ABELIANIZATION OF GROUPOIDS AND LIE ALGEBROIDS

BY

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DISSERTATION

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Abstract

This thesis discusses abelianization of Lie algebroids and groupoids in different categories. In Chapter 2 we find a sufficient and necessary condition for a Lie algebroid to admit an abelianization and give several examps and applications. In Chapter 3 we prove that every (locally subductive) diffeological groupoid has a (locally subductive) abelianization. We also find a sufficient condition for a Lie groupoid to admit a smooth abelianization and study the smoothness of the genus-integration, the set-theoretical abelianization of the Weinstein groupoid.

咸鱼看山

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List of Symbols

$\mathcal{A}$, $\mathcal{B}$	Lie algebroid
$\mathcal{G}$, $\mathcal{H}$	Groupoid
$\mathrm{Lie}(\mathcal{G})$	Lie algebroid of $\mathcal{G}$
$\mathcal{G}(\mathcal{A})$	The Weinsterin groupoid of $\mathcal{A}$
$\mathcal{G}_g(\mathcal{A})$	The genus-integration of $\mathcal{A}$
$\rho_{\mathcal{A}}$	The anchor map of $\mathcal{A}$
$\mathfrak{g}_x$	The isotopy Lie algebra at x
$\mathfrak{g}_M$	Bundle of Lie algebras, isotropy bundels
$[\mathfrak{g}_M, \mathfrak{g}_M]$	Commutator bundle of $\mathfrak{g}_M$ or $\mathcal{A} \to M$
$\mathcal{A}^{ab}$, $\mathfrak{g}_M^{ab}$	Abelianization of $\mathcal{A}$, $\mathfrak{g}_M$
$\mathcal{G}_x$	Isotropy Lie group of $\mathcal{G}$ at x
$(\mathcal{G}, \mathcal{G})$	Commutator bundle of $\mathcal{G}$
$\mathcal{G}^{ab}$	Abelianization of $\mathcal{G}$
$\nabla_{\mathcal{A}}$	The $T\mathcal{F}$-connection on $\mathfrak{g}_M$ for regular Lie algebroid $\mathcal{A} \to M$
$\mathcal{T}_0$	The coarse topology
$\mathcal{D}_0$	The coarse diffeology
$\mathcal{T}_{\mathcal{D}}$	The $\mathcal{D}$-topology of diffeology $\mathcal{D}$
$f_*(\mathcal{D})$	The pushforward diffeology of $\mathcal{D}$ by f

Introduction

Lie groupoids and Lie algebroids are a generalization of Lie groups and Lie algebras. The "Lie philosophy" of Lie groups and Lie algebras also makes sense in this more general setting: for any Lie groupoid there is an associated Lie algebroid and one can wonder when a given Lie algebroid comes from a Lie groupoid, in which case we say the Lie algebroid is *integrable*.

The integration problem of Lie algebras has a long history culminating in Lie's third theorem, stating that every finite dimensional Lie algebra integrates to a Lie group. There are various proofs of the theorem, including an algebraic proof in [1] and a geometric proof in [2] and [3]. Combining with the Lie's first and second theorem, one obtains a one-to-one correspondence between the categories of finite dimensional Lie algebras and simply-connected Lie groups. Lie's third theorem does not hold for Lie algebroids and the integration problem for Lie algebroids was solved in [4]. There, the authors introduced a construction called the Weinstein groupoid which, when smooth, is the unique source 1-connected Lie groupoid integrating the Lie algebroid. They also gave a neccessary and sufficient condition for the smoothness of the Weinstein groupoid. In particular, for the cotangent Lie algebroid coming from a Poisson structure, the integrability is equivalent to the integrability of the Poisson structure as shown in [5], and the monodromy groups in this case can be defined as in [6] using only the Poisson structure and are invariants of the Poisson manifold.

In [7], a modification of this process gives rise to a new groupoid called the genus-integration, which is the abelianization of the Weinstein groupoid in the category of sets. While the genus-integration itself is worthy of study, the abelianization, which can be thought of as the "maximal abelian part" of a groupoid or Lie algebroid, is also quite interesting. The existence is not guaranteed and it does not always "commute" with the integration.

In [8], crucial constructions for the integration of Lie algebroids were introduced in the setting of Lie-Rinehart algebras, allowing the generalization of many interesting problem to these objects. A generalized functor from "singular Lie groupoids" to Lie algebroids was discussed in [9], allowing us to consider integrating Lie algebroids to not only Lie groupoids, but certain diffeological groupoids. They showed that Lie's third theorem holds in this generalized setting.

In this thesis we study the abelianization of groupoids and Lie algebroids. We consider groupoids internal to several different categories, since the results differ in different settings.

Abelianization of Lie algebroids

A Lie algebroid is called *abelian* if its isotropy Lie algebras are all abelian. Given any Lie algebroid $\mathcal{A}$, one can wonder whether there exists an abelian Lie algebroid $\mathcal{A}^{ab}$ satisfying the universal property of an abelian object. We refer to $\mathcal{A}^{ab}$ as the *abelianization* of $\mathcal{A}$ (see Chapter 2 for the precise definition).

The notion of abelianization was first introduced in [7] in connection with the so-called genus integration of $\mathcal{A}$. The authors of [7] observed that for a transitive Lie algebroid, the abelianization always exists. They also provide examples of non-transitive Lie algebroids for which the abelianization exists, as well as examples for which it does not exist.

In this thesis we propose a new sheaf-theoretical definition of abelianization, which has better behavior than the one in [7].

Recall that for any Lie algebroid $\mathcal{A} \to M$, the kernel of the anchor at $x \in M$,

$$\mathfrak{g}_x := \ker \rho_x \subset \mathcal{A}_x,$$

forms a Lie algebra known as the *isotropy Lie algebra* at x. We denote the bundle of isotropy Lie algebras as

$$\mathfrak{g}_M := \bigcup_{x \in M} \mathfrak{g}_x \subset \mathcal{A}.$$

Note that, in general, the dimensions of the isotropy Lie algebras $\mathfrak{g}_x$ vary with x, so $\mathfrak{g}_M$ is not a vector subbundle. One can also define the commutator or derived bundle

$$[\mathfrak{g}_M, \mathfrak{g}_M] := \bigcup_{x \in M} [\mathfrak{g}_x, \mathfrak{g}_x] \subset \mathcal{A}.$$

which, again, may fail to be a vector subbundle. One of our main results can be stated as follows.

Theorem 1. *A Lie algebroid $\mathcal{A}$ has an abelianization if and only if the closure $\overline{[\mathfrak{g}_M, \mathfrak{g}_M]} \subset \mathcal{A}$ is a vector subbundle.*

Abelianization of groupoids

A groupoid is called *abelian* if its isotropy groups are all abelian. As before, we define the *abelianization of a groupoid* to be an abelian groupoid satisfying the universal peoperty of an abelian object (see Chapter 3 for the precise definition). We study the existence and properties of this construction.

The abelianization of a groupoid depends on the category. In general, a Lie groupoid might not admit an abelianization in the smooth category. And when it does, the smooth abelianization might differ from the set-theoretical abelianization.

Just as for Lie algebroids, the abelianization of groupoids is closely related to its isotropies, but it is more complicated as the commutator bundle can behave very differently. For example, the commutator of a Lie group can already fail to be closed.

We study the abelianization of groupoids in the category of smooth manifolds and obtained the following result.

Proposition 1. *Given a Lie groupoid $\mathcal{G}$, if the fiberwise closure $\overline{(\mathcal{G}, \mathcal{G})}^s$ is a closed submanifold, then $\mathcal{G}$ has abelianization $\mathcal{G}/\overline{(\mathcal{G}, \mathcal{G})}^s$.*

Diffeologies provide a more flexible setup to study differential geometry of singular spaces. For example, it has been used in solving integration problems of Lie algebroids as in [9], of singular Lie subalgebroids as in [10] and of singular foliations as in [11]. We study the abelianization of groupoids in the category of diffeological spaces and prove the following result for diffeological groupoids and locally subductive diffeological groupoids.

Proposition 2. *A (locally subductive) diffeological groupoid $\mathcal{G}$ has (locally subductive) abelianization $\mathcal{G}/(\mathcal{G}, \mathcal{G})$.*

Another interesting construction related to abelianization of groupoids is the genus-integration $\mathcal{G}_g(\mathcal{A})$ of a Lie algebroid $\mathcal{A}$. In [7], it is constructed using a similar idea as the construction of the Weinstein groupoid $\mathcal{G}(\mathcal{A})$ and is proved to be the set-theretical abelianization of the Weinstein groupoid. The kernel of the quotient map $q : \mathcal{G}(\mathcal{A}) \to \mathcal{G}_g(\mathcal{A})$ is studied in [7] for the integrable case, and we show that this integrability assumption can be removed.

Chapter 1

Backgrounds

In this chapter, we will recal some basic definitions and properties we will be using in the thesis.

1.1 Lie groupoids

Definition 1. *A groupoid $\mathcal{G}$ is a small category in which every arrow is invertible. To be more specific, it consist of the following data:*

- *a set of* arrows $\mathcal{G}_1$*;*
- *a set of* objects $\mathcal{G}_0$*;*
- *the* source *and* target *maps* $s, t : \mathcal{G}_1 \to \mathcal{G}_0$*;*
- *the multiplication* $m : \mathcal{G}_2 \to \mathcal{G}_1, (g, h) \mapsto gh$*, where*

$$\mathcal{G}_2 := \{(g, h) \in \mathcal{G}_1 \times \mathcal{G}_1 : s(g) = t(h)\}$$

  is the set of composable arrows;
- *the inverse map* $i : \mathcal{G}_1 \to \mathcal{G}_1, g \mapsto g^{-1}$*;*
- *the unit map* $u : \mathcal{G}_0 \to \mathcal{G}_1, x \mapsto 1_x$*,*

satisfying the following axioms:

- *for* g, h *such that* $t(h) = s(g)$*,* $s(gh) = s(h)$ *and* $t(gh) = t(g)$*;*
- *for* g, h, k *such that* $t(k) = s(h)$*,* $t(h) = s(g)$*,* $(gh)k = g(hk)$*;*

- *for* $x \xleftarrow{g} y$, $gg^{-1} = 1_x$ *and* $g^{-1}g = 1_y$*;*
- *for* $x \xleftarrow{g} y$, $1_x g = g 1_y = g$.

Definition 2. *A Lie groupoid is a groupoid where* $\mathcal{G}_1$ *and* $\mathcal{G}_0$ *are smooth manifolds, all the structure maps are smooth and the source and target maps are submersions.*

For a groupoid $\mathcal{G}$, when it does not cause confusion, we will denote the set of arrows by $\mathcal{G}$ and the set of objects by M.

For $x \in M$ we have the *isotropy group* at x, defined by $\mathcal{G}_x = s^{-1}(x) \cap t^{-1}(x)$. For a Lie groupoid, this is in fact a Lie group. There is also the *orbit* through x, defined by $O_x = t(s^{-1}(x))$. For a Lie groupoid, this is an immersed submanifold of M. The smooth structure is induced by the principal bundle $\mathcal{G}_x \circlearrowright s^{-1}(x) \xrightarrow{t} O_x$. When the dimension of the orbits is constant we call the groupoid *regular*. When there is only a single orbit, we say the groupoid is *transitive*. We call a Lie groupoid *source connected* if the source fibres are connected, and we call it *source 1-connected* if they are in addition simply connected.

Example 1. *A Lie group is a Lie groupoid over a point.*

Example 2. *Let a group* G *act on a set* M*. We can define the action groupoid* $G \ltimes M$ *as follows. We set* $\mathcal{G}_1 = G \times M$*,* $\mathcal{G}_0 = M$ *and define the source and target by* $s(g, x) = x$ *and* $t(g, x) = g \cdot x$*. The multiplication is given by* $(h, g \cdot x)(g, x) = (hg, x)$*. Of course, this example makes sense in the smooth category as well.*

Definition 3. *A (Lie) subgroupoid of* $\mathcal{G}$ *is a (Lie) groupoid* $\mathcal{H}$ *with an injective (smooth) groupoid morphism.* $i : \mathcal{H} \to \mathcal{G}$*.*

When the subgroupoid has the same space of objects, we say it is a *wide* subgroupoid. In this thesis, when we deal with a subgroupoid, we assume it is wide unless otherwise specified.

Proposition 3. *Let* $\mathcal{G}$ *be a Lie groupoid. If* $\mathcal{H}$ *is a closed normal Lie subgroupoid of* $\mathcal{G}$*, then* $\mathcal{G}/\mathcal{H}$ *is a Lie groupoid.*

As is well known, the tangent space at the identity of a Lie group has the structure of a Lie algebra and this Lie algebra captures a lot of information about the Lie group. The situation for Lie groupoids is similar: there is a vector bundle over the space of objects with extra structure, in the form of a *Lie bracket* on its space of sections and an *anchor map* to the tangent bundle of the space of objects. This data forms what is called a *Lie algebroid* (see Section 1.2). Let us know describe the "Lie functor" in this new context.

Given a Lie groupoid $\mathcal{G} \rightrightarrows M$, its Lie algebroid $\mathrm{Lie}(\mathcal{G})$ is a vector bundle $\mathcal{A}$ whose fiber at x is the tangent space of $s^{-1}(x)$ at 1_x, i.e. $\mathrm{Lie}(\mathcal{G}) := T^s\mathcal{G}|_M$. To define the bracket on $\Gamma(\mathcal{A})$, we will look at the right invariant vector fields of $\mathcal{G}$:

$$\mathfrak{X}^s_{inv}(\mathcal{G}) = \{X \in \Gamma(T^s(\mathcal{G})) : X_{hg} = R_g(X_h)\},$$

where $R_g : T^s_h\mathcal{G} \to T^s_{gh}\mathcal{G}$ is induced by the right multiplication by g. Similar to Lie algebras of Lie groups, there is a natural way to extend each section $\alpha \in \Gamma(\mathcal{A})$ to a right invariant vector field $\tilde{\alpha}$ on $\mathcal{G}$ defined by $\tilde{\alpha}_g := R_g(\alpha_{t(g)})$. This assignment $\alpha \mapsto \tilde{\alpha}$ actually induced an isomorphism $\Gamma(\mathcal{A}) \cong \mathfrak{X}^s_{inv}(\mathcal{G})$ and since $\mathfrak{X}^s_{inv}(\mathcal{G})$ is a Lie subalgebra of $\mathfrak{X}(\mathcal{G})$, we can use the bracket on $\mathfrak{X}(\mathcal{G})$ to obtain the Lie bracket on $\Gamma(\mathcal{A})$. The anchor map $\rho : \mathcal{A} \to TM$ is simply defined as the (restriction of the) differential of the target map $t : \mathcal{G} \to M$.

As the name implies, the Lie functor also applies to morphisms. Given a morphism $\Phi : \mathcal{G} \to \mathcal{H}$, $d\Phi$ restricted to $\mathrm{Lie}(\mathcal{G})$ induces a morphism $\Phi_* : \mathrm{Lie}(\mathcal{G}) \to \mathrm{Lie}(\mathcal{H})$.

1.2 Lie algebroids

Let us now turn to the general definition of Lie algebroids.

Definition 4. *A* Lie algebroid *over a manifold M is a vector bundle $\mathcal{A}$ over M equipped with a Lie bracket $[-,-]_{\mathcal{A}}$ on $\Gamma(A)$ and an anchor map $\rho_{\mathcal{A}} : \mathcal{A} \to TM$ such that the Leibniz identity is satisfied for all $\alpha, \beta \in \Gamma(\mathcal{A})$ and all $f \in C^\infty(M)$:*

$$[\alpha, f\beta]_{\mathcal{A}} = f[\alpha, \beta]_{\mathcal{A}} + \mathcal{L}_{\rho_{\mathcal{A}}(\alpha)}(f)\beta.$$

Equivalently, one can define a Lie algebroid by requiring the sheaf of local sections to have the structure of a sheaf of Lie algebras, as opposed to requiring a bracket on the space of global sections. This point of view is beneficial in certain cases when studying Lie algebroids. For example, when defining Lie algebroids in the holomorphic category, one is forced to use this approach since sections exist only locally.

A natural question is whether every Lie algebroid comes from some Lie groupoid. We turn to this question in Section 1.3.

We often denote a Lie algebroid by $\mathcal{A}$ or $\mathcal{A} \to M$. For $x \in M$, the *isotropy Lie algebra* is defined by $\mathfrak{g}_x = \ker(\rho_x)$, where the Lie bracket is induced by the Lie bracket on $\Gamma(\mathcal{A})$. We write

$$\mathfrak{g}_M = \bigcup_{x \in M} \mathfrak{g}_x \subset \mathcal{A}$$

for the bundle of isotropy Lie algebras. Note that, in general, this is not a subbundle since it might not have constant rank. The image of ρ is an integral distribution $T\mathcal{F} \subset TM$, although it is not regular in general. The (singular) foliation integrating this distribution is called the *foliation induced by* $\mathcal{A}$ and we call the leaves of the foliation the *leaves of* $\mathcal{A}$. When $\rho(\mathcal{A})$ has constant rank, we say that $\mathcal{A}$ is regular. Note that in this case, the induced foliation is in fact a regular foliation. When $\rho(\mathcal{A}) = TM$, we say that $\mathcal{A}$ is transitive. When $\rho = 0$, we call $\mathcal{A}$ a *bundle of Lie algebras.*

Note that when $\mathcal{A} = \text{Lie}(\mathcal{G})$, the terminology we introduced for Lie groupoids and Lie algebroids is compatible. The Lie algebras of the isotropy groups $\mathcal{G}_x$ are precisely the isotropy Lie algebras $\mathfrak{g}_x$, the leaves of $\mathcal{A}$ are the connected components of the orbits of $\mathcal{G}$, and if $\mathcal{G}$ is regular then so is $\mathcal{A}$.

Example 3. *Given an action $\rho : \mathfrak{g} \to \mathfrak{X}(M)$ of a Lie algebra $\mathfrak{g}$ on M, we can define the action Lie algebroid $\mathfrak{g} \ltimes M$ as follows. We set $\mathcal{A} = M \times \mathfrak{g}$ and define the anchor as $\rho_{\mathcal{A}}(\alpha) = \rho(\alpha)_x$ for $(x, \alpha) \in M \times \mathfrak{g}$. The Lie bracket is defined for constant sections by the Lie bracket on $\mathfrak{g}$, and then generalized to arbitrary sections by requiring the Leibniz identity.*

Let $\mathcal{A} \to M$ and $\mathcal{B} \to N$ be Lie algebroids, and consider a bundle map

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{B} \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array}$$

If f is a diffeomorphism, then one has an induced map at the level of sections $F : \Gamma(\mathcal{A}) \to \Gamma(B)$ and one says that $\phi = (F, f)$ is a Lie algebroid morphism if for any sections $s, s' \in \Gamma(\mathcal{A})$ one has:

(i) $\mathrm{d}f(\rho_{\mathcal{A}}(s)) = \rho_{\mathcal{B}}(F(s))$;

(ii) $F([s, s']_{\mathcal{A}}) = [F(s), F(s')]_{\mathcal{B}}$.

For a general bundle map there is no induced map at the level of sections, and one replaces (ii) by the following condition:

(ii)' for any sections $s, s' \in \Gamma(\mathcal{A})$ such that $F(s) = \sum_i a_i f^* s_i$ and $F(s') = \sum_j b_j f^* s_j$, one has

$$F([s, s']_{\mathcal{A}}) = \sum_{i,j} a_i b_i f^*[s_i, s_j]_{\mathcal{B}} + \sum_j (\mathrm{Lie}_{\rho(s)} b_i) f^*(s_j) + \sum_i (\mathrm{Lie}_{\rho(s')} a_i) f^*(s_i).$$

An alternative way to define algebroid morphisms is by using the algebroid de Rham differential on $\mathcal{A}$-forms. One defines k-forms to be the sections $\Omega^k(A) := \Gamma(\wedge^k A^*)$ and defines

$$\mathrm{d}_{\mathcal{A}} : \Omega^k(\mathcal{A}) \to \Omega^{k+1}(\mathcal{A})$$

by

$$\mathrm{d}_A \omega(s_0, \dots, s_k) := \sum_i (-1)^i \mathrm{Lie}_{\rho(s_i)} \omega(s_0, \dots, \widehat{s_i}, \dots, s_k) + \sum_{i<j} \omega([s_i, s_j], s_0, \dots, \widehat{s_i}, \dots, \widehat{s_j}, \dots, s_k).$$

A bundle map $\phi : \mathcal{A} \to \mathcal{B}$, as above, induces a pull-back map between such forms, and it is a Lie algebroid morphism if and only if

$$\phi^* \mathrm{d}_{\mathcal{B}} = \mathrm{d}_{\mathcal{A}} \phi^*.$$

1.3 Integrating Lie algebroids

Definition 5. *We say that a Lie groupoid $\mathcal{G} \rightrightarrows M$* integrates *a Lie algebroid $\mathcal{A} \to M$ if $\mathrm{Lie}(\mathcal{G}) \cong \mathcal{A}$. We say a Lie algebroid is* integrable *if there is a Lie groupoid integrating it.*

The integration problem for Lie algebroids has been studied extensively in pursuit of the classical results for Lie groups and Lie algebras. As it turns out, for Lie groupoids and Lie algebroids, we still have Lie's first and Lie's second theorems.

Theorem 2 (Lie's first theorem)**.** *Given a Lie groupoid $\mathcal{G}$, there exists a unique source 1-connected Lie groupoid $\tilde{\mathcal{G}}$ such that $Lie(\tilde{\mathcal{G}}) = Lie(\mathcal{G})$.*

Theorem 3 (Lie's second theorem)**.** *Given a Lie groupoid $\mathcal{G}$ and $\mathcal{H}$, where $\mathcal{G}$ is source 1-connected. For any morphism $\phi : Lie(\mathcal{G}) \to Lie(\mathcal{H})$, there exists a unique Lie groupoid morphism $\Phi : \mathcal{G} \to \mathcal{H}$ such that $\Phi_* = \phi$.*

However, Lie's third theorem does not hold anymore: it turns out that *there exist Lie algebroids that cannot be integrated by any Lie groupoid*. The integration problem for Lie algebroids was solved in [4], where a necessary and sufficient condition was found for integrability of Lie algebroids in terms of certain objects called the *monodromy groups*. We now explain this result in some detail.

1.3.1 The Weinstein groupoid

Given a Lie algebroid, one can construct a "candidate integration", called the Weinstein groupoid, as follows.

Definition 6. *Given a Lie algebroid $\mathcal{A}$, an $\mathcal{A}$-path is a pair (a, γ) consisting of a path $a : I \to \mathcal{A}$ and a path $\gamma : I \to M$ such that:*

- *$p \circ \mathcal{A} = \gamma$, where $p : \mathcal{A} \to M$ is the vector bundle projection;*
- *$\rho(a(t)) = \frac{d\gamma}{dt}(t)$, for all $t \in I$.*

We denote the set of $\mathcal{A}$-paths by $P(\mathcal{A})$.

Definition 7. *Given two $\mathcal{A}$-paths a_0 and a_1, we say a_0 is $\mathcal{A}$-homotopic to a_1 if there is a Lie algebroid morphism*

$$\begin{array}{ccc} T(I \times I) & \xrightarrow{a(t,\epsilon)dt + b(t,\epsilon)d\epsilon} & A \\ \downarrow & & \downarrow \\ I \times I & \xrightarrow{\gamma(t,\epsilon)} & M \end{array}$$

satisfying the boundary conditions $\gamma(i, \epsilon) = x_i, a(t, i) = a_i(t), b(i, \epsilon) = 0$ for $i = 0, 1$. We denote the equivalence relation on $P(\mathcal{A})$ defined by $\mathcal{A}$-homotopy by $\sim$.

Definition 8. *The Weinstein groupoid is defined as* $\mathcal{G}(\mathcal{A}) = P(\mathcal{A})/\sim$.

The groupoid structure on $\mathcal{G}(\mathcal{A})$ is as follows. The source and target maps are given by the start and end points of the base path, multiplication is given by "concatenation" of $\mathcal{A}$-paths, inversion is reversal of the orientation of an $\mathcal{A}$-path, and the unit map is given by constant zero paths. We refer to [4] for the details.

As mentioned above, $\mathcal{G}(\mathcal{A})$ is not always a Lie groupoid. However, it does always have the structure of a *topological* groupoid. This follows from the fact that $P(\mathcal{A})$ has the structure of a Banach manifold. It turns out that smoothness of $\mathcal{G}(\mathcal{A})$ is equivalent to integrability of $\mathcal{A}$, but there is a more useful way of determining integrability in terms of objects intrinsic to the Lie algebroid itself, which we now introduce.

1.3.2 (Ordinary) Monodromy groups

Let $\mathcal{A} \to M$ be a Lie algebroid and let $x \in M$. The *monodromy group* at x is a certain subgroup of $\mathcal{G}(\mathfrak{g}_x)$, the Lie group integrating the isotropy Lie algebra at x. The monodromy groups of $\mathcal{A}$ determine its integrability. Writing L for the leaf of $\mathcal{A}$ containing x, the monodromy group is defined in terms of the *monodromy map*, which is a homomorphism from the second homotopy group of L to $\mathcal{G}(\mathfrak{g}_x)$. We will define this map in its full generality, referring to [4] for the details.

Definition 9. *The (ordinary) monodromy group* $\mathcal{N}_x$ *is the image of*

$$\partial_x : \pi_2(L, x) \to \mathcal{G}(\mathfrak{g}_x).$$

Theorem 4 ([4] Crainic & Fernandes)**.** *For a Lie algebroid* $\mathcal{A}$*, the following statements are equivalent:*

1. $\mathcal{A}$ *is integrable.*

2. $\mathcal{G}(\mathcal{A})$ *is smooth.*

3. *the monodromy groups* $\mathcal{N}_x(\mathcal{A})$ *are locally uniformly discrete.*

Moreover, in this case, $\mathcal{G}(\mathcal{A})$ *is the unique source 1-connected Lie groupoid integrating* $\mathcal{A}$*.*

In general, the monodromy groups are hard to compute directly from the definition. Fortunately, there are more convenient ways to compute it in certain cases.

As above, let $x \in M$ and let L be the leaf of $\mathcal{A}$ through x. There is a short exact sequence

$$0 \longrightarrow \mathfrak{g}_L \longrightarrow \mathcal{A}_L \underset{\sigma}{\overset{\rho}{\rightleftarrows}} TL \longrightarrow 0$$

for which we choose a splitting $\sigma : TL \to \mathcal{A}_L$. Associated to this splitting there is the *curvature* $\Omega \in \Omega^2(L, \mathfrak{g}_L)$ defined by $\Omega(X,Y) = \sigma([X,Y]) - [\sigma(X), \sigma(Y)]$. When this curvature 2-form takes values in the center $Z(\mathfrak{g}_L) \subset \mathfrak{g}_L$, the monodromy map is given by

$$\partial_x([\gamma]) = \exp\left(\int_\gamma \Omega\right).$$

1.4 Cotangent Lie algebroid of a Poisson Manifold

Definition 10. *A Poisson manifold is a pair (M, π) consisting of a smooth manifold M and a bivector $\pi \in \mathfrak{X}^2(M)$ satisfying $[\pi, \pi] = 0$.*

A Poisson structure can also be defined as a Lie bracket on the space of smooth functions which is also a biderivation. However, the bivector perspective is more convenient for our purpose.

Example 4 (Linear poisson structure)**.** *Let $\mathfrak{g}$ be a Lie algebra with basis e_i and $\mathfrak{g}^*$ its dual with basis σ_i. Suppose the bracket on $\mathfrak{g}$ is defined by $[e_i, e_j] = \sum_{i,j,k} c_k^{ij} e_k$. Then $(\mathfrak{g}^*, \pi_{\mathfrak{g}}$ is a Poisson manifold, where $\pi_{\mathfrak{g}}$ is defined by:*

$$\pi_{\mathfrak{g}} = \frac{1}{2} \sum_{i,j,k} c_k^{ij} \sigma_k \frac{\partial}{\partial \sigma_i} \wedge \frac{\partial}{\partial \sigma_j}$$

Given a Poisson manifold (M, π), its cotangent bundle has an induced Lie algebroid structure, which we now describe. The anchor is given by the contraction map $\pi^\# : T^*M \to TM, \alpha \mapsto i_\alpha \pi$ and the Lie bracket on the space of one forms $\Omega^1(M)$ is given by:

$$[\alpha, \beta]_\pi := \mathcal{L}_{\pi^\sharp \alpha}(\beta) - \mathcal{L}_{\pi^\sharp \beta}(\alpha) - d(\pi(\alpha, \beta)).$$

We call $(T^*M, [\cdot,\cdot]_\pi, \pi^\#)$ the *cotangent Lie algebroid* associated to (M, π).

Thus the objects associated to Lie algebroids also appear for Poisson manifolds. In particular, $\mathrm{im}(\pi^\#) \subset TM$ integrates to a singular foliation, and for any $x \in M$ there is the isotropy Lie algebra $\ker(\rho_x) \subset T_x^*M$. When (M, π) is regular, meaning that the cotangent Lie algebroid is regular, the isotropy Lie algebra is exactly the conormal space to the foliation.

Lemma 1. *For a regular Poisson manifold, the isotropy Lie algebras are abelian.*

1.5 Diffeology

As mentioned in the introduction, diffeology has many applications in the study of Lie algebroids. We are particularly interested in diffeology because it allows us to study singular quotients of Lie groupoids and its potential infinitesimal version(s). In this section we give a brief introduction to diffeologies. We explain in Example 7 how diffeological spaces provide a useful middle ground between topological spaces and smooth manifolds.

A *parametrization* of a set X is a map from any open $U \subset \mathbb{R}^n$ to X.

Definition 11. *A diffeological space is a non-empty set X together with a set $\mathcal{D}$ of parametrizations $p : U \to X$, such that:*

- *For all $x \in X$ and $U \subset \mathbb{R}^n$, the constant parametrization $\mathbf{x} : U \to X, r \mapsto x$ is in $\mathcal{D}$.*
- *If $P : U \to X$ is a parametrization and for any $r \in U$ there exists open neighborhood V of r such that $P|_V \in \mathcal{D}$, then $U \in \mathcal{D}$.*
- *For any smooth parametrization $P : U \to X$ in $\mathcal{D}$ and smooth map $F \in C^\infty(V, U)$, $P \circ F$ is in $\mathcal{D}$.*

We call the elements of $\mathcal{D}$ the plots of the diffeological space.

A map $f : X \to X'$ is smooth if for each plot P of X, $f \circ P$ is a plot of X'.

Example 5. *1. Given a set X, the set of all parametrizations form a diffeology, called the* course *or* trivial diffeology*.*

2. *Given a set X, the set of all locally constant parametrizations form a diffeology, called the* fine *or* discrete diffeology*. For a plot $P : U \to X$, it is locally constant if for any $r \in U$, $P|_V$ is constant for some open neighborhood of r.*

3. *Given a smooth manifold M, the set of all smooth parametrizations $C^\infty(U, M)$ is a diffeology. This is called the* standard diffeology *on a manifold.*

Definition 12. *For a diffeological space $(X, \mathcal{D})$, its $\mathcal{D}$-*topology *$\mathcal{T}_\mathcal{D}$ is the finest topology making all the plots continuous, i.e.:*

$$\mathcal{T}_\mathcal{D} := \{O \subset \mathcal{X} : P^{-1}(O) \text{ is open}, \forall P \in \mathcal{D}\}.$$

Example 6. *It is easy to see the $\mathcal{D}$-topology for the diffeological space in example 5 are the trivial topology, discrete topology and the usual topology on manifolds respectively.*

For two diffeologies $\mathcal{D}$ and $\mathcal{D}'$ on X, we say $\mathcal{D}$ is finer than $\mathcal{D}'$ if $\mathcal{D} \subset \mathcal{D}'$. It is easy to see that when if $\mathcal{D}$ is finer than $\mathcal{D}'$ then $\mathcal{T}_\mathcal{D} \supset \mathcal{T}_{\mathcal{D}'}$. In other words, a finer diffeology has a finer $\mathcal{D}$-topology. However, the converse is not true. We will turn to this in Example 7 after introducing the quotient diffeology.

Definition 13. *Let $(X', \mathcal{D}')$ be a diffeological space and $f : X \to X'$ a map. The pullback diffeology $f^*(\mathcal{D}')$ on X consists of all the parametrizations of X such that $f \circ P$ is a plot of X'.*

The pull back diffeology is the coursest diffeology on X such that f is smooth.

Definition 14. *Given a family of diffeological spaces $\{(X_i, \mathcal{D}_i)\}_{i \in \mathcal{I}}$, the product diffeology of the product space $\prod_{i \in \mathcal{I}} X_i$ is defined as $\mathcal{D} = \bigcap_{i \in I} \pi^*(\mathcal{D}_i)$.*

The product diffeology is the finest diffeology making the projections smooth.

Definition 15. *Let $(X, \mathcal{D})$ be a diffeological space and $f : X \to X'$ a map. The pushforward diffeology $f_*(\mathcal{D})$ on $\mathcal{X}$ is defined as follows: $P \in f_*(\mathcal{D})$ if and only if for any $r \in U$ there exists an open neighborhood V of r such that either $P|_V$ is constant or $P|_V = f \circ Q$ for some $Q \in \mathcal{D}$.*

The pushforward diffeology is the finest diffeology making f smooth.

Definition 16. *Given a diffeological space* $(X, \mathcal{D})$ *and a quotient map* $q : X \to \tilde{X}$*, the quotient diffeology on* $\tilde{X}$ *is the pushforward of* $\mathcal{D}$ *by* p*.*

Proposition 4. *Given quotient map* $q : X \to X'$*, the* $\mathcal{D}$*-topology on* X' *is the same as the quotient topology of the* $\mathcal{D}$*-topology on* X*, i.e.* $\mathcal{T}_{q_*\mathcal{D}} = q_*\mathcal{T}_{\mathcal{D}}$*.*

The following example shows that diffeology can be studied on non-smooth spaces and carries more information than topology.

Example 7. *[12]) Consider the quotient* $\mathbb{R}/\mathbb{Q}$*, where* $\mathbb{R}$ *is equipped with the standard diffeology. The* $\mathcal{D}$*-topology is the usual topology. It is easy to see that the quotient topology is trivial. However, the quotient diffeology is strictly finer than the course diffeology on* $\mathbb{S}^1$*. Thus we have that* $\mathcal{T}_{\mathcal{D}_0}$ *is the same as (thus finer than)* $\mathcal{T}_{q_*\mathcal{D}}$*, but* $(\mathbb{S}^1, \mathcal{D}_0)$ *is not finer than* $(\mathbb{S}^1, q_*(\mathcal{D}))$

Definition 17. *A diffeological group is a group equipped with a diffeology such that the multiplication and the inversion are smooth.*

Any subgroup of a diffeological group is a diffeological subgroup (with the subspace diffeology). Given a diffeological group G and a normal subgroup H, the quotien G/H is a diffeological group with the quotient diffeology.

Definition 18. *A diffeological groupoid is a groupoid* $\mathcal{G}$ *where the set of morphisms* $\mathcal{G}_1$ *and objects* $\mathcal{G}_0$ *are equipped with diffeology such that all the structure maps are smooth.*

Chapter 2

Abelianization of Lie algebroids

2.1 Definition

We call a Lie algebroid *abelian* if all the isotropy Lie algebras are abelian.

Definition 19. *An* ***abelianization*** *of a Lie algebroid* $\mathcal{A} \to M$ *consists of*

1. *an abelian Lie algebroid* $\mathcal{A}^{ab} \to M$*, and*
2. *a surjective morphism* $p : \mathcal{A} \to \mathcal{A}^{ab}$ *covering the identity,*

such that for any open subset $U \subset M$*, any abelian Lie algebroid* $\mathcal{B} \to N$ *and morphism* $\phi : \mathcal{A}|_U \to \mathcal{B}$*, there is a unique morphism* $\tilde{\phi}$ *such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{A}|_U & \xrightarrow{\phi} & \mathcal{B}. \\ \downarrow p & \nearrow \exists ! \tilde{\phi} & \\ \mathcal{A}^{ab}|_U & & \end{array}$$

Clearly, if the abelianization of $\mathcal{A}$ exists, it is unique up to isomorphism. We will call $\mathcal{A}^{ab}$ the abelianization of $\mathcal{A}$ when it does not cause confusion. We will denote the anchor of $\mathcal{A}^{ab}$ by ρ^{ab} and its bracket by $[\cdot, \cdot]_{ab}$. We will use $(\mathfrak{g}^{ab})_x$ to denote the isotropy Lie algebras of $\mathcal{A}^{ab}$ and $\mathfrak{g}_x^{ab}$ to denote the abelianization of the isotropy Lie algebras $\mathfrak{g}_x$. In general, these two Lie algebras are different.

Example 8 (Lie algebras)**.** *Let* $\mathcal{A} = \mathfrak{g}$ *be a Lie algebra. Then its abelianization is*

$$\mathfrak{g}^{ab} = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}].$$

Example 9 (Transitive Lie algebroids [7]). *Let $\mathcal{A} \to M$ be a transitive Lie algebroid. Its isotropy bundle $\mathfrak{g}_M$ is then a vector subbundle of $\mathcal{A}$ and in fact an ideal. It is proved in [7] that the abelianization of $\mathcal{A}$ is the quotient algebroid*

$$\mathcal{A}^{ab} = \mathcal{A}/[\mathfrak{g}_M, \mathfrak{g}_M].$$

In particular, $\ker \rho_x^{ab} \simeq \mathfrak{g}_x^{ab}$ and $[\mathfrak{g}_x, \mathfrak{g}_x] \subset \ker(p_x)$, where $p : \mathcal{A} \to \mathcal{A}^{ab}$ is the quotient map.

Remark 1. *Our definition of abelianization has a sheaf flavor and hence differs from the one in [7]. For example, we can restrict $\mathcal{A}$ to any open set $O \subset M$ and we obviously have*

$$(\mathcal{A}|_O)^{ab} = (\mathcal{A}^{ab})|_O.$$

This is in line with the fact that one sometimes needs to think of Lie algebroids as sheaf-like objects. For instance, in the holomorphic category, where the bracket is only defined on the sheaf of holomorphic sections of $\mathcal{A}$, our definition still makes sense, while the one from [7] does not. However, for transitive (smooth) Lie algebroids, the two definitions are equivalent, so all the results in [7] still hold for our definition.

In the next sections we aim to investigate the existence of abelianization of general Lie algebroids, beginning with bundles of Lie algebras, progressing to the regular ones and finally the general case.

2.2 Bundle of Lie algebras

Bundles of Lie algebras, even when admitting an abelianization, can already have different properties from the transitive case.

Example 10 ([7]). *Consider $\mathcal{A} = \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ equipped with the bracket*

$$[e_1, e_2]_x = xe_1,$$

where e_1 and e_2 denote the constant sections $x \mapsto (1, 0, x)$ and $x \mapsto (0, 1, x)$. This bundle of Lie algebras has abelianization the bundle of abelian Lie algebras

$$\mathcal{A}^{ab} = \mathbb{R} \times \mathbb{R} \to \mathbb{R},$$

Notice that at $x = 0$, the isotropy of $\mathcal{A}$ is $\mathfrak{g}_0 = \mathfrak{g}_0^{ab} \simeq \mathbb{R}^2$, so abelian, but $\mathcal{A}^{ab}$ has smaller isotropy, namely $(\mathfrak{g}^{ab})_0 = \mathbb{R}$.

In general, a bundle of Lie algebras may not admit an abilianization. Before looking at more examples, let us first take a look at some basic properties of abelianizations. The following lemma was proved in [7], but it is easily seen to hold also for our definition of abelianization.

Lemma 2. *If $\mathcal{A}^{ab}$ is the abelianization of $\mathcal{A}$, then $\mathcal{I}(\rho) = \mathcal{I}(\rho^{ab})$ and $[\mathfrak{g}_M, \mathfrak{g}_M] \subset \ker p$.*

Hence, $\mathcal{A}$ and $\mathcal{A}^{ab}$ have the same foliation. Also, since $p : \mathcal{A} \to \mathcal{A}^{ab}$ has closed kernel, we also deduce that:

Corollary 1. *If $\mathcal{A}^{ab}$ is the abelianization of $\mathcal{A}$, then $\overline{[\mathfrak{g}_M, \mathfrak{g}_M]} \subset \ker p$.*

Example 11 (Bundles of Lie algebras with no abelianization)**.** *Let us replace the bracket in Example 10 by*

$$[e_1, e_2]_x := f(x)e_1,$$

where f vanishes on an open interval $I = (a, b) \subsetneq \mathbb{R}$. Then the resulting bundle no longer admits an abelianization. Indeed, a potential abelianization $\mathcal{A}^{ab}$ must have $\mathrm{rank}(\ker p) \geq 1$. But then

$$(\mathcal{A}^{ab})_I \subsetneq (\mathcal{A}_I)^{ab} \simeq \mathbb{R}^2 \times I,$$

a contradiction (cf. Remark 1).

One can also consider the bundle of Lie algebras $\mathcal{A} = \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ equipped with the bracket

$$[e_1, e_2]_{(x,y)} := xe_1 + ye_2.$$

Although $[\mathfrak{g}_x, \mathfrak{g}_x] = 0$ at origin and $\mathbb{R}$ otherwise, one actually has $\overline{[\mathfrak{g}_M, \mathfrak{g}_M]}_0 \simeq \mathbb{R}^2$. An argument using the rank, similar to the previous example, shows that this bundle does not admit an abelianization.

The previous example can be summarized by saying that if on an open set $O \subsetneq M$ the rank of $[\mathfrak{g}_M, \mathfrak{g}_M]$ is lower than the highest rank of $\overline{[\mathfrak{g}_M, \mathfrak{g}_M]}$, then the abelianization cannot exists since one would obtain

$$(\mathcal{A}^{ab})|_O \subsetneq (\mathcal{A}|_O)^{ab}.$$

This lead us to the following characterization of the bundles of Lie algebras that admit an abelianization:

Proposition 5. *A bundle of Lie algebras* $\mathfrak{g}_M$ *has an abelianization if and only if* $\overline{[\mathfrak{g}_M, \mathfrak{g}_M]}$ *is a subbundle of* $\mathfrak{g}_M$*. In this case,*

$$(\mathfrak{g}_M)^{ab} = \mathfrak{g}_M / \overline{[\mathfrak{g}_M, \mathfrak{g}_M]}.$$

Proof.

($\Leftarrow$) Let $\mathfrak{g}_M$ be a bundle of Lie algebras. If $\overline{[\mathfrak{g}_M, \mathfrak{g}_M]}$ is a subbundle of $\mathfrak{g}_M$, then clearly $\mathfrak{g}_M/\overline{[\mathfrak{g}_M, \mathfrak{g}_M]}$ is an abelian bundle of Lie algebras. Since the quotient map is a surjective morphism covering the identity, we only need to show that it satisfies the universal property. So let $U \subset M$ be an open set, $\mathcal{B} \to N$ an abelian Lie algebroid and $\phi : (\mathfrak{g}_M)|_U \to \mathcal{B}$ some morphism. Since $\ker \phi$ is closed in $(\mathfrak{g}_M)|_U$ and $[\mathfrak{g}_M, \mathfrak{g}_M]|_U \subset \ker \phi$, we must have $\overline{[\mathfrak{g}_M, \mathfrak{g}_M]}|_U \subset \ker(\phi)$. It follows that the universal property holds.

($\Rightarrow$) Suppose $\overline{[\mathfrak{g}_M, \mathfrak{g}_M]}$ is not a subbundle and $\mathfrak{g}_M$ has an abelianization $p : \mathfrak{g}_M \to (\mathfrak{g}_M)^{ab}$. Then $\overline{[\mathfrak{g}_M, \mathfrak{g}_M]} \subset \ker p$ and if we let n be the rank of $\ker p$ we claim that:

- there exists an open $O \subset M$ such that $(\overline{[\mathfrak{g}_M, \mathfrak{g}_M]})|_O$ has constant rank $k < n$.

Indeed, on the one hand, the set

$$\{x \in M : \mathrm{rank}_x \overline{[\mathfrak{g}_M, \mathfrak{g}_M]} = n\} = \{x \in M : \overline{[\mathfrak{g}_M, \mathfrak{g}_M]}_x = \ker p_x\}$$

is closed. So its complement is a non-empty, open subset $U \subset M$. On the other hand, the sets where rank $\overline{[\mathfrak{g}_M, \mathfrak{g}_M]}$ is constant $< n$ are finite and their union is U, so they cannot all have empty interior.

Let $\mathcal{B} \to O$ be a subbundle of $(\mathfrak{g}_M)|_O$ complementary to $(\overline{[\mathfrak{g}_M, \mathfrak{g}_M]})|_O$, which we view as a trivial bundle of Lie algebras. The projection $\phi : (\mathfrak{g}_M)|_O \to \mathcal{B}$ is a morphism of Lie algebroids which clearly does not factor through $p : \mathfrak{g}_M \to (\mathfrak{g}_M)^{ab}$, contradicting the universal property of the abelianization. □

Abelianization of a Lie algebroid is not preserved by quotients since sums of ideals are not always a subbundle.

Example 12. *Consider Lie algebra bundle $\mathfrak{g}_M$ over $\mathbb{R}$ with fiber $\mathfrak{g} = \mathbb{R}^4$ equipped with $[e_1, e_2] = e_3$, $[e_i, e_j] = 0$ otherwise. Clearly $\mathfrak{g}_M$ has abelianization where fibers are $< e_1, e_2, e_4 >$. Now take the ideal generated by $\mathfrak{h} = f(x)e_3 + (1 - f(x))e_4$, where $f(x) = 1$ on some interval I but $f(x) \neq 1$ outside. Then the quotient $\mathfrak{g}/\mathfrak{h}$ is abelian on I but non-abelian outside. Thus $\mathfrak{h}$ does not admit an abelianization.*

Before moving to more general cases, we prove the following technical but important fact.

Lemma 3. *If $\mathfrak{g}_M$ is a bundle of Lie algebras such that $\overline{[\mathfrak{g}_M, \mathfrak{g}_M]}$ is a subbundle, then there exist a dense, open, saturated subset $O \subset M$ such that $[\mathfrak{g}_M, \mathfrak{g}_M]|_O = \overline{[\mathfrak{g}_M, \mathfrak{g}_M]}|_O$.*

Proof. Consider the sets

$$S_k = \{x \in M \mid \dim([\mathfrak{g}_M, \mathfrak{g}_M]_x) = k\}.$$

and let $n = \max\{k \mid S_k \neq \emptyset\}$. Then $O := S_n$ is open in M and we have

$$[\mathfrak{g}_M, \mathfrak{g}_M]|_O = \overline{[\mathfrak{g}_M, \mathfrak{g}_M]}|_O.$$

Also, since $\overline{[\mathfrak{g}_M, \mathfrak{g}_M]}$ is a subbundle, it follows that $n = \operatorname{rank}\overline{[\mathfrak{g}_M, \mathfrak{g}_M]}$. We claim that $\overline{O} = M$. Indeed, if this fails then we can repeat the argument replacing M by $M' = M - \overline{O}$. We obtain sets S'_k, with $k < n$, and $n' = \max\{k \mid S'_k \neq \emptyset\} < n$ such that $n' = \operatorname{rank}\overline{[\mathfrak{g}_M, \mathfrak{g}_M]}$, contradicting that $\overline{[\mathfrak{g}_M, \mathfrak{g}_M]}$ is a subbundle.

The fact that O is saturated follows from the fact that the sets S_k are saturated. To see this, let x and y belong to the same leaf of $\mathcal{A}$. One can choose a compactly supported section $s \in \Gamma(A)$ such that the time-1 flow of the vector field $\rho(s)$ satisfies

$$\varphi^1_{\rho(s)}(x) = y.$$

Then the time-1 flow of the section s (see, e.g., [4]) is a Lie algebroid automorphism which maps $\mathfrak{g}_x$ to $\mathfrak{g}_y$ and hence also $[\mathfrak{g}_M, \mathfrak{g}_M]_x$ to $[\mathfrak{g}_M, \mathfrak{g}_M]_y$. Hence, if $x \in S_k$ we must have $y \in S_k$, so S_k is saturated. □

This already allows us to prove one half of Theorem 1.

Proposition 6. *Let $\mathcal{A}$ be a Lie algebroid with isotropy $\mathfrak{g}_M$ and assume that $\overline{[\mathfrak{g}_M, \mathfrak{g}_M]} \subset \mathcal{A}$ is a vector subbundle. Then $\mathcal{A}$ has abelianization the quotient*

$$p : \mathcal{A} \to \mathcal{A}/\overline{[\mathfrak{g}_M, \mathfrak{g}_M]}.$$

Proof. We claim that $\overline{[\mathfrak{g}_M, \mathfrak{g}_M]}$ is an ideal in $\mathcal{A}$, i.e., that

$$s \in \Gamma(\mathcal{A}),\ \xi \in \Gamma(\overline{[\mathfrak{g}_M, \mathfrak{g}_M]}) \quad \Longrightarrow \quad [s, \xi] \in \Gamma(\overline{[\mathfrak{g}_M, \mathfrak{g}_M]}).$$

To see this, note that if $\xi \in \Gamma(\overline{[\mathfrak{g}_M, \mathfrak{g}_M]})$ then over the open set O given by Lemma 3 there are sections $\xi_i \in \Gamma(\mathfrak{g}_M|_O)$ and real numbers a_{ij} such that

$$\xi|_O = \sum_{i<j} a_{ij} [\xi_i, \xi_j].$$

It follows that if $s \in \Gamma(\mathcal{A})$, one has

$$[s, \xi]|_O = \sum_{i<j} a_{ij} \left([[s, \xi_i], \xi_j] + [\xi_i, [s, \xi_j]]\right) \in [\mathfrak{g}_M, \mathfrak{g}_M]|_O.$$

Therefore, we must have $[s, \xi] \in \Gamma(\overline{[\mathfrak{g}_M, \mathfrak{g}_M]})$, and the claim follows.

Since $\overline{[\mathfrak{g}_M, \mathfrak{g}_M]}$ is an ideal in $\mathcal{A}$, it follows that there is a unique Lie algebroid structure such that quotient map

$$p : \mathcal{A} \to \mathcal{A}/\overline{[\mathfrak{g}_M, \mathfrak{g}_M]}$$

is a morphism of Lie algebroids. Moreover, this quotient has abelian isotropy $\mathfrak{g}_M/\overline{[\mathfrak{g}_M, \mathfrak{g}_M]}$, so we only need to check that the universal property holds.

Let $U \subset M$ be an open subset, $\mathcal{B} \to N$ an abelian algebroid and $\phi : \mathcal{A}|_U \to \mathcal{B}$ an algebroid morphism. Applying Lemma 3 again, $U \cap O$ is an open dense subset of U where

$$[\mathfrak{g}_M, \mathfrak{g}_M]|_{O \cap U} = \overline{[\mathfrak{g}_M, \mathfrak{g}_M]}|_{O \cap U}.$$

It follows that $\overline{[\mathfrak{g}_M, \mathfrak{g}_M]}|_U \subset \ker \phi$, so there is a unique morphism $\tilde{\phi}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{A}|_U & \xrightarrow{\phi} & \mathcal{B}. \\ \downarrow p & \nearrow \exists! \tilde{\phi} & \\ \mathcal{A}^{ab}|_U & & \end{array}$$

so the universal property holds. □

2.3 Regular Lie algebroids

We now consider the existence of abelianizations for arbitrary regular Lie algebroids. Note that a Lie algebroid $\mathcal{A} \to M$ is regular if and only if its isotropy $\mathfrak{g}_M \subset \mathcal{A}$ is a subbundle. Therefore, in this case, if $\{s_1, \ldots, s_n\}$ is a basis of local sections of $\mathfrak{g}_M$ over an open sen U, one has

$$[s_i, s_j](x) = [s_i(x), s_j(x)], \quad \forall x \in U.$$

It follows also that $[\mathfrak{g}_M, \mathfrak{g}_M]_x$ is generated by $[s_i, s_j](x)$.

2.3.1 The semi-direct product construction

In the sequel we will use the fact that one can recover a regular Lie algebroid from its foliation and isotropy bundle. We recall briefly how this works and refer, e.g., to [13] for details.

Let $\mathcal{A}$ be a regular Lie algebroid. A choice of a splitting of the short exact sequence defined by the anchor

$$0 \longrightarrow \mathfrak{g}_M \longrightarrow \mathcal{A} \underset{\sigma}{\overset{\rho}{\rightleftarrows}} T\mathcal{F} \longrightarrow 0$$

allows to identify $\mathcal{A}$ with $T\mathcal{F} \otimes \mathfrak{g}_M$ so that the anchor becomes the projection on $T\mathcal{F}$. On the other hand, the Lie bracket becomes

$$[(X, \xi), (Y, \eta)]_{\mathcal{A}} = ([X, Y], [\xi, \eta]_{\mathfrak{g}_M} + \nabla_X \eta - \nabla_Y \xi + \Omega(X, Y)), \qquad (2.1)$$

where

- ∇ is the $T\mathcal{F}$-connection on $\mathfrak{g}_M$ defined by

$$\nabla_X \xi := [\sigma(X), \xi]_{\mathcal{A}};$$

- $\Omega \in \Omega^2(T\mathcal{F}, \mathfrak{g}_M)$ is the curvature form of the splitting given by

$$\Omega(X, Y) := \sigma([X, Y]) - [\sigma(X), \sigma(Y)]_{\mathcal{A}}.$$

It is easy to check that under this isomorphism the Jacobi identity for $[\cdot, \cdot]_{\mathcal{A}}$ amounts to the following set of identities:

$$\begin{aligned} &\nabla_X[\xi, \eta]_{\mathfrak{g}_M} = [\nabla_X \xi, \eta]_{\mathfrak{g}_M} + [\xi, \nabla_X \eta]_{\mathfrak{g}_M}, \\ &[\Omega(X, Y), \xi]_{\mathfrak{g}_M} = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]} \xi, \\ &\bigodot_{X,Y,Z} \Big(\Omega([X, Y], Z) + \nabla_X(\Omega(Y, Z)) \Big) = 0, \end{aligned} \tag{2.2}$$

where $\xi, \eta \in \Gamma(\mathfrak{g}_M)$, $X, Y, Z \in \Gamma(T\mathcal{F})$, and the symbol $\odot$ denotes cyclic summation.

The converse also holds. Given a foliation $\mathcal{F}$ of M, a bundle of Lie algebras $\mathfrak{g}_m \to M$, a $T\mathcal{F}$-connection ∇ and a $\mathfrak{g}_M$-valued 2-form Ω satisfying identities (2.2), then one obtains a Lie algebroid structure on $T\mathcal{F} \oplus \mathfrak{g}_M$ with Lie bracket (2.1) and anchor $\rho = \mathrm{pr}_{T\mathcal{F}}$. We denote this Lie algebroid by $T\mathcal{F} \ltimes \mathfrak{g}_M$. The previous discussion shows that one has the following simple proposition.

Proposition 7. *Any regular Lie algebroid is isomorphic to $T\mathcal{F} \ltimes \mathfrak{g}_M$ for some quadruple $(\mathfrak{g}_M, T\mathcal{F}, \nabla, \Omega)$ satisfying (2.2).*

We also use the following notation. Given a collection of subspaces

$$E = \bigcup_{x \in M} E_x \subset \mathfrak{g}_M,$$

which is not necessarily a subbundle (e.g., $E = [\mathfrak{g}_M, \mathfrak{g}_M]$), we still denote the subspace of sections which take values in E by

$$\Gamma(E) := \{s \in \Gamma(g_M) : s(x) \in E_x \text{ for all } x \in M\}.$$

Also, given a $T\mathcal{F}$-connection ∇ on $\mathfrak{g}_M$ we will say that ∇ preserves E if for every $X \in \Gamma(T\mathcal{F})$ one has

$$s \in \Gamma(E) \quad \Longrightarrow \quad \nabla_X s \in \Gamma(E).$$

The following result will be useful in the sequel.

Lemma 4. *Given a bundle of Lie algebras $\mathfrak{g}_M$, any $T\mathcal{F}$-connection ∇ on $\mathfrak{g}_M$ satisfying (2.2), preserves $[\mathfrak{g}_M, \mathfrak{g}_M]$. Moreover, ∇ also preserves $\overline{[\mathfrak{g}_M, \mathfrak{g}_M]}$ provided the latter is a subbundle.*

Proof. For the first part, it is enough to show the result holds locally. Let $U \subset M$ be any trivializing open of $\mathfrak{g}_M$ and let $\{s_i\}$ be a local basis of sections. Then

$$\nabla_X[fs_i, gs_j] = \mathrm{Lie}_X(fg)[s_i, s_j] + fg[\nabla_X s_i, s_j] + fg[s_i, \nabla_X s_j].$$

The second part follows immediately from Lemma 3. □

2.3.2 Decomposition of morphisms

Consider a morphism of regular Lie algebroids $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ covering the identity. Then they share the same foliation $\mathcal{F}$ and we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{g}_M^1 & \longrightarrow & \mathcal{A}_1 & \underset{\overset{\sigma_1}{\dashleftarrow}}{\xrightarrow{\rho_1}} & T\mathcal{F} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi|_{\mathfrak{g}}} & & \downarrow{\scriptstyle \phi} & & \| {\scriptstyle \mathrm{Id}} & & \\ 0 & \longrightarrow & \mathfrak{g}_M^2 & \longrightarrow & \mathcal{A}_2 & \underset{\overset{\sigma_2}{\dashleftarrow}}{\xrightarrow{\rho_2}} & T\mathcal{F} & \longrightarrow & 0 \end{array}$$

If we choose choose a splitting $\sigma_1 : T\mathcal{F} \to \mathcal{A}_1$ of the anchor of $\mathcal{A}_1$, we obtain a splitting $\sigma_2 := \phi \circ \sigma_1 : T\mathcal{F} \to \mathcal{A}_2$ of the anchor of $\mathcal{A}_2$. These splittings give identifications

$$\mathcal{A}_i \simeq T\mathcal{F} \ltimes \mathfrak{g}_M^i.$$

We conclude that to specify the morphism $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ amounts to specifying the following data:

- $T\mathcal{F}$-connections ∇^i on $\mathfrak{g}_M^i$ and 2-forms $\Omega^i \in \Omega^2(M, \mathfrak{g}_M^i)$ satisfying (2.2);
- a morphism of Lie algebra bundles $\phi : \mathfrak{g}_M^1 \to \mathfrak{g}_M^2$ covering the identity

compatible with ϕ, i.e., satisfying

$$\phi \circ \nabla^1 = \nabla^2 \circ \phi, \quad \phi \circ \Omega^1 = \Omega^2. \tag{2.3}$$

Proposition 8. *Let $\phi : \mathfrak{g}^1_M \to \mathfrak{g}^2_M$ be a surjective morphism of Lie algebra bundles and let ∇^1 and Ω^1 be a $T\mathcal{F}$-connection and 2-form on $\mathfrak{g}^1_M$ satisfying (2.3). If ∇^1 preserves $\ker \phi$ then:*

1. *there exists a unique $T\mathcal{F}$-connection ∇^2 and a unique 2-form Ω^2 on $\mathfrak{g}^2_M$ satisfying (2.3), and*
2. *ϕ extends to a surjective Lie algebroid morphism*

$$(\mathrm{Id}, \phi) : T\mathcal{F} \ltimes \mathfrak{g}^1_M \to TF \ltimes \mathfrak{g}^2_M.$$

Proof. Let $\sigma : \mathfrak{g}^2_M \to \mathfrak{g}^1_M$ be a splitting of $\phi : \mathfrak{g}^1_M \to \mathfrak{g}^2_M$. Then the expression

$$\nabla^2_X \xi := \phi(\nabla^1_X(\sigma \circ \xi)),$$

defines a $T\mathcal{F}$-connection on $\mathfrak{g}^2_M$. Since ∇^1 preserves $\ker \phi$, if one sets $\Omega^2 := \phi \circ \Omega^1$, one checks easily that the pair (∇^2, Ω^2) satisfies (2.2) and that (2.3) holds. Both items should now be obvious. □

This can also be restated as follows.

Corollary 2. *Let $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a surjective morphism, covering the identity, between regular Lie algebroids. A choice of splitting of the anchor of $\mathcal{A}_1$ determines isomorphisms $\mathcal{A}_i \simeq T\mathcal{F} \ltimes \mathfrak{g}^i_M$, such that ϕ becomes*

$$\phi = (\mathrm{Id}, \phi|_{\mathfrak{g}}) : T\mathcal{F} \ltimes \mathfrak{g}^1_M \to T\mathcal{F} \ltimes \mathfrak{g}^2_M.$$

2.3.3 Existence of abelianizations for regular Lie algebroids

We are now ready to look into the abelianization of any regular Lie algebroid.

Proposition 9. *Let $\mathcal{A}$ be a regular Lie algebroid with isotropy $\mathfrak{g}_M$. Then $\mathcal{A}$ has an abelianization if and only if $\overline{[\mathfrak{g}_M, \mathfrak{g}_M]} \subset \mathcal{A}$ is a vector subbundle.*

Moreover, if $\mathcal{A} \simeq T\mathcal{F} \ltimes \mathfrak{g}_M$ *for a quadruple* $(\mathfrak{g}_M, T\mathcal{F}, \nabla, \Omega)$ *then* ∇ *and* Ω *induce a* $T\mathcal{F}$*-connection and a 2-form on*

$$\mathfrak{g}_M^{ab} := \mathfrak{g}_M / \overline{[\mathfrak{g}_M, \mathfrak{g}_M]}$$

and

$$\mathcal{A}^{ab} \simeq T\mathcal{F} \ltimes {\mathfrak{g}^{ab}}_M.$$

Proof. Assume first that $\overline{[\mathfrak{g}_M, \mathfrak{g}_M]} \subset \mathcal{A}$ is a vector subbundle. From Proposition 6 we already know that $\mathcal{A}$ has abelianization

$$\mathcal{A}^{ab} = \mathcal{A} / \overline{[\mathfrak{g}_M, \mathfrak{g}_M]}.$$

If we assume that $\mathcal{A} \simeq T\mathcal{F} \ltimes \mathfrak{g}_M$ for a quadruple $(\mathfrak{g}_M.T\mathcal{F}, \nabla, \Omega)$, it follows from Lemma 4 that ∇ preserves $\overline{[\mathfrak{g}_M, \mathfrak{g}_M]}$. By Proposition 8, ∇ and Ω induce a $T\mathcal{F}$-connection and a 2-form on $\mathfrak{g}_M^{ab} := \mathfrak{g}_M / \overline{[\mathfrak{g}_M, \mathfrak{g}_M]}$ for which we have

$$\mathcal{A}^{ab} \simeq T\mathcal{F} \ltimes {\mathfrak{g}^{ab}}_M.$$

Conversely, suppose $\mathcal{A}$ has abelianization $p : \mathcal{A} \to \mathcal{A}^{ab}$. We claim that $\mathfrak{g}_M$ also admits an abelianization, so by Proposition 5 we conclude that $\overline{[\mathfrak{g}_M, \mathfrak{g}_M]} \subset \mathcal{A}$ is a vector subbundle.

It remains to prove the claim. For that, observe that we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{g}_M & \longrightarrow & \mathcal{A} & \xrightarrow{\rho} & T\mathcal{F} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle p_{\mathfrak{g}}} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\ 0 & \longrightarrow & \mathfrak{g}_M^{ab} & \longrightarrow & \mathcal{A}^{ab} & \xrightarrow{\rho^{ab}} & T\mathcal{F} & \longrightarrow & 0, \end{array}$$

where $\mathfrak{g}_M^{ab}$ is the isotropy bundle of $\mathcal{A}^{ab}$ and $p_{\mathfrak{g}} = p|_{\mathfrak{g}_M}$. If $\mathfrak{g}_M$ does not admit an abelianization, then by Lemma 3, we know that $\overline{[\mathfrak{g}_M, \mathfrak{g}_M]}$ does not have constant rank. Furthermore, as we saw in the proof of Proposition 5, we can find an open $O \subset M$ such that

$$\mathrm{rank}([\mathfrak{g}_M, \mathfrak{g}_M]|_O) = k < n = \max_{x \in M} \left(\mathrm{rank} \overline{[\mathfrak{g}_M, \mathfrak{g}_M]}_x \right).$$

If we consider the Lie algebroid $\mathcal{A}|_O$, after choosing a splitting of its anchor, Corollary 2 yields a Lie algebroid morphism

$$\phi : \mathcal{A}|_O \to \mathcal{B} := \mathcal{A}|_O/[\mathfrak{g}_M, \mathfrak{g}_M]|_O,$$

where $\mathcal{B}$ is an abelian Lie algebroid. Clearly ϕ cannot be factored by $p : \mathcal{A} \to \mathcal{A}^{ab}$, which contradicts $\mathcal{A}^{ab}$ being the abelianization of A. Thus $\mathfrak{g}_M$ must have an abelianization, as claimed. □

2.4 Main theorem

We consider now arbitrary, possibly non-regular, Lie algebroids. Observe that the definition of the abelianization shows that if $\mathcal{A} \to M$ has abelianization $\mathcal{A}^{ab}$, then for any open set $U \subset M$ the restriction $\mathcal{A}|_U$ has abelianization $\mathcal{A}^{ab}|_U$. The following proposition gives a partial converse.

2.4.1 Locally regular Lie algebroids

Proposition 10. *Let $p : \mathcal{A} \to \mathcal{A}^{ab}$ be a surjective morphism of Lie algebroids over the identity, where $\mathcal{A}^{ab}$ is abelian. If there exists a dense open $O \in M$ such that $\mathcal{A}^{ab}|_O$ is the abelianization of $\mathcal{A}|_O$, then $\mathcal{A}^{ab}$ is the abelianization of $\mathcal{A}$.*

Proof. Since p is surjective, $\ker(p)$ is a subbundle of $\mathcal{A}$. Let $\phi : \mathcal{A}|_U \to \mathcal{B}$ be a morphism, where $\mathcal{B}$ is abelian. Since $\mathcal{A}^{ab}|_O$ is the abelianization of $\mathcal{A}|_O$, we have that $\ker(p|_{U\cap O}) \subset \ker(\phi|_{U\cap O})$. It follows that

$$\ker(p|_U) = \overline{\ker(p|_{U\cap O})} \subset \overline{\ker(\phi|_{U\cap O})} \subset \ker(\phi),$$

where we use that O is an open dense set and that $\ker(p)$ is a subbundle (here the closures are in $\mathcal{A}|_U$). Thus there is a unique induced algebroid morphism $\tilde{\phi} : \mathcal{A}^{ab}|_U \to \mathcal{B}$ such that $\tilde{\phi} \circ p = \phi$. So $\mathcal{A}^{ab}$ is the abelianization of $\mathcal{A}$. □

Given a Lie algebroid $\mathcal{A} \to M$, we consider the set of points where the rank of the anchor is locally constant

$$M_{reg} := \{x \in M : \exists \text{ open } V \ni x \text{ with } \operatorname{rank}(\rho_y) = \operatorname{rank}(\rho_x), \forall y \in V\}.$$

Since the rank takes only a finite number of values and it cannot drop locally, i.e., every $x \in M$ has a neighborhood U such that

$$\operatorname{rank}(\rho_x) \leq \operatorname{rank}(\rho_y), \quad \forall y \in U,$$

it follows that $M_{reg} \subset M$ is an open dense set.

2.4.2 Main theorem

Theorem 5. . *A Lie algebroid $\mathcal{A}$ has an abelianization if and only if $\overline{[\mathfrak{g}_M, \mathfrak{g}_M]} \subset \mathcal{A}$ is a vector subbundle.*

Proof. We already know that if $\overline{[\mathfrak{g}_M, \mathfrak{g}_M]} \subset \mathcal{A}$ is a vector subbundle then $\mathcal{A}$ admits an abelianization (cf. Proposition 6).

For the converse, suppose $\mathcal{A}$ admits an abelianization $p : \mathcal{A} \to \mathcal{A}^{ab}$. Let O be a connected components of M_{reg}. Since $\mathcal{A}|_O$ is regular with abelianization $\mathcal{A}^{ab}|_O$, it follows from Proposition 9 and Lemma 3 that there is an open dense subset $O' \subset O$ where

$$\ker(p_x) = [\mathfrak{g}_x, \mathfrak{g}_x], \quad \forall x \in O'.$$

The union of the sets O', where O varies in the collection of all connected components on M_{reg}, is an open dense set $M' \subset M$ where

$$\ker(p|_{M'}) = [\mathfrak{g}_M, \mathfrak{g}_M]|_{M'}.$$

Since $\ker(p)$ is a subbundle, we must have $\overline{[\mathfrak{g}_M, \mathfrak{g}_M]} = \ker(p)$, so the result follows. □

2.5 Applications

2.5.1 Functoriality

Proposition 11. *Let $f : M \to N$ be a submersion and $\mathcal{A} \to N$ a Lie algebroid with abelianization $\mathcal{A}^{ab}$. Then $f^!\mathcal{A}$ admits an abelianization and one has*

$$(f^!\mathcal{A})^{ab} \simeq f^!(\mathcal{A}^{ab}).$$

Proof. Since the image of a submersion is an open set, we can restrict $\mathcal{A}$ to $f(M)$. Hence, without loss of generality, we can assume that f is surjective.

Let $p : \mathcal{A} \to \mathcal{A}^{ab}$ be an abelianization of $\mathcal{A}$. Then $f^!p : f^!\mathcal{A} \to f^!\mathcal{A}^{ab}$ is a surjective morphism onto an abelian algebroid. Also, if $\mathfrak{g}_N$ is the isotropy bundle of $\mathcal{A}$, then the isotropy bundle of $f^!\mathcal{A}$ is

$$\mathfrak{g}_M := \{(a, 0_x) \in \mathcal{A} \times TM : \rho(a) = 0_{f(x)}\} = f^*\mathfrak{g}_N.$$

It follows that

$$[\mathfrak{g}_M, \mathfrak{g}_M] = f^*[\mathfrak{g}_N, \mathfrak{g}_N].$$

By Lemma 3, there is an open dense set $O \subset N$ where

$$\ker(p)|_O = [\mathfrak{g}_N, \mathfrak{g}_N]|_O.$$

Since f is a surjective submersion, $O' := f^{-1}(O)$ is an open dense set in M where we have

$$\ker(f^!p)|_{O'} = [\mathfrak{g}_M, \mathfrak{g}_M]|_{O'}.$$

This implies that

$$\overline{[\mathfrak{g}_M, \mathfrak{g}_M]} = \ker(f^!p),$$

so $\overline{[\mathfrak{g}_M, \mathfrak{g}_M]} \subset f^!\mathcal{A}$ is a vector subbundle. By Theorem **??**, $f^!\mathcal{A}$ has abelianization

$$(f^!\mathcal{A})^{ab} = (f^!\mathcal{A})/\overline{[\mathfrak{g}_M, \mathfrak{g}_M]} = f^!(\mathcal{A}/\overline{[\mathfrak{g}_N, \mathfrak{g}_N]}) = f^!(\mathcal{A}^{ab}).$$

□

In the previous proof the assumption that $f : M \to N$ is a submersion was used to guarantee that the preimage of a dense open subset $O \subset f(M)$ is an open dense subset of M. This property still holds if $f : M \to N$ is a surjective map transverse to a (non-singular) foliation $\mathcal{F}$ of N and the open dense set $O \subset M$ is *saturated*. Hence, for regular Lie algebroids we have the following result.

Proposition 12. *Let $\mathcal{A} \to N$ be a regular Lie algebroid and let $f : M \to N$ be a surjective map transverse to the anchor $\rho_{\mathcal{A}}$. If $\mathcal{A}$ has an abelianization*

$\mathcal{A}^{ab}$*, then* $f^!\mathcal{A}$ *admits an abelianization and one has*

$$(f^!\mathcal{A})^{ab} \simeq f^!(\mathcal{A}^{ab}).$$

Proof. Since f is transverse to the anchor $\rho_{\mathcal{A}}$, the pullback algebroid $f^!\mathcal{A}$ is well defined. The proof of the previous proposition holds word-by-word if one can show that $f^{-1}(O)$ is open and dense in M where $O \subset N$ is the dense, saturated, open set given by Lemma 3.

To see this, cover M by foliated charts (U_i, ϕ_i) for $T\mathcal{F} = \mathrm{im}(\rho_{\mathcal{A}})$. So $\phi_i : U_i \to \mathbb{R}^q$ is a submersion such that the plaques of $\mathcal{F}$ in U_i are the level sets $\phi_i^{-1}(p)$. Since $f : M \to N$ is transverse to $\mathcal{F}$, the composition

$$f_i := \phi_i \circ f : f^{-1}(U_i) \to \mathbb{R}^q,$$

is a submersion. Since O is saturated, we see that

$$f^{-1}(O \cap U_i) = f_i^{-1}(\phi_i(O \cap U_i))$$

is open and dense in $f^{-1}(U_i)$. Since f is surjective and $N = \bigcup_I U_i$, it follows that $f^{-1}(O)$ is open and dense in M, as claimed. □

Example 13. *Let* $M = \mathbb{T}^2 \simeq (\mathbb{R}/2\pi\mathbb{Z})^2$ *and consider the Lie algebroid* $\mathcal{A} \to \mathbb{T}^2$ *with underlying bundle the trivial rank 3 vector bundle, anchor*

$$\rho(e_1) := \frac{\partial}{\partial \theta^1}, \quad \rho(e_2) = \rho(e_3) := 0,$$

and Lie bracket defined by

$$[e_2, e_3]_{(\theta^1,\theta^2)} := \sin(\theta^2)\, e_2, \quad [e_1, e_2] = [e_1, e_3] := 0.$$

This is a regular Lie algebroid with isotropy bundle

$$\mathfrak{g}_M = \mathbb{R}e_2 \oplus \mathbb{R}e_3,$$

and commutator bundle

$$[\mathfrak{g}_M, \mathfrak{g}_M]_{(\theta^1,\theta^2)} = \begin{cases} \mathbb{R}e_1, & \textit{if } \theta^2 \neq 0 \\ 0, & \textit{if } \theta^2 = 0. \end{cases}$$

This has closure $\overline{[\mathfrak{g}_M, \mathfrak{g}_M]} = \mathbb{R}e_1$, a vector subbundle of $\mathcal{A}$, so the abelianization $\mathcal{A}^{ab}$ exists (and has rank 2).

Now consider the surjective map $f : \mathbb{T}^2 \to \mathbb{T}^2$, $f(\theta^1, \theta^2) = (s(\theta^1), \theta^2)$, where $s : \mathbb{S}^1 \to \mathbb{S}^1$ is a surjective map which equals 0 on some open interval $I \subset \mathbb{T}$. The map f is transverse to the anchor, so by Proposition 12 the abelianization of $f^!\mathcal{A}$ exists.

2.5.2 Cotangent Lie algebroid of a Poisson Structure

Proposition 13. *Any non-abelian cotangent Lie algebroid of a Poisson manifold does not admit an abelianization.*

Proof. Let T^*M a non-abelian cotangent Lie algebroid of a Poisson structure (M, π), then there exist $x \in M$ such that $[\mathfrak{g}_x, \mathfrak{g}_x] \neq 0$. Now take locally regular $O \subset M$. On each connected componentO_i of O, the Poisson stucture O_i, π_{O_i} is regular, thus T^*O_i is abelian, i.e. $[\mathfrak{g}, \mathfrak{g}] = 0$ on O_i. Thus $\overline{[\mathfrak{g}, \mathfrak{g}]}$ is not a subbundle and by Theorem 5, T^*M admits no abelianization. □

Example 14. *Consider a linear Poisson structure $(\mathfrak{g}^*, \pi_{\mathfrak{g}})$ corresponding to a Lie algebra $\mathfrak{g}$. It is easy to see that if $\mathfrak{g}$ is abelian, then the cotangent Lie algebroid is abelian. If $\mathfrak{g}$ is non-abelian, notice $\pi^\sharp$ is trivial at 0, the isotropy $\mathfrak{g}_0$ is isomorphic to $T_0^*\mathfrak{g}^* \simeq \mathfrak{g}$, thus is not abelian. So by Propostion 13, it has no abelianizaiton.*

Chapter 3

Abelianization of groupoids

3.1 Definition

A groupoid is called *abelian* if all the isotropy groups are abelian.

Definition 20. *The abelianization of* $\mathcal{G} \rightrightarrows M$ *consists of:*

- *An abelian groupoid* $\mathcal{G}^{ab} \rightrightarrows M$*;*
- *a surjective morphism* $p : \mathcal{G} \to \mathcal{G}^{ab}$ *covering* Id_M*;*

such that for any open subset $U \subset M$*, any abelian groupoid* $\mathcal{H} \rightrightarrows N$ *and morphism* $\psi : \mathcal{G} \to \mathcal{H}$*, we have:*

$$\begin{array}{ccc} \mathcal{G}|_{\mathcal{U}} & \xrightarrow{\psi} & \mathcal{H} \\ \downarrow p & \nearrow \exists! \tilde{\psi} & \\ \mathcal{G}^{ab}|_U & & \end{array} .$$

Just as in the case of Lie algebroids, the abelianization is unique, if it exists.

The abelianization of groupoids is heavily dependent on the category we are working in. The existence of abelianizations differs greatly between different categories, and even if it exists in two categories, the abelianizations themselves can be different. The following examples illustrate these phenomena.

Example 15. *A set-theoretical groupoid always has an abelianization* $\mathcal{G}^{ab} = \mathcal{G}/(\mathcal{G}_M, \mathcal{G}_M)$*. In particular, the set-theoretical abelianization of a group is* $G/(G, G)$*.*

Example 16. *A Lie group always has an abelianization* $G^{ab} = G/\overline{(G,G)}$.

So we see that already for a group, the abelianizations can differ between categories, since in general $(G,G) \subsetneq \overline{(G,G)}$.

Example 17. *[14] Consider* $G = (\widetilde{SL(2,\mathbb{R})} \times \mathbb{S}^1/)\mathbb{Z}$*, where* $\mathbb{Z}$ *is embedded as the covering group of* $\widetilde{SL}(2,\mathbb{R})$ *as the universal cover of* $SL(2,\mathbb{R})$*, and as a non-discrete subgroup of* $\mathbb{S}^1$*. Note that since* $(SL(2,\mathbb{R}), SL(2,\mathbb{R})) = (SL(2,\mathbb{R}))$*, the commutator* (G,G) *is actually dense in* G*, thus* $\overline{(G,G)} = G \neq (G,G)$.

A Lie groupoid does not always admit a smooth abelianization.

Example 18. *[7] Consider the action Lie groupoid* $\mathcal{G} = SO(3) \times \mathbb{R}^3 \rightrightarrows \mathbb{R}^3$ *of* $SO(3)$ *acting on* $\mathbb{R}^3$ *by rotation. The isotropy* $\mathcal{G}_0$ *at* 0 *is* $SO(3)$*, and so is its commutator* $(\mathcal{G}_0, \mathcal{G}_0)$*. Any potential abelianization will need to quotient out* $(\mathcal{G}_0, \mathcal{G}_0)$ *and thus can not recover the natural map from this groupoid to the pair groupoid* $\mathbb{R}^3 \times \mathbb{R}^3$ *away from* 0.

3.2 Abelianization in the diffeological category

Proposition 14. *A diffeological groupoid* $\mathcal{G}$ *has abelianization* $\mathcal{G}/(\mathcal{G},\mathcal{G})$.

Proof. We just need to show that all the structure maps are smooth. The identity map $\tilde{u} = q \circ u$ is clearly smooth. The rest follows from the following commutative diagrams:

inverse: $\begin{array}{ccc} \mathcal{G}_1 & \xrightarrow{i} & \mathcal{G}_1 \\ \downarrow q & & \downarrow q \\ \tilde{\mathcal{G}}_1 & \xrightarrow{\tilde{i}} & \tilde{\mathcal{G}}_1 \end{array}$, multiplication: $\begin{array}{ccc} \mathcal{G}_2 & \xrightarrow{m} & \mathcal{G}_1 \\ \downarrow q\times q & & \downarrow q \\ \tilde{\mathcal{G}}_2 & \xrightarrow{\tilde{m}} & \tilde{\mathcal{G}}_1 \end{array}$,

source and target: $\begin{array}{ccc} \mathcal{G} & & \\ \downarrow q & \searrow s & \\ \tilde{\mathcal{G}}_1 & \xrightarrow{\tilde{s}} & \mathcal{G}_0 \end{array}$, $\begin{array}{ccc} \mathcal{G} & & \\ \downarrow q & \searrow t & \\ \tilde{\mathcal{G}} & \xrightarrow{\tilde{t}} & \mathcal{G}_0 \end{array}$ □

We might want to consider a subcategory with stronger assumptions.

Definition 21. *A smooth map* $f : X \to X'$ *is a subduction if the map is surjective and the diffeology* $\mathcal{D}'$ *on* X' *is the pushforward of the diffeology* $\mathcal{D}$ *on* X.

The source and target maps of a diffeological groupoid are automatically a subduction.

Definition 22. *A smooth map $f : X \to X'$ is locally subductive at $x \in X$ if for any plot P of X' such that $P(0) = f(x)$, there exist an open neighborhood V of 0 and a plot $Q : V \to X$ of X such that $Q(0) = x$ and $f \circ Q = P|_V$. We call f a local subduction if it is locally subductive at all $x \in X$.*

Example 19. *Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(0) = 1$ and f vanishes when $|x| > 1$. Then $g : \mathbb{R}^2 \to \mathbb{R} = f(x) \cdot y$ is a subduction but not a local subduction.*

A diffeological groupoids is called locally subductive if the source and target maps are local subductions.

Proposition 15. *A locally subductive diffeological groupoid $\mathcal{G}$ has abelianization $\mathcal{G}^{ab} = \mathcal{G}/(\mathcal{G}, \mathcal{G})$.*

Proof. Let $[g] \in \mathcal{G}_1^{ab}$ and P be a plot of $\mathcal{G}_0 = \tilde{\mathcal{G}}_0$ such that $P(0) = s^{ab}(\tilde{g})$. Then since s is locally subductive, we have open neighborhood V of 0 and Q in plot of $\mathcal{G}_1$ such that $Q(0) = g$ and $s \circ Q = P|_V$. Now take $\tilde{Q} = q \circ Q$, which will be a plot of $\mathcal{G}_1^{ab}$. We have $\tilde{Q}(0) = [g]$ and $\tilde{s} \circ \tilde{Q} = s \circ Q = P|_V$. Thus $\tilde{s}$ is a local subduction. □

3.3 Abelianization in the smooth category

We already saw that a Lie groupoid does not always admit a smooth abelianization. But for transitive Lie groupoids, we can always find an abelianization as in the following proposition.

Proposition 16. *[7] A transitive Lie groupoid has abelianization $\mathcal{G}^{ab} = \mathcal{G}/\overline{(\mathcal{G}, \mathcal{G})}$.*

Let $\overline{(\mathcal{G}, \mathcal{G})}$ be the closure of the commutator $(\mathcal{G}, \mathcal{G})$ in $\mathcal{G}$. Let $\overline{(\mathcal{G}, \mathcal{G})}^s = \bigcup \overline{(\mathcal{G}_x, \mathcal{G}_x)}^s$ be the union of all "fiberwise closures" of $\overline{(\mathcal{G}_x, \mathcal{G}_x)}$. By "fiberwise closure", we mean that $\overline{(\mathcal{G}_x, \mathcal{G}_x)}^s$ is the closure of the commutator of $\mathcal{G}_x$ in the source fiber. In general, these two closures are not the same.

Example 20 ([7])**.** *Let us look at the groupoid version of Example 10. There $\overline{(\mathcal{G}, \mathcal{G})}$ is the trivial line bundle over $\mathbb{R}$, while $\overline{(\mathcal{G}, \mathcal{G})}^s$ is 0-dimensional at 0.*

It is immediate from Proposition 3 that if $\overline{(\mathcal{G},\mathcal{G})}$ is a normal Lie subgroupoid, then $\mathcal{G}/\overline{(\mathcal{G},\mathcal{G})}$ is the abelianization of $\mathcal{G}$.

Lemma 5. *If $\overline{(\mathcal{G},\mathcal{G})}$ is a subgroupoid of $\mathcal{G}$, then $\overline{(\mathcal{G},\mathcal{G})}$ is normal.*

Proof. Consider $l \circ \mathfrak{g} \circ l^{-1}$ where $g \in \overline{(\mathcal{G},\mathcal{G})}$ and $l \in \mathcal{G}$ such that $s(l) = t(g)$. Let $g_i \in (\mathcal{G},\mathcal{G})$ such that $g_i \to g$. We can pick $l_i \to l$ such that $s(l_i) = t(g_i)$. Now we have $l_i \circ g_i \circ l_i^{-1} \to l \circ g \circ l^{-1}$. A simple computation shows that $l_i \circ \mathfrak{g}_i \circ l_i^{-1} \in (\mathcal{G},\mathcal{G})$, which implies that $l \circ \mathfrak{g} \circ l^{-1} \in \overline{(\mathcal{G},\mathcal{G})}$ and thus that $\in \overline{(\mathcal{G},\mathcal{G})}$ is normal. □

Proposition 17. *Given a Lie groupoid $\mathcal{G}$, if $\overline{(\mathcal{G},\mathcal{G})}^s$ is a closed submanifold, then $\mathcal{G}$ has abelianization $\mathcal{G}/\overline{(\mathcal{G},\mathcal{G})}^s$.*

Proof. If $\overline{(\mathcal{G},\mathcal{G})}^s$ is closed, then $\overline{(\mathcal{G},\mathcal{G})}^s = \overline{(\mathcal{G},\mathcal{G})}$. Since it is contained in the isotropy and on each source fiber it is a subgroup, it is a subgroupoid. Thus by lemma 5, it is a normal Lie subgroupoid. So the quotient $\mathcal{G}/\overline{(\mathcal{G},\mathcal{G})}^s$ is the abelianization of $\mathcal{G}$.

□

3.4 The genus-integration

3.4.1 $\mathcal{A}$-homology

Recall that the Weinstein groupoid is the quotient of $P(\mathcal{A})$ by $\mathcal{A}$-homotopies. If we generalize this idea and replace the equivelence relation by $\mathcal{A}$-homology, we can obtain another interesting construction.

Let us first recall that $\mathcal{A}$-homotopy between two $\mathcal{A}$-paths is a Lie algebroid morphism from $T(I \times I)$ to $\mathcal{A}$ satisfying certain boundary conditions. We can generalize this notion by allowing the unit square to be Σ, a square with genus. To be more precise, a *square with genus n* is constructed as follows: First we remove an open disk from the unit square and also from a compact surface with genus n. This creates a boundary ∂D on both of the surface. Gluing them together along the boundary gives us a *square with genus n*. Since the boundary $\partial\Sigma$ has a neighborhood diffeomorphic to a neighborhood of $\partial(I \times I)$, the boundary condition for $\mathcal{A}$-homotopy is well-defined on this neighborhood.

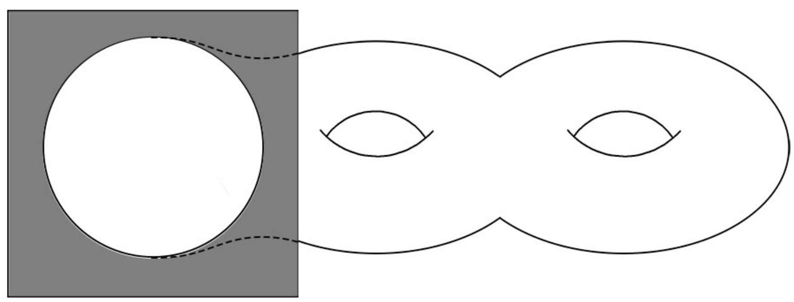
Definition 23. *Given two $\mathcal{A}$-paths a_0 and a_1, we say a_0 is $\mathcal{A}$-homologous to a_1 if there is a Lie algebroid morphism*

$$\begin{array}{ccc} T(\Sigma) & \xrightarrow{h} & A \\ \downarrow & & \downarrow \\ \Sigma & \xrightarrow{\gamma} & M \end{array} .$$

satisfying the boundary conditions $h|_U = a(t,\epsilon)dt + b(t,\epsilon)d\epsilon$, $\gamma(i,\epsilon) = x_i, a(t,i) = a_i(t), b(i,\epsilon) = 0$ for $i = 0, 1$.

We denote the equivalence relation on $P(\mathcal{A})$ defined by $\mathcal{A}$-homology by $\approx$.

Figure 3.1: Square with genus



Now we can construct a new groupoid called the genus-integration.

Definition 24. *The genus-integration of $\mathcal{A}$ is defined as $\mathcal{G}_g(\mathcal{A}) = P(\mathcal{A})/\approx$.*

The equivalence relation $\approx$ is courser than the relation $\sim$ defined by $\mathcal{A}$-homotopy and thus there is a natural quotient map $\mathcal{G}(\mathcal{A}) \to \mathcal{G}_g(\mathcal{A})$. Actually, we have the following proposition:

Proposition 18. *[7] The genus-integration is the abelianization of the Weinstein groupoid in the category of sets.*

It is clear from the proposition 18 that if the genus-integration is smooth, it will be the abelianization of the Weinstein groupoid in the smooth category. In general, given a integrable Lie algebroid, its genus-integration does not need to be smooth, i.e., smoothness of Weinstein groupoid does not imply the smoothness of the genus-integration. The converse is also not true, but we will come back to this after first looking into the obstructions to the smoothness of the genus-integration.

3.4.2 Extended monodromy groups

Given a Lie algebroid $\mathcal{A} \to M$, for any $x \in M$, take L to be the leaf containing x. Since $\mathcal{A}$ restricted to this leaf has abelianization $\mathcal{A}_L^{ab}$, we have the following short exact sequence:

$$0 \longrightarrow \mathfrak{g}_L^{ab} \longrightarrow \mathcal{A}_L^{ab} \underset{\sigma^{ab}}{\overset{\rho}{\rightleftarrows}} TL \longrightarrow 0.$$

Choosing a splitting $\sigma^{ab} \to \mathcal{A}_L^{ab}$, we can define a closed 2-form $\omega \in \Omega^2(TL, \mathfrak{g}^{ab})$ by $\omega^{ab} := \sigma^{ab}([X,Y]) - [\sigma^{ab}(X), \sigma^{ab}(Y)]$. We can also obtain a flat TL-connection ∇^{ab} on $\mathfrak{g}^{ab}$ by $\nabla_X^{ab} s = [\sigma^{ab}(X), s]$. Letting $q : \tilde{L}^h \to L$ denote the holonomy cover of L relative to ∇^{ab}, we can define the extended monodromy as follows.

Definition 25. *The extended monodromy group $\mathcal{N}_x^{ext}$ of $\mathcal{A}$ at $x \in M$ is the image of the extended monodromy homomorphism:*

$$\begin{aligned} \partial_x^{ext} : H_2(\tilde{L}^h, \mathbb{Z}) &\to \mathcal{G}(\mathfrak{g}_x^{ab}) \\ [\gamma] &\mapsto \exp\left(\int_\gamma q^*\Omega^{ab}\right). \end{aligned}$$

The extended monodromy groups and the ordinary monodromy groups are related by the following proposition.

Proposition 19. *[7] The extended and ordinary monodromy homomorphisms of a Lie algebroid fit into a commutative diagram:*

$$\begin{array}{ccc} \pi_2(L,x) & \xrightarrow{\partial_x} & \mathcal{G}(\mathfrak{g}_x) \\ \downarrow{\scriptstyle h_2} & & \downarrow \\ H_2(\tilde{L}^h, \mathbb{Z}) & \xrightarrow{\partial_x^{ext}} & \mathcal{G}(\mathfrak{g}_x^{ab}) \end{array},$$

where h_2 is the Hurewicz map.

Theorem 6. *[7] Let $\mathcal{A} \to M$ be a transitive Lie algebroid with trivial holonomy. The following statements are equivalent:*

(a) the extended monodromy groups are discrete;

(b) the genus integraton $\mathcal{G}_g(\mathcal{A})$ is smooth;

(c) the abelianization $\mathcal{A}^{ab}$ has an abelian integration.

If any of these hold then $\mathcal{G}_g(\mathcal{A})$ has Lie algebroid isomorphic to $\mathcal{A}^{ab}$.

Theorem 6 together with proposition 19 show that for a transitive abelian Lie algebroid, the smoothness of genus integraion implies the smoothness of Weinstein groupoid.

The following examples help understanding the relationship between smoothness of the Weinstein groupoid and the genus-integration. For that, recall from 2 that one can construct regular Lie algebroids from bundles of Lie algebras, connections and 2-forms. In particular, given a Lie algebra $\mathfrak{g}$ and a two form $\omega \in \Omega^2_{cl}(M; Z(\mathfrak{g}))$, we can construct $\mathcal{A}_\omega = TM \oplus \mathfrak{g}$ by taking the connection to be the Lie derivative:

- $\rho = pr_{TM}$
- $[(X,u)(Y,v)] = ([X,Y], [u,v]_{\mathfrak{g}} + Ł_X(v) - Ł_Y(u) + \omega(X,Y))$

In this case, we can compute $\mathcal{N}_x(\mathcal{A})$ and $\mathcal{N}_x(\mathcal{A})$ as follows:

$$\mathcal{N}_x(\mathcal{A}) = \{\exp\left(\int_\gamma \omega\right) : \gamma \in \pi_2(M,x)\},$$

$$\mathcal{N}_x^{ext}(\mathcal{A}) = \{\exp\left(\int_\gamma \omega\right) : \gamma \in H_2(M,\mathbb{Z})\}.$$

Example 21. *[7] Let $\mathfrak{g}$ be a 4-dimensional vector space equipped with the bracket: $[e_2, e_3] = e_1$, $[e_i, e_j] = 0$ otherwise. We have that $[\mathfrak{g},\mathfrak{g}] =< e_1 >$, and the center is $Z(\mathfrak{g}) =< e_1, e_4 >$.*

1. *Let $M = \mathbb{S}^2 \times \mathbb{S}^2$ and $\omega = pr_1^*\omega_{\mathbb{S}^2}e_1 + \lambda pr_2^*\omega_{\mathbb{S}^2}e_1$. Here $\omega_{\mathbb{S}^2}$ is the volume form on $\mathbb{S}^2$, pr_1 and pr_2 be the two projection respectively. We can compute that:*

   $$\mathcal{N}_x(\mathcal{A}) = exp((n_1 + \lambda n_2)e_1), \mathcal{N}_x(\mathcal{A}^{ab}) = \mathcal{N}_x^{ext} = \{1\}.$$

   Thus we obtain a non-integrable Lie algebroid with smooth genus-integration and whose abelianization is integrable.

2. *Change the form in the previous example and define* $\omega = pr_1^*\omega_{\mathbb{S}^2}(e_1 + e_4) + \lambda pr_2^*\omega_{\mathbb{S}^2}(e_1 - e_4)$*. We can compute that:*

$$\mathcal{N}_x(\mathcal{A}) = exp(n_1(e_1 + e_4) + \lambda n_2(e_1 - e_4),$$

$$\mathcal{N}_x(\mathcal{A}^{ab}) = \mathcal{N}_x^{ext} = exp((n_1 - \lambda n_2)e_4)$$

Thus we have an integrable Lie algebroid whose abelianization is non-integrable and whose genus-integration is not smooth.

3. *Now replace the manifold by* $M = \mathbb{T}^2 \times \mathbb{T}^2$ *and let the form be* $\omega = pr_1^*\omega_{\mathbb{T}^2}(e_1 + e_4) + \lambda pr_2^*\omega_{\mathbb{T}^2}(e_1 - e_4)$*, where* $\omega_{\mathbb{T}^2}$ *is the volume form on* $\mathbb{T}^2$*. We can compute that*

$$\mathcal{N}_x(\mathcal{A}) = \mathcal{N}_x(\mathcal{A}^{ab}) = \{1\}, \mathcal{N}_x^{ext} = exp((n_1 - \lambda n_2)e_4).$$

Thus both this Lie algebroid and its abelianization are integrable, but their genus-integraion is not smooth.

3.4.3 The kernel of the quotient map

As a quotient of the Weinstein groupoid, we already saw that the kernel of the quotient map $q : \mathcal{G}(\mathcal{A}) \to \mathcal{G}_g(\mathcal{A})$ is $(\mathcal{G}(\mathcal{A}), \mathcal{G}(\mathcal{A}))$. We are still interested in understanding this kernel using the structure of $\mathcal{A}$.

Proposition 20. *[7] Let* $\mathcal{A} \to L$ *be an integrable abelian transitive Lie algebroid. The kernel of the morphism* $\phi_x : \mathcal{G}(\mathfrak{g}_x) \to \mathcal{G}_g(A)_x^0$ *is given by*

$$\operatorname{Ker}\phi_x = \left\{ \exp\left(\int_{\tilde{\gamma}} \Omega\right) : \tilde{\gamma} \textit{ a lift to } \tilde{L}^h \textit{ of a compact surface } \gamma \textit{ in } L \right\}.$$

Adjusting the technique of the proof of propostion 5.2 in [7], we see that the integrability assumption can be removed. Note that both the Weinstein groupoid and the genus-integration in some sense respects the foliation of the Lie algebroid. To be more precise, we have $\mathcal{G}(\mathcal{A})_L = \mathcal{G}(\mathcal{A}_L)$. Since $\mathcal{G}_g(\mathcal{A})$ is the set-theoretical abelianization of the Weinstein groupoid, we also have $\mathcal{G}_g(\mathcal{A})_L = \mathcal{G}_g(\mathcal{A}_L)$. Thus the kernel of this quotient is completely determined by the structure on each leaf and we obtain the following proposition.

Proposition 21. *Let* $\mathcal{A} \to M$ *be an abelian Lie algebroid. Let* L *be the leaf* *containing x, the kernel of the morphism $\phi_x : \mathcal{G}(\mathfrak{g}_x) \to \mathcal{G}_g(A)^0_x$ is given by*

$$\operatorname{Ker}\phi_x = \left\{ \exp\left(\int_{\tilde{\gamma}} \Omega \right) : \tilde{\gamma} \textit{ a lift to } \tilde{L}^h \textit{ of a compact surface } \gamma \textit{ in } L \right\}.$$

Proof. Recall $h = adt + bd\epsilon$ is an $\mathcal{A}$-homotopy if and only if

$$\partial_t b - \partial_\epsilon a = T_\nabla(\star).$$

First note that for any transitive Lie algebroid $\mathcal{A} \simeq TL \ltimes \mathfrak{g}_L$ over L, we have $q^*\mathcal{A} \simeq T\tilde{L} \ltimes \mathfrak{g}_{\tilde{L}}$, where $q : \tilde{L} \to L$ is the holonomy cover of L. Let σ, $\tilde{\sigma}$ be the corresponding splitting, we have $\tilde{\sigma}(X_x)(x) = \sigma(d_x q(X_x))(q(x))$. The induced connection satisfies

$$\nabla^{\tilde{\sigma}}_{X_x} s(x) = \sum \nabla^{\sigma}_{d_x q(X_x)} s_i(q(x)),\ \forall s = \sum q^* s_i \in \Gamma(\mathfrak{g}_{\tilde{L}}).$$

$\Rightarrow$ Now, suppose $\exp(v) \in \operatorname{Ker}\phi$, then v is $\mathcal{A}$-homologous to 0_x via some $\mathcal{A}$-homology with base map $\gamma : \Sigma \to L$. Let $\{\gamma_i, \eta_i\}$ be the generator of the fundamental group of Σ. We can view the $\mathcal{A}$-homology as an $\mathcal{A}$-homotopy h between $\mathcal{A}$-paths over x and $\tau = \Pi(\gamma_i, \eta_i)$. Choosing a splitting $\sigma : TL \to A$ such that $\sigma(\frac{d\tau}{dt}) = h|_\tau$ we can identify $\mathcal{A}$ with $TL \ltimes \mathfrak{g}_L$, where the bracket is

$$[(X, v), (Y, w)] = ([X, Y], [v, w] + \nabla^\sigma_X(w) - \nabla^\sigma_Y(v) - \Omega(X, Y)]).$$

Now the homotopy can be written as $h(t, \epsilon) = (\frac{d\gamma}{dt}, \varphi)dt + (\frac{d\gamma}{d\epsilon}, \psi)d\epsilon$.
We can also identify q^*A with $T\tilde{L} \ltimes \mathfrak{g}_{\tilde{L}}$ using $\tilde{\sigma}$, where the bracket is

$$[(X, v), (Y, w)] = ([X, Y], [v, w] + \nabla^{\tilde{\sigma}}_X(w) - \nabla^{\tilde{\sigma}}_Y(v) - (q^*\Omega)(X, Y)]).$$

Notice that γ is a compact surface since 0_x and v have trivial base maps. If we now lift γ to $\tilde{\gamma}$ on $\tilde{L}$ and let $\tilde{\tau}$ be the corresponding lift of τ. We get an $q^*\mathcal{A}$-homotopy $\tilde{h}(t, \epsilon) = (\frac{d\tilde{\gamma}}{dt}, q^*\phi)dt + (\frac{d\tilde{\gamma}}{d\epsilon}, q^*\psi)d\epsilon$. Now let $\nabla^{\tilde{L}}$ be a connection on $\tilde{L}$ then $\nabla = (\nabla^{\tilde{L}}, \nabla^{\tilde{\sigma}})$ is a connection on $q^*\mathcal{A}$. Recall that $\mathfrak{g}$ is abelian, thus we can compute the torsion of ∇:

$$T_\nabla((X, v), (Y, w)) = (T_{\nabla^{\tilde{L}}}(X, Y), q^*\Omega(X, Y)).$$

Integrating the equation $\partial_t(q^*\psi) - \partial_\epsilon(q^*\phi) = q^*\Omega(\frac{d\tilde{\gamma}}{dt}, \frac{d\tilde{\gamma}}{d\epsilon})$ twice and using the boundary conditions, we get $v = \int_{\tilde{\gamma}} q^*\Omega$.

$\Leftarrow$ Now suppose we have some compact surface γ in L. We can cut along its generators to get an ordinary homotopy between x and $\tau = \Pi(\gamma_i, \eta_i)$. Choose a splitting $\sigma : TL \to A$ and define $b(t, \epsilon) = \sigma(\frac{d\tau}{d\epsilon})(t, \epsilon)$. Solving $(\star)$ with initial condition $a(t, 0) = 0$, we get an $\mathcal{A}$-homotopy h which can be viewed as an $\mathcal{A}$-homology between 0_x and $a(t, 1)$. If we identify $\mathcal{A}$ with $TL \ltimes \mathfrak{g}_L$ using σ and $q^*\mathcal{A}$ with $T\tilde{L} \ltimes \mathfrak{g}_{\tilde{L}}$ using $\tilde{\sigma}$, a similar argument as before shows that:

$$\int_0^r q^*a(t,1)dt = \int_0^r \int_0^1 q^*\Omega(\frac{d\tilde{\gamma}}{dt}, \frac{d\tilde{\gamma}}{d\epsilon}).$$

Thus we get a $q^*\mathcal{A}$ homotopy h' between $q^*a(t, 1)$ and its average

$$\int_0^1 q^*a(t,1)dt = \int_0^1 \int_0^1 q^*\Omega(\frac{d\tilde{\gamma}}{dt}, \frac{d\tilde{\gamma}}{d\epsilon}) = \int_{\tilde{\gamma}} q^*\Omega.$$

Thus $0_x \approx a(t, 1) \overset{h' \circ q}{\sim} \displaystyle\int_{\tilde{\gamma}} q^*\Omega$. □

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