DUISTERMAAT-HECKMAN MEASURES FOR HAMILTONIAN GROUPOID ACTIONS

BY

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DISSERTATION

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Abstract

This thesis treats two problems related to Poisson manifolds of compact types: the existence of Poisson manifolds of strong compact type, and the generalisation of classical Duistermaat-Heckman results to the setting of Hamiltonian actions of symplectic groupoids. In Chapter 4 we prove that all strongly affine circles and 2-tori appear as the leaf space of a regular Poisson manifold of strong compact type. These Poisson manifolds are all fibrations over their leaf space with symplectic leaves diffeomorphic to the smooth manifold underlying a K3 surface. In Chapter 5 we show that for a Hamiltonian action of a regular, source proper symplectic groupoid with sufficiently nice properties there is an analogue of the Duistermaat-Heckman measure which is a *polynomial* measure with respect to the natural integral affine structure.

To my family

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Chapter 1

Introduction

Poisson manifolds of compact type (PMCTs) are the "compact objects" in Poisson geometry. They were first introduced in [1] and their role in the theory is analogous to the one played by compact Lie algebras in Lie theory. Unlike general Poisson manifolds, PMCTs have a rich geometry transverse to their associated symplectic foliation. For example, the leaf space of a regular PMCT inherits the structure of an integral affine orbifold. Roughly speaking this means that the leaf space has an orbifold atlas where the transitions are integral affine maps. The precise statement can be found in [2], [3], where many other properties of PMCTs are discussed.

We recall that a Poisson manifold (M, π) is called *integrable* if there exists some Lie groupoid $\mathcal{G} \rightrightarrows M$ carrying a multiplicative symplectic form $\Omega \in \Omega^2(\mathcal{G})$ for which the target map $\mathbf{t} : (\mathcal{G}, \Omega) \to (M, \pi)$ is a Poisson map. PMCTs are defined as those Poisson manifolds that are integrated by a source connected, Hausdorff symplectic groupoid having a certain compactness property. Contrary to the case of Lie groups and Lie algebras, there are multiple notions of compactness for Lie groupoids, namely a Lie groupoid $\mathcal{G} \rightrightarrows M$ is called

- *proper* if the anchor map $(\mathbf{s}, \mathbf{t}) : \mathcal{G} \to M \times M$ is proper;
- *source proper*, or $\mathbf{s}$-proper, if the source map is proper;
- *compact* if the space of arrows $\mathcal{G}$ is compact.

Accordingly, we say that (M, π) is of *proper/source proper/compact type* if it admits a source connected, Hausdorff symplectic groupoid of proper/source proper/compact type, respectively.

The types just defined depend on the choice of integration of (M, π). However, just like for Lie groups, there is a unique "largest" integration, namely the one with 1-connected source fibers. This is often called the *Weinstein groupoid*. We say that an integrable Poisson manifold has *strong* proper/source proper/compact type if its Weinstein groupoid is Hausdorff and has the corresponding type.

In this thesis, we deal with two topics related to PMCTs. The first concerns the existence of the strongest type of PMCT: Poisson manifolds of strong compact type. The second is the study of *Hamiltonian actions* in the context of symplectic groupoids, and more importantly the generalisation of the classical Duistermaat-Heckman theorems to this setting.

1.1 Poisson manifolds of strong compact type

Just as there is the special class of compact *semisimple* Lie algebras among compact Lie algebras, there is an important distinguished class among PMCTs, namely that of Poisson manifolds of *strong* compact type (PMSCTs). A simple class of examples of PMSCTs is given by compact symplectic manifolds with finite fundamental group, but it is difficult to construct examples that are not symplectic. The first example of a PMSCT that is not symplectic was given by Martinez-Torres in [4]. The construction there is inspired by the work of Kotschick [5], where non-trivial results on the geometry of K3 surfaces are used to construct a free symplectic circle action with contractible orbits. The orbit space of such an action is a PMSCT with smooth leaf space a circle endowed with its standard integral affine structure (that is, the one it inherits as a quotient of $\mathbb{R}$ by $\mathbb{Z}$ acting by translations). In general, it is not known whether any compact integral affine orbifold can appear as the leaf space of a PMSCT. On the one hand constructing strong PMCTs is a difficult problem on its own, and on the other not much is known about the classification of compact integral affine manifolds in dimension greater than two. The integral affine structures on a circle are easily classified, and the classification of integral affine structures on compact 2-dimensional manifolds was obtained in [6], [7]. The main result of Chapter 4 is the following.

Main Theorem 1. *Any strongly integral affine circle or two-dimensional torus can be realised as the leaf space of a PMSCT.*

Here by a *strongly integral affine structure* we mean an integral affine structure with integral translational part (see [8, Remark 5.10] and Remark 4.16).

1.2 Duistermaat-Heckman measures

The study of symmetries has a long history in classical mechanics and its mathematical formalisations. A particularly powerful instance of this is the theory of Hamiltonian actions, where one can perform a "double reduction" using the symmetry (the action) and the conserved quantities (the moment map) [9], [10]. This classical notion of Hamiltonian actions of Lie groups on symplectic manifolds has been thoroughly studied and many remarkable results have been obtained, such as singular reduction [11], convexity [12], [13] and localisation [14], [15]. A special case of the localisation formula appears also in [16], where Hamiltonian actions of a torus are studied and a sequence of results is obtained that culminates in the aforementioned formula. First, the *linear variation theorem* asserts that the symplectic forms on the reduced spaces vary linearly in cohomology. Next it is proved that the pushforward of the Liouville measure by the moment map is a *polynomial* measure, which then results in an integration formula for an oscillatory integral. Several variations of Hamiltonian actions have also been introduced, such as quasi-Hamiltonian actions [17], Hamiltonian actions of Poisson-Lie groups [18] and group-valued moment maps [19]. A more general notion of Hamiltonian action that unifies the ones mentioned above can be formulated using symplectic groupoids. An action of $(\mathcal{G}, \Omega) \rightrightarrows (M, \pi)$ on a symplectic manifold (X, ω) along $\mu : X \to M$ is called *Hamiltonian* if the multiplicativity condition $A^*\omega = \mathrm{pr}_1^*\Omega + \mathrm{pr}_2^*\omega \in \Omega^2(\mathcal{G}\,{}_{\mathbf{s}}\!\times_{\mu} X)$ holds. In Chapter 5 we consider source proper, source connected, regular symplectic groupoids. There is a nice theory of measures on the leaf space (see [20]): they are in some sense "transverse measures" on M, with respect to the orbit foliation. The integral affine structure on the leaf space induces a measure on it which we refer to as the *affine measure*. It plays the role of the Lebesgue measure on the dual of the Lie algebra in the classical setting. There is also the *Duistermaat-Heckman* *measure*, which we define as in the classical case as the pushforward of the Liouville measure on X along the moment map and the quotient map to the leaf space.

Main Theorem 2. *If the action is locally free and effective and the moment is proper and has connected fibers, the Duistermaat-Heckman measure is equal to a polynomial function times the affine measure.*

Just as in the classical case, we have an exact interpretation of the polynomial function. It is, up to integer factors having to do with the isotropy of $\mathcal{G}$, the function that assigns to an orbit the symplectic volume of the orbit times the symplectic volume of the associated reduced space.

Chapter 2

Groupoids in Poisson geometry

We introduce the basics and establish notation for the main objects of study in this thesis, Hamiltonian groupoid actions, and their preliminaries. For a textbook account of this material, see [21].

2.1 Poisson structures

Definition 2.1 (Poisson structures)**.** A *Poisson structure* on a manifold M is a bivector $\pi \in \mathfrak{X}^2(M)$ satisfying $[\pi, \pi] = 0$.

A *Poisson map* between (M_1, π_1) and (M_2, π_2) is a smooth map $\varphi : M_1 \to M_2$ such that π_1 and π_2 are f-related. ♢

Remark 2.2. Equivalently, a Poisson structure can be defined as a Lie bracket $\{\cdot, \cdot\}$ on the space of smooth functions $C^\infty(M)$ which is also a biderivation: $\{f, gh\} = g\{f, h\} + \{f, g\}h$. One can pass between two notions by the relation $\{f, g\} = \pi(df, dg)$. In this formulation, a Poisson map is simply a map that intertwines the brackets.

The cotangent bundle of a Poisson manifold (M, π) is endowed with several interesting structures. Firstly, there is the contraction map

$$\pi^\# : T^*M \to TM, \alpha \mapsto i_\alpha \pi. \tag{2.1}$$

This bundle map is useful for stating certain properties of Poisson manifolds. For instance, a map $\varphi : (M_1, \pi_1) \to (M_2, \pi_2)$ is Poisson if and only if the diagram

$$\begin{tikzcd} T_p^*M_1 \arrow[d, "\pi_1^\#"'] & T_p^*M_2 \arrow[l, "(d_p\varphi)^*"'] \arrow[d, "\pi_2^\#"] \\ T_pM_1 \arrow[r, "d_p\varphi"] & T_pM_2 \end{tikzcd}$$

commutes for all $p \in M$. There are also certain objects attached to Poisson manifolds that we can define in terms of this map. For $p \in M$, one can show that $\ker(\pi_p^\#) \subset T_p^*M$ has the structure of a Lie algebra: the bracket is defined as $[d_pf, d_pg] := d_p\{f, g\}$.

Definition 2.3. The *isotropy Lie algebra* of (M, π) at p is $\mathfrak{g}_p := \ker(\pi_p^\#)$ endowed with the bracket defined above. ♢

In general, $\mathrm{im}(\pi^\#)$ is not a subbundle of TM, but one can show that it still has leaves, in the following sense.

Theorem 2.4 ([21, Theorem 4.1])**.** *There is a partition of M into immersed submanifolds S satisfying*

$$T_pS = \mathrm{im}(\pi_p^\#) \tag{2.2}$$

for all $p \in S$. Furthermore, S carries a symplectic structure $\omega_S \in \Omega^2(S)$ induced by the Poisson bracket:

$$\omega_{S,p}(\pi_p^\#(\alpha), \pi_p^\#(\beta)) := -\pi_p(\alpha, \beta). \tag{2.3}$$

We call these (S, ω_S) the *symplectic leaves* of the Poisson manifold (M, π), and the collection $\{(S, \omega_S)\}$ the *symplectic foliation*. We usually denote it by $\mathcal{F}_\pi$. We call a Poisson manifold *regular* if its leaves have constant dimension (and thus form a regular foliation). In this case, the symplectic forms on the leaves patch together into a foliated symplectic form $\omega_{\mathcal{F}_\pi} \in \Omega^2(T\mathcal{F}_\pi)$.

Lemma 2.5. *Let (M, π) be a regular Poisson manifold. Then the isotropy Lie algebras are abelian.*

Proof. Let $p \in M$. Note that $\mathfrak{g}_p = \nu_p^*(\mathcal{F}_\pi)$. Since π is regular, we can represent two arbitrary elements in the conormal space as d_pf and d_pg, where f and g are functions that are constant along $\mathcal{F}_\pi$ in a neighbourhood of p. This means that $\{f, g\} = 0$ on this neighbourhood, which implies $[d_pf, d_pg] = d_p\{f, g\} = 0$. □

It turns out that some of these notions – the isotropy Lie algebra and the existence of "singular leaves" – are part of a more general picture. The space of 1-forms $\Omega^1(M)$ of a Poisson manifold (M, π) is equipped with a bracket given by

$$[\alpha, \beta]_\pi := \mathcal{L}_{\pi^\#(\alpha)}(\beta) - \mathcal{L}_{\pi^\#(\beta)}(\alpha) - d\big(\pi(\alpha, \beta)\big). \tag{2.4}$$

One can show that this is in fact a Lie bracket, and that it satisfies the Leibniz identity

$$[\alpha, f\beta]_\pi = f[\alpha, \beta]_{\mathcal{A}} + \mathcal{L}_{\pi^\#(\alpha)}(f)\beta. \tag{2.5}$$

This means that the tuple $(T^*M, \pi^\#, [\cdot, \cdot]_\pi)$ is a so-called *Lie algebroid*. We discuss these in the next section.

Example 2.6. Any manifold M admits the *zero Poisson structure* $\pi = 0$. △

Example 2.7. Any symplectic manifold is a Poisson manifold. In fact, let $\pi \in \mathfrak{X}^2(M)$ be any *nondegenerate* bivector. We obtain a (nondegenerate) 2-form $\omega = \pi^{-1} \in \Omega^2(M)$ by $\omega^\flat = (\pi^\#)^{-1}$, and one can show that $[\pi, \pi] = 0$ iff $d\omega = 0$. It follows that the symplectic manifolds are precisely the nondegenerate Poisson manifolds. △

Example 2.8. Let $\mathfrak{g}$ be a Lie algebra. There is an induced Poisson structure π_{lin} on its dual $\mathfrak{g}^*$, called the *linear* Poisson structure, described as follows. For $f, g \in C^\infty(\mathfrak{g}^*)$ and $\xi \in \mathfrak{g}^*$ we identify $d_\xi f, d_\xi g \in \mathfrak{g}$ and set $\{f, g\}(\xi) := \xi([d_\xi f, d_\xi g])$. The symplectic leaves of $(\mathfrak{g}^*, \pi_{\mathrm{lin}})$ are the coadjoint orbits. △

2.2 Lie groupoids & algebroids

2.2.1 Lie algebroids

Definition 2.9 (Lie algebroids)**.** A *Lie algebroid* is a triple $(\mathcal{A} \to M, \rho, [\cdot, \cdot]_{\mathcal{A}})$ consisting of:

- a vector bundle $\mathcal{A} \to M$;
- a vector bundle map $\rho : \mathcal{A} \to TM$ covering the identity;
- a Lie bracket $[\cdot,\cdot]_{\mathcal{A}}$ on the space of sections $\Gamma(\mathcal{A})$

satisfying the Leibniz identity

$$[\alpha, f\beta]_{\mathcal{A}} = f[\alpha,\beta]_{\mathcal{A}} + \mathcal{L}_{\rho(\alpha)}(f)\beta \tag{2.6}$$

for all $\alpha, \beta \in \Gamma(\mathcal{A})$ and $f \in C^\infty(M)$.

A *Lie algebroid morphism* between $(\mathcal{A}_1 \to M_1, \rho_1, [\cdot,\cdot]_{\mathcal{A}_1})$ and $(\mathcal{A}_2 \to M_2, \rho_2, [\cdot,\cdot]_{\mathcal{A}_2})$ is a bundle map (ϕ, φ) that intertwines the anchors and brackets. ♢

Remark 2.10. The precise meaning of "intertwining the brackets" is quite complicated, unless φ is a diffeomorphism. We don't need these details here, so we omit them.

We will usually refer to a Lie algebroid simply by $\mathcal{A} \to M$ or just $\mathcal{A}$. Associated to a Lie algebroid we have at each $p \in M$ the *isotropy Lie algebra* at p, defined as $\mathfrak{g}_p := \ker(\rho_p)$: the bracket $[\cdot,\cdot]_{\mathcal{A}}$ restricts to a Lie algebra structure on $\mathfrak{g}_p$. The image of the anchor gives a singular foliation, and we call its leaves the *orbits* of $\mathcal{A}$. When the anchor has constant rank, this is in fact a regular foliation and in this case we say that the Lie algebroid is *regular*.

Example 2.11. One of the simplest examples of a Lie algebroid is the tangent bundle $TM \to M$, with $\rho = \mathrm{id}$ and the bracket given by the Lie bracket of vector fields. △

Example 2.12. As we saw in Section 2.1, for any Poisson manifold (M, π) we have a Lie algebroid $(T^*M, \pi^\#, [\cdot,\cdot]_\pi)$. This is often called the *cotangent algebroid*. △

Example 2.13. Any Lie algebra $\mathfrak{g}$ is a Lie algebroid over $M = \mathrm{pt}$. One can think of Lie algebroids as a generalisation of Lie algebras in this way. △

Example 2.14. Given a Lie algebra action $\mathfrak{g} \to \mathfrak{X}(M)$, we can form the action algebroid $\mathfrak{g} \ltimes M$ as follows. As a bundle, it is just the trivial bundle $\mathfrak{g} \times M$. The bracket is given on constant sections by $[\cdot,\cdot]_{\mathfrak{g}}$ and extended to general sections by the Leibniz rule. The anchor map is induced by the action map itself. △

As we know, there is a rich Lie theory associated to Lie algebras. They are the "infinitesimal objects" and have Lie groups as their "global" counterparts. It turns out that a similar story holds for Lie algebroids. This leads us to Lie groupoids, which provide a "global" counterpart to Lie algebroids, and thus also to Poisson manifolds.

2.2.2 Lie groupoids

Definition 2.15 ((Lie) groupoids)**.** A *groupoid* is a small category in which all arrows are invertible. Specifically, it consists of

- a set $\mathcal{G}$ of *arrows*;
- a set M of *objects*;
- maps $\mathbf{s}, \mathbf{t} : \mathcal{G} \to M$ called the *source* and *target* maps, respectively;

- the *multiplication* map $m : \mathcal{G}^{(2)} \to \mathcal{G}, (g,h) \mapsto gh$, defined on the set $\mathcal{G}^{(2)} = \mathcal{G} \,{}_{\mathbf{s}}\!\times_{\mathbf{t}} \mathcal{G}$ of *composable arrows*;
- the *unit* map $u : M \to \mathcal{G}, p \mapsto 1_p$;
- the *inversion* map $i : \mathcal{G} \to \mathcal{G}, g \mapsto g^{-1}$;

subject to the following axioms:

- *composition:* for all $(g,h) \in \mathcal{G}^{(2)}$, $s(gh) = s(h)$ and $t(gh) = t(g)$;
- *associativity:* for all $g,h,k \in \mathcal{G}$ such that $(g,h) \in \mathcal{G}^{(2)}$ and $(h,k) \in \mathcal{G}^{(2)}$, $(gh)k = g(hk)$;
- *units:* for all $g \in G$, $g1_{s(g)} = g$ and $1_{t(g)}g = g$;
- *inverses:* for all $g \in G$, $gg^{-1} = 1_{t(g)}$ and $g^{-1}g = 1_{s(g)}$.

A *Lie groupoid* is a groupoid where $\mathcal{G}$ and M are smooth manifolds, the maps $\mathbf{s}, \mathbf{t}, m, u, i$ are smooth and $\mathbf{s}$ and $\mathbf{t}$ are submersions.

A *Lie groupoid morphism* between $\mathcal{G}_1 \rightrightarrows M_1$ and $\mathcal{G}_2 \rightrightarrows M_2$ is a pair (Φ, φ) of maps $\Phi : \mathcal{G}_1 \to \mathcal{G}_2$ and $\varphi : M_1 \to M_2$ that intertwine the structure maps. $\diamondsuit$

Remark 2.16. Note that since $\mathbf{s}$ and $\mathbf{t}$ are submersions, the space $\mathcal{G}^{(2)}$ inherits the structure of a smooth manifold. The smoothness of m is with respect to this structure.

Remark 2.17. Often in the study of Lie groupoids the space of arrows $\mathcal{G}$ is not required to be Hausdorff. In this thesis, we exclusively work with Lie groupoids of compact type, which we *do* require to be Hausdorff.

We will often denote a Lie groupoid by $\mathcal{G} \rightrightarrows M$. There are several objects associated to a (Lie) groupoid. For all $p \in M$ we have

- the source and target fibres $\mathbf{s}^{-1}(p)$ and $\mathbf{t}^{-1}(p)$: these are submanifolds of $\mathcal{G}$;
- the *isotropy group* at p, defined as $\mathcal{G}_p := \mathbf{s}^{-1}(p) \cap \mathbf{t}^{-1}(p)$: the multiplication on $\mathcal{G}$ restricts to a group structure on $\mathcal{G}_p$, and this makes it into a Lie group;
- the *orbit* through x, defined as $\mathcal{O}_p := \mathbf{t}(\mathbf{s}^{-1}(p))$: this inherits a smooth structure as the base of the principal bundle $\mathcal{G}_p \circlearrowright \mathbf{s}^{-1}(p) \xrightarrow{\mathbf{t}} \mathcal{O}_p$, making it into an immersed submanifold of M.

The *leaf space* of $\mathcal{G} \rightrightarrows M$, written as $B = M/\mathcal{G}$, is defined as the set of orbits $\{\mathcal{O}_p \mid p \in M\}$. In general, this is just a topological space. When the dimension of the orbits is constant, we say the groupoid is *regular.*

For an arrow $g \in \mathcal{G}$, left and right multiplication are only defined on target and source fibres, respectively. More precisely, we have

$$L_g : \mathbf{t}^{-1}(\mathbf{s}(g)) \to \mathbf{t}^{-1}(\mathbf{t}(g)), h \mapsto gh, \qquad R_g : \mathbf{s}^{-1}(\mathbf{t}(g)) \to \mathbf{s}^{-1}(\mathbf{s}(g)), h \mapsto hg. \tag{2.7}$$

Example 2.18. For any manifold M we can form the *pair groupoid* $M \times M \rightrightarrows M$. The structure maps are given by $\mathbf{s}(p_1,p_2) = p_2$, $\mathbf{t}(p_1,p_2) = p_1$, $u(p) = (p,p)$, $(p_1,p_2)^{-1} = (p_2,p_1)$ and $(p_1,p_2)(p_2,p_3) = (p_1,p_3)$. Essentially we can think of (p_1,p_2) as an arrow from p_2 to p_1. $\triangle$

Example 2.19. Following the spirit of the previous example, we have the *fundamental groupoid* $\Pi_1(M) \rightrightarrows M$. The space of arrows $\Pi_1(M)$ consists of equivalence classes of (smooth) paths $[0,1] \to M$, where the equivalence is homotopy of paths with fixed end points. The source and target are given by the start and end point of the path, the unit map sends $p \in M$ to the constant path at p, inversion reverses the orientation of a path, and multiplication is concatenation. From the definition is follows that the isotropy groups are the fundamental groups (as discrete groups). △

Example 2.20. Any Lie group G is a Lie groupoid over $M = \mathrm{pt}$. △

Example 2.21. Given a Lie group action $G \circlearrowright X$ we can form the *action groupoid* $G \ltimes X \rightrightarrows X$ as follows: $G \ltimes X := G \times X$, $\mathbf{s} := \mathrm{pr}_2$, $\mathbf{t}$ is the action map, $u(x) := (e_G, x)$ and multiplication is given by

$$(g, hx)(h, x) := (gh, x). \tag{2.8}$$

The groupoid terminology is compatible with the action terminology: the isotropy groups & orbits of $G \ltimes X$ are the isotropy groups & orbits of the action. △

2.2.3 The Lie algebroid of a Lie groupoid

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. There is an associated Lie algebroid that we describe as follows. The vector bundle is $\mathcal{A} := \ker(d\mathbf{t})|_M$, and the anchor is given by $\rho := d\mathbf{s}|_{\mathcal{A}}$. To describe the bracket, we note that we can identify sections of $\mathcal{A}$ with *left invariant* vector fields on $\mathcal{G}$.

Definition 2.22. A vector field $V \in \mathfrak{X}(\mathcal{G})$ is called *left invariant* if

(i) $V \in \Gamma(\ker(d\mathbf{t}))$;

(ii) $d_h L_g(V_h) = V_{gh}$ for all $(g, h) \in \mathcal{G}^{(2)}$. ♢

It is now straightforward to see that the assignment $V \mapsto V|_M$ establishes a one-to-one correspondence between left invariant vector fields and sections of $\mathcal{A}$, and thus the Lie bracket of vector fields induces a bracket $[\cdot,\cdot]_{\mathcal{A}}$ on $\Gamma(\mathcal{A})$. It is easy to check that this bracket satisfies the Leibniz identity and thus that $(\mathcal{A}, \rho, [\cdot,\cdot]_{\mathcal{A}})$ is a Lie algebroid. We call it the Lie algebroid of the groupoid $\mathcal{G} \rightrightarrows M$, often denoted $\mathcal{A} = \mathrm{Lie}(\mathcal{G})$.

Remark 2.23. Often in the literature the Lie algebroid of a Lie groupoid is defined using *right invariant* vector fields instead. One can use the inversion map to move between these two definitions. Usually the difference between the two conventions is obvious, with occasionally a minus sign in the formulas. For more details, see [21, Appendix D].

There are several obvious relations between objects associated to a Lie groupoid and its algebroid. For $p \in M$, the isotropy Lie algebra $\mathfrak{g}_p$ of the algebroid is the Lie algebra of the isotropy Lie group $\mathcal{G}_p$, and the orbits of the Lie algebroid are the connected components of the orbits of $\mathcal{G}$. Furthermore, a Lie groupoid morphism $(\Phi, \varphi) : (\mathcal{G}_1 \rightrightarrows M_1) \to (\mathcal{G}_2 \rightrightarrows M_2)$ induces a map $\mathrm{Lie}(\mathcal{G}_1) \to \mathrm{Lie}(\mathcal{G}_2)$ by differentiating Φ at the units: $\mathrm{Lie}(\Phi, \varphi) = (d\Phi|_{\mathrm{Lie}(\mathcal{G}_1)}, \varphi)$.

Example 2.24. The Lie algebroid associated to both the pair groupoid $M \times M$ and the fundamental groupoid $\Pi_1(M)$ is the tangent bundle TM. △

Example 2.25. It is clear from the construction that the Lie functor for Lie groupoids applied to a Lie group yields the classical Lie functor for Lie groups. △

Example 2.26. Given a Lie group action $G \circlearrowleft X$, we have $\mathrm{Lie}(G \ltimes X) = \mathfrak{g} \ltimes X$, the action algebroid of the induced infinitesimal action of $\mathfrak{g} = \mathrm{Lie}(G)$. △

2.3 Symplectic groupoids

Let us now turn to the additional structure one can put on a Lie groupoid to ensure its Lie algebroid comes from a Poisson structure.

Definition 2.27 (Symplectic groupoids)**.** A *symplectic groupoid* is a pair $(\mathcal{G}, \Omega)$ consisting of a Lie groupoid $\mathcal{G} \rightrightarrows M$ and a multiplicative symplectic form $\Omega \in \Omega^2(\mathcal{G})$. ♢

Remark 2.28. Recall that a form $\alpha \in \Omega^\bullet(\mathcal{G})$ is called *multiplicative* if

$$m^*\alpha = \mathrm{pr}_1^*\alpha + \mathrm{pr}_2^*\alpha,$$

where $m, \mathrm{pr}_1, \mathrm{pr}_2 : \mathcal{G}^{(2)} \to \mathcal{G}$ denote the multiplication and projection maps.

Lemma 2.29 ([21, Proposition 14.9])**.** *Let $(\mathcal{G}, \Omega)$ be a symplectic groupoid. Then we have*

$$\ker(d\mathbf{s})^\Omega = \ker(d\mathbf{t}), \tag{2.9}$$

$$TM^\Omega = TM. \tag{2.10}$$

In particular, this tells us that $\dim(\mathcal{G}) = 2\dim(M)$.

Symplectic groupoids are the "global objects" corresponding to Poisson manifolds, in the sense of the following result.

Theorem 2.30 ([21, Theorem 14.10])**.** *Let $(\mathcal{G}, \Omega)$ be a symplectic groupoid. There is a unique Poisson structure π on the space of objects M such that $\mathbf{t} : (\mathcal{G}, \Omega) \to (M, \pi)$ is a Poisson map.*

In this case, the Lie algebroid of $\mathcal{G} \rightrightarrows M$ is isomorphic to the cotangent algebroid associated to (M, π). The symplectic form Ω induces the isomorphism $\sigma_\Omega : \mathrm{Lie}(\mathcal{G}) \to T^*M$ given by

$$\sigma_\Omega(\alpha) := -\Omega(\alpha, \cdot)|_M. \tag{2.11}$$

Example 2.31. Consider T^*M viewed as a Lie groupoid over M with both source and target given by the projection $T^*M \to M$ and fiberwise addition as multiplication. The canonical symplectic form ω_{can} is multiplicative, so that $(T^*M, \omega_{\mathrm{can}})$ is a symplectic groupoid. The induced Poisson structure on M is the zero Poisson structure. △

Example 2.32. Given a symplectic manifold (M, ω), the pair groupoid becomes a symplectic groupoid endowed with the symplectic form $\omega \oplus -\omega = \mathbf{t}^*\omega - \mathbf{s}^*\omega$. The induced Poisson structure is of course $\pi = \omega^{-1}$. In fact, the same holds for the fundamental groupoid $(\Pi_1(M), \mathbf{t}^*\omega - \mathbf{s}^*\omega)$. △

Example 2.33. Let G be a Lie group and $\mathfrak{g} = \mathrm{Lie}(G)$ its a Lie algebra. Let $G \ltimes \mathfrak{g}^* \rightrightarrows \mathfrak{g}^*$ be the action groupoid associated to the coadjoint action $G \circlearrowleft \mathfrak{g}^*$. Since $G \times \mathfrak{g}^* \cong T^*G$, this groupoid inherits a symplectic form $-\omega_{\mathrm{can}}$ from the canonical symplectic form on the cotangent bundle. It can be shown that this form is multiplicative and thus that $(G \ltimes \mathfrak{g}^*, -\omega_{\mathrm{can}})$ is a symplectic groupoid. The induced Poisson structure on $\mathfrak{g}^*$ is π_{lin}. △

2.4 Hamiltonian actions

Definition 2.34 (Groupoid actions)**.** A *(left) action* of a groupoid $\mathcal{G} \rightrightarrows M$ on a set X along a map $\mu : X \to M$ is a map $A : \mathcal{G}\ {}_{\mathbf{s}}\!\times_{\mu} X \to X, (g, x) \mapsto gx$ satisfying the following properties:

- for all $(g, x) \in \mathcal{G}\ {}_{\mathbf{s}}\!\times_{\mu} X$, $\mu(gx) = \mathbf{t}(g)$;
- for all $g, h \in \mathcal{G}$ and $x \in X$ such that $(g, h) \in \mathcal{G}^{(2)}$ and $(h, x) \in \mathcal{G}\ {}_{\mathbf{s}}\!\times_{\mu} X$, $g(hx) = (gh)x$;
- for all $x \in X$, $1_{\mu(x)}x = x$.

If $\mathcal{G} \rightrightarrows M$ is a Lie groupoid, X is a manifold and μ a smooth map, we say the action is *smooth* if the map A is smooth. ♢

The familiar notions associated to (Lie) group actions also make sense in this context: for $x \in X$ we have the *isotropy group* at x

$$\mathcal{G}_x := \{g \in \mathbf{s}^{-1}(\mu(x)) \mid gx = x\} \subset \mathcal{G}_{\mu(x)} \tag{2.12}$$

and the *orbit* through x

$$\mathcal{G} \cdot x := \{gx \mid g \in \mathbf{s}^{-1}(\mu(x))\}. \tag{2.13}$$

The quotient space $X/\mathcal{G}$ is defined as usual as the collection of orbits. We can form the action groupoid $\mathcal{G} \ltimes X \rightrightarrows X$ just as in Example 2.21, with some slight modifications: we set $\mathcal{G} \ltimes X := \mathcal{G}\ {}_{\mathbf{s}}\!\times_{\mu} X$, $\mathbf{s} := \mathrm{pr}_2$, $\mathbf{t} = A$, $u(x) := (1_{\mu(x)}, x)$ and multiplication as

$$(g, hx)(h, x) := (gh, x). \tag{2.14}$$

The isotropy groups, orbits and quotient space defined above can be recognised as the isotropy groups, orbits and leaf space of this groupoid. In particular, in the smooth case the isotropy groups and orbits inherit smooth structure as described in Section 2.2.2.

We say that the action is *free* if the isotropy groups are trivial and *locally free* if they are discrete. We say that the action is *proper* if the map $\mathcal{G}\ {}_{\mathbf{s}}\!\times_{\mu} X \to X \times X$ given by $(g, x) \mapsto (gx, x)$ is proper.

Lemma 2.35. *If a Lie groupoid $\mathcal{G} \rightrightarrows M$ acts freely and properly on a manifold X, the quotient $X/\mathcal{G}$ inherits a unique smooth structure making the quotient map $X \to X/\mathcal{G}$ into submersion.*

Proof. This follows from the "Godement Criterion" for quotient manifolds. □

Definition 2.36 (Lie algebroid actions)**.** An *action* of a Lie algebroid $\mathcal{A} \to M$ on a manifold X along a map $\mu : X \to M$ is a Lie algebra morphism $a : (\Gamma(\mathcal{A}), [\cdot, \cdot]_{\mathcal{A}}) \to (\mathfrak{X}(X), [\cdot, \cdot])$ satisfying

(i) $d_x\mu(a(\alpha)_x) = \rho(\alpha)_{\mu(x)}$ for all $\alpha \in \Gamma(\mathcal{A})$ and $x \in X$;

(ii) $a(f\alpha) = \mu^*(f)a(\alpha)$ for all $\alpha \in \Gamma(\mathcal{A})$ and $f \in C^\infty(M)$. ♢

Remark 2.37. We usually denote the action by $a(\alpha) = \alpha^X \in \mathfrak{X}(X)$.

Associated to a Lie algebroid action is the action Lie algebroid $\mathcal{A} \ltimes X \to X$, with $\mathcal{A} \ltimes X := \mu^*\mathcal{A}$, the anchor map being the bundle map induced by the action, and the (unique) bracket determined by $[\mu^*\alpha, \mu^*\beta] = \mu^*[\alpha, \beta]_{\mathcal{A}}$. This gives us for every $x \in M$ the isotropy Lie algebra $\mathfrak{g}_x$ and the orbit $\mathcal{A} \cdot x$.

Given a Lie groupoid $\mathcal{G} \rightrightarrows M$ acting on $\mu : X \to M$, there is an induced infinitesimal action of $\mathcal{A} = \mathrm{Lie}(\mathcal{G})$ which we describe as follows. For fixed $x \in X$, we consider the map $\mathbf{t}^{-1}(\mu(x)) \to X$ given by $g \mapsto g^{-1}x$. Taking its derivative at $1_{\mu(x)}$ yields a map $\mathcal{A}_{\mu(x)} \to T_xX$ and these maps fit together into a bundle map $\mu^*\mathcal{A} \to TX$. The induced map on sections $a : \Gamma(\mathcal{A}) \to \mathfrak{X}(X)$ can be checked to give a Lie algebroid action. In this case, just as in Example 2.26 we have $\mathrm{Lie}(\mathcal{G} \ltimes X) = \mathcal{A} \ltimes X$ and hence the obvious relations between isotropy and orbits.

Remark 2.38. The use of the inverse here is a consequence of our conventions in defining the Lie algebroid of a Lie groupoid.

Definition 2.39 (Hamiltonian actions)**.** An action A of a symplectic groupoid $(\mathcal{G}, \Omega) \rightrightarrows (M, \pi)$ on a symplectic manifold (X, ω) along $\mu : X \to M$ is called *Hamiltonian* if

$$A^*\omega = \mathrm{pr}_1^*\Omega + \mathrm{pr}_2^*\omega, \tag{2.15}$$

where $\mathrm{pr}_1 : \mathcal{G}\ {}_{\mathbf{s}}\!\times_{\mu} X \to \mathcal{G}$ and $\mathrm{pr}_2 : \mathcal{G}\ {}_{\mathbf{s}}\!\times_{\mu} X \to X$ denote the projection maps. ♢

In this context, we call μ the *moment map* of the action. It is a Poisson map and satisfies the "moment map condition"

$$i_{\alpha^X}\omega = \mu^*(\sigma_\Omega(\alpha)). \tag{2.16}$$

When $\mathcal{G}$ is source connected, this is in fact equivalent to (2.15).

We can display the data of a Hamiltonian action in a diagram as follows:

$$\begin{tikzcd}
(\mathcal{G}, \Omega) \arrow[d, shift left] \arrow[d, shift right] \arrow[r, phantom, "\circlearrowright"] & (X, \omega) \arrow[dl, "\mu"] \\
(M, \pi) &
\end{tikzcd}$$

Lemma 2.40. *For $x \in X$ we have*

$$T_x(\mathcal{G} \cdot x) = \ker(d_x\mu)^\omega, \tag{2.17}$$

$$\sigma_\Omega(\mathfrak{g}_x) = \mathrm{im}(d_x\mu)^\circ. \tag{2.18}$$

Proof. For the first equality, note that using the moment map condition (2.16) we have

$$\begin{aligned}
v \in T_x(\mathcal{G} \cdot x)^\omega &\iff \omega_x(\alpha^X, v) = 0 \text{ for all } \alpha \in \mathcal{A}_{\mu(x)} \\
&\iff \sigma_\Omega(\alpha)_{\mu(x)}(d_x\mu(v)) = 0 \text{ for all } \alpha \in \mathcal{A}_{\mu(x)} \\
&\iff \Omega(\alpha_{\mu(x)}, d_x\mu(v)) = 0 \text{ for all } \alpha \in \mathcal{A}_{\mu(x)} \\
&\iff d_x\mu(v) = 0.
\end{aligned}$$

Here in the last equivalence we use (2.10).

For the second equality, note first that for any $\alpha \in \mathfrak{g}_x$ we have $\alpha^X = 0$ and thus by (2.16)

$$\sigma_\Omega(\alpha)(d_x\mu(v)) = 0 \text{ for all } v \in T_xX.$$

This shows that $\sigma_\Omega(\mathfrak{g}_x) \subset \mathrm{im}(d_x\mu)^\circ$. Then a simple dimension count shows that the dimensions of the left and right side spaces coincide. $\square$

The following example shows how this notion of Hamiltonian action generalises the classical one.

Example 2.41. Let $G \circlearrowleft (X, \omega) \xrightarrow{\mu} \mathfrak{g}^*$ be a classical Hamiltonian action. The symplectic groupoid $(G \ltimes \mathfrak{g}^*, -\omega_{\mathrm{can}})$ acts on (X, ω) along μ as $(g, \xi) \cdot x := g \cdot x$, where the right hand side denotes the action $G \circlearrowleft X$. It is easily verified that this action is Hamiltonian in the groupoid sense. $\triangle$

Chapter 3

Poisson manifolds of compact types and their leaf spaces

In this chapter we introduce Poisson manifolds of compact type and explain two fundamental results: the existence of integral affine structure on their leaf space in Section 3.5 and the linear variation theorem in Section 3.6.

3.1 Integration of Lie algebroids & Poisson manifolds

In Section 2.2.3 we describe the *Lie functor* in the context of Lie groupoids and algebroids. Just as in the context of Lie groups and algebras, one can wonder about "inverting" this functor.

Definition 3.1. A Lie algebroid $\mathcal{A} \to M$ is *integrable* if there exists a Lie groupoid $\mathcal{G} \rightrightarrows M$ such that $\mathrm{Lie}(\mathcal{G}) = \mathcal{A}$. In this case $\mathcal{G} \rightrightarrows M$ is called an *integration* of $\mathcal{A}$. $\diamondsuit$

There are analogues of Lie's Three Theorems in this context.

Definition 3.2. A Lie groupoid $\mathcal{G} \rightrightarrows M$ is called *source connected*, or **s**-connected, if the source fibres are connected. It is called *source 1-connected* if the source fibres are connected and simply connected. $\diamondsuit$

Theorem 3.3 (Lie's First Theorem)**.** *An integrable Lie algebroid has, up to isomorphism, a unique source 1-connected integration.*

The unique source 1-connected integration is often called the *Weinstein groupoid* of $\mathcal{A}$.

Theorem 3.4 (Lie's Second Theorem)**.** *Let $\mathcal{G}_1 \rightrightarrows M_1$ and $\mathcal{G}_2 \rightrightarrows M_2$ be Lie groupoids and suppose that $\mathcal{G}_1 \rightrightarrows M_1$ is source 1-connected. Then for any Lie algebroid morphism $(\phi, \varphi) : \mathrm{Lie}(\mathcal{G}_1) \to \mathrm{Lie}(\mathcal{G}_2)$ there exists a unique morphism $(\Phi, \varphi) : (\mathcal{G}_1 \rightrightarrows M_1) \to (\mathcal{G}_2 \rightrightarrows M_2)$ such that $\mathrm{Lie}(\Phi, \varphi) = (\phi, \varphi)$.*

Contrary to the case of Lie groups and algebras, Lie's Third Theorem does not hold for groupoids and algebroids. However, the obstructions are well understood [22]. They are formulated in terms of the *monodromy groups*. For $p \in M$ the monodromy group at p, denoted $N_p(\mathcal{A})$, is a certain additive subgroup of the isotropy at p. We will not go into the general definition, but we will explain a way of computing the monodromy groups for regular Poisson manifolds in Section 3.1.1.

Theorem 3.5 ([22, Theorem 4.1])**.** *A Lie algebroid $\mathcal{A} \to M$ is integrable if and only if the monodromy groups are uniformly discrete.*

Here *uniformly discrete* means that each $N_p(\mathcal{A})$ is discrete and additionally that there is a neighbourhood U of the zero section $M \subset \mathcal{A}$ such that $U \cap N_p(\mathcal{A}) = 0$ for all $p \in M$.

Definition 3.6. A Poisson manifold (M, π) is called *integrable* if there is a symplectic groupoid $(\mathcal{G}, \Omega) \rightrightarrows M$ whose induced Poisson structure is π. ◊

We know that in this case the groupoid $\mathcal{G} \rightrightarrows M$ integrates the cotangent algebroid T^*M through the isomorphism σ_Ω. In general, it is not true that any integration of T^*M can be made into a symplectic groupoid integrating (M, π). However, if the integration is source 1-connected we have the following result.

Theorem 3.7 ([21], [23])**.** *Let (M, π) be a Poisson manifold and $\mathcal{G} \rightrightarrows M$ a source 1-connected groupoid which integrates T^*M through an isomorphism $\sigma : \mathrm{Lie}(\mathcal{G}) \to T^*M$. Then there is a unique multiplicative form $\Omega \in \Omega^2(\mathcal{G})$ such that $(\mathcal{G}, \Omega)$ integrates (M, π) and $\sigma = \sigma_\Omega$.*

The Weinstein groupoid of a Poisson manifold, as a symplectic groupoid, is denoted $\Sigma(M, \pi)$.

The above result implies that a Poisson manifold is integrable if and only if its cotangent algebroid is and in particular that integrability is governed by its monodromy groups. For a Poisson manifold (M, π), we denote the monodromy group at $p \in M$ by $N_p(M, \pi)$.

3.1.1 Integration of regular Poisson manifolds

Let (M, π) be a regular Poisson manifold and fix $p \in M$. We will give a description of $N_p(M, \pi)$ as "variation of symplectic area". For a proof of this result, see [24, Section 6].

Let us write $S = S_p$ for the symplectic leaf through p. We will describe the *monodromy map* $\partial_p : \pi_2(S, p) \to \nu_p^*(S)$ whose image equals the monodromy group $N_p(M, \pi)$. Let $[\sigma] \in \pi_2(S, p)$, represented by a smooth based map $\sigma : (S^2, p_N) \to (S, p)$, and let $v \in \nu_p(S)$. One can show that there exists a path $t \mapsto p_t$ and a smooth family of maps $\sigma_t : (S^2, p_N) \to (S_t, p_t)$ such that $p_0 = p$, $[\dot{p}_0] = v$ and $\sigma_0 \sigma$. Here we write (S_t, ω_t) for the symplectic leaf through p_t. Then we have

$$\partial_p([\sigma])(v) = \left.\frac{d}{dt}\right|_{t=0} \int_{S^2} \sigma_t^* \omega_t. \tag{3.1}$$

3.2 Poisson manifolds of compact types

Definition 3.8. A Lie groupoid $\mathcal{G} \rightrightarrows M$ is called

- *proper* if the map $(\mathbf{s}, \mathbf{t}) : \mathcal{G} \to M \times M$ is proper;
- *source proper*, or $\mathbf{s}$-proper, if $\mathbf{s} : \mathcal{G} \to M$ is proper;
- *compact* if $\mathcal{G}$ is compact. ◊

Definition 3.9. Let $\mathcal{C} \in \{\text{proper}, \mathbf{s}{-}\text{proper}, \text{compact}\}$. We say that a Poisson manifold is *of $\mathcal{C}$ type* if it can be integrated by a Hausdorff $\mathbf{s}$-connected symplectic groupoid having property $\mathcal{C}$. We say it is *of strong $\mathcal{C}$ type* if its source 1-connected integration is smooth, Hausdorff and has property $\mathcal{C}$. ◊

In general, any of the above are referred to as *Poisson manifolds of compact types*, or PMCTs for short. Below we describe in detail one of the fundamental properties of regular PMCTs, namely the existence of an induced integral affine orbifold structure on the leaf space.

Example 3.10. As we saw in Example 2.31, the Weinstein groupoid of $(M, 0)$ is given by the cotangent bundle $(T^*M, \omega_{\mathrm{can}})$, which is not of proper type. However, it can be shown that given an integral affine structure $\Lambda \subset T^*M$ (see Section 3.4 below), the quotient T^*M/Λ provides a source proper integration. △

Example 3.11. In Example 2.32 we saw two integrations of a symplectic manifold (M, ω), namely the pair groupoid and the fundamental groupoid. In fact, the latter is the Weinstein groupoid of (M, ω) (the source fibres are the universal cover of M). Since $M \times M$ is always proper, we see that (M, ω) is always of proper type, and that it is of compact type if M is compact. Since the isotropy groups of the fundamental groupoid are just the fundamental groups of M, it follows that (M, ω) is of strong proper type if $\pi_1(M)$ is finite, and of strong compact type if in addition M is compact. △

Example 3.12. Let $(\mathfrak{g}^*, \pi_{\mathrm{lin}})$ be a linear Poisson manifold. From Example 2.33 we know that any Lie group G integrating $\mathfrak{g}$ induces an integration $G \ltimes \mathfrak{g}^*$. Thus $(\mathfrak{g}^*, \pi_{\mathrm{lin}})$ is of source proper type if $\mathfrak{g}$ admits a compact integration and of strong source proper type if $\mathfrak{g}$ admits a compact, 1-connected integration. △

3.3 Orbifolds

Definition 3.13. An *orbifold atlas* on a topological space B is a proper foliation groupoid $\mathcal{B} \rightrightarrows M$ together with a homeomorphism $q : M/\mathcal{B} \to B$. An *orbifold* is a triple $(B, \mathcal{B} \rightrightarrows M, q)$ of such data. An *equivalence* between orbifolds $(B, \mathcal{B} \rightrightarrows M, q)$ and $(B', \mathcal{B}' \rightrightarrows M', q')$ is a Morita equivalence $(\mathcal{B} \rightrightarrows M) \cong (\mathcal{B}' \rightrightarrows M')$ whose induced homeomorphism $B \cong B'$ intertwines q and q'. ♢

Remark 3.14. Recall that a Lie groupoid $\mathcal{B} \rightrightarrows M$ is called a *foliation groupoid* if the isotropy groups $\mathcal{B}_p$ are discrete for all $p \in M$. A proper foliation groupoid is also called an *orbifold groupoid.*

In this thesis, the orbifolds we are dealing with are actually quotients of groupoids on the nose. Thus Definition 3.13 simplifies significantly and we can essentially just think of orbifolds & equivalences as orbifold groupoids & Morita equivalences.

3.4 Integral affine structures

Definition 3.15. An *integral affine structure* on a manifold M is a maximal atlas such that the transition functions are integral affine maps. ♢

Remark 3.16. Recall that an integral affine map is a map $\mathbb{R}^q \to \mathbb{R}^q$ of the form $x \mapsto Ax + b$, where $A \in \mathrm{GL}_{\mathbb{Z}}(\mathbb{R}^q)$ and $b \in \mathbb{R}^q$. We denote the group of integral affine maps by $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^q)$.

Lemma 3.17. *An integral affine structure on a manifold M is the same thing as a lattice $\Lambda \subset T^*M$ that is locally spanned by closed 1-forms. In this correspondence, the lattice is given locally in a chart $(U, \varphi = (x^1, \dots, x^q))$ by*

$$\Lambda|_U = \mathbb{Z}\, dx^1 + \cdots + \mathbb{Z}\, dx^q$$

The lattice point of view is usually the more convenient one to work with, but we will use the two notions interchangeably.

Definition 3.18. A *transverse integral affine structure* on a regular foliated manifold $(M, \mathcal{F})$ is a foliation atlas for $\mathcal{F}$ such that the transition functions are integral affine maps. $\diamondsuit$

Remark 3.19. To clarify, we consider the charts of a foliation atlas to be submersions whose fibres are the (restrictions of) leaves of the foliation.

Lemma 3.20. *A transverse integral affine structure on $(M, \mathcal{F})$ is the same thing as a lattice $\Lambda \subset \nu^*(\mathcal{F})$ that is locally spanned by $\mathcal{F}$-basic closed 1-forms. In this correspondence, the lattice is given locally in a foliation chart $(U, \varphi = (x^1, \ldots, x^q))$ by*

$$\Lambda|_U = \mathbb{Z}\,dx^1 + \cdots + \mathbb{Z}\,dx^q$$

Definition 3.21. Let $(B, \mathcal{B} \rightrightarrows M, q)$ be an orbifold and write $\mathcal{F}$ for the foliation on M induced by $\mathcal{B}$. An *integral affine structure* on $(B, \mathcal{B} \rightrightarrows M, q)$ is a $\mathcal{B}$-invariant transverse integral affine structure on $(M, \mathcal{F})$. $\diamondsuit$

Remark 3.22. If $\mathcal{B}$ is $\mathbf{s}$-connected, the invariance condition is automatic.

3.5 The leaf space of regular PMCTs

Let $\mathcal{G} \rightrightarrows M$ be a regular proper Lie groupoid. For $p \in M$, we consider $\mathcal{G}_p^\circ$, the connected component of the isotropy group $\mathcal{G}_p$ containing the identity, and form

$$\mathcal{T}(\mathcal{G}) = \bigsqcup_{p \in M} \mathcal{G}_p^\circ. \tag{3.2}$$

One can show that this is a Lie subgroupoid and thus that the quotient

$$\mathcal{B}(\mathcal{G}) := \mathcal{G}/\mathcal{T}(\mathcal{G}) \tag{3.3}$$

is a Lie groupoid. Moreover, since $\mathcal{G}$ is proper, this groupoid will be Hausdorff and also proper. Of course, by construction $\mathcal{B}(\mathcal{G})$ is a foliation groupoid and has the same leaf space as $\mathcal{G}$, meaning that it defines an orbifold structure on the leaf space B.

Now suppose that $(\mathcal{G}, \Omega) \rightrightarrows (M, \pi)$ is a regular $\mathbf{s}$-connected proper symplectic groupoid. We write $\mathcal{F}_\pi$ for the induced symplectic foliation. In this case, there is an induced integral affine structure on the leaf space, obtained as follows. For $p \in M$:

(i) the isotropy Lie algebra $\mathfrak{g}_p$ is abelian (Lemma 2.5), and thus we obtain a lattice $\ker(\exp : \mathfrak{g}_p \to \mathcal{G}_p)$;

(ii) the isomorphism $\sigma_\Omega : \mathfrak{g}_p \cong \nu^*(\mathcal{F}_\pi)$ (see equation 2.11) allows us to transport it to a lattice $\Lambda_p \subset \nu^*(\mathcal{F}_\pi)$.

One can show that together these form a transverse integral affine structure $\Lambda \subset \nu^*(\mathcal{F}_\pi)$, and thus an integral affine structure on the leaf space [3, Section 3].

3.6 The linear variation theorem

We describe here another major result concerning regular PMCTs: the linear variation theorem [3, Sections 4-5]. This is a generalisation of the classical result of Duistermaat & Heckman [16].

Let $(\mathcal{G},\Omega) \rightrightarrows (M,\pi)$ be regular and source proper. We also assume that the leaves of (M,π) are 1-connected, so that the leaf space is smooth. Denoting the symplectic leaf corresponding to $b \in B$ by (S_b,ω_b), we form the vector bundle

$$\mathcal{H}^2 := \bigsqcup_{b\in B} H^2(S_b,\mathbb{R}) \to B$$

and the lattice

$$\mathcal{H}^2_{\mathbb{Z}} := \bigsqcup_{b\in B} \mathrm{im}\big(H^2(S_b,\mathbb{Z}) \to H^2(S_b,\mathbb{R})\big)$$

inside it. Associated to this we have the *Gauss-Manin connection* ∇ on $\mathcal{H}^2$, uniquely determined by requiring the sections of $\mathcal{H}^2_{\mathbb{Z}}$ to be parallel. Note that π gives us a section $\varpi \in \Gamma(\mathcal{H}^2)$, $b \mapsto [\omega_b]$.

The Gauss-Manin connection allows us to study the variation of ϖ: parallel transport makes $\mathcal{H}^2$ into a $\Pi_1(B)$-representation and we define the *variation map* $\mathrm{var}_\varpi : \Pi_1(B) \to \mathcal{H}^2$ to be

$$[\gamma] \mapsto \gamma_*(\varpi_{\gamma(0)}) \in \mathcal{H}^2_{\gamma(1)}.$$

On the other hand, we also have the *linear variation map* $\mathrm{var}^{\mathrm{lin}}_\varpi : TB \to \mathcal{H}^2$ given by

$$v \mapsto \nabla_v \varpi$$

and the *affine variation map* $\mathrm{var}^{\mathrm{aff}}_\varpi := \varpi + \mathrm{var}^{\mathrm{lin}}_\varpi$.

The linear variation theorem relates the variation and affine variation maps by means of the *developing map* associated to the integral affine manifold (B,Λ). Associated to the lattice $\Lambda^* \subset TB$ we have a canonical flat connection on TB (not to be confused with ∇ above). This makes TB into a TB-representation, and since the connection is torsion-free the identity map $TB \to TB$ is an algebroid cocycle. The developing map is defined to be the groupoid cocycle $\mathrm{dev} : \Pi_1(B) \to TB$ integrating it.

Remark 3.23. One can show that after fixing $b \in B$ and a basis of Λ_b this boils down to the classical notion of developing map defined on the universal covering space (see [3, Section 4.2]):

$$\mathrm{dev}_b : \widetilde{B} \to T_bB \simeq \mathbb{R}^q.$$

We can now state the linear variation theorem as follows.

Theorem 3.24 ([3, Theorem 4.4.2])**.** *One has a commutative diagram*

$$\begin{tikzcd} \Pi_1(B) \arrow[rr, "\mathrm{var}_\varpi"] \arrow[dr, "\mathrm{dev}"'] & & \mathcal{H}^2 \\ & TB \arrow[ur, "\mathrm{var}^{\mathrm{aff}}_\varpi"'] & \end{tikzcd}$$

This rather abstract formulation can locally be made explicit. Let $b_0 \in B$ and choose an integral affine chart (U,φ) centered at b_0 such that $\varphi(U)$ is convex and such that $M \to B$ trivialises over U. This induces a trivialisation $\Phi : \mathcal{H}^2|_U \cong U \times H^2(S_{b_0},\mathbb{R})$. The chart induces an identification $T_{b_0}B \cong \mathbb{R}^q$ and allows us to consider "straight line" paths from $b \in U$ to b_0. Restricting to such paths the above diagram becomes

$$\begin{tikzcd}
U \arrow[rr, "b \mapsto \Phi([\omega_b])"] \arrow[dr, "\varphi"'] & & H^2(S_{b_0}, \mathbb{R}) \\
& \mathbb{R}^q \arrow[ur, "v \mapsto [\omega_{b_0}] + \sum_i v_i c_i"'] &
\end{tikzcd}$$

where $c_i \in H^2(S_{b_0}, \mathbb{Z})$ are the Chern classes of the torus bundle $\mathbf{s}^{-1}(x) \to S_{b_0}$, where $x \in S_{b_0}$ (see [3, Corollary 4.4.4]). This local formulation is reminiscent of the linear variation theorem from [16]. In other words, Theorem 3.24 can be viewed as a global formulation and generalisation of the classical Duistermaat-Heckman theorem.

Chapter 4

Poisson manifolds of strong compact type over 2-tori

In this chapter we prove Main Theorem 1. In Section 4.1 we explain a general strategy for constructing PMSCTs inspired by the linear variation theorem (see Section 3.6) and in Section 4.2 we go over the basic theory of K3 surfaces and use it to endow a universal family with a regular Poisson structure. In Section 4.3 we put these two sections together to contruct the examples that prove the theorem.

4.1 General construction of PMSCTs

The construction we give below is based on two results on PMCTs:

(a) the leaf space carries an integral affine orbifold structure (Section 3.5), and

(b) the *linear variation theorem* (Section 3.6).

Here we only need to consider the case of 1-connected leaves. In this case the leaf space is smooth, since this assumption implies that the monodromy groupoid of the symplectic foliation is proper and has trivial isotropy groups. This means that the transverse integral affine structure on the symplectic foliation descends to an honest integral affine structure on the leaf space.

The construction we describe in this section yields a PMSCT with 1-connected symplectic leaves, whose leaf space is a complete integral affine manifold. This means that the leaf space is a quotient of $\mathbb{R}^q$ by a free and proper action of a discrete group of integral affine transformations. Note that if the Markus conjecture holds true, then in fact every compact integral affine manifold is of this type (see [25, Section 8.6]). This allows us to give an explicit formulation of the linear variation, similar to the discussion following Theorem 3.24. The setup is as follows.

Let $E \to \mathbb{R}^q$ be a fibre bundle with typical fibre S, a compact 1-connected manifold, and assume that E admits a Poisson structure π_E whose symplectic leaves are precisely the fibres of this bundle. As in Section 3.6 we have (i) the vector bundle $\mathcal{H}^2 \to \mathbb{R}^q$ whose fibers are the degree two cohomology groups of the symplectic leaves, (ii) the lattice $\mathcal{H}^2_{\mathbb{Z}} \subset \mathcal{H}^2$ of integral cohomology, (iii) the associated Gauss-Manin connection ∇ and (iv) the section $\varpi \in \Gamma(\mathcal{H}^2)$ induced by π_E.

Next, let $\Gamma \subset \mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^q) = \{x \mapsto Ax + v \mid A \in \mathrm{GL}(q, \mathbb{Z}), v \in \mathbb{R}^q\}$ be a discrete group of integral affine transformations acting freely and properly on $\mathbb{R}^q$, and assume that there is a Poisson action of Γ on (E, π_E)

making the projection $E \to \mathbb{R}^q$ equivariant. Then setting $M := E/\Gamma$ and $B := \mathbb{R}^q/\Gamma$, we get a (smooth) fibre bundle $\mathrm{pr} : M \to B$, again with typical fibre S, and a Poisson structure π on M whose leaves are the fibres of pr. In other words, (M, π) is a regular Poisson manifold with leaf space B. Note also that B, being a quotient $\mathbb{R}^q/\Gamma$, naturally inherits an integral affine structure.

We can now state the general method of constructing PMSCTs. It is a reformulation of [3, Proposition 4.4.6].

Proposition 4.1. *Let $(M = E/\Gamma, \pi)$ be constructed as above. Assume that there exists a ∇-flat section $s \in \Gamma(\mathcal{H}^2)$ and linearly independent sections $c_1, \dots, c_q \in \Gamma(\mathcal{H}^2_{\mathbb{Z}})$ such that*

$$\varpi = s + \sum_{i=1}^{q} \mathrm{pr}^i \cdot c_i, \tag{4.1}$$

where $\mathrm{pr}^i : \mathbb{R}^q \to \mathbb{R}$ denotes projection onto the i-th coordinate. Then (M, π) is of strong $\mathbf{s}$-proper type and the induced integral affine structure on B agrees with the one coming from the quotient $\mathbb{R}^q/\Gamma$. In particular, if B is compact then (M, π) is a PMSCT.

Proof. Pulling back the integral affine structure on B along $\mathrm{pr} : M \to B$ yields a transverse integral affine structure on the symplectic foliation $\mathcal{F}_\pi$, *i.e.* a lattice in its conormal bundle. We denote this lattice by $\widetilde{\Lambda} \subset \nu^*(\mathcal{F}_\pi)$. The main point is that for all $p \in M$, the monodromy group $N_p(M, \pi)$ is equal to the lattice $\widetilde{\Lambda}_p$. In fact, using the description of the monodromy groups for regular Poisson manifolds as the variation of symplectic areas (see Section 3.1.1) this follows directly from equation (4.1). The integrability criteria for Poisson manifolds then imply that (M, π) is integrable. Furthermore, since S has trivial fundamental group, the isotropy groups of the Weinstein groupoid $\Sigma(M, \pi)$ fit into the exact sequence

$$\cdots \to \pi_2(S, p) \xrightarrow{\partial_p} \nu_p^*(\mathcal{F}_\pi) \to \Sigma_p(M, \pi) \to 0,$$

where ∂_p is the monodromy map at p. Therefore, from our previous discussion, it follows that $\Sigma_p(M, \pi) \simeq \nu_p^*(\mathcal{F}_\pi)/\widetilde{\Lambda}_p$, *i.e.* that the isotropy group at p is compact. Since this holds for all $p \in M$ and since S is also compact, this shows that the Weinstein groupoid is $\mathbf{s}$-proper.

Finally, since $\widetilde{\Lambda} \subset \nu^*(\mathcal{F}_\pi)$ is closed, Hausdorffness of the Weinstein groupoid follows from [26, Theorem 1.1]. $\square$

4.2 Background on K3 surfaces and the Poisson structure on the universal family

We start by listing some definitions and results concerning K3 surfaces, after which we describe the moduli spaces and universal families for K3 surfaces. These results can be found in [27]. Finally, following [4], we use the Calabi-Yau theorem to turn the universal family into a Poisson manifold and the strong Torelli theorem to establish a Poisson action on it, setting us up to apply our construction.

Definition 4.2. A K3 surface is a compact, 1-connected complex surface with trivial canonical bundle. $\diamondsuit$

Every K3 surface is Kähler (see [28]). All K3 surfaces have the same underlying smooth manifold S (see [27, Corollary VIII.8.6]); this will be the model fibre used in Proposition 4.1. The intersection form on $H^2(S, \mathbb{Z})$ turns it into a lattice and this lattice is isomorphic to the aptly named *K3 lattice*, which we denote by $(L,(\cdot,\cdot))$. It is the unique even, unimodular lattice of signature $(3,19)$ (see [27, Proposition VIII.3.2 (ii)]). Explicitly, we have $L = U^{\oplus 3} \oplus (-E_8)^{\oplus 2}$, where $U = \mathbb{Z}^{\oplus 2}$ with form given by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $E_8 = \mathbb{Z}^{\oplus 8}$ with form given by the Cartan matrix of E_8; it is important for us that this form is positive definite. We also set $L_{\mathbb{R}} := L \otimes \mathbb{R}$ and $L_{\mathbb{C}} := L \otimes \mathbb{C}$; note that these are models for the real and complex cohomology, respectively.

4.2.1 The Torelli theorem

Definition 4.3. Let X, X' be K3 surfaces. A $\mathbb{Z}$-module isomorphism $H^2(X',\mathbb{Z}) \to H^2(X,\mathbb{Z})$ is a *Hodge isometry* if

(i) it preserves the intersection form;

(ii) its $\mathbb{C}$-linear extension preserves the Hodge decomposition.

A Hodge isometry is called *effective* if its $\mathbb{R}$-linear extension maps some Kähler class of X' to one of X. $\diamondsuit$

Effectiveness of a Hodge isometry is equivalent to requiring it to map the Kähler cone of X' to that of X (see [27, Proposition VIII.3.10]).

Theorem 4.4 (Torelli [27, Corollary VIII.11.4])**.** *Let X, X' be K3 surfaces. Then for any effective Hodge isometry $\varphi : H^2(X',\mathbb{Z}) \to H^2(X,\mathbb{Z})$ there exists a unique biholomorphism $f : X \to X'$ such that $f^* = \varphi$.*

This result is ultimately used to obtain the action in Proposition 4.1.

4.2.2 Moduli spaces and universal families

There are two moduli spaces and corresponding families for K3 surfaces: one takes into account the Kähler structure and the other only considers the complex structure. We start now with the latter.

Definition 4.5. A *marked K3 surface* is a pair (X,φ) consisting of a K3 surface X and a *marking* φ, *i.e.* an isometry $\varphi : H^2(X,\mathbb{Z}) \to L$. Two marked K3 surfaces are *equivalent* if there is a bihomolorphism between them intertwining the markings. The *moduli space of marked K3 surfaces* is the set of equivalence classes:

$$M_1 := \{(X,\varphi)\}/\sim . \qquad \diamondsuit$$

It follows immediately from the definition that any K3 surface admits, up to scalar multiplication, a unique nowhere vanishing holomorphic 2-form. In fact, one can show that, again up to scalar multiplication, there is a bijection between complex structures on S and closed, complex 2-forms $\sigma \in \Omega^2(S,\mathbb{C})$ satisfying $\sigma \wedge \sigma = 0$ and $\sigma \wedge \bar{\sigma} > 0$. This motivates the following definitions. We will use the same letter to denote a marking $\varphi : H^2(X,\mathbb{Z}) \to L$ and the induced maps $\varphi : H^2(X,\mathbb{R}) \to L_{\mathbb{R}}$ and $\varphi : H^2(X,\mathbb{C}) \to L_{\mathbb{C}}$.

Definition 4.6. The *period domain* is given by

$$\Omega := \{[\sigma] \in \mathbb{P}(L_{\mathbb{C}}) \mid (\sigma,\sigma) = 0, (\sigma,\bar{\sigma}) > 0\}.$$

We define the *period map* $\tau_1 : M_1 \to \Omega$ by

$$[(X,\varphi)] \mapsto [\varphi(\sigma_X)],$$

where σ_X is a nowhere vanishing holomorphic 2-form on X. $\diamondsuit$

Theorem 4.7 ([27, Theorem VIII.12.1]). *The moduli space M_1 admits the structure of a 20-dimensional complex manifold such that the period map $\tau_1 : M_1 \to \Omega$ becomes a surjective local biholomorphism. Furthermore, there exists a universal family $\mathcal{U} \to M_1$ of marked K3 surfaces.*

Remark 4.8. Recall that a family is universal if any other family is locally the pullback of it by a unique map (see [27, Section I.10]). The fibre of the universal family $\mathcal{U} \to M_1$ over any $t \in M_1$ is a marked K3 surface (X_t, φ_t) such that $[(X_t, \varphi_t)] = t$. Furthermore, these markings vary smoothly in the sense that they induce local trivialisations of the bundle $\cup_{t\in M_1} H^2(X_t, \mathbb{R})$.

There are still some inconveniences present here. It can be shown that M_1 is not Hausdorff, and that the period map τ_1 is not injective (see [27, Remark VIII.12.2]). These problems disappear when taking into account the Kähler structure.

Definition 4.9. We define M_2 to be the subset of the bundle

$$\bigsqcup_{t\in M_1} H^2(X_t, \mathbb{C})$$

consisting of all Kähler classes. ♢

It can be shown that M_2 is a real-analytic manifold of dimension 60 (see [27, Lemma VIII.9.3] and its proof). One should think of a point in M_2 as an equivalence class of marked K3 surfaces together with a specified Kähler class. Note that there is a projection map $\mathrm{pr} : M_2 \to M_1$.

Inspired by some analysis of the Kähler cone of K3 surfaces (see [27, Section VIII.3 and VIII.9]) one makes the following definitions.

Definition 4.10. Set

$$K\Omega := \{(k, [\sigma]) \in L_{\mathbb{R}} \times \Omega \mid (k, k) > 0, (k, \sigma) = 0\}.$$

The *refined period domain* is then given by

$$K\Omega^0 := \{(k, [\sigma]) \in K\Omega \mid (k, d) \neq 0 \text{ for all } d \in L \text{ such that } (d, d) = -2 \text{ and } (d, \sigma) = 0\}.$$

The *refined period map* $\tau_2 : M_2 \to K\Omega^0$ is defined as

$$(t, k) \mapsto (\varphi_t(k), \tau_1(t)).$$ ♢

Theorem 4.11 ([27, Theorem VIII.12.3 and VIII.14.1]). *The refined period map is a diffeomorphism.*

We set $K\mathcal{U} := (\mathrm{pr} \circ \tau_2^{-1})^*\mathcal{U}$. This is a real-analytic family (*i.e.* fibre bundle) over $K\Omega^0$ with extra data attached: the fibre over $(k, [\sigma])$ is a triple (X, φ, ω) consisting of a K3 surface X, a marking $\varphi : H^2(X, \mathbb{Z}) \to L$ and a Kähler class $\omega \in H^2(X, \mathbb{R})$ such that $\varphi(\omega) = k$. These markings vary smoothly in the same sense as before, and hence so do the Kähler classes.

The family $K\mathcal{U} \to K\Omega^0$ is universal for real-analytic "marked Kähler K3 families", *i.e.* real-analytic families of K3 surfaces equipped with smoothly varying markings and Kähler classes.

4.2.3 The Poisson structure

Recall the following special version of the Calabi-Yau theorem (see e.g. [27, Theorem I.15.1]).

Theorem 4.12. *Let X be a compact complex manifold with vanishing first Chern class. Then for any Kähler class $\omega \in H^2(X, \mathbb{R})$ there exists a unique Ricci flat Kähler metric whose Kähler form belongs to ω.*

This theorem applies in particular to K3 surfaces, and thus we can use it to endow the fibres of $K\mathcal{U} \to K\Omega^0$ with smoothly varying Kähler forms, turning it into a Poisson manifold (see also [4, Secion 2.1.3]).

Corollary 4.13. *The family $K\mathcal{U}$ admits a regular Poisson structure $\pi_{K\mathcal{U}}$ whose symplectic leaves are the fibres of $K\mathcal{U} \to K\Omega^0$. Moreover the symplectic form on the fibre X over $(k, [\sigma])$ with marking φ is the Kähler form associated to the unique Ricci flat Kähler metric on X with Kähler class $\varphi^{-1}(k)$.*

4.2.4 The action

We will construct an action on $K\mathcal{U}$ by the group $O(L)$ of isometries of the K3 lattice. Note that there is an obvious induced action of $O(L)$ on $K\Omega^0$.

Proposition 4.14. *There is a Poisson action of $O(L)$ on $(K\mathcal{U}, \pi_{K\mathcal{U}})$ with respect to which the projection $K\mathcal{U} \to K\Omega^0$ is equivariant.*

Proof. Fix $\gamma \in O(L)$ and $p \in K\Omega^0$. Using the notation from above, denote the triple over p by $(X_p, \varphi_p, \omega_p)$ and similarly for $\gamma(p)$. It is easy to see that

$$\varphi_p^{-1} \circ \gamma^{-1} \circ \varphi_{\gamma(p)} : H^2(X_{\gamma(p)}, \mathbb{Z}) \to H^2(X_p, \mathbb{Z})$$

is an effective Hodge isometry, so that by Theorem 4.4 we obtain a biholomorphism $f_\gamma^p : X_p \to X_{\gamma(p)}$. The universality of the family then gives neighbourhoods U and V of p and of $\gamma(p)$ respectively and an isomorphism $(\Psi, \psi) : K\mathcal{U}|_U \to K\mathcal{U}|_V$ extending f_γ^p: through the biholomorphism f_γ^p, $K\mathcal{U}$ becomes a deformation of X_p at two basepoints, p and $\gamma(p)$. Since $K\mathcal{U}$ is universal, these two deformations are locally isomorphic. Writing $\Psi_q : X_q \to X_{\psi(q)}$ for the fiberwise maps, it then follows that for all $q \in U$ we have that

$$\Psi_q^* = \varphi_q^{-1} \circ \gamma^{-1} \circ \varphi_{\psi(q)} : H^2(X_{\psi(q)}, \mathbb{Z}) \to H^2(X_q, \mathbb{Z}).$$

This implies first of all that $\psi = \gamma|_U$, from which it follows that $\Psi_q = f_\gamma^q$, since biholomorphisms of K3 surfaces are uniquely determined by their induced maps on degree 2 integral cohomology (see [27, Proposition VIII.11.3]). Thus these fibrewise biholomorphisms f_γ^p, $p \in K\Omega^0$, together form an automorphism $F_\gamma : K\mathcal{U} \to K\mathcal{U}$. It is immediate from the above construction that $F_{\mathrm{id}} = \mathrm{id}$, and from the uniqueness part of Theorem 4.4 it follows that $F_{\gamma\circ\gamma'} = F_\gamma \circ F_{\gamma'}$ for all $\gamma, \gamma' \in O(L)$, meaning that we have an action of $O(L)$ on $K\mathcal{U}$. This action makes $K\mathcal{U} \to K\Omega^0$ equivariant by construction. Finally, from the uniqueness part of the Calabi-Yau theorem it follows that each f_γ^p preserves the symplectic forms on the fibres, meaning that the action is by Poisson maps. ☐

4.3 The examples

From our work in Section 4.2 we have a Poisson manifold $(K\mathcal{U}, \pi_{K\mathcal{U}})$ with leaf space $K\Omega^0$ such that:

(i) the cohomology classes of the symplectic forms on the leaves are described in terms of the leaf space $K\Omega^0$ (Corollary 4.13);

(ii) the natural action of $O(L)$ on $K\Omega^0$ lifts to a Poisson action on $(K\mathcal{U}, \pi_{K\mathcal{U}})$ (Proposition 4.14).

In order to apply the construction described in Section 4.1, we need to find a suitable embedding $\mathbb{R}^q \hookrightarrow K\Omega^0$ and a suitable subgroup $\Gamma \subset O(L)$. We rephrase Proposition 4.1 in the current setting in order to make this more precise. For a different version of this result see also [4, Theorem 1].

Corollary 4.15. *Assume that we have an embedding $f : \mathbb{R}^q \to K\Omega^0$ and a subgroup $\Gamma \subset O(L)$ such that*

(i) there exist $a \in L_{\mathbb{R}}$ and linearly independent $a_1, \ldots, a_q \in L$ such that the $L_{\mathbb{R}}$-component of f has the form

$$(x_1, \ldots, x_q) \mapsto a + \sum_{i=1}^{q} x_i a_i;$$

(ii) the action of Γ on $K\Omega^0$ preserves the image of f;

(iii) the induced action on $\mathbb{R}^q$ is free, proper and by integral affine maps.

Then $M := f^* K\mathcal{U}/\Gamma$ with the Poisson structure induced from $\pi_{K\mathcal{U}}$ is a Poisson manifold of strong $\mathbf{s}$-proper type with leaf space $B := \mathbb{R}^q/\Gamma$. If B is compact, M is a PMSCT.

Remark 4.16. We can now explain why our construction leads to PMSCTs with *strongly* integral affine leaf spaces. On the one hand, because of Theorem 3.24, we are forced to consider embeddings with integral variation, *i.e.* the a_i must lie in the integral lattice L. On the other hand, to apply Theorem 4.4 we need to consider isometries of integral cohomology, *i.e.* we need to act by elements of $O(L)$. These two technical limitations together only allow for strongly integral affine leaf spaces in the examples.

Remark 4.17. At the level of the symplectic groupoid, one can see that the leaf space being strongly integral affine implies that the restriction of the symplectic form to the identity component of the isotropy (a torus bundle) lies in the integral cohomology. See [8, Remark 5.10].

We now recall the classification of strongly integral affine structures for S^1 and $\mathbb{T}^2$.

Theorem 4.18. *The strongly integral affine circles are, up to isomorphism, the quotients $\mathbb{R}/\mathbb{Z}$ where the $\mathbb{Z}$-action is generated by $x \mapsto x + p$, for a fixed $p \in \mathbb{Z}_{\geq 1}$.*

Proof. It is easy to see that all integral affine circles are complete. Hence, it suffices to classify, up to conjugation, embeddings $\mathbb{Z} \to \mathrm{Aff}_{\mathbb{Z}}(\mathbb{R})$ inducing free and proper actions. These are precisely the actions generated by $x \mapsto x + a$ with $a > 0$. Restricting to *strongly* integral affine circles yields the result. □

Theorem 4.19. *The strongly integral affine 2-tori, up to isomorphism, are quotients $\mathbb{R}^2/\mathbb{Z}^2$, where the $\mathbb{Z}^2$-actions fall into one of the following types:*

(I) an action generated by $(x, y) \mapsto (x + p, y)$ and $(x, y) \mapsto (x, y + q)$, where $p, q \in \mathbb{Z}_{\geq 1}$ and $p|q$;

(II) an action generated generated by $(x, y) \mapsto (x + p, y)$ and $(x, y) \mapsto (x + ny, y + q)$, where $n, p, q \in \mathbb{Z}_{\geq 1}$.

Proof. The classification of all integral affine structures on 2-tori is given in [6, Theorem A]. Restricting to strongly integral affine structures and using the Smith normal form for matrices with integer entries to simplify the possibilities from type (I) yields the above classification. □

Remark 4.20. The integral affine 2-tori of type (I) are (isomorphic to) products of integral affine circles. Thus to find examples of PMSCTs with leaf space of this type one can simply take products of PMSCTs with leaf space S^1, constructed in Section 4.3.1. This yields Poisson manifolds of dimension 10 whose leaves are products of K3 surfaces. However, the examples we construct in Section 4.3.2 are six-dimensional Poisson manifolds with K3 surfaces as symplectic leaves and thus result in "smaller" examples.

Remark 4.21. Continuing the previous remark, note that by taking products we can also realise *some* higher dimensional integral affine tori as the leaf space of a PMSCT, namely those that are isomorphic to a product of some of the integral affine circles and 2-tori classified above.

Before we move on to the examples, we establish some notation. Recall that $L = U^{\oplus 3} \oplus (-E_8)^{\oplus 2}$. We denote the standard bases of the three copies of U by $\{u, v\}$, $\{x, y\}$ and $\{z, t\}$, so that $(u, v) = (x, y) = (z, t) = 1$ with all other combinations yielding zero. Recall also that $-E_8$ is even and negative definite. Finally, let $\{e_1, \ldots, e_8\}$ be a set of real numbers such that the set

$$\{1, e_1, \ldots, e_8, e_1^2, e_1e_2, \ldots, e_7^2, e_7e_8, e_8^2\}$$

consisting of $1, e_1, \ldots, e_8$ and their pairwise products is linearly independent over the integers, or equivalently the rationals. The existence of such a set is guaranteed by [29]. We then set $e := (e_1, \ldots, e_8) \in (-E_8)_{\mathbb{R}}$, scaling if necessary such that $|(e, e)| \leq \frac{1}{2}$, and we set $a := (0, e), b := (e, 0) \in (-E_8)_{\mathbb{R}}^{\oplus 2} \subset L_{\mathbb{R}}$.

Let us outline the strategy for the examples below. In each case, we start by defining f and Γ. It is fairly straightforward to check items ((ii)) and ((iii)) from Corollary 4.15 and that the image of f is contained in $K\Omega$. It then remains to show that it is actually contained in $K\Omega^0$. This is the more involved part of the computations.

4.3.1 The PMSCTs with leaf space the circle

We will construct a PMSCT whose leaf space is a strongly integral affine circle, *i.e.* we want the action of $\mathbb{Z}$ on $\mathbb{R}$ generated by $x \mapsto x + p$ with $p \in \mathbb{Z}_{\geq 1}$. The case $p = 1$ is the one treated in [4] and the computations carried out below for general p are an obvious generalisation of the computations there.

Consider the map $f : \mathbb{R} \to L_{\mathbb{R}} \times \mathbb{P}(L_{\mathbb{C}})$ defined by

$$s \mapsto (2u + v + sy, [x - su + 2y + a + i(z + 2t + b)])$$

and the map $\varphi : L \to L$ defined by $u \mapsto u, v \mapsto v + py$, $x \mapsto x - pu, y \mapsto y$ on the first two copies of U and as the identity on the other summands of L. It is easily checked that φ is an isometry and that

$$\varphi \cdot f(s) = f(s + p).$$

This implies that the image of f is invariant under the action of $\Gamma := \langle \varphi \rangle$, and also that the induced action on $\mathbb{R}$ is the one we need.

To show that the image of f is contained in $K\Omega$, let $s \in \mathbb{R}$. Setting $f_1(s) = 2u+v+sy$, $f_2(s) = x-su+2y+a$

and $f_3(s) = z + 2t + b$, we see that

$$\begin{aligned}
(f_2(s), f_2(s)) &= (x - su + 2y + a, x - su + 2y + a) \\
&= 4(x, y) + (a, a) \\
&= 4 + (e, e) \geq 3\frac{1}{2} > 0, \\
(f_3(s), f_3(s)) &= (z + 2t + b, z + 2t + b) \\
&= 4(z, t) + (b, b) \\
&= 4 + (e, e) \geq 3\frac{1}{2} > 0, \\
(f_2(s), f_3(s)) &= (x - su + 2y + a, z + 2t + b) \\
&= 0.
\end{aligned}$$

These computations imply that $[f_2(s) + i f_3(s)] \in \Omega$. Since

$$\begin{aligned}
(f_1(s), f_1(s)) &= (2u + v + sy, 2u + v + sy) = (2u, v) + (v, 2u) = 4 > 0, \\
(f_1(s), f_2(s)) &= (2u + v + sy, x - su + 2y + a) = -s(v, u) + s(y, x) = -s + s = 0, \\
(f_1(s), f_3(s)) &= (2u + v + sy, z + 2t + b) = 0,
\end{aligned}$$

we see that $f(s) \in K\Omega$.

It remains to check that $f(s) \in K\Omega^0$ for all $s \in \mathbb{R}$.

Proof. Assume that we have $d \in L$ such that $(d, d) = -2$ and $(d, f_1(s)) = (d, f_2(s)) = (d, f_3(s)) = 0$. We need to find a contradiction. Let us write

$$d = Au + Bv + Cx + Dy + Ez + Ft + d_1 + d_2,$$

with $A, \ldots, F \in \mathbb{Z}$ and d_i in the i-th copy of $-E_8$. Since E_8 is even and positive definite, we can write $(d_i, d_i) = -2n_i$, for $n_i \in \mathbb{Z}_{\geq 0}$. The above conditions then translate into three equations:

$$AB + CD + EF = n_1 + n_2 - 1, \tag{4.2}$$

$$2B + A + Cs = 0, \tag{4.3}$$

$$D - Bs + 2C + (d_2, e) = 0, \tag{4.4}$$

$$F + 2E + (d_1, e) = 0. \tag{4.5}$$

This is where the seemingly strange choice of e comes in. There exist $k_1, \ldots, k_8 \in \mathbb{Z}$ such that $(d_1, e) = \sum_i k_i e_i$ and since $\{1, e_1, \ldots, e_8\}$ is linearly independent over the integers by choice of e, it follows from (4.5) that we must have $F + 2E = k_1 = \cdots = k_8 = 0$. Since the bilinear form on $-E_8$ is nondegenerate, it follows that $d_1 = 0$ and thus that $n_1 = 0$.

Case $C = 0$: Equation (4.3) yields $2B + A = 0$, and (4.2) becomes

$$2B^2 + 2E^2 = 1 - n_2.$$

This implies that $B = E = 0$ and $n_2 = 1$. But then $d_2 \neq 0$ and (4.4) becomes

$$D + (d_2, e) = 0,$$

which together with $d_2 \neq 0$ contradicts the "linear independence" assumption on e.

Case $C \neq 0$: From (4.3) we get

$$s = -\frac{2B + A}{C},$$

and substituting this into (4.4) yields

$$AB + CD = -2C^2 - 2B^2 - (d_2, e).$$

Combining this with (4.2) gives

$$2B^2 + 2C^2 + 2E^2 + C(d_2, e) = 1 - n_2.$$

From the properties of e we get $Cd_2 = 0$, implying that $d_2 = 0$ and thus also that $n_2 = 0$, so that we are left with

$$2B^2 + 2C^2 + 2E^2 = 1,$$

which is absurd since $B, C, E \in \mathbb{Z}$. □

4.3.2 The PMSCTs with leaf space a torus of type (I)

Here we construct a PMSCT with leaf space the torus $\mathbb{T}^2$ with an integral affine structure of type (I). This means that we want the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ generated by $(x, y) \mapsto (x + p, y)$ and $(x, y) \mapsto (x, y + q)$, with $p, q \in \mathbb{Z}_{\geq 1}$.

Consider the map $f : \mathbb{R}^2 \to L_{\mathbb{R}} \times \mathbb{P}(L_{\mathbb{C}})$ defined by

$$(s, r) \mapsto (2u + v + sy + rt, [x - su + 2y + a + i(z - ru + 2t + b)]),$$

the map $\varphi : L \to L$ as in the previous example and the map $\psi : L \to L$ defined by $u \mapsto u, v \mapsto v + qt$, $x \mapsto x$, $y \mapsto y$, $z \mapsto z - qu, t \mapsto t$ on two copies of U and as the identity on the other summands of L. It is easily checked that these are isometries and that

$$\begin{aligned}\varphi \cdot f(s, r) &= f(s + p, r),\\ \psi \cdot f(s, r) &= f(s, r + q).\end{aligned}$$

This implies that the image of f is invariant under the action of $\Gamma := \langle \varphi, \psi \rangle$, and also that the induced action on $\mathbb{R}^2$ is as desired.

To show that the image of f is contained in $K\Omega$, let f_1, f_2, f_3 be the three "components" of f, as before, and let $(s,r) \in \mathbb{R}^2$. We compute

$$\begin{aligned}
(f_2(s,r), f_2(s,r)) &= (x - su + 2y + a, x - su + 2y + a) \\
&= 4(x,y) + (a,a) \\
&= 4 + (e,e) \geq 3\frac{1}{2} > 0, \\
(f_3(s,r), f_3(s,r)) &= (z - ru + 2t + b, z - ru + 2t + b) \\
&= 4(z,t) + (b,b) \\
&= 4 + (e,e) \geq 3\frac{1}{2} > 0, \\
(f_2(s,r), f_3(s,r)) &= (x - su + 2y + a, z - ru + 2t + b) \\
&= 0
\end{aligned}$$

and conclude that $[f_2(s,r) + i f_3(s,r)] \in \Omega$. Also,

$$\begin{aligned}
(f_1(s,r), f_1(s,r)) &= (2u + v + sy + rt, 2u + v + sy + rt) = (2u, v) + (v, 2u) = 4 > 0, \\
(f_1(s,r), f_2(s,r)) &= (2u + v + sy + rt, x - su + 2y + a) = -s(u,v) + s(x,y) = -s + s = 0, \\
(f_1(s,r), f_3(s,r)) &= (2u + v + sy + rt, z - ru + 2t + b) = -r(u,v) + r(z,t) = -r + r = 0
\end{aligned}$$

implies that $f(s,r) \in K\Omega$.

It remains to check that $f(s,r) \in K\Omega^0$ for all $(s,r) \in \mathbb{R}^2$.

Proof. Let $d \in L$ such that $(d,d) = -2$ and $(d, f_1(s,r)) = (d, f_2(s,r)) = (d, f_3(s,r)) = 0$ and as before write

$$d = Au + Bv + Cx + Dy + Ez + Ft + d_1 + d_2,$$

and $(d_i, d_i) = -2n_i$ for $n_i \in \mathbb{Z}_{\geq 0}$. We need to find a contradiction. The relevant equations now become

$$AB + CD + EF = n_1 + n_2 - 1, \tag{4.6}$$

$$2B + A + Cs + Er = 0, \tag{4.7}$$

$$D - Bs + 2C + (d_2, e) = 0, \tag{4.8}$$

$$F - Br + 2E + (d_1, e) = 0. \tag{4.9}$$

Case $B = 0$**:** The assumptions on e, together with (4.8) and (4.9), imply that $D + 2C = F + 2E = 0$ and $d_1 = d_2 = 0$, so that $n_1 = n_2 = 0$. But then (4.6) becomes

$$2C^2 + 2E^2 = 1,$$

which is impossible.

Case $B \neq 0$**:** From (4.8) and (4.9) we get

$$s = \frac{D + 2C + (d_2, e)}{B}, \qquad r = \frac{F + 2E + (d_1, e)}{B}.$$

Substituting this into (4.7) gives

$$AB + CD + EF = -2B^2 - 2C^2 - 2E^2 - C(d_2, e) - E(d_1, e),$$

and combining this with (4.6) we obtain

$$2B^2 + 2C^2 + 2E^2 + C(d_2, e) + E(d_1, e) = 1 - n_1 - n_2.$$

The assumptions on e imply that $Cd_2 + Ed_1 = 0$, so that this becomes

$$2B^2 + 2C^2 + 2E^2 = 1 - n_1 - n_2.$$

This is impossible under the assumption $B \neq 0$, since $n_i \in \mathbb{Z}_{\geq 0}$. $\square$

4.3.3 The PMSCTs with leaf space a torus of type (II)

In this example we will construct a PMSCT whose leaf space is a torus with an induced integral affine structure of type (II), namely one induced by the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ generated by $(x, y) \mapsto (x + p, y)$ and $(x, y) \mapsto (x + ny, y + q)$, where $n, p, q \in \mathbb{Z}_{\geq 1}$.

Consider the map $f : \mathbb{R}^2 \to L_{\mathbb{R}} \times \mathbb{P}(L_{\mathbb{C}})$ defined by

$$(s, r) \mapsto (2u + v + sy + rt, [qx + (nr^2 - qs)u - nrz + 2qy + a + i(z - ru + 2q^2t + 2nqry + b)]),$$

the map $\varphi : L \to L$ defined as before and the map $\psi : L \to L$ defined by $u \mapsto u, v \mapsto v + qt$, $x \mapsto x - nz + qnu, y \mapsto y$, $z \mapsto z - qu, t \mapsto t + ny$ on the copies of U and the identity on the other summands of L. It is easily checked that these are isometries and that

$$\begin{aligned}
\varphi \cdot f(s, r) &= f(s + p, r), \\
\psi \cdot f(s, r) &= f(s + nr, r + q).
\end{aligned}$$

This implies that the image of f is invariant under the action of $\Gamma := \langle \varphi, \psi \rangle$, and also that the induced action on $\mathbb{R}^2$ is the desired one. To show that the image of f is contained in $K\Omega$, denote once more by f_1, f_2, f_3 the "components" of f, and let $(s, r) \in \mathbb{R}^2$. Since

$$\begin{aligned}
(f_2(s, r), f_2(s, r)) &= (qx + (nr^2 - qs)u - nrz + 2qy + a, qx + (nr^2 - qs)u - nrz + 2qy + a) \\
&= 4q^2(x, y) + (a, a) \\
&= 4q^2 + (e, e) \geq 3\frac{1}{2} > 0, \\
(f_3(s, r), f_3(s, r)) &= (z - ru + 2q^2t + 2nqry + b, z - ru + 2q^2t + 2nqry + b) \\
&= 4q^2(z, t) + (b, b) \\
&= 4q^2 + (e, e) \geq 3\frac{1}{2} > 0, \\
(f_2(s, r), f_3(s, r)) &= (qx + (nr^2 - qs)u - nrz + 2qy + a, z - ru + 2q^2t + 2nqry + b) \\
&= 2nq^2r(x, y) - 2nq^2r(z, t) = 2nq^2r - 2nq^2r = 0.
\end{aligned}$$

we get that $[f_2(s,r)+if_3(s,r)]\in\Omega$. The computations

$$\begin{aligned}
(f_1(s,r),f_1(s,r)) &= (2u+v+sy+rt,2u+v+sy+rt)=(2u,v)+(v,2u)=4>0,\\
(f_1(s,r),f_2(s,r)) &= (2u+v+sy+rt,qx+(nr^2-qs)u-nrz+2qy+a)\\
&= (nr^2-qs)(u,v)+qs(x,y)-nr^2(z,t)=nr^2-qs+qs-nr^2=0,\\
(f_1(s,r),f_3(s,r)) &= (2u+v+sy+rt,z-ru+2q^2t+2nqry+b)\\
&= -r(u,v)+r(z,t)=-r+r=0
\end{aligned}$$

show that $f(s,r)\in K\Omega$.

It remains to show that $f(s,r)\in K\Omega^0$ for all $(s,r)\in\mathbb{R}^2$.

Proof. Let $d\in L$ such that $(d,d)=-2$ and $(d,f_1(s))=(d,f_2(s))=(d,f_3(s))=0$. Like before we write

$$d=Au+Bv+Cx+Dy+Ez+Ft+d_1+d_2,$$

and we set $(d_i,d_i)=-2n_i$ with $n_i\in\mathbb{Z}_{\geq 0}$. The goal is to find a contradiction. The main equations are now

$$AB+CD+EF=n_1+n_2-1, \tag{4.10}$$

$$2B+A+Cs+Er=0, \tag{4.11}$$

$$Dq+B(nr^2-qs)-Fnr+2Cq+(d_2,e)=0, \tag{4.12}$$

$$F-Br+2Eq^2+2Cnqr+(d_1,e)=0. \tag{4.13}$$

Case $B-2Cnq=0$: Equation (4.13) tells us that $d_1=0$ and $F+2Eq^2=0$.

Subcase $C=0$: This implies that $B=0$, so that (4.10) becomes

$$2E^2q^2=1-n_2.$$

This is only possible if $E=0$ and $n_2=1$, but then also $F=0$ and (4.12) becomes

$$Dq+(d_2,e)=0,$$

which would imply that $d_2=0$, contradicting $n_2=1$.

Subcase $C\neq 0$: Equation (4.11) tells us that

$$s=-\frac{2B+A+Er}{C},$$

and with (4.12) we obtain

$$2Cn^2qr^2-2Fnr+2Anq^2+C(8n^2q^3+2q)+Dq+(d_2,e)=0.$$

Since $C,n,q\neq 0$ and $r\in\mathbb{R}$, we must have that

$$F^2\geq 2Cq\big[2Anq^2+C(8n^2q^3+2q)+Dq+(d_2,e)\big].$$

But $F = -2Eq^2$ and $B = 2Cnq$, so combining this with (4.10) yields

$$q(1 - n_2) \geq C^2(8n^2q^3 + 2q) + C(d_2, e).$$

Since $C \neq 0$, this is certainly impossible when C and (d_2, e) have the same parity. So let us assume that they have opposite parity, so that the equation becomes

$$q(1 - n_2) \geq C^2(8n^2q^3 + 2q) - |C| \cdot |(d_2, e)|. \tag{4.14}$$

Now both d_2 and e lie in the same copy of $-E_8$, and since $(\cdot, \cdot)$ is negative definite on $-E_8$ we can use the Cauchy-Schwarz inequality to obtain

$$|(d_2, e)| \leq \sqrt{|(d_2, d_2)| \cdot |(e, e)|} = \sqrt{2 \cdot |(e, e)| n_2} \leq \sqrt{n_2},$$

using that we chose e such that $|(e, e)| \leq \frac{1}{2}$. Now, in order for (4.14) to hold we certainly must have

$$C^2(8n^2q^3 + 2q) - \sqrt{n_2} \cdot |C| + qn_2 - q \leq 0$$

and it is easily seen that this is not possible for $0 \neq C \in \mathbb{Z}$.

Case $B - 2Cnq \neq 0$**:** We immediately distinguish two cases: $B = 0$ and $B \neq 0$.

Subcase $B = 0$**:** We claim that $F \neq 0$. Indeed, if we had $F = 0$, (4.12) would become

$$Dq + 2Cq + (d_2, e) = 0,$$

meaning that $d_2 = 0$, so $n_2 = 0$, and $D + 2C = 0$. But then (4.10) becomes

$$2C^2 = 1 - n_1,$$

which can only hold if $C = 0$ and $n_1 = 1$. But then (4.13) becomes

$$2Eq^2 + (d_1, e) = 0,$$

which implies $d_1 = 0$, contradicting $n_1 = 1$. So we see indeed that $F \neq 0$. But then (4.12) and (4.13) yield

$$r = -\frac{F + 2Eq^2 + (d_1, e)}{2Cnq} = \frac{Dq + 2Cq + (d_2, e)}{Fn}.$$

This becomes

$$2CDnq^2 + 4C^2nq^2 + 2Cnq(d_2, e) + F^2n + 2EFnq^2 + Fn(d_1, e) = 0,$$

and the assumptions on e imply that $2Cqd_2 + Fd_1 = 0$ and

$$2CDnq^2 + 4C^2nq^2 + F^2n + 2EFnq^2 = 0.$$

Since $B = 0$, combining this with (4.10) we obtain

$$4C^2nq^2 + F^2n = 2nq^2(1 - n_1 - n_2).$$

Both C and F are nonzero, meaning that this is impossible.

Subcase $B \neq 0$: We can write

$$r = \frac{F + 2Eq^2 + (d_1, e)}{B - 2Cnq}, \qquad s = \frac{Dq + Bnr^2 - Fnr + 2Cq + (d_2, e)}{Bq}.$$

This yields

$$\begin{aligned} s = &\frac{(B-2Cnq)^2(2Cq + Dq + (d_2, e)) + Bn(F + 2Eq^2 + (d_1, e))^2}{Bq(B-2Cnq)^2} \\ &- \frac{Fn(B-2Cnq)(F + 2Eq^2 + (d_1, e))}{Bq(B-2Cnq)^2} \end{aligned}$$

and substituting this into (4.11) and using the assumptions on e (actually, finally using them to their full potential), this reduces to

$$\begin{aligned} 0 = &\, 2B^2q(B-2Cnq)^2 + ABq(B-2Cnq)^2 \\ &+ C\Big((B-2Cnq)^2(2Cq + Dq) + Bn(F + 2Eq^2)^2 - Fn(B-2Cnq)(F+2Eq^2)\Big) \\ &+ BEq(B-2Cnq)(F+2Eq^2). \end{aligned}$$

Some rewriting turns this into

$$\begin{aligned} 0 = &\, q(B-2Cnq)^2\big(2B^2 + 2C^2 + AB + CD\big) \\ &+ BCn(F+2Eq^2)^2 - CFn(B-2Cnq)(F+2Eq^2) + BEq(B-2Cnq)(F+2Eq^2), \end{aligned}$$

and some easy computations show that the second line is equal to

$$EFq(B-2Cnq)^2 + 2q(BEq + CFn)^2,$$

so that altogether we obtain

$$q(B-2Cnq)^2\big(2B^2 + 2C^2 + AB + CD + EF\big) + 2q(BEq + CFn)^2 = 0.$$

Combining this with (4.10) we get

$$2\Big((B-2Cnq)^2(B^2 + C^2) + (BEq + CFn)^2\Big) = (B-2Cnq)^2(1 - n_1 - n_2).$$

But this is impossible, since $B - 2Cnq \neq 0$, $B \neq 0$ and $n_1, n_2 \geq 0$, giving us the desired contradiction. □

Chapter 5

Duistermaat-Heckman measures

In this chapter we prove Main Theorem 2. In Section 5.1 we explain some general theory regarding measures on leaf spaces and in Section 5.2 we apply this theory to define both the *affine measure* and the *Duistermaat-Heckman measure* associated to Hamiltonian actions of symplectic groupoids. In Section 5.3 we state our main result and in Section 5.4 we give its proof.

5.1 Measures on leaf spaces

A theory of measures on differentiable stacks is laid out in [20]. We present a selection of the results in the specific case of measures on leaf spaces of source proper, regular groupoids, for which the theory simplifies significantly.

5.1.1 Measures on manifolds

Let us first make precise what we mean by "measures" in the context of smooth manifolds.

Definition 5.1. Let X be a smooth manifold. A *measure* μ on X is a linear functional $\mu : C_c^\infty(X) \to \mathbb{R}$ satisfying $\mu(f) \geq 0$ for all $0 \leq f \in C_c^\infty(X)$. $\diamondsuit$

Remark 5.2. This is essentially the definition of a *Radon measure*, which makes sense for any locally compact, Hausdorff space (replacing $C_c^\infty(X)$ by $C_c(X)$). For manifolds, the two definitions are equivalent.

A special class of measures consists of those arising from densities. Measures of this type are often called *geometric measures*. Recall that a *density* on a manifold X is a section of the density bundle $\mathcal{D}_{TX}$. We denote the set of densities by $\mathcal{D}(X)$. Any differential form $\alpha \in \Omega^{\text{top}}(X)$ induces a density $|\alpha| \in \mathcal{D}(X)$. There is a canonical integration map

$$\int_X : \mathcal{D}(X) \to \mathbb{R}$$

and this integration generalises that of differential forms. Any positive density $\rho \in \mathcal{D}(X)$ induces a measure μ_ρ, defined by

$$\mu_\rho(f) := \int_X f \cdot \rho, \qquad f \in C_c^\infty(M).$$

Geometric measures can be pushed forward along proper submersions using fiber integration. For $q : X \to Y$ a proper submersion and $\rho \in \mathcal{D}(X)$, the *pushforward* of ρ along q is denoted $q_!(\rho) \in \mathcal{D}(Y)$.

General measures, even between locally compact, Hausdorff spaces, can be pushed forward along any proper map and we use similar notation. For geometric measures, these two notions of pushforward are compatible.

5.1.2 Measures on leaf spaces

We now fix a source proper, regular groupoid $\mathcal{G} \rightrightarrows X$. We write B for its leaf space, $q : X \to B$ for the quotient map and $\mathcal{B} = \mathcal{B}(\mathcal{G})$ for the underlying orbifold groupoid. Since B is locally compact and Hausdorff, we have the notion of Radon measure on it. However, it turns out that, similar to the case of manifolds, it is equivalent to define a measure as a positive linear functional on

$$C_c^\infty(B) := \{f \in C_c(B) \mid f \circ q \in C^\infty(X)\}.$$

We will describe two ways of obtaining measures on B. The first is simple: since q is a proper map, we can push measures on X forward to B. The second way is more complicated: it describes measures induced by *transverse densities*.

Let us write $A = \mathrm{Lie}(\mathcal{G})$ for the Lie algebroid of $\mathcal{G}$, $\mathfrak{g} \subset A$ for the isotropy subbundle, $T\mathcal{F} \subset TX$ for the foliation induced by $\mathcal{G}$ and $\nu(\mathcal{F})$ for the associated normal bundle. Note that $\mathfrak{g}$ and $\nu(\mathcal{F})$ are $\mathcal{G}$-representations.

Definition 5.3. A *transverse density* on $\mathcal{G}$ is a $\mathcal{G}$-invariant positive density $\rho_\nu \in \Gamma(\mathcal{D}_{\nu(\mathcal{F})})$. $\diamondsuit$

To see how this induces a measure on B, we need to choose a strictly positive density $\rho_{\mathcal{F}} \in \Gamma(\mathcal{D}_{T\mathcal{F}})$. We then set $\rho_X := \rho_{\mathcal{F}} \otimes \rho_\nu \in \mathcal{D}(X)$ and we can define a measure μ_{ρ_ν} on B by the formula

$$\int_B f(b)\, d\mu_{\rho_\nu}(b) := \int_X \frac{f(q(x))}{\iota(x) \cdot \mathrm{vol}(\mathcal{O}_x, \rho_{\mathcal{F}})}\, d\mu_{\rho_X}(x), \qquad f \in C_c^\infty(B). \tag{5.1}$$

Here $\mathcal{O}_x$ denotes the orbit of $\mathcal{G}$ through $x \in X$, $\mathrm{vol}(\mathcal{O}_x, \rho_{\mathcal{F}})$ is its volume with respect to the density $\rho_{\mathcal{F}}$ (restricted to the orbit) and $\iota(x)$ is the number of connected components of the isotropy group $\mathcal{G}_x$. This definition does not depend on the choice of $\rho_{\mathcal{F}}$. We also have the "fiber integration formula"

$$\int_X f(x)\, d\mu_{\rho_X}(x) := \int_B \iota(b) \cdot \left(\int_{\mathcal{O}_b} f(x)\, d\mu_{\rho_{\mathcal{F}}}(x) \right) d\mu_{\rho_\nu}(x), \qquad f \in C_c^\infty(X). \tag{5.2}$$

Here $\mathcal{O}_b = q^{-1}(b)$ is the orbit associated to $b \in B$ and $\iota : B \to \mathbb{Z}_{\geq 1}$ is defined as above.

Remark 5.4. As mentioned before, these definitions only work in the specific case of source proper, regular groupoids. For the more general theory, and for a detailed account on how it reduces to the above in our case, see [20].

5.2 Measures associated to Hamiltonian actions

For the remainder of this chapter, let

$$\begin{tikzcd}
(\mathcal{G}, \Omega) \arrow[r, bend left] \arrow[d, shift left] \arrow[d, shift right] & (X, \omega) \arrow[dl, "\mu"] \\
(M, \pi) &
\end{tikzcd}$$

be a locally free, effective Hamiltonian action of a source connected, source proper, regular symplectic groupoid and assume that μ is proper and has connected fibres. We denote the leaf space by B, the projection by $q : M \to B$ and the symplectic foliation by $\mathcal{F}_\pi$. With these assumptions we can assume without loss of generality that the moment map is surjective.

5.2.1 The affine measure

From Section 3.4 we get the lattice $\Lambda \subset \nu^*(\mathcal{F}_\pi)$, which induces a transverse density (Definition 5.3) as follows. Pick any local frame $\{\lambda_1, \ldots, \lambda_q\}$ of Λ, and define (locally)

$$\rho_\nu := |\lambda_1 \wedge \cdots \wedge \lambda_q|.$$

It is easily checked that this gives a well-defined transverse density $\rho_\nu \in \Gamma(\mathcal{D}_{\nu(\mathcal{F}_\pi)})$. As described in Section 5.1.2 we obtain an induced measure on B, which we call the *affine measure* and denote by μ_{aff}.

In this context, there is a nice candidate for the foliated density $\rho_{\mathcal{F}_\pi}$. Since the leaves of the foliation are equipped with symplectic forms, we have a "foliated Liouville form"

$$\rho_{\mathcal{F}_\pi} := \left| \frac{\omega_{\mathcal{F}_\pi}^{\mathrm{top}}}{\mathrm{top}!} \right|.$$

The density on M that we work with in computations becomes

$$\rho_M := \left| \frac{\omega_{\mathcal{F}_\pi}^{\mathrm{top}}}{\mathrm{top}!} \right| \otimes \rho_\nu.$$

5.2.2 The Duistermaat-Heckman measure

Definition 5.5. The *Duistermaat-Heckman measure* is defined as the pushforward of the Liouville measure on X:

$$\mu_{\mathrm{DH}} := (q \circ \mu)_* \left(\frac{\omega^{\mathrm{top}}}{\mathrm{top}!} \right). \qquad \diamondsuit$$

We can describe this measure in an alternate way, which gives us an opportunity to analyse the quotient $X/\mathcal{G}$ in some more detail as well. Since the action is locally free, $X/\mathcal{G}$ is an orbifold with atlas $\mathcal{G} \ltimes X \rightrightarrows X$. In fact, one can view it as a "Poisson orbifold", the Poisson structure appearing on the level of X as the (regular) Dirac structure

$$\mathbb{L}_\omega = \{v + w + i_w\omega \mid v \in \mathcal{G} \cdot TX, w \in \ker(d\mu)\}.$$

The leaf space of this Dirac structure can be thought of as the leaf space of $X/\mathcal{G}$. We could also define the Duistermaat-Heckman measure as the pushforward of the Liouville measure to this leaf space. These two definitions are compatible, as we now explain.

The pullback groupoid $\mu^*\mathcal{G} = X\ {}_\mu\!\times_{\mathbf{t}} \mathcal{G}\ {}_{\mathbf{s}}\!\times_\mu X$ equipped with the 2-form $\omega \oplus -\Omega \oplus -\omega$ is a presymplectic groupoid integrating $\mathbb{L}_\omega$ and the standard Morita equivalence $\mu^*\mathcal{G} \cong \mathcal{G}$ establishes an isomorphism between the leaf space of $X/\mathcal{G}$ and B. Clearly, this isomorphism interwines the two definitions of the Duistermaat-Heckman measure.

Remark 5.6. The presymplectic integration of $\mathbb{L}_\omega$ also induces a transverse integral affine structure and thus we get an "affine measure" on the leaf space of $X/\mathcal{G}$ in the same way as before (see Section 5.2.1). Since the isomorphism induced by $\mu^*\mathcal{G} \cong \mathcal{G}$ preserves the integral affine structure on the leaf spaces, it also intertwines the two affine measures. All in all, there are no issues working solely on the leaf space B.

5.2.3 The volume of the reduced spaces

For Hamiltonian groupoid actions, there is still the notion of reduced spaces. For $p \in M$, $\mathcal{G}_p$ acts on $\mu^{-1}(p)$ and in our case this action is locally free and proper, meaning that the quotient $\mu^{-1}(p)/\mathcal{G}_p$ inherits an orbifold structure. In fact, it is a *symplectic orbifold*, the symplectic structure ω_{red} being induced by presymplectic form $\omega|_{\mu^{-1}(p)}$. Of course, the same holds true when we just consider the action of $\mathcal{G}_p^0$, the connected component of the identity of $\mathcal{G}_p$. It is this action that we will consider now.

Using the theory from Section 5.1.2 there is a well-defined volume associated to these reduced spaces. Taking the top (nonzero) power of $\omega|_{\mu^{-1}(p)}$ yields a transverse measure (see Definition 5.3) and combining this with the Haar measure on $\mathcal{G}_p^0$, transported by the action to X as a foliated density, gives the formula

$$\operatorname{vol}(\mu_{-1}(p)/\mathcal{G}_p^0, \omega_{\text{red}}) = \int_{\mu^{-1}(p)} \rho_{\text{Haar}} \otimes \left| \frac{\left(\omega|_{\mu^{-1}(p)}\right)^{\text{top}}}{\text{top}!} \right|. \tag{5.3}$$

Here ρ_{Haar} is normalised according to $\mathcal{G}_p^0$.

Remark 5.7. Strictly speaking, the integrand in equation (5.3) needs to be modified by a function similar to ι, associating to $x \in \mu^{-1}(p)$ the number of connected components of the isotropy group of the action of $\mathcal{G}_p^0$. However, this function is only not equal to 1 on a set of measure zero (see Section 5.4.2) so the equation still holds in the above form.

5.3 The result

Before we state the theorem, we need one more bit of notation. Recall that we have a function $\iota : B \to \mathbb{Z}_{\geq 1}$ associating to some $b \in B$ the number of connected components of $\mathcal{G}_p$ for any $p \in \mathcal{O}_b$. We denote by $\operatorname{vol} : B \to \mathbb{R}$ the function

$$b \mapsto \iota(b) \cdot \operatorname{vol}(\mathcal{O}_b, \omega_{\mathcal{F}_\pi}).$$

Similarly, we denote by $\operatorname{vol}_{\text{red}} : B \to \mathbb{R}$ the function

$$b \mapsto \operatorname{vol}(\mu^{-1}(p)/\mathcal{G}_p^0, \omega_{\text{red}}).$$

Here we take any $p \in \mathcal{O}_b$ and $\operatorname{vol}(\mu^{-1}(p)/\mathcal{G}_p^0, \omega_{\text{red}})$ is the volume of the reduced space as described in Section 5.2.2.

Theorem 5.8. *The Duistermaat-Heckman measure is related to the affine measure by the formula*

$$\mu_{\text{DH}} = \operatorname{vol} \cdot \operatorname{vol}_{\text{red}} \cdot \mu_{\text{aff}}. \tag{5.4}$$

Moreover, vol *and* $\operatorname{vol}_{\text{red}}$ *are* polynomial *functions on the leaf space.*

Remark 5.9. The notion of a polynomial function makes sense on integral affine manifolds (and orbifolds): since integral affine maps preserve polynomials, one can simply require the function to be polynomial in every integral affine (foliation) chart.

Remark 5.10. Suppose $(\mathcal{G},\Omega) \rightrightarrows (M,\pi)$ is a source connected, source proper, regular symplectic groupoid acting effectively on a symplectic manifold (X,ω) with proper moment map $\mu : X \to M$ having connected fibers. If the isotropy groups of $\mathcal{G}$ are connected, then it follows from the reasoning in Section 5.4.2 that the action is actually *free* on an invariant, open dense subset $O \subset X$. Indeed, since the action of $\mathcal{T}(\mathcal{G})$ is locally just a classical Hamiltonian torus action, this follows directly from [16, Lemma 3.1]. Since $\mu|_O$ is a submersion, $\mu(O) \subset M$ is open and invariant and we can apply Theorem 5.8 to the restricted action of $(\mathcal{G}|_{\mu(O)},\Omega) \rightrightarrows (\mu(O),\pi)$ on (O,ω).

Example 5.11 (The classical case)**.** Consider a locally free Hamiltonian torus action $\mathbb{T} \circlearrowright (X,\omega)$ with proper moment map $\mu : X \to \mathfrak{t}^*$. The groupoid is now a bundle of tori $\mathbb{T} \ltimes \mathfrak{t}^* \rightrightarrows \mathfrak{t}^*$ and thus $B = \mathfrak{t}^*$ is smooth. The affine measure is Lebesgue measure on $\mathfrak{t}^* \cong \mathbb{R}^q$ and the Duistermaat-Heckman measure is the classical one as in [16]. The function vol has constant value 1 and $\mathrm{vol}_{\mathrm{red}}$ gives the volume of the reduced spaces. Thus Theorem 5.8 reduces to the classical Duistermaat-Heckman theorem in this case. △

Example 5.12 (The free case)**.** If the action is free, the situation simplifies significantly. In this case, the quotient $X_{\mathrm{red}} := X/\mathcal{G}$ is a smooth manifold endowed with a Poisson structure π_{red} induced from ω. In fact, $(X_{\mathrm{red}},\pi_{\mathrm{red}})$ is again a Poisson manifold of source proper type: the gauge groupoid $\big((X\ {}_{\mu}\!\times_{\mu} X)/\mathcal{G}, \omega \oplus -\omega\big)$ provides a source connected, source proper symplectic integration. Moreover, (X,ω) gives a symplectic Morita equivalence with $(\mathcal{G},\Omega) \rightrightarrows (M,\pi)$. It is not hard to show that for any symplectic Morita equivalence

$$\begin{tikzcd}
(\mathcal{G}_1,\Omega_1) \arrow[r, phantom, "\circlearrowright"] \arrow[d, shift left] \arrow[d, shift right] & (X,\omega) \arrow[dl, "\mu_1"'] \arrow[dr, "\mu_2"] & (\mathcal{G}_2,\Omega_2) \arrow[l, phantom, "\circlearrowleft"] \arrow[d, shift left] \arrow[d, shift right] \\
(M_1,\pi_1) \arrow[dr, "q_1"'] & & (M_2,\pi_2) \arrow[dl, "q_2"] \\
& B &
\end{tikzcd}$$

between regular, source connected, source proper symplectic groupoids we have the formula

$$\mu_{\mathrm{DH}} = \mathrm{vol}_1 \cdot \mathrm{vol}_2 \cdot \mu_{\mathrm{aff}}, \tag{5.5}$$

where vol_i is the vol-function associated to $(\mathcal{G}_i,\Omega_i) \rightrightarrows (M_i,\pi_i)$. (Note that μ_{DH} and μ_{aff} can be defined using either side of the diagram without change.) In the free case vol_2 is just the function $\mathrm{vol}_{\mathrm{red}}$ and thus Theorem 5.8 reduces to (5.5). △

Example 5.13. Consider the special case of $\mathcal{G}$ acting on itself by left translation. This brings us to the situation in Example 5.12, where the quotient is just (M,π) and the integration is $(\mathcal{G},\Omega)$ (with the same groupoid structure). The Duistermaat-Heckman measure is now the one as defined in [3, Section 6.3] and equation (5.5) becomes

$$\mu_{\mathrm{DH}} = \mathrm{vol}^2 \cdot \mu_{\mathrm{aff}}$$

which is exactly [3, Theorem 6.3.1]. △

5.4 The proof

We first prove equation (5.4) and then the polynomial nature of vol and $\mathrm{vol}_{\mathrm{red}}$.

5.4.1 Proof of equation (5.4)

From the definitions of the affine and Duistermaat-Heckman measures (see Sections 5.2.1 and 5.2.2 respectively) and equations (5.2) and (5.3) it follows that we need to prove the formula

$$\left|\frac{\omega^{\text{top}}}{\text{top!}}\right| = \rho_{\text{Haar}} \otimes \left|\frac{\left(\omega|_{\mu^{-1}(p)}\right)^{\text{top}}}{\text{top!}}\right| \otimes \mu^* \left(\left|\frac{\omega_{\mathcal{F}_\pi}^{\text{top}}}{\text{top!}}\right| \otimes |\lambda_1 \wedge \cdots \wedge \lambda_q|\right). \tag{5.6}$$

To prove (5.6), let us fix $x \in X$ and choose a convenient basis of T_xX. We write $m = \dim(M)$ and $n = \dim(X)$. Recall also that $q = \dim(\mathcal{G}_p)$. Let $\{\alpha_1, \ldots, \alpha_q\}$ be the basis of $\mathfrak{g}_{\mu(x)}$ that, through the identifications of Section 3.4, is dual to $\{\lambda_1, \ldots, \lambda_q\}$, and complete it to a basis $\{\alpha_1, \ldots, \alpha_m\}$ of $A_{\mu(x)}$. The infinitesimal action gives us associated vectors $\{\alpha_i^X\}_i$ which form a basis of $\mathcal{G} \cdot T_xX$. Next, let $\{v_1, \ldots, v_{n-m-q}\}$ be a basis for a complement of $\mathcal{G}_p \cdot T_xX$ in $\ker(d_x\mu)$. Finally, let $\{w_1, \ldots, w_q\}$ be dual to $\{\mu^*\lambda_1, \ldots, \mu^*\lambda_q\}$. Let us write

$$\begin{aligned}
V_1 &= \operatorname{span}\{\alpha_1^X, \ldots, \alpha_q^X\}, \\
V_2 &= \operatorname{span}\{v_1, \ldots, v_{n-m-q}\}, \\
V_3 &= \operatorname{span}\{\alpha_{q+1}^X, \ldots, \alpha_m^X\}, \\
V_4 &= \operatorname{span}\{w_1, \ldots, w_q\}
\end{aligned}$$

consistent with the decomposition in equation (5.6).

Note that $V_1 \oplus V_2 = \ker(d_x\mu)$ and $V_1 \oplus V_3 = \mathcal{G} \cdot T_xX$. Since $\ker(d_x\mu)^\omega = \mathcal{G} \cdot T_xX$, it follows that for any $1 \leq i \leq q$ the 1-form $i_{\alpha_i^X}\omega$ is automatically zero on $V_1 \oplus V_2 \oplus V_3$. We now compute

$$\omega_x(\alpha_i^X, w_j) = (\mu^*\lambda_i)(w_j) = \delta_{ij}$$

using the moment map condition (2.16). Note also that by definition of the Haar measure we have

$$\rho_{\text{Haar}}(\alpha_1^X, \ldots, \alpha_q^X) = 1.$$

Next, let us analyse $\omega|_{V_3}$. We compute

$$\begin{aligned}
\omega_x(\alpha_{q+i}^X, \alpha_{q+j}^X) &= -\Omega_{1_{\mu(x)}}(\alpha_{q+i}, d_x\mu(\alpha_{q+j}^X)) \\
&= -\Omega_{1_{\mu(x)}}(\alpha_{q+i}, d_{1_{\mu(x)}}\mathbf{s}(\alpha_{q+j})) \\
&= -\Omega_{1_{\mu(x)}}(\alpha_{q+i}, \alpha_{q+j}),
\end{aligned}$$

where we use that the $\mathbf{s}$- and $\mathbf{t}$-fibers are Ω-orthogonal. On the other hand, we have

$$\begin{aligned}
(\mu^*\omega_{\mathcal{F}_\pi})_x(\alpha_{q+i}^X, \alpha_{q+j}^X) &= (\omega_{\mathcal{F}_\pi})_{\mu(x)}(d_x\mu(\alpha_{q+i}^X), d_x\mu(\alpha_{q+j}^X)) \\
&= (\omega_{\mathcal{F}_\pi})_{\mu(x)}(d_{1_{\mu(x)}}\mathbf{s}(\alpha_{q+i}), d_{1_{\mu(x)}}\mathbf{s}(\alpha_{q+j})) \\
&= \Omega_{1_{\mu(x)}}(\alpha_{q+i}, \alpha_{q+j})
\end{aligned}$$

so that we can conclude that

$$\omega_x(\alpha_{q+i}^X, \alpha_{q+j}^X) = -(\mu^*\omega_{\mathcal{F}_\pi})_x(\alpha_{q+i}^X, \alpha_{q+j}^X).$$

Combining the above equations and doing the necessary combinatorics we arrive precisely at equation (5.6).

Remark 5.14. In [30] a local model is given for Hamiltonian actions of proper symplectic groupoids, which can be used to give an alternate proof of equation (5.6).

5.4.2 Proof of polynomial nature

The polynomial nature of vol is well-known (see [3, Theorem 6.3.1]), so it remains to show that $\mathrm{vol}_{\mathrm{red}}$ is a polynomial. To this end, let us study the what the action of $\mathcal{T}(\mathcal{G})$ looks like locally.

Fix a transverse integral affine chart (U, φ) and let $T = \varphi^{-1}(\{0\} \times \mathbb{R}^q)$ be the associated transversal. The chart trivialises the conormal bundle and the lattice $\Lambda \subset \nu^*(\mathcal{F}_\pi)$ inside, and thus also the bundle $\mathcal{T}(\mathcal{G})|_T$. Thus we can consider it as a single torus $\mathbb{T}$ acting on $\mu^{-1}(T)$. The latter is a symplectic submanifold of (X, ω) and the map $\widetilde{\mu} := \varphi \circ \mu : \mu^{-1}(T) \to \mathbb{R}^q$ is easily verified to make the resulting data $\mathbb{T} \circlearrowright (\mu^{-1}(T), \omega|_{\mu^{-1}(T)}) \xrightarrow{\widetilde{\mu}} \mathbb{R}^q$ into a classical Hamiltonian torus action. It then follows from [16, Corollary 3.3] that the volumes of the associated reduced spaces vary in a polynomial way. These volumes precisly form the function $\mathrm{vol}_{\mathrm{red}}$, meaning that the latter is a polynomial.

References

[1] M. Crainic and R. L. Fernandes, "Rigidity and flexibility in Poisson geometry," in *Travaux mathématiques. Fasc. XVI*, ser. Trav. Math. Vol. 16, Univ. Luxemb., Luxembourg, 2005, pp. 53–68.

[2] M. Crainic, R. L. Fernandes, and D. M. Torres, "Poisson manifolds of compact types (PMCT 1)," *Journal für die reine und angewandte Mathematik*, vol. 2019, no. 756, pp. 101–149, 2019. DOI: https://doi.org/10.1515/crelle-2017-0006. [Online]. Available: https://www.degruyter.com/view/journals/crll/2019/756/article-p101.xml.

[3] M. Crainic, R. L. Fernandes, and D. Martínez Torres, "Regular Poisson manifolds of compact types (PMCT 2)," English (US), *Asterisque*, vol. 413, pp. 1–166, Jan. 2019, ISSN: 0303-1179. DOI: 10.24033/ast.1079.

[4] D. Martínez Torres, "A Poisson manifold of strong compact type," *Indagationes Mathematicae*, vol. 25, no. 5, pp. 1154–1159, Dec. 2013.

[5] D. Kotschick, "Free circle actions with contractible orbits on symplectic manifolds," *Mathematische Zeitschrift*, vol. 252, no. 1, pp. 19–25, 2006.

[6] K. Mishachev, "The classification of Lagrangian bundles over surfaces," *Differential Geometry and its Applications*, vol. 6, no. 4, pp. 301–320, 1996, ISSN: 0926-2245. DOI: https://doi.org/10.1016/S0926-2245(96)00024-1. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0926224596000241.

[7] D. Sepe, "Classification of Lagrangian fibrations over a Klein bottle," *Geom. Dedicata*, vol. 149, pp. 347–362, 2010, ISSN: 0046-5755. DOI: 10.1007/s10711-010-9485-8. [Online]. Available: https://doi.org/10.1007/s10711-010-9485-8.

[8] D. Sepe, "Universal Lagrangian bundles," *Geometriae Dedicata*, vol. 165, Aug. 2013. DOI: 10.1007/s10711-013-9860-3.

[9] J. Marsden and A. Weinstein, "Reduction of symplectic manifolds with symmetry," *Reports on Mathematical Physics*, vol. 5, no. 1, pp. 121–130, 1974, ISSN: 0034-4877. DOI: https://doi.org/10.1016/0034-4877(74)90021-4. [Online]. Available: https://www.sciencedirect.com/science/article/pii/0034487774900214.

[10] K. R. Meyer, "Symmetries and integrals in mechanics," in *Dynamical systems (Proc. Sympos., Univ. Bahia, Salvador, 1971)*, Academic Press, New York-London, 1973, pp. 259–272.

[11] R. Sjamaar and E. Lerman, "Stratified symplectic spaces and reduction," *Annals of Mathematics*, vol. 134, no. 2, pp. 375–422, 1991, ISSN: 0003486X. [Online]. Available: http://www.jstor.org/stable/2944350 (visited on 10/30/2023).

[12] V. Guillemin and S. Sternberg, "Convexity properties of the moment mapping," *Invent. Math.*, vol. 67, no. 3, pp. 491–513, 1982, ISSN: 0020-9910,1432-1297.

[13] M. F. Atiyah, "Convexity and commuting Hamiltonians," *Bull. London Math. Soc.*, vol. 14, no. 1, pp. 1–15, 1982, ISSN: 0024-6093,1469-2120. DOI: 10.1112/blms/14.1.1. [Online]. Available: https://doi.org/10.1112/blms/14.1.1.

[14] M. Atiyah and R. Bott, "The moment map and equivariant cohomology," *Topology*, vol. 23, no. 1, pp. 1–28, 1984, ISSN: 0040-9383. DOI: https://doi.org/10.1016/0040-9383(84)90021-1. [Online]. Available: http://www.sciencedirect.com/science/article/pii/0040938384900211.

[15] N. Berline and M. Vergne, "Classes caractéristiques équivariantes. formule de localisation en cohomologie équivariante," *CR Acad. Sci. Paris*, vol. 295, no. 2, pp. 539–541, 1982.

[16] J. J. Duistermaat and G. J. Heckman, "On the variation in the cohomology of the symplectic form of the reduced phase space," *Inventiones mathematicae*, vol. 69, no. 2, pp. 259–268, 1982.

[17] D. McDuff, "The moment map for circle actions on symplectic manifolds," *J. Geom. Phys.*, vol. 5, no. 2, pp. 149–160, 1988, ISSN: 0393-0440. DOI: 10.1016/0393-0440(88)90001-0. [Online]. Available: https://doi.org/10.1016/0393-0440(88)90001-0.

[18] J.-H. Lu, "Momentum mappings and reduction of Poisson actions," in *Symplectic geometry, groupoids, and integrable systems (Berkeley, CA, 1989)*, ser. Math. Sci. Res. Inst. Publ. Vol. 20, Springer, New York, 1991, pp. 209–226, ISBN: 0-387-97526-8. DOI: 10.1007/978-1-4613-9719-9_15. [Online]. Available: https://doi.org/10.1007/978-1-4613-9719-9_15.

[19] A. Alekseev, A. Malkin, and E. Meinrenken, "Lie group valued moment maps," *J. Differential Geom.*, vol. 48, no. 3, pp. 445–495, 1998, ISSN: 0022-040X,1945-743X. [Online]. Available: http://projecteuclid.org/euclid.jdg/1214460860.

[20] M. Crainic and J. N. Mestre, "Measures on differentiable stacks," *J. Noncommut. Geom.*, vol. 13, no. 4, pp. 1235–1294, 2019, ISSN: 1661-6952,1661-6960. DOI: 10.4171/jncg/362. [Online]. Available: https://doi.org/10.4171/jncg/362.

[21] M. Crainic, R. L. Fernandes, and I. Mărcuţ, *Lectures on Poisson geometry* (Graduate Studies in Mathematics). American Mathematical Society, Providence, RI, [2021] ©2021, vol. 217, pp. xix+479, ISBN: 978-1-4704-6430-1. DOI: 10.1090/gsm/217. [Online]. Available: https://doi.org/10.1090/gsm/217.

[22] M. Crainic and R. L. Fernandes, "Integrability of Lie brackets," *Annals of Mathematics*, vol. 157, no. 2, pp. 575–620, 2003.

[23] K. C. Mackenzie and P. Xu, "Integration of lie bialgebroids," *Topology*, vol. 39, no. 3, pp. 445–467, 2000, ISSN: 0040-9383. DOI: https://doi.org/10.1016/S0040-9383(98)00069-X. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S004093839800069X.

[24] M. Crainic and R. L. Fernandes, "Integrability of Poisson brackets," *J. Differential Geom.*, vol. 66, no. 1, pp. 71–137, Jan. 2004. DOI: 10.4310/jdg/1090415030. [Online]. Available: https://doi.org/10.4310/jdg/1090415030.

[25] W. M. Goldman, *Geometric structures on manifolds* (Graduate Studies in Mathematics). American Mathematical Society, Providence, RI, [2022] ©2022, vol. 227, pp. li+409, ISBN: [9781470471033]; [9781470471989]; [9781470471972]. DOI: 10.1090/gsm/227. [Online]. Available: https://doi.org/10.1090/gsm/227.

[26] F. Alcalde-Cuesta and G. Hector, "Intégration symplectique des variétés de poisson régulières," *Israel Journal of Math*, vol. 90, no. 1-3, pp. 125–165, 1995.

[27] W. Barth, C. Peters, and A. Ven, *Compact complex surfaces* (Ergebnisse der Mathematik und ihrer Grenzgebiete). Springer, 1984, ISBN: 9783540121725. [Online]. Available: https://books.google.nl/books?id=LsAPAQAAMAAJ.

[28] Y.-T. Siu, "Every K3 surface is Kähler," *Inventiones mathematicae*, vol. 73, no. 1, pp. 139–150, Feb. 1983, ISSN: 1432-1297. DOI: 10.1007/BF01393829. [Online]. Available: https://doi.org/10.1007/BF01393829.

[29] L. J. Mordell, "On the linear independence of algebraic numbers.," *Pacific Journal of Mathematics*, vol. 3, no. 3, pp. 625–630, 1953. DOI: pjm/1103051334. [Online]. Available: https://doi.org/.

[30] M. Mol, *Stratification of the transverse momentum map*, 2021. arXiv: 2109.13874 [math.SG].