal{G}_S}$ is also the pull-back of $\mathcal{N}_S$ by s:

$$\mathcal{N}_{\mathcal{G}_S} = \{(g, v) \in \mathcal{G}_S \times \mathcal{N}_S | s(g) = \pi(v)\}$$

$\mathcal{G}_S$ acts on S by normal representation. The groupoid structure on $\mathcal{N}_{\mathcal{G}_S}$ is given by the action groupoid $\mathcal{G}_S \ltimes \mathcal{N}_S$.

Definition 2.1.1. Let $\mathcal{G} \rightrightarrows M$ be an (analytic) Lie groupoid and S be a saturated submanifold of M. We say that:

1. $\mathcal{G}$ is (analytically) linearizable at S if there exist an open neighborhoods of S, $U \subset M$ and $V \subset \mathcal{N}_S$, such that the there is an (analytic) groupoid isomorphism:

   $$\mathcal{G}|_U \cong \mathcal{N}_{\mathcal{G}_S}|_V$$

   which is identity on $\mathcal{G}_S$.

2. $\mathcal{G}$ invariant (analytically) linearizable if we can choose open neighborhoods U and V to be invariant.

Definition 2.1.2. We say that a Lie groupoid $\mathcal{G}$ is:

- **proper** if the map $(s, t) : \mathcal{G} \times \mathcal{G} \to M \times M$ is proper,
- **s-proper** if the the source map $s : \mathcal{G} \to M$ is proper.

In the smooth category, proper groupoids can be linearized.

Theorem 2.1.3 ([4], [5], [1], [2], [3])**.** *If a groupoid $\mathcal{G}$ is proper, then it is linearizable at any saturated submanifold S. If we further assume that $\mathcal{G}$ is s-proper, then it is invariantly linearizable.*

In the remainder of this chapter, we will mainly focus on the analytic category. We will show that when $\mathcal{G}$ is analytic and s-proper, the linearization can be made analytic.

2.2 Riemannian Submersions

Let (E, η^E) and (B, η^B) be Riemannian manifolds. Let $p : E \to B$ be a submersion, we say that p is a Riemannian submersion if for any $e \in E$, the differential of p at e restricts to an isomortry between the horizontal space at e and the tangent space of B at $b = p(e)$:

$$d_e p : (kerdp)_e^{\perp} \to T_b B$$

In that case, $\eta^B = p_* \eta^E$ is completely determined by the metric η^E, called the push-forward metric. We say that the metric η^E is p-transverse if for any two points $e, e' \in E$ with $p(e) = p(e')$, the composition

$$(kerdp)_e^{\perp} \to T_b B \leftarrow (kerdp)_{e'}^{\perp}$$

is an isometry. When η^E is p-transverse, it induces the push-forward metric $\eta^B = p_* \eta^E$ on B, which makes $p : E \to B$ a Riemannian submersion. If we further assume that η is analytic and p is analytic, then the push-forward metric $p_* \eta$ is also analytic.

Lemma 2.2.1. *Let $p : E \to B$ be an analytic submersion. There exists a p-transversal analytic metric η_E on E.*

Proof. We have the following exact sequence on E

$$0 \longrightarrow ker(dp) \longrightarrow TE \underset{\sigma}{\overset{s_*}{\rightleftarrows}} p^*(TB) \longrightarrow 0.$$

Choose an analytic metric g_E on E. Let $\sigma : TB \to (Ker(dp))^{\perp}$ be the projection onto the orthogonal complement of $Ker(dp)$, which induces an isomorphism

$$TE \cong p^* TB \oplus Ker(dp)$$

Choose an analytic metric g_B on the manifold B, which induces a metric $p^* g_B$ on $p^*(TB)$. Define $\eta_E = g_E \mid_{Ker(dp)} + p^* g_B$. We have η_E is p-transversal and analytic. □

Lemma 2.2.2 ([6])**.** *Let $q : \tilde{E} \to E$ and $p : E \to M$ be surjective submersions. Let η be a q-transverse metric on $\tilde{E}$. Then η is $q \circ p$-transverse if and only if $q_* \eta$ is p-transverse. In that case, $(q \circ p)_* \eta = p_*(q_* \eta)$.*

2.3 Quasi-action on Groupoids

Let $\theta : \mathcal{G} \curvearrowright E$ be an analytic groupoid action with momentum map $q : E \to M$. Let $\mathcal{O}$ be an orbit of the action. Restricting the action groupoid $\mathcal{G} \ltimes E$ to the orbit $\mathcal{O}$, we get the analytic groupoid

$$\mathcal{G} \ltimes E|_{\mathcal{O}} = \mathcal{G} \ltimes \mathcal{O} = \{(g,e) \in \mathcal{G} \times \mathcal{O} | s(g) = q(e)\}.$$

$\mathcal{G} \ltimes \mathcal{O}$ acts linearly on the normal bundle $\mathcal{N}_{\mathcal{O}} = TE/T\mathcal{O}$. Any $(g,e) \in \mathcal{G} \ltimes \mathcal{O}$ induces a linear isomorphism:

$$(g,e) : \mathcal{N}_e \to \mathcal{N}_{ge},$$

which is defined as follows. Given $v \in \mathcal{N}_e$, choose curves $g(t)$ in $\mathcal{G}$ and $e(t)$ in E such that $g(0) = g$, $e(0) = e$, $e'(0)$ represents v, and $s(g(t)) = q(e(t))$. Then $(g,e)v$ is represented by $\frac{d}{dt}\big|_{t=0}(g(t)e(t))$. One can check it is independent of choice of curves, hence defines a groupoid action, called the **normal representation**, denoted by $\mathcal{N}_{\mathcal{O}}(\theta)$.

We can dualize the normal representation to define the **conormal representation** $\mathcal{N}^*_{\mathcal{O}}(\theta) : \mathcal{G} \ltimes \mathcal{O} \curvearrowright \mathcal{N}^*_{\mathcal{O}}$. Identifying $\mathcal{N}^*_{O} \cong (T\mathcal{O})^\circ$, for any $\alpha \in (T\mathcal{O})^\circ$, the action is given by:

$$\mathcal{N}^*_{\mathcal{O}}(\theta)_{(g,e)}(\alpha) = \alpha \circ \mathcal{N}_{\mathcal{O}}(\theta)_{(g^{-1},ge)}.$$

To explicitly express the normal and conormal representations, let A be the Lie algebroid of $\mathcal{G}$ and let σ be a right-splitting of the following exact sequence.

$$0 \longrightarrow t^*A \longrightarrow T\mathcal{G} \underset{\sigma}{\overset{s_*}{\rightleftarrows}} s^*TM \longrightarrow 0. \tag{2.1}$$

By right-splitting of the exact sequence, we mean that the bundle map $\sigma : s^*TM \to T\mathcal{G}$ satisfies $s_* \circ \sigma = id_{s^*TM}$. Notice such σ always exists and can be analytic. Given an analytic Riemannian metric on $\mathcal{G}$, let $(t^*A)^\perp$ be the orthogonal complement of $t^*A \subset T\mathcal{G}$. Restricting to the orthogonal complement, $s_*|_{(t^*A)^\perp} : (t^*A)^\perp \to s^*TM$ is an isomorphism. Let $\sigma : s^*TM \to T\mathcal{G}$ be the inverse of the isomorphism. We have σ is an analytic right-splitting.

Given a vector $v \in T_eE$, which represents a normal vector $[v] \in \mathcal{N}_{\mathcal{O},e}$, the normal representation can be written as:

$$\mathcal{N}_{\mathcal{O}}(\theta)_{(g,e)}([v]) = [d\theta(\sigma_g(d_eq(v), v)] \tag{2.2}$$

For $\alpha \in (T\mathcal{O})^\circ_e$ and $v \in T_{ge}E$, the conormal representation is:

$$\langle \mathcal{N}^*_{\mathcal{O}}(\theta)_{(g,e)}(\alpha), v\rangle = \langle \alpha, d\theta(\sigma_{g^{-1}}(d_{ge}q(v), v)\rangle \tag{2.3}$$

The above expressions does not depend on the choice of our splitting σ.

Given an analytic metric η on TE, we can identify the normal bundle $\mathcal{N}_{\mathcal{O}} \cong (T\mathcal{O})^\perp$. η induces an analytic metric η^* on the cotangent bundle T^*E. Restricting the metric η on $(T\mathcal{O})^\perp \cong \mathcal{N}^*_{O}$ and η^* to $(T\mathcal{O})^\circ \cong \mathcal{N}^*_{O}$, we get metrics on $\mathcal{N}_{\mathcal{O}}$ and $\mathcal{N}^*_{\mathcal{O}}$ respectively.

Definition 2.3.1. Let Let $\theta : \mathcal{G} \curvearrowright E$ be a groupoid action. We say that a metric η on E is **transversely θ-invariant** if for any orbit $\mathcal{O}$, the normal representation $\mathcal{N}_{\mathcal{O}}(\theta) : \mathcal{G} \ltimes \mathcal{O} \curvearrowright \mathcal{N}_{\mathcal{O}}$ acts by isometry, or equivalently, if the conormal representation $\mathcal{N}^*_{\mathcal{O}}(\theta) : \mathcal{G} \ltimes \mathcal{O} \curvearrowright \mathcal{N}^*_{\mathcal{O}}$ acts by isometry.

Lemma 2.3.2 ([6]). *Let $\theta : \mathcal{G} \curvearrowright E$ be free and proper groupoid action with the quotient map $\pi : E \to E/\mathcal{G}$. The metric η on E is transversely θ-invariant if and only if η is π-transverse. In that case, $E/\mathcal{G}$ has a push-forward metric $\pi_*\eta$, which makes π a Riemanina submersion.*

Definition 2.3.3. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid E be a manifold with a map $q : E \to M$. Denote $\mathcal{G} \times_M E = \{(g, e) \in \mathcal{G} \times E \mid s(g) = q(e)\}$. We say that $\theta : \mathcal{G} \times_M E \to E$ is a **quasi-action** if $q(\theta(g, e)) = t(g)$, denoted by $\theta : \mathcal{G} \tilde{\curvearrowright} E$.

For any arrow $g : y \leftarrow x$, the quasi-action θ gives a map $\theta_g : E_x \to E_y$ between the fibers of the moment map q. A groupoid action $\theta : \mathcal{G} \curvearrowright E$ may not be lifted to a groupoid action on the tangent bundle TE, but we can always lift it to have a quasi-action.

With the splitting map, we can list the action θ to the tangent space of E.

Definition 2.3.4. Let $\sigma : s^*TM \to T\mathcal{G}$ be a splitting of the exact sequence 2.1 and $\theta : \mathcal{G} \curvearrowright E$ be a groupoid action with moment map $q : E \to M$. The **tangent lift** of θ is the groupoid quasi-action $T_\sigma\theta : \mathcal{G} \ltimes E \tilde{\curvearrowright} TE$ with the moment map the projection $p : TE \to E$ defined by:

$$T_\sigma\theta_{(g,e)}(v) = d\theta(\sigma_g(d_e q(v), v) \tag{2.4}$$

We can also define the cotangent lift $T^*_\sigma\theta : \mathcal{G} \ltimes E \tilde{\curvearrowright} T^*E$ by:

$$\langle T^*_\sigma\theta_{(g,e)}(\alpha), v\rangle = \langle \alpha, T_\sigma\theta_{(g^{-1},ge)}(v)\rangle \tag{2.5}$$

In the analytic setting, we have the tangent lift $T_\sigma\theta$ and the cotangent lift $T^*_\sigma\theta$ are analytic maps.

The following theorem generalizes [6] on the analytic context. When restricting to the conormal bundle, the cotangent lift is independent of the choice of the splitting.

Proposition 2.3.5. *Given a (analytic) right-splitting of the exact sequence 2.1 and a groupoid action $\theta : \mathcal{G} \curvearrowright E$ with moment map $q : E \to M$, let $\mathcal{O}$ be an orbit of the action. $(T\mathcal{O})^\circ$ is invariant under the cotangent lift quasi-action $T^*_\sigma\theta : \mathcal{G} \ltimes E \tilde{\curvearrowright} T^*E$. Moreover, when restricting to the conormal bundle $(T\mathcal{O})^\circ$, the cotangent lift and the conormal representation agree with each other, namely, $T^*_\sigma\theta \mid_{(T\mathcal{O})^\circ} = \mathcal{N}^*_{\mathcal{O}}(\theta)$.*

Proof. When restricting to the orbit $\mathcal{O}$,

$$\theta \mid_{\mathcal{O}} : \mathcal{G} \times_M \mathcal{O} \to \mathcal{O}.$$

Taking differentiation on both sides, we have

$$d\theta \mid_{\mathcal{O}} : T\mathcal{G} \times_{TM} T\mathcal{O} \to T\mathcal{O}$$

For any $v \in T_e\mathcal{O}$, $T_\sigma\theta_{(g,e)}(v) = d\theta(\sigma_g(d_e q(v), v) \in T_{ge}\mathcal{O}$. Hence $T\mathcal{O}$ is invariant under the quasi-action $T_\sigma\theta$. By equation 2.5, $(T\mathcal{O})^\circ$ is invariant under the cotangent quasi-action $T^*_\sigma\theta$. When restricting to $(T\mathcal{O})^\circ$, the conormal representation 2.3 and cotangent lift 2.5 gives the same formula. ☐

2.4 Analytic Haar Density

It is well known that every proper groupoid admits a Haar density (cf. [20], [21]), which can be used to average functions or sections of vector bundles. We will extend this result to the analytic setting.

For the reminder of this chapter, we always assume that that the groupoid $\mathcal{G}$ is analytic and s-proper. In this section, we will demonstrate the existence of a Haar density that varies analytically across the source fibers.

Let V be a vector space of dimension n. A density on V is a function

$$\mu : \underbrace{V \times \cdots \times V}_{n \text{ copies}} \to \mathbb{R}$$

such that $\mu(Tv_1, \cdots, Tv_n) = |det(T)|\mu(v_1, \ldots, v_n)$ for any linear map T. $\mathcal{D}(V)$ denote the set of all densities on V. We say that μ is positive if $\mu(v_1, \cdots, v_n) > 0$ for any linearly independent vectors $v_1, \cdots, v_n$. Let E be a vector bundle on M. The set of all densities on each fiber $\mathcal{D}(E) = \bigsqcup_{p \in M} \mathcal{D}(E_p)$ gives a line bundle over M, called density bundle of E. Let $e_1, \cdots, e_n$ be a local basis of E, then $|e_1 \wedge \cdots \wedge e_n|$ is local basis of $\mathcal{D}(E)$. A smooth section μ of $\mathcal{D}(E)$ is called a density of E. We say that μ is positive if it is positive at every point. In this paper, we only consider positive densities. When the bundle E is analytic, the line bundle $\mathcal{D}(E)$ is analytic. We say that a density μ is **analytic** if it is an analytic section of $\mathcal{D}(E)$. We denote $\mathcal{D}(M) = \mathcal{D}(T^*M)$ and say μ is a density on M if it is a density on its cotangent bundle. When the manifold M is oriented, a volume form ω induces a positive density $|\omega|$ by requiring $\omega_p = |\omega_p|$ for every p in M. Not every manifold is oriented and admits a volume form, but every manifold admits a density and we can further assume it to be analytic when M is analytic.

The following lemma is a well-known criterion for analytic functions [22].

Lemma 2.4.1. *A smooth function $f : U \to \mathbb{R}$ is an analytic function on an open set $U \subset \mathbb{R}^n$ if and only if for any compact set $K \subset U$, there exists a constant $C > 0$ depending on K such that for all $\alpha \in \mathbb{Z}_{\leq 0}$ and all $x \in K$,*

$$\left| \frac{\partial^\alpha f}{\partial x^\alpha} \right| \leq C^{|\alpha|+1} \alpha!$$

The above lemma immediately yeilds the following.

Lemma 2.4.2. *If f is a positive analytic function, then $\sqrt{f}$ is also analytic.*

By the result of Morrey and Grauert, every analytic bundle E admits an analytic Riemannian metric $\{g_{\alpha\beta}\}$. We can define a positive analytic density by the local formula

$$\mu = \sqrt{det(g_{\alpha\beta})} |e_1 \wedge \cdots \wedge e_n|.$$

According to the above lemma, μ is analytic. Hence we have the following theorem.

Theorem 2.4.3. *Every analytic vector bundle has a positive analytic density.*

Lemma 2.4.4. *Let $s : E \to M$ be an proper analytic submersion between two analytic manifolds. Given an analytic measure on the vector bundle $(Ker(ds))^* \to E$, it induces a family of analytic measure μ^x on each fiber $s^{-1}(x)$. For any analytic function f on E, we have the function*

$$x \mapsto \int_{s^{-1}(x)} f(g) \mu^x(g)$$

is analytic on M.

Proof. By analytic version of Ehresmann theorem, s is a locally trivial fibration. Choosing some open neighborhood U of x, we can just assume $E|_U = U \times s^{-1}(x)$ and $s = \mathrm{pr}_U$. Let $\{V_i\}$ be a finite open cover of $s^{-1}(x)$ and $\{\rho_i\}$ be a partition of unity subordinate to it.

$$\int_{s^{-1}(x)} f(g)\mu^x(g) = \sum_i \int_{V_i} \rho_i f(g)\mu^x(g).$$

It suffices to prove that $g_i(x) = \int_{s^{-1}(x)\cap V_i} \rho_i f(g)\mu^x(g)$ is analytic. In local coordinates, write $g = (x, y)$ and $\mu^x(g) = \lambda(x, y)dy$, we have

$$g_i(x) = \int_{V_i} \rho_i(y) f(x, y)\lambda(x, y)dy.$$

Since f and λ are analytic, so is $f\lambda$. By Lemma 2.4.1, for any compact set $K \subset U \times V_i$, there exists constant C such that $\left|\frac{\partial^\alpha (f\lambda)}{\partial x^\alpha}(x, y)\right| \leq C^{|\alpha|+1}\alpha!$. Therefore

$$\left|\frac{\partial^\alpha g_i}{\partial x^\alpha}(x)\right| \leq \int_{\mathrm{supp}(\rho_i)} \left|\frac{\partial^\alpha (f\lambda)}{\partial x^\alpha}(x, y)\right| dy \leq |\mathrm{supp}(\rho_i)| C^{|\alpha|+1}\alpha!.$$

□

Definition 2.4.5. We say that a family of positive density $\{\mu^x\}_{x\in M}$ on s-fibers of $\mathcal{G}$ is an **analytic normalized Haar density** if the following condition are satisfied:

1. **(Analyticity)** The function
$$x \mapsto \int_{s^{-1}(x)} f(g)\mu^x(g)$$
is analytic on M for any analytic function $f \in C^\omega(\mathcal{G})$.
2. **(Right-invariance)** For any arrow $y \xleftarrow{h} x$ and $f \in C^\omega(s^{-1}(x))$, we have
$$\int_{s^{-1}(y)} f(gh)\mu^y(g) = \int_{s^{-1}(x)} f(g)\mu^x(g).$$
3. **(Normalization)** The support of μ^x is compact and for all $x \in M$
$$\int_{s^{-1}(x)} \mu^x(g) = 1.$$

Let $\mathcal{G}$ be an s-proper analytic groupoid and A be its algebroid. Let μ be a positive analytic density on the dual bundle of Lie algebra A^*. Define the map $\phi : Ker(ds) \to \mathcal{G}$ by $\phi(g, v) = (t(g), \mathrm{d}R_{g^{-1}}(v))$. We have the following commutative diagram

$$\begin{array}{ccc} Ker(ds) & \xrightarrow{\phi} & A \\ \downarrow & & \downarrow \\ \mathcal{G} & \xrightarrow{t} & M. \end{array}$$

Pulling back an analytic density μ through ϕ gives an analytic density $\tilde{\mu}$ on $(Ker(ds))^*$, which induces a family of analytic densities $\{\tilde{\mu}^x\}$ on each fiber $s^{-1}(x)$. According to Lemma 2.4.4, $\{\tilde{\mu}^x\}$ varies analytically across the fiber. It is easy to check that for any arrow $y \xleftarrow{h} x$, $\tilde{\mu}^y = R_h^*\tilde{\mu}^x$. So $\{\tilde{\mu}^x\}$ is right-invariant. We define the following function:

$$c(x) = \int_{s^{-1}(x)} \tilde{\mu}^x(g).$$

By Lemma 2.4.4, $c(x)$ is analytic. Replacing $\tilde{\mu}^x$ by $\frac{1}{c(x)}\tilde{\mu}^x$, we have an analytic normalized density and also call it $\tilde{\mu}$. Now we have the following theorem.

Theorem 2.4.6. *Every s-proper groupoid $\mathcal{G}$ admits an analytic normalized Haar density.*

2.5 Analytic Metric on Groupoids

Given an analytic right-splitting σ, we can lift the groupoid action θ to obtain cotangent quasi-action $T^*_\sigma\theta$. For simplicity, we denote $ge = \theta_g(e)$ and $g\alpha = T^*_\sigma\theta_{(g,e)}(\alpha)$. Let μ be an analytic normalized Haar density on $\mathcal{G}$, which always exists by theorem 2.4.6. Given an analytic metric η on E, we can take average of the metric by the following method defined on [6].

Definition 2.5.1. Let $\mathcal{G}$ be an s-proper analytic groupoid and $\theta : \mathcal{G} \curvearrowright E$ be a groupoid action and η be an analytic metric on E. The **cotangent average** $\tilde{\eta} \in \Gamma(E, S^2(T^*E))$ is an analytic metric on E obtained by averaging η^*, given by

$$(\tilde{\eta})^*_e(\alpha, \beta) := \int_{G(-,x)} \eta^*_{ge}(g\alpha, g\beta)\, \mu^x(g),$$

where $x = q(e)$, $y \xleftarrow{g} x$, and $\alpha, \beta \in T^*_e E$.

Theorem 2.5.2. *[6] Given groupoid action $\theta : \mathcal{G} \curvearrowright E$ and a metric η, let $\tilde{\eta}$ be the cotangent average metric , then $\tilde{\eta}$ is transversally θ-invariant.*

Proof. It suffices to show that for any orbit $\mathcal{O} \subset E$, the conormal representation $\mathcal{N}^*_{\mathcal{O}}(\theta) : \mathcal{G} \curvearrowright (T\mathcal{O})^\circ$ acts by isometry respect to the metric $\tilde{\eta}^*|_{(T\mathcal{O})^\circ \times T\mathcal{O})^\circ}$. According to theorem 2.3.5, for any $\alpha \in (T\mathcal{O})^\circ$,

$$\mathcal{N}^*_{\mathcal{O}}(\theta)_{(g,e)}(\alpha) = g\alpha$$

$$h(g\alpha) = \mathcal{N}^*_{\mathcal{O}}(\theta)_{(h,ge)}\mathcal{N}^*_{\mathcal{O}}(\theta)_{(g,e)}(\alpha) = \mathcal{N}^*_{\mathcal{O}}(\theta)_{(hg,e)}(\alpha) = (hg)\alpha$$

$$\begin{aligned}
&(\tilde{\eta})^*_{ge}(\mathcal{N}^*_{\mathcal{O}}(\theta)_{(g,e)}(\alpha), \mathcal{N}^*_{\mathcal{O}}(\theta)_{(g,e)}(\beta))\\
=&(\tilde{\eta})^*_{ge}(g\alpha, g\beta)\\
=&\int_{G(-,y)} \eta^*_{h(ge)}(h(g\alpha), h(g\beta))\, \mu^y(h)\\
=&\int_{G(-,y)} \eta^*_{(hg)e}((hg)\alpha, (hg)\beta)\, \mu^y(h)\\
=&\int_{G(-,y)} \eta^*_{he}(h\alpha, h\beta)\, \mu^y(h)\\
=&(\tilde{\eta})^*_e(\alpha, \beta).
\end{aligned}$$

□

Let $\mathcal{G}^{[k]}$ be k-tuple of arrows in $\mathcal{G}$ with the same target. Let $\mathcal{G}^{(k)}$ be the k-tuple of composable arrows. $\mathcal{G}$ acts on $\mathcal{G}^{[k]}$ by left multiplication $g(h_1, \cdots, h_k) = (gh_1, \cdots, gh_k)$. The action is free and proper, with quotient map

$$\pi^{(k)} : \mathcal{G}^{[k]} \to \mathcal{G}^{(k)} \cong \mathcal{G}^{[k]}/\mathcal{G}$$

where $\pi^{(k)}(g_1, \cdots, g_k) = (g_1^{-1}g_2, \cdots, g_{k-1}^{-1}g_k)$.

Definition 2.5.3. We say that an analytic Riemannian metric $\eta^{(2)}$ on $\mathcal{G}^{(2)}$ is an **analytic 2-metric** if $\eta^{(2)}$ is π_1, π_2, m-transverse and $(\pi_1)_*\eta^{(2)} = (\pi_2)_*\eta^{(2)} = m_*\eta^{(2)}$.

Theorem 2.5.4. *Every Hausdorff analytic s-proper groupoid admits an analytic 2-metric* $\eta^{(2)}$*.*

Proof. Let η be an analytic t-transverse metric on $\mathcal{G}$, which induces the pull-back metric $\eta^{[k]}$ on $\mathcal{G}^{[k]}$. By taking the cotangent average, we obtain $\tilde{\eta}^{[k]}$ on $\mathcal{G}^{[k]}$, which is transversely invariant for left multiplication. $\tilde{\eta}^{[k]}$ is transverse for the quotient map $\pi^{(k)} : \mathcal{G}^{[k]} \to \mathcal{G}^{(k)}$, hence induces the push-forward metric $\tilde{\eta}^{(k)} = \pi_*^{(k)}\tilde{\eta}^{[k]}$ on $\mathcal{G}^{(k)}$ (cf. Lemma 2.3.2). We claim that $\tilde{\eta}^{(2)}$ is an analytic 2-metric on $\mathcal{G}^{(2)}$.

We have the following commutative diagram:

$$\begin{array}{ccccc}
\cdots\mathcal{G}^{[3]} & \rightrightarrows & \mathcal{G}^{[2]} & \rightrightarrows & \mathcal{G} \\
{\scriptstyle \pi^{(2)}}\downarrow & & {\scriptstyle \pi^{(1)}}\downarrow & & \downarrow{\scriptstyle s} \\
\cdots\mathcal{G}^{(2)} & \xrightarrow[\pi_2,\ m]{\pi_1} & \mathcal{G} & \underset{s}{\overset{t}{\rightrightarrows}} & M
\end{array}$$

where all the arrows are Riemannian submersions. Since the graph is commutative, we have $(\pi_1)_*\tilde{\eta}^{(2)} = (\pi_2)_*\tilde{\eta}^{(2)} = m_*\tilde{\eta}^{(2)} = \pi_*^{(1)}\tilde{\eta}^{[2]}$. ☐

Theorem 2.5.5 ([6])**.** *Let* $\mathcal{G}$ *be a proper groupoid and* S *be a saturated groupoid. Let* $\eta^{(2)}$ *be a 2-metric on* $\mathcal{G}^{(2)}$*, which induces a metric* $\eta^{(1)} = m_*\eta^{(2)}$ *on* $\mathcal{G}$*. Then the exponential map of* $\eta^{(1)}$ *gives a Linearization of* $\mathcal{G}$ *at* S*. Namely, there exist an open neighborhoods of* S*,* $U \subset M$ *and* $V \subset \mathcal{N}_S$*, such that the following map is an isomorphism:*

$$\begin{array}{ccc}
\mathcal{N}_{\mathcal{G}_S}|_V & \xrightarrow{\exp} & \mathcal{G}|_U \\
\downarrow & & \downarrow \\
V & \xrightarrow{\exp} & U
\end{array}$$

If we further assume that $\mathcal{G}$ *is s-proper, then* U *and* V *can be chosen to be invariant so that* $\mathcal{G}$ *is invariant linearizable.*

If we assume that the 2-metric is analytic, then the induced metric is also analytic. Hence we have the exponential map gives an analytic groupoid isomorphism.

Theorem 2.5.6. *Let* $\mathcal{G}$ *be an analytic s-proper groupoid and* S *be a saturated groupoid. Then* $\mathcal{G}$ *can be analytically invariant linearizable at* S*.*

Chapter 3

Complexification of Algebroids and Groupoids

3.1 Complexification of Manifolds

Definition 3.1.1. Let N be a complex manifold and $M \subset N$ be an embedded analytic submanifold of N, we say that M is a **totally real submanifold** of N if

$$T_xM \oplus J(T_xM) = T_xN$$

for any $x \in M$

Remark. In this case, we also call $M_{\mathbb{C}}$ a **complexification** of M.

We will see that the embedding $M \to M_{\mathbb{C}}$ looks like the standard embedding $\mathbb{R}^n \to \mathbb{C}^n$ on local charts.

Lemma 3.1.2. *Let N be a complex manifold and $M \subset N$ be an embedded analytic submanifold of N. The following are equivalent:*

1. *M is a totally real submanifold of N*
2. *For any $x \in M$, there is a coordinate chart $U \subset N$ containing x, such that*

$$M \cap U = \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid x_i \in \mathbb{R} \quad i = 1, \dots, n\}$$

3. *There is an open neighborhood $U \subset N$ containing M and an antiholomorphic involution $\sigma : U \to U$ with $Fix(\sigma) = M$*

Proof. Let M be a totally real submanifold of N. Let $x \in N$. In local coordinate, then analytic embedding $i : M \to N$ can be represented by a embedding $f : \mathbb{R}^n \to \mathbb{C}^n$. Since f is analytic, we can extend f to a holomorphic map $\tilde{f} : V \to \mathbb{C}^n$, where V is an open set of $\mathbb{C}^n$ containing $\mathbb{R}^n$. Since M is totally real, we have $rank_x d\tilde{f} = n$. $\tilde{f}$ gives a holomorphic local chart of N near x, which satisfies condition 2.

At any point $x \in M$, let $U \subset N$ be a local chart satisfying condition 2, define $\sigma_U : U \to U$ by $\sigma(z_1, \cdots, z_n) = (\bar{z_1}, \cdots, \bar{z_n})$. We have the fixed point of $Fix(\sigma_U) = U \cap M$. Since $\sigma_U|_{U\cap V\cap N} = \sigma_V|_{U\cap V\cap N}$

for any such charts $U, V \subset N$. By analytic continuation, $\sigma_U|_{U\cap V} = \sigma_V|_{U\cap V}$. We can glue σ_U to get an antiholomorphic involution $\sigma : W \to W$, where $W \subset N$ is an open neighborhood of M in N.

Assuming *3*, let $x \in M$. Since $(d_x\sigma)^2 = id$, T_xN decomposes to $+1$ and -1 eigenspace of $d_x\sigma$. The $+1$ eigenspace is T_xM. Since $d\sigma \circ J = -J \circ d\sigma$, the -1 eigenspace is given by $J(T_xM)$. Hence we have $T_xM \oplus J(T_xM) = T_xN$ for any $x \in M$. □

Example 3.1.3. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g} \otimes \mathbb{C}$ be its complexification. Let G and $G_{\mathbb{C}}$ be the simply connected Lie groups integrating $\mathfrak{g}$ and $\mathfrak{g} \otimes \mathbb{C}$ respectively. Then $G \to G_{\mathbb{C}}$ is a totally real submanifold.*

Example 3.1.4. *Assume that $X \subset \mathbb{R}^n$ is an smooth affine variety, which is defined by polynomials $f_1, \cdots, f_k \in \mathbb{R}[x_1, \cdots, x_n]$. We can view $f_1, \cdots, f_k$ as polynomials in $\mathbb{C}[x_1, \cdots, x_n]$. Hence they define a smooth affine variety $X_{\mathbb{C}} \subset \mathbb{C}^n$. $X \to X_{\mathbb{C}}$ is a totally real submanifold.*

The following theorem shows that every analytic manifold can be realized as a totally real submanifold.

Theorem 3.1.5. *[7][23] Any analytic manifold M can be analytically embeded to an complex manifold $M_{\mathbb{C}}$ as a totally real submanifold.*

In this case, we say that $M_{\mathbb{C}}$ is a complexification of M, and M is a real form of $M_{\mathbb{C}}$. If N and N' are two complexification of M, then N and N' are isomorphic in a neighborhood of M.

Lemma 3.1.6. *Let M be a totally real submanifold of $M_{\mathbb{C}}$. Then any analytic function on M extends to a holomorphic function on a neighborhood of M in $M_{\mathbb{C}}$*

Proof. For any point $x \in M$, take a local normal coordinate around x as in Lemma 3.1.2. Any analytic function f can be written as its Taylor expansion

$$f = \sum_{i_1, i_2, \cdots, i_k} a_{i_1, i_2, \cdots, i_k} x^{i_1} x^{i_2} \cdots x^{i_k}$$

We define the holomorphic extension of f to be

$$\tilde{f} = \sum_{i_1, i_2, \cdots, i_k} a_{i_1, i_2, \cdots, i_k} z^{i_1} z^{i_2} \cdots z^{i_k}$$

By analytic continuation, the extension is unique around x. Glue all those $\tilde{f}$, we get a holomorphic function around M extending f. □

Proposition 3.1.7. *Let M be a totally real submanifold of $M_{\mathbb{C}}$. Let N be a complex manifold and $\phi : M \to N$ be a real analytic map. Then there exists neighborhood $U \subset M_{\mathbb{C}}$ containing M and a holomorphic map $\tilde{\phi} : U \to N$ such that $\tilde{\phi}|_M = \phi$. Moreover, we have $rank_{\mathbb{C}} d_x\tilde{\phi} = rank_{\mathbb{R}} d_x\phi$ for any $x \in X$.*

Proof. Similar to 3.1.6, we can always extend ϕ by extending its Taylor expansion in each coordinate. We have $d_x\tilde{\phi}$ is the complexification of $d_x\phi$ for all x. Hence they have the same rank. □

We also have the following proposition by analytic continuation.

Proposition 3.1.8. *Suppose that a complex manifold $M_{\mathbb{C}}$ is connected. Let M be a totally real submanifold of $M_{\mathbb{C}}$ and f be an (complex or real) analytic function on $M_{\mathbb{C}}$. If f vanishes on M then it must vanish on $M_{\mathbb{C}}$.*

Let M be a smooth manifold. By Whitney's theorem [23], we can always find an analytic coordinate chart for M which is compatible with the given smooth structure. Then we can embed M be a complex manifold $M_{\mathbb{C}}$ as a totally real submanifold. Hence any smooth manifold M admits a complexification $M_{\mathbb{C}}$.

Let (M, π) be a Poisson manifold. We can always give M an analytic structure. But the problem is, in general, π may not be analytic on coordinate charts. When π is nondegenerate, we can always find an analytic structure on M, such that π is analytic.

Theorem 3.1.9. *[24] Let (M, ω) be a symplectic structure. There exists an analytic structure on M compatible with the smooth structure, such that the symplectic form ω is analytic on coordinate charts.*

3.2 Complexification of Analytic Algebroids

Definition 3.2.1. Let $(A, \rho, [\cdot,\cdot])$ be an analytic Lie algebroid over an analytic manifold M. We say that a holomorphic algeborid $(A_{\mathbb{C}}, \rho_{\mathbb{C}}, [\cdot,\cdot]_{\mathbb{C}})$ over $M_{\mathbb{C}}$ is a **complexification** of A or A is a **real form** of $A_{\mathbb{C}}$ if the following holds:

1. M is a totally real submanifold of $M_{\mathbb{C}}$,
2. $A_{\mathbb{C}}|_M = A \oplus J(A)$, where J is the complex structure on $A_{\mathbb{C}}$,
3. We have the following commutative diagram:

$$\begin{array}{ccc} A & \longrightarrow & A_{\mathbb{C}} \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho_{\mathbb{C}}} \\ TM & \longrightarrow & TM_{\mathbb{C}}, \end{array} \tag{3.1}$$

4. For any analytic section α, β of A and any point $x \in M$, we have

$$[\alpha, \beta](x) = [\tilde{\alpha}, \tilde{\beta}]_{\mathbb{C}}(x)$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are holomorphic sections extending α and β near x.

Example 3.2.2. *Every Lie algebra $\mathfrak{g}$ can be viewed as a Lie algebroid over a point. Its complexification is just the usual complexification of Lie algebra $\mathfrak{g} \otimes \mathbb{C}$.*

Example 3.2.3. *Let $TM \to M$ be an analytic algebroid with identity anchor map. Let $M_{\mathbb{C}}$ be a complexification of M. Then $TM_{\mathbb{C}} \to M_{\mathbb{C}}$ with the identity anchor map is a complexification of the algebroid $TM \to M$.*

Example 3.2.4. *Let $(\mathbb{R}^n, \pi)$ be an analytic Poisson manifold. Write*

$$\pi = \sum_{i,j=1}^{n} \pi_{ij}(x) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$$

Embed $\mathbb{R}^n$ to $\mathbb{C}^n$ by standard embedding. Since π_{ij} is analytic, we can extend π_{ij} to get holomorphic functions $\tilde{\pi}_{ij}$ near an open neighborhood $U \subset \mathbb{C}^n$ of $\mathbb{R}^n$. We have a holomorphic Poisson manifold on U,

$$\pi_{\mathbb{C}} = \sum_{i,j=1}^{n} \tilde{\pi}_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}.$$

Then $(T^*U, \pi_{\mathbb{C}}^{\#}, [\cdot,\cdot]_{\pi_{\mathbb{C}}})$ *is complexification of* $(T^*\mathbb{R}^n, \pi^{\#}, [\cdot,\cdot]_{\pi})$.

Proposition 3.2.5. *Every analytic algebroid* $A \to M$ *admits a complexification.*

Proof. According to [7], [25], [26], there exists a holomorphic vector bundle $A_{\mathbb{C}} \to M_{\mathbb{C}}$ such that M is a totally real submanifold of $M_{\mathbb{C}}$ and $A_{\mathbb{C}}|_M = A \oplus J(A)$. Locally we can choose trivialization such that the embedding $A \to A_{\mathbb{C}}$ looks like the standard embedding of $\mathbb{R}^n$ to $\mathbb{C}^n$:

$$\begin{array}{ccc} A|_{U\cap M} & \xrightarrow{\ i\ } & A_{\mathbb{C}}|_U \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\phi} \\ \mathbb{R}^n \times \mathbb{R}^k & \xrightarrow{\ j\ } & \mathbb{C}^n \times \mathbb{C}^k \end{array}$$

Locally ρ can be expressed as a function $\mathbb{R}^n \to M_{n\times k}(\mathbb{R})$, which can be extended to a holomorphic function $U \to M_{n\times k}(\mathbb{C})$, where $U \subset \mathbb{C}^n$ is an open neighborhood of $\mathbb{C}^n$. Possibly by shrinking $A_{\mathbb{C}}$ to an open neighborhood of M, we can glue those extensions to get a holomorphic anchor map $\rho_{\mathbb{C}} : A_{\mathbb{C}} \to TM_{\mathbb{C}}$. Similarly, we can define the bracket $[\cdot,\cdot]_{\mathbb{C}}$ by holomorphically extending $[\cdot,\cdot]$. □

Remark. The complexification of an algebroid A is not unique, but is unique near A, meaning that if A_1 and A_2 are two complexification of A, then they are isomorphic when restricted to an open neighborhood of M.

Proposition 3.2.6. *Let* $x \in M$. *Let* $\mathfrak{g}_x$ *be the isotropy Lie algebra of* A *at* x *and* $\mathfrak{g}_{\mathbb{C},x}$ *be the isotropy Lie algebra of* $A_{\mathbb{C}}$ *at* x, *we have*

$$\mathfrak{g}_{\mathbb{C},x} \cong \mathfrak{g}_x \otimes \mathbb{C} \text{ as Lie algebras.}$$

Proof. According to the Definition 3.2.1, $A_{\mathbb{C},x} \cong A_x \oplus J(A_x)$ for all $x \in M$. Let $w = u + J(v) \in \mathfrak{g}_{\mathbb{C},x}$, where $u, v \in A_x$. Then $\rho_{\mathbb{C}}(w) = \rho_{\mathbb{C}}(u) + \rho_{\mathbb{C}}(Jv) = \rho(u) + J \circ \rho(v) = 0$. We have $u \in Ker(\rho)$ and $v \in J(Ker(\rho))$. Hence $\mathfrak{g}_{\mathbb{C},x} \cong \mathfrak{g}_x \otimes \mathbb{C}$ as vector spaces. They are isomorphic as Lie algebra since the bracket in $\mathfrak{g}_{\mathbb{C},x}$ extends the bracket in $\mathfrak{g}_x$ by $\mathbb{C}$-linearity. □

Proposition 3.2.7. *If* A *is regular then there is an open neighborhood* $U \subset M_{\mathbb{C}}$ *containing* M *such that* $A_{\mathbb{C}}|_U$ *is regular. Moreover, in this case, we have* $\operatorname{rank}_{\mathbb{R}} \rho = \operatorname{rank}_{\mathbb{C}} \rho_{\mathbb{C}}|_U$

Proof. By choosing local trivialization, locally ρ can be represented by a function $\mathbb{R}^n \to M_{n\times k}(\mathbb{R}), x \mapsto \rho_{ij}(x)$. Assume that $\operatorname{rank}_{\mathbb{R}} \rho = l$. There exists a non-zero $l \times l$ minor of the matrix (ρ_{ij}). And every $(l+1) \times (l+1)$ minor is zero. Since $\rho_{\mathbb{C}}$ can be represented by a $n \times k$ matrix $(\tilde{\rho}_{ij})$, where $\tilde{\rho}_{ij}$ is the holomorphic extension of ρ_{ij}. By analytic continuation, we have the same $l \times l$ minor of $(\tilde{\rho}_{ij})$ is non-zero in an open neighborhood and every every $(l+1) \times (l+1)$ minor is zero. □

Definition 3.2.8. Let $\mathcal{G}_{\mathbb{C}} \rightrightarrows M_{\mathbb{C}}$ be a holomorphic groupoid and $\mathcal{G} \rightrightarrows M$ be an analytic subgroupoid of $\mathcal{G}_{\mathbb{C}}$. We say that $\mathcal{G}$ is a **real form** of $\mathcal{G}_{\mathbb{C}}$ if it is a totally real submanifold of $\mathcal{G}_{\mathbb{C}}$. In this case, we also say that $\mathcal{G}_{\mathbb{C}}$ is a **complexification** of $\mathcal{G}$.

Remark. As in the case of Lie algebroids, the complexification of a Lie groupoid $\mathcal{G}$ is not unique. However, the groupoid structure on $\mathcal{G}_{\mathbb{C}}$ is uniquely determined in a neighborhood of $\mathcal{G}$. The structure maps $s_{\mathbb{C}}, t_{\mathbb{C}}, m_{\mathbb{C}}$ of the complexified groupoid $\mathcal{G}_{\mathbb{C}}$ are holomorphic extensions of the analytic maps s, t, m, and by the uniqueness of analytic continuation, they are uniquely determined near $\mathcal{G}$.

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \mathcal{G}_{\mathbb{C}} \\ {\scriptstyle t}\downdownarrows{\scriptstyle s} & & {\scriptstyle t_{\mathbb{C}}}\downdownarrows{\scriptstyle s_{\mathbb{C}}} \\ M & \longrightarrow & M_{\mathbb{C}}, \end{array}$$

Now we have a notion of the real form of holomorphic algebroids and the real form of holomorphic groupoids. They are related in the following way.

Proposition 3.2.9. *Let $\mathcal{G}_{\mathbb{C}}$ be a holomorphic groupoid over a complex manifold $M_{\mathbb{C}}$. Let $\mathcal{G}$ be a real form of $\mathcal{G}_{\mathbb{C}}$. Then $Lie(\mathcal{G})$ is a real form of $Lie(\mathcal{G}_{\mathbb{C}})$.*

Proof. We have the following commutative diagram:

$$\begin{array}{ccccc} M & \xrightarrow{\iota} & \mathcal{G} & \overset{s}{\underset{t}{\rightrightarrows}} & M_{\mathbb{C}} \\ \downarrow & & \downarrow & & \downarrow \\ M_{\mathbb{C}} & \xrightarrow[\iota_{\mathbb{C}}]{} & \mathcal{G}_{\mathbb{C}} & \overset{s_{\mathbb{C}}}{\underset{t_{\mathbb{C}}}{\rightrightarrows}} & M_{\mathbb{C}} \end{array}$$

For all $x \in M$, we have

$$\begin{aligned} T_x M_{\mathbb{C}} &= ds_{\mathbb{C}} d\iota_{\mathbb{C}}(T_x M_{\mathbb{C}}) \\ &\subset ds_{\mathbb{C}}(T_x \mathcal{G}_{\mathbb{C}}) \\ &= ds_{\mathbb{C}}(T_x \mathcal{G} \oplus J(T_x \mathcal{G})) \\ &= ds_{\mathbb{C}}(T_x \mathcal{G}) \oplus J(ds_{\mathbb{C}}(T_x \mathcal{G})) \\ &= ds(T_x \mathcal{G}) \oplus J(ds(T_x \mathcal{G})) \subset T_x M \oplus J(T_x M) \end{aligned}$$

$$\begin{aligned} T_x M \oplus J(T_x M) &= dsd\iota(T_x M) \oplus dsd\iota(J(T_x M)) \\ &= ds_{\mathbb{C}} d\iota_{\mathbb{C}}(T_x M) \oplus ds_{\mathbb{C}} d\iota_{\mathbb{C}}(J(T_x M)) \\ &= ds_{\mathbb{C}} d\iota_{\mathbb{C}}(T_x M) \oplus J(ds_{\mathbb{C}} d\iota_{\mathbb{C}}(T_x M)) \subset T_x M_{\mathbb{C}} \end{aligned}$$

We have $TM_{\mathbb{C}}|_M = TM \oplus J(TM)$. Write $A_{\mathbb{C}} = Lie(\mathcal{G}_{\mathbb{C}})$ and $A = Lie(\mathcal{G})$. By the similar argument, we can show that $A_{\mathbb{C}}|_M = A \oplus J(A)$. The commutative diagram (3.1) comes from the fact that $\mathcal{G}$ is a subgroupoid of $\mathcal{G}_{\mathbb{C}}$. Let $\alpha, \beta \in \Gamma(A)$ be sections of A. Extending α, β to the right-invariant vector fields of $\mathcal{G}$, then the Lie bracket $[\alpha, \beta]$ is the usual Lie bracket of the vector fields in $T\mathcal{G}$. Extending α and β holomorphically, we get $\tilde{\alpha}, \tilde{\beta}$ as the right-invariant vector field of $T\mathcal{G}_{\mathbb{C}}$. We have the holomorphic extension of $[\alpha, \beta]$ is $[\tilde{\alpha}, \tilde{\beta}]_{\mathbb{C}}$. ☐

Proposition 3.2.10. *Let $(A_{\mathbb{C}}, \rho_{\mathbb{C}}, [\cdot,\cdot]_{\mathbb{C}}) \to M_{\mathbb{C}}$ be a holomorphic Lie algebroid. Let A_R be the underlying real Lie algebroid. Let $A \to M$ be an analytic subbundle of A_R. Then A is a real form of $A_{\mathbb{C}}$ if and only if there exists an open neighborhood V containing M and an antiholomorphic Lie algebroid involution $\sigma : A_R|_V \to A_R|_V$ with $Fix(\sigma) = A$.*

Proof. Assume A is a real form of $A_{\mathbb{C}}$. At each point $x \in M$, we can always find a local trivialization of $A_{\mathbb{C}}$ and A such that we have the following commutative diagram:

$$\begin{array}{ccc} A|_{U \cap M} & \xrightarrow{i} & A_{\mathbb{C}}|_U \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \phi} \\ \mathbb{R}^n \times \mathbb{R}^k & \xrightarrow{j} & \mathbb{C}^n \times \mathbb{C}^k \end{array}$$

where ψ and ϕ are local trivialization of A and $A_{\mathbb{C}}$ around x, U is an open set in $M_{\mathbb{C}}$ containing x, and j is the standard embedding of $\mathbb{R}^n \times \mathbb{R}^k$ to $\mathbb{C}^n \times \mathbb{C}^k$. Let σ_U be defined by requiring $\sigma_U(z_1, \cdots, z_n, u_1, \cdots, u_k) =$

$(\bar{z}_1, \cdots, \bar{z}_n, \bar{u}_1, \cdots, \bar{u}_k)$. By analytic continuation, we can glue such σ_U to get an anti-holomorphic involution σ on a neighborhood of M with $Fix(\sigma) = M$.

It suffices to show that σ is a Lie algebroid morphism. Let $\lambda : V \to V$ be the anti-holomorphic involution covered by σ. Then $d\lambda \circ \rho_R \circ \sigma : A_R \to TV$ is a holomorphic map which is identity when restricting to A is the same as $\rho_{\mathbb{C}}$. By analytic continuation, $d\lambda \circ \rho_R \circ \sigma = \rho_{\mathbb{C}}$, i.e., $d\lambda \circ \rho_R = \rho_{\mathbb{C}} \circ \sigma$. Any smooth section of A can be locally written as linear combination $\sum_k f_k e_k$, where e_k is a holomorphic section and f_k is a complex-valued smooth function. It suffices to show that for any smooth complex valued function f and g,

$$[\sigma_*(fe_i), \sigma_*(ge_j)]_R = \sigma_*[fe_i, ge_j]_R$$

where $\sigma_*(e) = \sigma \circ e \circ \lambda^{-1}$.

$$\begin{aligned}
\sigma_*[fe_i, ge_j]_R(z) &= \overline{[fe_i, ge_j]_R(\bar{z})} \\
&= \overline{f(\bar{z})g(\bar{z})}[e_i, e_j] + \overline{\rho_R(e_i)(g)(\bar{z})} \cdot \overline{f(\bar{z})}e_j - \overline{\rho_R(e_j)(f)(\bar{z})} \cdot \overline{g(\bar{z})}e_i \\
&= \overline{f(\bar{z})g(\bar{z})}[e_i, e_j] + \rho_R(e_i)(\overline{g(\bar{z})}) \cdot \overline{f(\bar{z})}e_j - \rho_R(e_j)(\overline{f(\bar{z})}) \cdot \overline{g(\bar{z})}e_i \\
&= [\overline{f(\bar{z})}e_i, \overline{g(\bar{z})}e_j]_R \\
&= [\sigma_*(fe_i), \sigma_*(ge_j)]_R
\end{aligned}$$

□

Theorem 3.2.11. *Let $A_{\mathbb{C}}$ be a complexification of A. If $A_{\mathbb{C}}$ is integrable, then A is analytically integrable. Moreover, there is an open neighborhood $U \subset M_{\mathbb{C}}$ containing M, such that A integrates to a real form of $\mathcal{G}_{\mathbb{C}}$, where $\mathcal{G}_{\mathbb{C}}$ is the source simply connected integration of $A_{\mathbb{C}}|_U$.*

Proof. According to Proposition 3.2.10, there exists an open neighborhood $U \subset M_{\mathbb{C}}$ and a Lie algebroid involution $\sigma : A_{\mathbb{C}}|_U \to A_{\mathbb{C}}|_U$. Let $\mathcal{G}_{\mathbb{C}}$ be the simply connected integration of $A_{\mathbb{C}}|_U$. Integrating the antiholomorphic Lie algebroid morphism σ, we have an antiholomorphic involution $\Sigma : \mathcal{G}_{\mathbb{C}} \to \mathcal{G}_{\mathbb{C}}$ (cf. [16]). Let $\mathcal{G}$ be the fixed point set of $\mathcal{G}_{\mathbb{C}}$. Restricting the structure maps to $\mathcal{G}$, we can give $\mathcal{G}$ a Lie groupoid structure. By Lemma 3.1.2, we have that $\mathcal{G}$ is a real form of $\mathcal{G}_{\mathbb{C}}$. □

3.3 Integration of the Complexification

Theorem 3.2.11 shows that integrability of the complexification implies the integrability of A. In this section, we are going to show that the converse also holds in the following cases:

1. The anchor map is zero;
2. The anchor map $\rho : A \to TM$ is injective at some point;
3. A is transitive, meaning the anchor map is surjective.

Let A be an analytic algebroid with zero anchor map. Then it is an analytic bundle of Lie algebras with fibers $\mathfrak{g}_x$. Its complexification $A_{\mathbb{C}}$ is a bundle of complex Lie algebras with fibers $\mathfrak{g}_x \otimes \mathbb{C}$. Let G_x and $G_{\mathbb{C},x}$ be the simply connected Lie groups integrating $\mathfrak{g}_x$ and $\mathfrak{g}_x \otimes \mathbb{C}$ respectively. According to Corollary 1.2.12, both A and $A_{\mathbb{C}}$ can be integrated to bundles of Lie groups, whose fibers are G_x and $G_{\mathbb{C},x}$ respectively.

For the Lie algebroid with "almost injective" anchor map, we have the following theorem:

Theorem 3.3.1 ([27],[9])**.** *If a Lie algebroid A has injective anchor map on a dense open set, then it is integrable.*

If the anchor map of a holomorphic algebroid is injective at one point, then it is injective on a "large set".

Theorem 3.3.2. *Let $A_{\mathbb{C}}$ be a holomorphic algebroid over a connected complex manifold $M_{\mathbb{C}}$. If the anchor map $\rho_{\mathbb{C}}$ is injective at some point, then $A_{\mathbb{C}}$ is integrable.*

Proof. According to Theorem 3.3.1, it suffices to show the set

$$\{x \in M_{\mathbb{C}} | \rho_{\mathbb{C},x} \text{ is injective}\}$$

is open and dense. Let k be the rank of the vector bundle $A_{\mathbb{C}}$ and n be the dimension of $M_{\mathbb{C}}$. Since the anchor map $\rho_{\mathbb{C}}$ is injective at some point, we have $k \leq n$. Let x be a point such that $\rho_{\mathbb{C},x}$ is injective. $\square$

Corollary 3.3.3. *Let A be an analytic algebroid over a connected analytic manifold M. If the anchor map of A is injective at one point, then A is analytically integrable.*

Proof. Let $A_{\mathbb{C}} \to M_{\mathbb{C}}$ be a complexification of A. We may assume that $M_{\mathbb{C}}$ is connected. Assume that the anchor map is injective at some point x, by Proposition 3.2.6, the anchor map $\rho_{\mathbb{C}}$ of $A_{\mathbb{C}}$ is also injective at x. We have $A_{\mathbb{C}}$ is integrable, which implies that A is analytically integrable (cf. Theorem 3.2.11). $\square$

Let A be a transitive analytic algebroid and let $A_{\mathbb{C}} \to M_{\mathbb{C}}$ be a complexification of A. According to Proposition 3.2.7, possibly by shrinking $M_{\mathbb{C}}$, we may assume that $A_{\mathbb{C}}$ is also transitive. By shrinking $M_{\mathbb{C}}$ to a tubular neighborhood of M in $M_{\mathbb{C}}$, we may assume that M and $M_{\mathbb{C}}$ are homotopy equivalent. We will show that the integrability of A implies the integrability of $A_{\mathbb{C}}$.

Recall that a transitive Lie algebroid is integrable if and only if the monodromy group $\tilde{N}_x(A)$ is closed for some x (cf. Corollary 1.2.10). Recall from Proposition 1.2.11, we have the following exact sequence:

$$\pi_2(M,x) \xrightarrow{\partial_x} \mathcal{G}(\mathfrak{g}_x) \xrightarrow{\mathcal{G}(i)} \mathcal{G}(A)_x \xrightarrow{p_x} \pi_1(M,x) \to 1.$$

The image of ∂_x is the monodromy group $\mathcal{N}_x(A)$. We will use the exact sequence to compare the monodromy groups for A and $A_{\mathbb{C}}$.

Theorem 3.3.4. *Let A be a transitive integrable analytic Lie algebroid. Let $A_{\mathbb{C}} \to M_{\mathbb{C}}$ be a complexification of A, then there is an open neighborhood $U \subset M_{\mathbb{C}}$ containing M such that $A_{\mathbb{C}}|_U$ is integrable.*

Proof. By shrinking $M_{\mathbb{C}}$, we may assume that $M_{\mathbb{C}}$ has the same homotopy type of M and $A_{\mathbb{C}}$ is transitive. We will prove that $A_{\mathbb{C}}$ is integrable in this case. It suffices to show that its underlying real Lie algebroid is integrable (cf. [17]). By abuse of notation, we will use the same notation $A_{\mathbb{C}}$ for the underlying real Lie algebroid. The inclusion $A \to A_{\mathbb{C}}$ is a morphism of real Lie algebroids, which gives a commutative diagram:

$$\begin{array}{ccccc}
\pi_2(M,x) & \xrightarrow{\partial_x} & \mathcal{G}(\mathfrak{g}_x) & \longrightarrow & \mathcal{G}(A)_x \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \mathcal{G}(\phi)} & & \downarrow \\
\pi_2(M_{\mathbb{C}},x) & \xrightarrow{\partial_{\mathbb{C},x}} & \mathcal{G}(\mathfrak{g}_{\mathbb{C}},x) & \longrightarrow & \mathcal{G}(A_{\mathbb{C}})_x.
\end{array}$$

j is induced by the inclusion $M \to M_{\mathbb{C}}$. Since $M \to M_{\mathbb{C}}$ gives a homotopy equivalence, we have $j : \pi_2(M,x) \to \pi_2(M_{\mathbb{C}},x)$ is an isomorphism. According to Proposition 3.2.6, $\mathfrak{g}_{\mathbb{C},x} \cong \mathfrak{g} \otimes \mathbb{C}$. Let $\phi : \mathfrak{g}_x \to \mathfrak{g}_{\mathbb{C},x}$ be the inclusion. We have ϕ is a real Lie algebroid morphism, hence can be integrated to a Lie algebra morphism $\mathcal{G}(\phi) : \mathcal{G}(\mathfrak{g}_x) \to \mathcal{G}(\mathfrak{g}_{\mathbb{C},x})$. Let $\varphi : \mathfrak{g}_{\mathbb{C},x} \to \mathfrak{g}$ be the projection to its real part. φ is a morphism of real Lie algebras and we obtain $\mathcal{G}(\varphi) : \mathcal{G}(\mathfrak{g}_{\mathbb{C},x}) \to \mathcal{G}(\mathfrak{g})$. We have $\mathcal{G}(\varphi) \circ \mathcal{G}(\phi) = id$. Since $\mathcal{G}(\phi) \circ \partial_x = \partial_{\mathbb{C},x} \circ j$, we have

$$\partial_x = \mathcal{G}(\varphi)\mathcal{G}(\phi) \circ \partial_x = \mathcal{G}(\varphi) \circ \partial_{\mathbb{C},x} \circ j.$$

Since j is an isomorphism, we have

$$(\mathcal{G}(\varphi))^{-1}(\mathcal{N}_x(A)) = \mathcal{N}_x(A_{\mathbb{C}}).$$

Since $\mathcal{N}_x(A_{\mathbb{C}})$ is closed, we have $\mathcal{N}_x(A_{\mathbb{C}})$ is also closed. □

For transitive Lie algebroids, there is only one orbit, and thus all monodromy groups are isomorphic. In such cases, understanding the monodromy groups of A provides information about the monodromy groups of its complexification $A_{\mathbb{C}}$. However, in the general (non-transitive) case, it is not straightforward to relate the monodromy groups of A and $A_{\mathbb{C}}$. In particular, there may exist orbits of $A_{\mathbb{C}}$ in $M_{\mathbb{C}}$ that lie arbitrarily close to M but do not intersect it.

Example 3.3.5. *Consider the linear Poisson manifold $(\mathfrak{so}(3)^*, \pi)$. With respect to the basis E_{23}, E_{31}, E_{12}, we can identify $\mathfrak{so}(3) \cong \mathbb{R}^3$. Using coordinates x_1, x_2, x_3, the Poisson bivector is given by:*

$$\pi = x_1 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} + x_2 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}.$$

The symplectic leaves (orbits) of this Poisson structure are $\{0\}$ and the spheres

$$S_r^2 = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \;\middle|\; x_1^2 + x_2^2 + x_3^2 = r^2\right\} \quad \text{for } r > 0.$$

The complexification of $\mathfrak{so}(3)$ is

$$\mathfrak{so}(3, \mathbb{C}) = \mathfrak{so}(3) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{sl}(2, \mathbb{C}),$$

and the complexified Poisson manifold is $(\mathfrak{so}(3, \mathbb{C})^*, \pi_{\mathbb{C}})$. The complex Poisson bivector takes the form:

$$\pi_{\mathbb{C}} = z_1 \frac{\partial}{\partial z_2} \wedge \frac{\partial}{\partial z_3} + z_2 \frac{\partial}{\partial z_3} \wedge \frac{\partial}{\partial z_1} + z_3 \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2}.$$

The symplectic leaves are given by the level sets:

$$\mathcal{O}_\lambda = \left\{z \in \mathfrak{so}(3, \mathbb{C})^* \;\middle|\; z_1^2 + z_2^2 + z_3^2 = \lambda\right\}, \quad \lambda \in \mathbb{C}.$$

When λ is sufficiently close to a real number, the orbit $\mathcal{O}_\lambda$ may contain points arbitrarily close to $\mathfrak{so}(3)^* \subset \mathfrak{so}(3, \mathbb{C})^*$, but it does not intersect the real form unless λ is itself real.

3.4 Complexification of Source Proper Groupoids

Definition 3.4.1. Let G be a Lie group. We say that a complex Lie group $G_{\mathbb{C}}$ is a **complexification** of G if there is an injective group homomorphism

$$\iota : G \to G_{\mathbb{C}}$$

satisfying the universal property: for any complex Lie group H and any Lie group homomorphism $\phi : G \to H$, there exists a holomorphic Lie group homomorphism $\phi_{\mathbb{C}} : G_{\mathbb{C}} \to H$ such that $\phi = \phi_{\mathbb{C}} \circ \iota$.

Every compact Lie group G admits a complexification $G_{\mathbb{C}}$ (cf. [28], [29]). In that case, the Lie algebra of $G_{\mathbb{C}}$ is the complexification of Lie algebra of G and G is a totally real submanifold of $G_{\mathbb{C}}$. Moreover, the complexification of a compact group G is unique up to isomorphism.

A principal G-bundle is determined by an open over $\{U_i\}$ and transition functions:

$$g_{ij} : U_{ij} := U_i \cap U_j \to G$$

which satisfy the cocycle conditions.

Assume that P is an analytic principal bundle over an analytic manifold M. Let $M_{\mathbb{C}}$ be a complexification of M. Then we can holomorphically extend the transition functions to

$$\tilde{g}_{ij} : \tilde{U}_{ij} \to G_{\mathbb{C}}$$

where $\tilde{U}_{ij} \subset M_{\mathbb{C}}$ is an open neighborhood of U_{ij}. Then we can glue charts by the transition functions to obtain a principal $G_{\mathbb{C}}$ bundle. First we need the following lemma.

Lemma 3.4.2. *Let $M \to N$ be an embedding of smooth manifold. Let U be an open set in M. There exists an open set V on N such that $V \cap M = U$ and*

$$\overline{V} \cap M = \overline{V \cap M}$$

Proof. There is a tubular open neighborhood $W \subset N$ of M and a smooth retraction:

$$r : W \to X$$

such that $r|_X = id_X$. Let $V = r^{-1}(U) \cap W$. Since U is open in M and r is continuous, we have V is open in N. We claim that V is the desired open set.

We have $V \cap M = r^{-1}(U) \cap W \cap M = U$. It is clear that $\overline{V \cap M} \subset \overline{V} \cap M$. To see that $\overline{V} \cap M \subset \overline{V \cap M}$, let $x \in \overline{V} \cap M$. There is a sequence of points $\{y_k\} \subset V$, such that $y_k \to x$. Then we have $r(y_k) \to r(x) = x$. Hence $x \in \overline{r(V)} \subset \overline{V \cap M}$. ☐

The following theorem gives a special case for [30, Theorem 1].

Theorem 3.4.3. *Let G be a compact group and P be an analytic principal G-bundle over an analytic manifold M. Let $G_{\mathbb{C}}$ be the complexification of G. There exists a holomorphic principal $G_{\mathbb{C}}$-bundle $P_{\mathbb{C}}$ such that P is a totally real submanifold of $P_{\mathbb{C}}$ and the embedding $P \to P_{\mathbb{C}}$ is G equivariant.*

Proof. Let $\{U_i\}$ be an open cover of M with transition functions $g_{ij} : U_{ij} = U_i \cap U_j \to G$ for the principal bundle P. Let $M_{\mathbb{C}}$ be a complexification of M. By analytic continuation, extending g_{ij}, we get a holomorphic function

$$\tilde{g}_{ij} : \tilde{U}_{ij} \subset M_{\mathbb{C}} \to G_{\mathbb{C}},$$

where $\tilde{U}_{ij}$ is an open set in $M_{\mathbb{C}}$ containing U_{ij}. We may also assume that $\tilde{g}_{ij}$ satisfies the cocycle conditions.

We can find an refinement $\{V_\alpha\}$ of the open cover $\{U_i\}$, such that $\overline{V}_\alpha \subset U_{i(\alpha)}$ is compact and V_α only intersects finite many V_β's. Replacing $\{U_i\}$ by $\{V_\alpha\}$, we may assume that $\overline{U_i} \cap \overline{U_j} \subset \tilde{U}_{ij}$, where the closure is taken inside M.

According to Lemma 3.4.2, we can choose open sets $\tilde{U}_i$ in $M_{\mathbb{C}}$ such that $M \cap \tilde{U}_i = U_i$ and $\overline{\tilde{U}_i} \cap M = \overline{U_i}$. Possibly by taking a refinement, we may assume that each $\tilde{U}_i$ only intersects finitely U_j's. By modifying the open cover $\{\tilde{U}_i\}$, we want to find a set of open sets $\{\tilde{W}_i\}$ in $M_{\mathbb{C}}$, which satisfying the following conditions:

1. $\{\tilde{W}_i\}$ covers M, i.e, $M \subset \cup_i W_i$;

2. $\tilde{W}_i \cap \tilde{W}_j = \emptyset$ if $\tilde{U}_i \cap \tilde{U}_j = \emptyset$;

3. $\tilde{W}_i \cap \tilde{W}_j \subset \tilde{U}_{ij}$ if $\tilde{U}_i \cap \tilde{U}_j \neq \emptyset$.

Suppose that $\{\tilde{W}_i\}$ exists. We can restrict the function $\tilde{g}_{ij}$ on $\tilde{W}_i \cap \tilde{W}_j$ to obtain holomorphic transition functions:

$$\tilde{g}_{ij} : \tilde{W}_i \cap \tilde{W}_j \to G_{\mathbb{C}}.$$

Write $\tilde{W} = \cup_i \tilde{W}_i$. Define a principal bundle over $\tilde{W}$ by:

$$P_{\mathbb{C}} = \bigsqcup_i \tilde{W}_i \times G_{\mathbb{C}} / \sim,$$

where the equivalence relation is given by $(x, \lambda) \sim (x, \tilde{g}_{ij}\lambda)$. Then $P_{\mathbb{C}}$ is the desired holomorphic $G_{\mathbb{C}}$ principal bundle.

Now it suffices to show that such an open cover exists. Take

$$\tilde{V}_{ij} = (\tilde{U}_i - \tilde{U}_i \cap \tilde{U}_j) \cup \tilde{U}_{ij}.$$

Let $\mathring{\tilde{V}}_{ij}$ be the interior of $\tilde{V}_{ij}$. We claim that $\mathring{\tilde{V}}_{ij}$ is an open neighborhood of U_i. For any point $x \in U_i$, if $x \in \overline{U_i} \cap \overline{U_j}$, since $\overline{U_i} \cap \overline{U_j} \subset \tilde{U}_{ij}$, we have $x \in \tilde{U}_{ij}$ is an interior point of $\tilde{V}_{ij}$. Since $\overline{\tilde{U}_i} \cap M \subset \overline{U_i}$, we have

$$\overline{\tilde{U}_i \cap \tilde{U}_j} \cap M \subset \overline{\tilde{U}_i} \cap \overline{\tilde{U}_j} \cap M \subset \overline{U_i} \cap \overline{U_j}$$

. If $x \notin \overline{U_i} \cap \overline{U_j}$, then it is not in the set $\overline{\tilde{U}_i \cap \tilde{U}_j}$, hence it is an interior point of $\tilde{U}_i - \tilde{U}_i \cap \tilde{U}_j$.

Set

$$\tilde{W}_i = \left(\bigcap_{\tilde{U}_i \cap \tilde{U}_j \neq \emptyset} \mathring{\tilde{V}}_{ij} \right) \cap \tilde{U}_i.$$

We have $\tilde{W}_i$ is an open neighborhood of U_i. Then $\cup_i \tilde{W}_i$ is an open neighborhood of M. If $\tilde{U}_i \cap \tilde{U}_j = \emptyset$, then $\tilde{W}_i \cap \tilde{W}_j = \emptyset$. If $\tilde{U}_i \cap \tilde{U}_j \neq \emptyset$,

$$\tilde{W}_i \cap \tilde{W}_j \subset \tilde{V}_{ij} \cap \tilde{U}_i \cap \tilde{U}_j \subset \tilde{U}_{ij}$$

□

Definition 3.4.4. Let A be an analytic Lie algebroid. We say that A is of s-proper type if there exists an analytic s-proper groupoid $\mathcal{G}$ integrating A.

Assume that $\mathcal{G}$ is an analytic s-proper groupoid integrating A. Let $\mathcal{O}$ be an orbit of the groupoid $\mathcal{G}$. Let $x \in O$ and $P = s^{-1}(x)$. We have $t : P \to \mathcal{O}$ is a principal bundle over $\mathcal{O}$ with the structure group $G = s^{-1}(x) \cap t^{-1}(x)$. Let $N = T_xM/T_x\mathcal{O}$ be the normal bundle at x. We have G acts on N by normal representation. The linear model of $\mathcal{G}$ can be realized as follows (cf. [4]).

- The normal bundle of $\mathcal{O}$ is isomorphic to the associated vector bundle

$$\mathcal{N}_{\mathcal{O}} \cong (P \times N)/G,$$

where $G \curvearrowright P \times N$ by $\lambda(g,v) = (g\lambda^{-1}, \lambda v)$.

- The normal bundle of $\mathcal{G}_{\mathcal{O}} = s^{-1}(\mathcal{O})$ is:

$$\mathcal{N}_{\mathcal{G}_{\mathcal{O}}} \cong (P \times P \times N)/G,$$

where $G \curvearrowright P \times P \times N$ by $\lambda(g,h,v) = (g\lambda^{-1}, h\lambda^{-1}, \lambda v)$.

- The structure of the linear model is given by:

$$s([g,h,v]) = [h,v], t([g,h,v]) = [g,v], [g,h,v] \cdot [h,k,v] = [g,k,v] \tag{3.2}$$

Now we can complexify the linear model to obtain a holomorphic groupoid. Let $P_{\mathbb{C}}$ be a complexification of P in Theorem 3.4.3. Since G acts on N by normal representation $\rho : G \to GL(N)$, by the universal property of $G_{\mathbb{C}}$, we can extend ρ to a holomorphic representation $\rho_{\mathbb{C}} : G_{\mathbb{C}} \to GL(N \otimes \mathbb{C})$. Denote $N_{\mathbb{C}} = N \otimes \mathbb{C}$. We can define a holomorphic groupoid

$$(P_{\mathbb{C}} \times P_{\mathbb{C}} \times N_{\mathbb{C}})/G_{\mathbb{C}} \rightrightarrows (P_{\mathbb{C}} \times N_{\mathbb{C}})/G_{\mathbb{C}},$$

whose structure maps are define in the same way of (3.2). To show that the groupoid is actually a complexification of the linear model, we need the following lemma about quotient map.

Lemma 3.4.5. *Let $f : M \to N$ be an embedding. Let G be a Lie group acting N and $H \subset G$ be a subgroup acting on M. Both actions are free and proper. Assume that:*

- *f is H-equivariant,*
- *the induced map $\overline{f} : M/H \to N/G$ is injective,*

Then $\overline{f} : M/H \to N/G$ is an embedding.

Proof. $\overline{f}$ is injective means that every orbit of G in N restricts to an orbit of H in M:

$$f^{-1}(G \cdot f(m)) = H \cdot m \text{ for all } m \in M.$$

We have the following commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow{\scriptstyle \pi_M} & & \downarrow{\scriptstyle \pi_N} \\ M/H & \xrightarrow{\bar{f}} & N/G \end{array}$$

Let $v = d\pi_M(w) \in Ker(d\overline{f})$,

$$d\bar{f}_{[m]}(v) = d\bar{f}_{[m]} \circ d\pi_M|_m(w) = d\pi_N|_{f(m)} \circ df|_m(w) = d\pi_M(w) = 0$$

We have

$$df|_m(w) \in T_{f(m)}(G \cdot f(m)) \Rightarrow w \in T_m(H \cdot m) \Rightarrow v = d\pi_M(w) = 0.$$

We have f is an injective immersion. Let $U \subset M/H$ be an open set. Then $\pi_M^{-1}(U) \subset M$. Since f is an embedding, $f(\pi_M^{-1}(U)) \subset M$ is open in the subspace topology of $f(M)$. Then

$$\pi_N(f(\pi_M^{-1}(U))) = \overline{f}(U)$$

is open in the image of $\overline{f}$ in the subspace topology. □

Proposition 3.4.6. *Let $M_{\mathbb{C}}$ be a complex manifold and M be a totally submanifold. Let G be a compact Lie group acting on M and $G_{\mathbb{C}}$ be the complexification. $G_{\mathbb{C}}$ acts holomorphically on $M_{\mathbb{C}}$. Assume that:*

- *both G and $G_{\mathbb{C}}$ actions are free and proper,*
- *the embedding $M \to M_{\mathbb{C}}$ is G-equivariant.*

Then $M/G \to M_{\mathbb{C}}/G_{\mathbb{C}}$ is a totally real submanifold.

Proof. Fix $x \in M$, define the orbit map:

$$\Phi_x : G \to G \cdot x, \quad g \mapsto gx.$$

Taking the differential, since the action is free, we obtain the following isomorphism:

$$d\Phi_x|_e : \mathfrak{g} \to T_x(G \cdot x), \quad v \mapsto \left.\frac{d}{dt}\right|_{t=0} \exp(tv) \cdot x.$$

Taking differential of the following commutative diagram:

$$\begin{array}{ccc} G & \longrightarrow & G \cdot x \\ \downarrow & & \downarrow \\ G_{\mathbb{C}} & \longrightarrow & G_{\mathbb{C}} \cdot x, \end{array}$$

we obtain:

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & T_x(G \cdot x) \\ \downarrow & & \downarrow \\ \mathfrak{g}_{\mathbb{C}} \cong \mathfrak{g} \oplus i\mathfrak{g} & \longrightarrow & T_x(G_{\mathbb{C}} \cdot x), \end{array}$$

where the bottom map is holomorphic. We have

$$T_x(G_{\mathbb{C}} \cdot x) \cong \mathfrak{g} \oplus i\mathfrak{g} \cong T_x(G \cdot x) \oplus J(T_x(G \cdot x)),$$

which shows the following:

$$\begin{aligned} &T_x(M_{\mathbb{C}}/G_{\mathbb{C}}) \cong T_x M_{\mathbb{C}}/T_x(G_{\mathbb{C}} \cdot x) \\ \cong &(T_x M \oplus J(T_x M))/(T_x(G \cdot x) \oplus J(T_x(G \cdot x))) \\ \cong &T_x(M/G) \oplus J(T_x(M/G)) \end{aligned}$$

□

Proposition 3.4.7. *The holomorphic groupoid* $(P_{\mathbb{C}} \times P_{\mathbb{C}} \times N_{\mathbb{C}})/G_{\mathbb{C}} \rightrightarrows (P_{\mathbb{C}} \times N_{\mathbb{C}})/G_{\mathbb{C}}$ *defined above is a complexification of the linear model* $(P \times P \times N)/G \rightrightarrows (P \times N)/G$.

Proof. Let $(g, h, v), (g', h', v') \in P \times P \times N$. If they belongs to the same orbit in $P_{\mathbb{C}} \times P_{\mathbb{C}} \times N_{\mathbb{C}}$, then there exist $\lambda \in G_{\mathbb{C}}$, such that

$$(g\lambda^{-1}, h\lambda^{-1}, \lambda v) = (g', h', v')$$

Since $g = g'\lambda$, we have g and g' belongs to the same fiber of P. Then there exist a unique element $\xi \in G$, such that $g = g'\xi$. We have $\lambda \in G$. This shows that the induced map between the quotient spaces $(P \times P \times N)/G \to (P_{\mathbb{C}} \times P_{\mathbb{C}} \times N_{\mathbb{C}})/G_{\mathbb{C}}$ is injective.

By Lemma 3.4.5, the standard embedding $P \times P \times N \to P_{\mathbb{C}} \times P_{\mathbb{C}} \times N_{\mathbb{C}}$ induces the following embedding between groupoids:

$$\begin{array}{ccc} (P \times P \times N)/G & \to & (P_{\mathbb{C}} \times P_{\mathbb{C}} \times N_{\mathbb{C}})/G_{\mathbb{C}} \\ \downarrow\downarrow & & \downarrow\downarrow \\ (P \times N)/G & \to & (P_{\mathbb{C}} \times N_{\mathbb{C}})/G_{\mathbb{C}} \end{array},$$

where all the structure maps of groupoids are preserved. According to Proposition 3.4.6, $(P \times P \times N)/G$ is a totally real submanifold of $(P_{\mathbb{C}} \times P_{\mathbb{C}} \times N_{\mathbb{C}})/G_{\mathbb{C}}$.

□

We say that a Lie algebroid is **integrable** if it is a Lie algebroid of some Lie groupoid. We say that a Lie algebroid $A \to M$ is **locally integrable** if for every point $x \in M$, there is an open neighborhood U of x, such that $A|_U$ is integrable.

Remark. Integrability is not a local problem. There are Lie algebroid that are locally integrable but not globally integrable (cf. [9, Example 4.3 and 4.5]). There are also algebroids which are not locally integrable ((cf. [9, Example 4.4])).

Theorem 3.4.8. *Let* A *be an analytic algebroid of* s*-proper type and let* $A_{\mathbb{C}}$ *be its complexification. Then* $A_{\mathbb{C}}$ *is locally integrable around* M *(i.e. for every point* $x \in M$*, there is an open neighborhood* $U \subset M$ *containing* x*, such that* $A_{\mathbb{C}}|_U$ *is integrable).*

Proof. It suffices to show A is integrable around each point of $x \in M$. Let $\mathcal{G}$ be an s-proper analytic groupoid integrating A. Let $\mathcal{O}$ be an orbit. According to Theorem 2.5.6, $\mathcal{G}$ can be analytically linearizable around $\mathcal{O}$. There exists an open neighborhoods V and W, such that $\mathcal{G}|_V \cong (P \times P \times N)/G|_W$. According to Proposition 3.2.9 and 3.4.7, the Lie algebroid of $(P_{\mathbb{C}} \times P_{\mathbb{C}} \times N_{\mathbb{C}})/G_{\mathbb{C}}$ is a complexification of $A|_U$. Then there is an open neighborhood $\tilde{W} \subset (P_{\mathbb{C}} \times N_{\mathbb{C}})/G_{\mathbb{C}}$ and $\tilde{V} \subset M_{\mathbb{C}}$, such that $(P_{\mathbb{C}} \times P_{\mathbb{C}} \times N_{\mathbb{C}})/G_{\mathbb{C}}|_{\tilde{W}}$ integrates $A_{\mathbb{C}}|_{\tilde{V}}$. □

Chapter 4

Poisson Cohomology of Projective Spaces

4.1 Poisson Structure on Projective Spaces

Let X be a complex manifold. Recall that a holomorphic Poisson structure on X is a holomorphic bivector field $\pi \in H^0(X, \bigwedge^2 \mathcal{T}_X)$ which satisfies the Schouten bracket $[\pi, \pi] = 0$.

Definition 4.1.1. Let π be a holomorphic Poisson structure on $\mathbb{C}^{n+1}$, we can write

$$\pi = \sum_{0 \leq i < j \leq n} \pi_{ij} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$$

We say that π is a **quadratic Poisson structure** if π_{ij} are homogeneous quadratics.

Definition 4.1.2. Let $v \in H^0(\wedge^k \mathcal{T}_{\mathbb{C}^{n+1}})$ be a multivector field. We say that v is **homogeneous of degree** k if v can be written as:

$$v = \sum_{0 \leq j_1 < j_2 <, \cdots, < j_k \leq n} f_{j_1, \cdots, j_k} \frac{\partial}{\partial z_{j_1}} \wedge \cdots \wedge \frac{\partial}{\partial z_{j_k}}, \tag{4.1}$$

where $f_{j_1, \cdots, j_k}$'s are homogeneous polynomials of degree k.

Let $p : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ be the quotient map. Let $v \in H^0(\wedge^k \mathcal{T}_{\mathbb{C}^{n+1}})$ be a multivector field of homogeneous degree k. Since v is invariant for the $\mathbb{C}^*$ action $\lambda : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}^{n+1} \setminus \{0\}$ $v \mapsto \lambda v$, v descends to a multivector field $p_* v \in H^0(\wedge^k T\mathbb{P}^n)$. Actually, any multivector field in $\mathbb{P}^n$ comes from a homogeneous multivector field in $\mathbb{C}^{n+1}$.

Lemma 4.1.3 ([31])**.** *Any multivector field $w \in H^0(X, \wedge^k \mathcal{T}_{\mathbb{P}^n})$ can be written as*

$$w = p_* v,$$

where $v \in H^0(\wedge^k \mathcal{T}_{\mathbb{C}^{n+1}})$ is homogeneous of degree k. Two multivector fields v, v' in $\mathbb{C}^{n+1}$ present the same multivector fields in $\mathbb{P}^n$ if and only if:

$$v - v' = E \wedge u,$$

where $E = \sum_i z_i \frac{\partial}{\partial z_i}$ is the Euler vector field and $u \in H^0(\wedge^{k-1} \mathcal{T}_{\mathbb{C}^{n+1}})$ is homogeneous of degree $k-1$.

Definition 4.1.4. Let $\phi : X \to Y$ be a smooth map between smooth manifolds. We say that two multivector fields $\tilde{v} \in \Gamma(\wedge^p TX)$ and $v \in \Gamma(\wedge^p TX)$ are **ϕ-related** if and only if, for every point $x \in X$ and for every choice of covectors $\alpha_1, \ldots, \alpha_p \in T^*_{\phi(x)}Y$, we have:

$$v_{f(x)}(\alpha_1, \ldots, \alpha_p) = \tilde{v}_x(\phi^*_x(\alpha_1), \ldots, \phi^*_x(\alpha_p)).$$

In this case, we denote $v = \phi_*\tilde{v}$.

Lemma 4.1.5 ([31])**.** *Let $\phi : X \to Y$ be a smooth map between smooth manifolds. Let $\tilde{v}, \tilde{w}$ be two multivector fields on X and v, w be multivector fields on Y such that $\phi_*\tilde{v} = v$ and $\phi_*\tilde{w} = w$, then we have the Schouten bracket commutes with ϕ in the following way:*

$$[v, w] = \phi_*[\tilde{v}, \tilde{w}]$$

By the above lemmas, every Poisson structure on $\mathbb{P}^n$ comes from a quadratic Poisson structure on $\mathbb{C}^{n+1}$:

$$\pi = \sum_{0 \le i < j \le n} \pi_{ij} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$$

where π_{ij} are homogeneous quadratics. Two Poisson structures π and π' on $\mathbb{C}^{n+1}$ descend to the same Poisson structure on $\mathbb{P}^n$ if and only if a vector field Z such that

$$\pi - \pi' = E \wedge Z,$$

where $E = \sum_{i=1}^{0} z_i \frac{\partial}{\partial z_i}$ is the Euler vector field.

Example 4.1.6. *Any bivector field of the form:*

$$\pi = \sum_{0 \le i < j \le n} a_{ij} z_i z_j \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$$

descends to a Poisson structure on $\mathbb{P}^n$, called a ***diagonal Poisson structure****.*

4.2 Poisson Cohomology of $\mathbb{P}^n$

Let (X, π) be a holomorphic Poisson manifold of dimension n. Recall that the Poisson cohomology $H^\bullet_\pi(X)$ is the cohomology group of the complex of sheaves

$$0 \to \mathcal{O}_X \xrightarrow{d_\pi} \mathcal{T}_X \xrightarrow{d_\pi} \cdots \xrightarrow{d_\pi} \wedge^i \mathcal{T}_X \xrightarrow{d_\pi} \cdots \xrightarrow{d_\pi} \wedge^n \mathcal{T}_X \to 0$$

Lemma 4.2.1. *[10] The Poisson cohomology of a holomorphic Poisson manifold (X, π) is isomorphic to the total cohomology of the double complex*

$$\begin{array}{ccccccc}
\cdots & & \cdots & & \cdots & & \\
d_\pi \uparrow & & d_\pi \uparrow & & d_\pi \uparrow & & \\
\Omega^{0,0}(X, T^{2,0}X) & \xrightarrow{\bar\partial} & \Omega^{0,1}(X, T^{2,0}X) & \xrightarrow{\bar\partial} & \Omega^{0,2}(X, T^{2,0}X) & \xrightarrow{\bar\partial} & \cdots \\
d_\pi \uparrow & & d_\pi \uparrow & & d_\pi \uparrow & & \\
\Omega^{0,0}(X, T^{1,0}X) & \xrightarrow{\bar\partial} & \Omega^{0,1}(X, T^{1,0}X) & \xrightarrow{\bar\partial} & \Omega^{0,2}(X, T^{1,0}X) & \xrightarrow{\bar\partial} & \cdots \\
d_\pi \uparrow & & d_\pi \uparrow & & d_\pi \uparrow & & \\
\Omega^{0,0}(X, T^{0,0}X) & \xrightarrow{\bar\partial} & \Omega^{0,1}(X, T^{0,0}X) & \xrightarrow{\bar\partial} & \Omega^{0,2}(X, T^{0,0}X) & \xrightarrow{\bar\partial} & \cdots
\end{array}$$

where $\Omega^{0,i}(X, T^{j,0}X) = \Gamma(\wedge^i (T^*X)^{0,1} \otimes \wedge^j (T^{1,0}X))$.

Lemma 4.2.2. *Let* (X, π) *be a holomorphic Poisson manifold. If all the higher cohomology groups* $H^i(X, \wedge^j \mathcal{T}_X)$ *vanish for* $i > 0$*, then the Poisson cohomology* $H^\bullet_\pi(X)$ *is isomorphic to the cohomology of the complex*

$$0 \to H^0(X, \mathcal{O}_X) \xrightarrow{d_\pi} H^0(X, \mathcal{T}_X) \xrightarrow{d_\pi} \cdots \xrightarrow{d_\pi} H^0(X, \wedge^n \mathcal{T}_X) \to 0 \tag{4.2}$$

According to Bott formula and Serre duality, we can compute the following cohomology for projective spaces.

Theorem 4.2.3. *The Betti number for the multivector sheaf is given by:*

$$h^q(\mathbb{P}^n, \wedge^k \mathcal{T}_{\mathbb{P}^n}) = \begin{cases} \binom{k+n+1}{k}\binom{n}{k}, & \text{if } q = 0,\ 0 \le k \le n, \\ 0, & \text{otherwise.} \end{cases}$$

To compute the Poisson cohomology of projective spaces, we observe that each vector space in the sequence (4.2) is finite-dimensional. Therefore, it suffices to determine how the Poisson differential acts as a linear map between these vector spaces.

Let π be a Poisson structure on $\mathbb{P}^n$. There exist a quadratic Poisson structure $\tilde\pi$ such that $\pi = p_*\tilde\pi$. Define V_k as the vector space of all homogeneous multivector fields of degree k in $\mathbb{C}^{n+1}$. Let $W_k = H^0(\mathbb{P}^n, \wedge^k \mathcal{T}_{\mathbb{P}^n})$. According to Lemma 4.1.3, $W_k \cong V_k/(E \wedge V_{k-1})$. We have the following commutative diagram:

$$\begin{array}{ccc}
V_k & \xrightarrow{d_{\tilde\pi}} & V_{k+1} \\
\downarrow & & \downarrow \\
W_k \cong V_k/(E \wedge V_{k-1}) & \xrightarrow{d_\pi} & W_{k+1} \cong V_{k+1}/(E \wedge V_k),
\end{array}$$

For any positive integer k and multi-index $I = (i_1, \cdots, i_k)$ and $J = (j_1, \cdots, j_k)$, write

$$w^k_{I,J} = z_{i_1} \cdots z_{i_k} \frac{\partial}{\partial z_{j_1}} \wedge \cdots \wedge \frac{\partial}{\partial z_{j_k}}$$

Then $S^k = \{w^k_{I,J} \mid 0 \le i_1 \le i_2 \le \cdots \le i_k \le n, 0 \le j_1 < j_2 < \cdots < j_k \le n\}$ is a basis of V_k. Write

$$\tilde\pi = \sum_{0 \le i < j \le n} \tilde\pi_{ij} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j},$$

where $\tilde\pi_{ij}$ is a homogeneous quadratic. Write $f_I(z) = z_{i_1} \cdots z_{i_k}$. By direct computation, we have:

$$
\begin{aligned}
d_{\tilde{\pi}}\big(w^k_{I,J}\big) &= [\tilde{\pi},\ f_I(z)\,\partial_J] \\
&= \big[\tilde{\pi},\ f_I\big] \wedge \partial_J \ +\ f_I(z)\,\big[\tilde{\pi},\ \partial_J\big] \\
&= \sum_{a=0}^{n}\sum_{b=0}^{n} \tilde{\pi}_{ab}(z)\,\frac{\partial f_I}{\partial z_b}\,\partial_a \wedge \partial_{j_1} \wedge \cdots \wedge \partial_{j_k} \\
&\quad + \sum_{p=1}^{k}\sum_{a<b} (-1)^{p-1}\, z_{i_1}\cdots z_{i_k}\,\frac{\partial \tilde{\pi}_{ab}(z)}{\partial z_{j_p}}\,\partial_a \wedge \partial_b \wedge \partial_{j_1} \wedge \cdots \widehat{\partial_{j_p}} \cdots \wedge \partial_{j_k}.
\end{aligned}
$$

With the basis, the Poisson differential $d^k_{\tilde{\pi}}$ can be represented by a $\dim V_{k+1} \times \dim V_k$ matrix involving b^{uv}_{ij}. Then the rank of d^k_π is given by:

$$\operatorname{rank} d^k_\pi = \dim(\operatorname{Im}(d^k_{\tilde{\pi}}) + E \wedge V_{k+1}) - \dim(E \wedge V_{k+1})$$

Then the dimension of the holomorphic Poisson cohomology for π in $\mathbb{P}^n$ is given by

$$\dim H^k_\pi(\mathbb{P}^n) = \dim H^0(\mathbb{P}^n, \wedge^k \mathcal{T}_{\mathbb{P}^n}) - \operatorname{rank} d^k_\pi - \operatorname{rank} d^{k-1}_\pi,$$

which is computable by the code in Appendix A.

4.3 Applications

4.3.1 Poisson Cohomology of $\mathbb{P}^2$

A Poisson structure on $\mathbb{P}^2$ is given by a holomorphic bivector field $\pi \in H^0(\mathbb{P}^2, \wedge^2 \mathcal{T}_{\mathbb{P}^2}) \cong H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$. Since the Schouten bracket $[\pi, \pi]$ lies in $H^0(\mathbb{P}^2, \wedge^2 \mathcal{T}_{\mathbb{P}^2}) = 0$, every holomorphic bivector is automatically Poisson. Hence the Poisson structure on $\mathbb{P}^2$ is classified by the homogeneous polynomail of degree 3. Let

$$F(z_0, z_1, z_2) = \sum_{i+j+k=3} a_{ijk}\, z_0^i z_1^j z_2^k.$$

It determines a Poisson structure on $\mathbb{P}^2$ given by:

$$\pi_F = p_*\left(\frac{\partial F}{\partial z_0}\,\frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2} \ +\ \frac{\partial F}{\partial z_1}\,\frac{\partial}{\partial z_2} \wedge \frac{\partial}{\partial z_0} \ +\ \frac{\partial F}{\partial z_2}\,\frac{\partial}{\partial z_0} \wedge \frac{\partial}{\partial z_1}\right).$$

For the purpose of computation, we write:

$$F \ =\ a_1 z_0^3 + a_2 z_0^2 z_1 + a_3 z_0^2 z_2 + a_4 z_0 z_1^2 + a_5 z_0 z_1 z_2 + a_6 z_0 z_2^2 + a_7 z_1^3 + a_8 z_1^2 z_2 + a_9 z_1 z_2^2 + a_{10} z_2^3,$$

which gives the Poisson structure:

$$\tilde{\pi} = p_*\Big(\big(3a_1 z_0^2 + 2a_2 z_0 z_1 + 2a_3 z_0 z_2 + a_4 z_1^2 + a_5 z_1 z_2 + a_6 z_2^2\big)\,\partial_{z_1} \wedge \partial_{z_2}$$

$$+\left(a_2z_0^2+2a_4z_0z_1+a_5z_0z_2+3a_7z_1^2+2a_8z_1z_2+a_9z_2^2\right)\partial_{z_2}\wedge\partial_{z_0}$$

$$+\left(a_3z_0^2+a_5z_0z_1+2a_6z_0z_2+a_8z_1^2+2a_9z_1z_2+3a_{10}z_2^2\right)\partial_{z_0}\wedge\partial_{z_1}\Big).$$

We can use our code to compute the Poisson cohomology for any Poisson structure on $\mathbb{P}^2$. The Poisson cochain (4.2) on $\mathbb{P}^2$ is:

$$H^0(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2})\cong\mathbb{C}\xrightarrow{d_\pi^0=0}H^0(\mathbb{P}^2,\mathcal{T}_{\mathbb{P}^2})\xrightarrow{d_\pi^1}H^0(\mathbb{P}^2,\wedge^2\mathcal{T}_{\mathbb{P}^2})\to 0$$

We have the Poisson cohomology of π is given by:

$$\dim H^i_\pi(\mathbb{P}^2)=\begin{cases}1 & i=0,\\ 8-\operatorname{rank}(d_\pi^1) & i=1,\\ 10-\operatorname{rank}(d_\pi^1) & i=2,\\ 0 & i>2.\end{cases}$$

The moduli space of Poisson structures on $\mathbb{P}^2$ is $\mathbb{C}^{10}$ with coordinates $a_1,\cdots,a_{10}$. By the computation, we find that d_π^1 is of full rank in an open dense subset of $\mathbb{C}^{10}$. Therefore, we have the following proposition.

Proposition 4.3.1. *For the generic Poisson structure on $\mathbb{P}^2$, we have the Poisson cohomology is given by:*

$$\dim H^i_\pi(\mathbb{P}^2)=\begin{cases}1 & i=0,\\ 0 & i=1,\\ 2 & i=2,\\ 0 & i>2.\end{cases}$$

We also compute Poisson cohomology for some specific Poisson structure on $\mathbb{P}^2$, which recovers the result in [32].

Example 4.3.2.

- *For $F(z_0,z_1,z_2)=z_0^3$, $\pi=p_*\left(3\,z_0^2\,\frac{\partial}{\partial z_1}\wedge\frac{\partial}{\partial z_2}\right)$, we have:*

$$\dim H^1_\pi(\mathbb{P}^2)=5,\quad \dim H^2_\pi(\mathbb{P}^2)=7.$$

- *For $F(z_0,z_1,z_2)=z_0^3$, $\pi=p_*\left(2\,z_0z_1\,\frac{\partial}{\partial z_1}\wedge\frac{\partial}{\partial z_2}-z_0^2\,\frac{\partial}{\partial z_0}\wedge\frac{\partial}{\partial z_2}\right)$, we have:*

$$\dim H^1_\pi(\mathbb{P}^2)=3,\quad \dim H^2_\pi(\mathbb{P}^2)=5.$$

- *For $F(z_0,z_1,z_2)=z_0^3+z_0^2z_1-7z_1^2z_2$, $\pi=p_*\left((2z_0z_1+3z_0^2)\,\frac{\partial}{\partial z_1}\wedge\frac{\partial}{\partial z_2}+(-z_0^2+14z_1z_2)\,\frac{\partial}{\partial z_0}\wedge\frac{\partial}{\partial z_2}\right)$, we have:*

$$\dim H^1_\pi(\mathbb{P}^2)=0,\quad \dim H^2_\pi(\mathbb{P}^2)=2.$$

4.3.2 Poisson Cohomology of $\mathbb{P}^3$

Theorem 4.3.3 ([31], [33]). *Let π be a quadratic Poisson structure on $\mathbb{C}^{n+1}$. Then π has a unique decomposition*

$$\pi = \pi_0 + Z \wedge E,$$

where π_0 is a unimodular Poisson structure and E is the Euler vector field.

Every Poisson structure on $\mathbb{P}^3$ comes from the quadratic Poisson structure on $\mathbb{P}^3$. The space of unimodular Poisson structures on $\mathbb{C}^4$ has six irreducible components shown in the Table 4.1 (cf. [11], [12], [34]), which also classified the Poisson structures on $\mathbb{P}^3$. Here we compute Poisson cohomology for some examples in $\mathbb{P}^3$.

Example 4.3.4. *1. For the Poisson structure π in $E(3)$, the Poisson cohomology is:*

$$\dim H^i_\pi(\mathbb{P}^3) = \begin{cases} 1 & i = 0, \\ 1 & i = 1, \\ 0 & i = 2, \\ 4 & i = 3, \\ 0 & i > 3. \end{cases}$$

2. Let $\pi = 2\,x_0x_2 \frac{\partial}{\partial z_0} \wedge \frac{\partial}{\partial z_2} - 2\,x_1x_2 \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2} + 2\,x_1x_3 \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_3} - 2\,x_0x_3 \frac{\partial}{\partial z_0} \wedge \frac{\partial}{\partial z_3}$ in L(1,1,1,1). The Poisson cohomology is:

$$\dim H^i_\pi(\mathbb{P}^3) = \begin{cases} 1 & i = 0, \\ 3 & i = 1, \\ 9 & i = 2, \\ 11 & i = 3, \\ 0 & i > 3. \end{cases}$$

3. For

$$\begin{aligned} \pi = -5\,x_2x_3 \frac{\partial}{\partial x_0} \wedge \frac{\partial}{\partial x_1} + 4\,x_1x_2 \frac{\partial}{\partial x_0} \wedge \frac{\partial}{\partial x_2} + x_1x_2 \frac{\partial}{\partial x_0} \wedge \frac{\partial}{\partial x_3} \\ -x_0x_3 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + x_0x_2 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3} - x_0x_1 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} \end{aligned}$$

in R(2,2), the Poisson cohomology is:

$$\dim H^i_\pi(\mathbb{P}^3) = \begin{cases} 1 & i = 0, \\ 0 & i = 1, \\ 0 & i = 2, \\ 5 & i = 3, \\ 0 & i > 3. \end{cases}$$

Table 4.1: Normal forms for generic unimodular quadratic Poisson structures on $\mathbb{C}^4$

Type	**Poisson brackets**	**Parameters**
L(1,1,1,1)	$\{x_i, x_{i+1}\} = (-1)^i(a_{i+3} - a_{i+2})x_i x_{i+1},$ $\{x_i, x_{i+2}\} = (-1)^i(a_{i+1} - a_{i+3})x_i x_{i+2},$ $i \in \mathbb{Z}/4\mathbb{Z}$	$a_0 + a_1 + a_2 + a_3 = 0$
L(1,1,2)	$\{x_0, x_1\} = 0, \quad \{x_1, x_2\} = -c_1 x_1 x_2,$ $\{x_0, x_2\} = c_0 x_0 x_2, \quad \{x_0, x_3\} = -c_0 x_0 x_3,$ $\{x_1, x_3\} = c_1 x_1 x_3,$ $\{x_2, x_3\} = (c_0 - c_1)(x_0^2 + \lambda x_0 x_1 + x_1^2 + x_2 x_3$ $+ 2c_0 x_0^2 - 2c_1 x_1^2)$	$c_0, c_1, \lambda \in \mathbb{C}$
R(2,2)	$\{x_0, x_1\} = (a_3 - a_2)x_2 x_3, \quad \{x_2, x_1\} = x_0 x_3,$ $\{x_0, x_2\} = (a_1 - a_3)x_2 x_1, \quad \{x_3, x_2\} = x_0 x_1,$ $\{x_0, x_3\} = (a_2 - a_1)x_1 x_2, \quad \{x_1, x_3\} = x_0 x_2$	$a_1 + a_2 + a_3 = 0$
R(1,3)	$\{x_3, x_i\} = 0,$ $\{x_{i+1}, x_i\} = \nu x_{i+2}^2 - \lambda x_{i+1} x_i + \sum_{j=0}^{2} b_{ij} x_j x_3,$ $i \in \mathbb{Z}/3\mathbb{Z}$	$b_{ij} = b_{ji},$ $i, j \in \mathbb{Z}/3\mathbb{Z},$ $\det(b_{ij}) = 1, \ \lambda \in \mathbb{C}$
S(2,3)	$\{x_i, x_3\} = x_i^2 + x_i(b_i x_{i+1} + c_i x_{i-1}) + d_i x_{i+1} x_{i-1},$ $\{x_i, x_j\} = 0, \quad i, j \in \mathbb{Z}/3\mathbb{Z}$	$b_{i-1} + c_{i+1} + 2 = 0,$ $d_i \in \mathbb{C}$ $i \in \mathbb{Z}/3\mathbb{Z}$
E(3)	$\{x_0, x_1\} = 5x_0^2, \quad \{x_1, x_2\} = x_1^2 + 3x_0 x_2,$ $\{x_0, x_2\} = 5x_0 x_1, \quad \{x_1, x_3\} = x_1 x_2 + 7x_0 x_3,$ $\{x_0, x_3\} = 5x_0 x_2, \quad \{x_2, x_3\} = 7x_1 x_3 - 3x_2^2$	none

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Appendix A

Code for Computing Poisson Cohomology

```
import itertools
import sympy as sp
import math
from sympy import expand

def wedge_permutation_sign(from_list, to_list):
    """
    Both lists have the same length, same distinct entries.
    Returns +1 or -1 depending on parity of the permutation sending
    from_list -> to_list.
    """
    perm = []
    used = [False]*len(from_list)
    for val in to_list:
        for j, w in enumerate(from_list):
            if (not used[j]) and (w == val):
                perm.append(j)
                used[j] = True
                break
    inversions = 0
    for i in range(len(perm)):
        for j in range(i+1, len(perm)):
            if perm[i] > perm[j]:
                inversions += 1
    return 1 if (inversions % 2 == 0) else -1

# Get exponents (multi-indices)
def get_multi_index(expr, var_list):
    powers = expr.as_powers_dict()
```

```
    index_list = []
    for i, var in enumerate(var_list):
        exp = powers.get(var, 0)
        index_list.extend([i] * exp)
    return tuple(index_list)

def compute_poisson_differential(n: int, b_dict: dict, k: int = 1) -> sp.Matrix:
    Z  = sp.symbols(f'z0:{n+1}')
    # pi^{ab}(z)
    pi = [[0]*(n+1) for _ in range(n+1)]
    for (p,q), uv in b_dict.items():
        poly = sum(c*Z[u]*Z[v] for (u,v), c in uv)
        pi[p][q] =  poly
        pi[q][p] = -poly

    # bases
    ms   = lambda r: itertools.combinations_with_replacement(range(n+1), r)
    inc  = lambda r: itertools.combinations(range(n+1), r)
    S_k  = [(I,J) for I in ms(k)   for J in inc(k)]
    S_k1 = [(I,J) for I in ms(k+1) for J in inc(k+1)]
    row  = {key:i for i,key in enumerate(S_k1)}

    M = sp.zeros(len(S_k1), len(S_k))
    for col,(I,J) in enumerate(S_k):
        f = sp.prod(Z[i] for i in I)
        #for a,b in b_dict.keys():
        for a in range(n+1):
            for b in range(n+1):
                polyno = pi[a][b] * sp.diff(f, Z[b])
                if (a not in J) and polyno:
                    monomials = expand(polyno).as_ordered_terms()
                    for term in monomials:
                        coeff, _ = term.as_coeff_mul()
                        new_I = get_multi_index(term, Z)
                        new_J_unsorted = (a,) + J
                        new_J = tuple(sorted(new_J_unsorted))
                        sign = wedge_permutation_sign(new_J_unsorted, new_J)
                        M[row[(new_I,new_J)], col] += sign*coeff

        for p in range(k):
            #for a,b in b_dict.keys():
            for a in range(n+1):
                for b in range(a+1, n+1):
                    polyno = f * sp.diff(pi[a][b], Z[J[p]])
```

```
                    if a < b and polyno:
                        monomials = expand(polyno).as_ordered_terms()
                        for term in monomials:
                            coeff, _ = term.as_coeff_mul()
                            new_I = get_multi_index(term, Z)
                            J_rest = J[:p] + J[p+1:]
                            if (a in J_rest) or (b in J_rest):
                                continue
                            new_J_unsorted = (a,b) + J_rest
                            new_J = tuple(sorted(new_J_unsorted))
                            sign = -1 * ((-1)**p) * wedge_permutation_sign(new_J_unsorted, new_J)
                            M[row[(new_I,new_J)], col] += sign*coeff
    return M

def compute_gE_matrix(n: int, k: int):
    ms  = lambda r: list(itertools.combinations_with_replacement(range(n+1), r))
    inc = lambda r: list(itertools.combinations(range(n+1), r))

    S_k   = [(I, J) for I in ms(k)   for J in inc(k)]
    S_k1  = [(I, J) for I in ms(k+1) for J in inc(k+1)]
    row   = {key: r for r, key in enumerate(S_k1)}

    M = sp.zeros(len(S_k1), len(S_k))

    for col, (I, J) in enumerate(S_k):
        I = list(I);  J = list(J)
        for r in range(n + 1):
            if r in J:          # partial_r already present -> wedge vanishes
                continue
            # monomial part
            I_prime = tuple(sorted(I + [r]))
            # sign = (-1)^{#(j in J with j < r)}
            #sign = -1 if sum(j < r for j in J) % 2 else 1
            sign = wedge_permutation_sign([r] + J, sorted([r] + J))
            # wedge index part
            J_prime = tuple(sorted([r] + J))
            M[row[(I_prime, J_prime)], col] += sign
    return M

def Poisson_differential_proj(n, b_dict, k):
    """
    Compute dim(image(d_pi^k) + image(g_E^k)) - dim(image(g_E^k)).
    """
    if k <=0 or k >= n:
```

```python
        return 0
    # Compute full matrices
    M_d = compute_poisson_differential(n, b_dict, k)
    M_g = compute_gE_matrix(n, k)

    # Concatenate column-wise: [M_d | M_g]
    M_concat = M_d.row_join(M_g)

    # Compute ranks
    rank_concat = M_concat.rank()
    rank_g = M_g.rank()

    return rank_concat - rank_g

def Poisson_cohomology(n, b_dict, k):
    """
    Compute the Poisson cohomology H^k of P^n .

    The input is the dimension n and a dictionary b_dict that contains
    the Poisson bivector coefficients in the form:
    b_dict = {(p,q): [(u,v, c), ...]} where c is the coefficient of
    the monomial z_u * z_v in the Poisson bivector pi^{pq}.
    The output is the dimension of the Poisson cohomology H^k(P^n).
    """
    if k >  n or k < 0:
        return 0
    # First term: binomial product
    term1 = math.comb(n + 1 + k, k) * math.comb(n, k)
    # rank(d_pi^k)
    rank_k = Poisson_differential_proj(n, b_dict, k)
    # rank(d_pi^(k-1))
    rank_prev = Poisson_differential_proj(n, b_dict, k - 1)

    return term1 - rank_k - rank_prev
```