# REMARKS ON NON-FORMAL DEFORMATION QUANTIZATION OF POISSON MANIFOLDS

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ABSTRACT. We survey the problem of non-formal deformation quantization of Poisson manifolds. This survey is a written version of the talk given by the second author at the Bahia meeting and includes ongoing, unpublished work, by the authors.

## 1. INTRODUCTION

Heuristically, the term *quantization* refers to a variety of procedures in which one aims to associate to a classical system the data underlying a quantum system. In particular, to classical observables described by functions on a Poisson manifold (see Section 2) one aims to associate quantum observables consisting of operators on a Hilbert space. This correspondence, denoted

$$f \mapsto Q_\hbar(f),$$

depends on Planck's constant $\hbar$, which maybe interpreted as a scale or a deformation parameter. The notion of *deformation quantization* arises from considering a *star product* structure $\star_\hbar$ defined by the relation

$$Q_\hbar(f_1) \circ Q_\hbar(f_2) = Q_\hbar(f_1 \star_\hbar f_2).$$

In paradigmatic cases, the star product $\star_\hbar$ defines a family of associative algebras $(A_\hbar, \star_\hbar)$, depending on the parameter $\hbar$, and admitting embeddings $C^\infty(M) \hookrightarrow A_\hbar$.

Thinking of $\hbar$ as a deformation parameter, one can interpret $(A_\hbar, \star_\hbar)$ as a (non-commutative) deformation of the usual (commutative) algebra of functions $(C^\infty(M), \cdot)$. At this point one may forget about the original quantization functor $Q_\hbar$ and look for any such deformation with the property that

$$\{f,g\} = \lim_{\hbar \to 0} \frac{1}{\hbar} \left( f \star_\hbar g - g \star_\hbar f \right). \tag{1}$$

This equation is known as the *classical correspondence principle* and one says that $\star_\hbar$ is a deformation of the usual product in the direction of the Poisson bracket $\{\cdot,\cdot\}$.

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One way to make this heuristic discussion precise is to consider **formal star products** deforming $\{\cdot,\cdot\}$ which are given by formal power series

$$f_1 \star_\hbar f_2 := \sum_{k=0}^{\infty} B_k(f_1, f_2)\hbar^k,$$

and satisfy a set of axioms (see Section 3). The formality conjecture stated and proved by Kontsevich [20] implies that every Poisson bracket admits a a formal star product and that they can be classified.

While formal deformation quantization, after Kontsevich's work, is a remarkably successful theory, it only partially captures the original features that were intended to be made mathematically precise in the transition from classical to quantum mechanics. For example, Kontsevich noted in [20], "It is not clear whether 'deformation quantization' is natural for quantum mechanics (...) A topological open string theory seems to be more relevant." Additionally, Gukov and Witten in [16] emphasize that "(formal) deformation quantization is not quantization." Consequently, there is a significant interest in non-formal deformation quantization, of which much less is known.

A particular case of such non-formal quantizations uses $C^*$-algebras and is called *strict deformation quantizations* (see, e.g., [25] for a precise formulation). An analogue of Kontsevich's result for strict deformation quantization is not known and Rieffel in [26] poses the following questions.

**Fundamental Problem.** *When does a Poisson manifold admit a strict deformation quantization? In particular, what cohomological obstructions are there to having a strict deformation quantization?*

There are several variations on the notion of non-formal deformation quantization – see, e.g., [27] for a recent survey. In recent unpublished work we propose a new approach to non-formal star products $\star_\hbar$. We consider star products given by the sort of *semi-classical Fourier integral operators (SCFIO)* studied in semi-classical analysis – see, e.g., [15, 21, 33]. Two reasons for this approach are:

(i) the need to consider products $\star_\hbar$ defined on $\hbar$-dependent families of smooth functions that may become singular as $\hbar \to 0$, such as highly oscillatory functions $e^{\frac{i}{\hbar}\langle\cdot,\xi_0\rangle}$ and
(ii) several successful quantization schemes, such as the Weyl quantization of cotangent bundles [15] or Rieffel's quantization of linear Poisson brackets [25], can be expressed in terms of semi-classical Fourier integral operators.

The use of semi-classical analysis then suggests that the axioms of a star product should only hold microlocally. These ideas have many precedents (see [19, 30, 7] and references therein). Also, oscillatory integrals have been considered in the study of strict deformation quantization (see, e.g., [2, 22] and the survey [27]) but, as far as the authors know, ours is the first systematic treatment which incorporates all the semi-classical and microlocal features into play and establishes a detailed relation to the Lie theory of the underlying Poisson manifold.

In this note we sketch our work in this direction, including a provisional definition of non-formal deformation quantization. We describe some conjectural results, attempting to extend results valid in the formal setting to the non-formal setting. We will explain why we believe in the following:

**Motto.** *If a Poisson manifold* $(M,\{\cdot,\cdot\})$ *admits an* associative non-formal quantization*, then it must be integrable by a symplectic groupoid.*

This note is organized as follows. In Section 2, we recall some basic facts about Poisson manifolds and symplectic groupoids. In Section 3, we review formal deformation quantization. In Section 4, we give a tentative definition of non-formal deformation quantization, using SCFIOs, we state some conjectural results and we explain what led us to our motto above.

## 2. Poisson geometry

The geometry underlying many fundamental classical physical systems is *Poisson geometry*, whose objects of study are Poisson manifolds, i.e., manifolds equipped with a Poisson bracket. On a Poisson manifold, a choice of a function – the so-called hamiltonian – determines the Hamiltonian dynamics and mechanical observables are also given by functions. Therefore, the ultimate goal is to quantize a Poisson manifold together with all functions of interest. In this section we provide a brief sketch of Poisson geometry and we refer to the monograph [13] for a in-depth introduction to the subject.

2.1. **Poisson manifolds.** A **Poisson bracket** on a manifold $M$ is a Lie bracket on the vector space of smooth functions

$$\{\ ,\ \} : C^\infty(M)\times C^\infty(M)\to C^\infty(M),$$

which satisfies the Leibniz type identity

$$\{f,f_1f_2\}=\{f,f_1\}f_2+f_1\{f,f_2\}. \tag{2}$$

The pair $(M,\{\cdot,\cdot\})$ is called a **Poisson manifold**.

On a Poisson manifold $(M,\{\cdot,\cdot\})$, to each function $h\in C^\infty(M)$ is assigned a vector field $X_h$ on $M$ by setting

$$X_h(f):=\{h,f\}.$$

One calls $h$ a **hamiltonian function** and $X_h$ the associated **hamiltonian vector field**.

In the language of classical mechanics, the Poisson manifold $(M,\{\cdot,\cdot\})$ is identified with the *phase space* (or "state space") and the equations for the trajectories of the hamiltonian vector field $X_h$ are called the *Hamilton's equations of the motion*. In local coordinates $(x^1,\dots,x^m)$, the Poisson bracket can be written has

$$\{f_1,f_2\}(x)=\sum_{i,j=1}^{n}\pi^{ij}(x)\frac{\partial f_1}{\partial x^i}\frac{\partial f_2}{\partial x^i}, \tag{3}$$

and a hamiltonian vector field $X_h$ has the expression

$$X_h = \sum_{i,j=1}^{n} \pi^{ij} \frac{\partial h}{\partial x^i} \frac{\partial}{\partial x^i}.$$

Hamilton's equations are then

$$\dot{x}^k = \{h, x^k\} = \sum_{i=1}^{n} \pi^{ik} \frac{\partial h}{\partial x^i}, \quad (k = 1, \ldots, m). \tag{4}$$

Let us illustrate this with several examples.

2.2. **Symplectic Poisson brackets.** The motion of a particle $q(t) \in \mathbb{R}^d$ with mass $m$ subject to a conservative force with potential $V : \mathbb{R}^d \to \mathbb{R}$ is described by Newton's equations

$$m\ddot{q}^k(t) = -\frac{\partial V}{\partial q^k}, \quad (k = 1, \ldots, d),$$

which can be rewritten as the first order system

$$\begin{cases} \dot{q}^k = \frac{p_k}{m_k} \\ \dot{p}_k = -\frac{\partial V}{\partial q^k} \end{cases}, \quad (k = 1, \ldots, d). \tag{5}$$

To interpret it in the language of Poisson geometry, we consider $M = \mathbb{R}^{2d}$ with coordinates $(q^k, p_k)$ and a Poisson bracket defined by

$$\{f_1, f_2\} := \sum_{k=1}^{d} \left( \frac{\partial f_1}{\partial p_k} \frac{\partial f_2}{\partial q^k} - \frac{\partial f_1}{\partial q^k} \frac{\partial f_2}{\partial p_k} \right). \tag{6}$$

Given any function $h : \mathbb{R}^{2n} \to \mathbb{R}$, the corresponding Hamilton's equations (4) become

$$\begin{cases} \dot{q}^k = \frac{\partial h}{\partial p_k} \\ \dot{p}_k = -\frac{\partial h}{\partial q_k} \end{cases}, \quad (k = 1, \ldots, d).$$

If we let

$$h = \sum_{k=1}^{d} \frac{p_k^2}{2m_k} + V(q),$$

Hamilton's equations reduce to Newton's equations in the form (5).

The Poisson bracket in this example belongs to an important class of Poisson brackets, called *symplectic*. In general, a symplectic form on a manifold $M$ of even dimension $n = 2d$ is a 2-form $\omega \in \Omega^2(M)$ satisfying

$$\mathrm{d}\omega = 0, \qquad \underbrace{\omega \wedge \cdots \wedge \omega}_{d\text{-times}} \neq 0$$

Given a symplectic form, each choice of a function $f \in C^\infty(M)$ determines a vector field $X_f$ by

$$i_{X_f}\omega = \mathrm{d}f.$$

It is easy to check that the expression

$$\{f, g\}_\omega := \omega(X_f, X_g)$$

defines a Poisson bracket on $M$. Any Poisson bracket arising in this way is called a symplectic Poisson bracket.

The previous example is symplectic because it arises from the symplectic form on $\mathbb{R}^{2d}$ given by

$$\omega = \sum_{k=1}^{d} \mathrm{d}q^k \wedge \mathrm{d}p_k.$$

Symplectic Poisson brackets can be characterized as follows

**Lemma 1.** *A Poisson bracket $\{\cdot,\cdot\}$ on $M$ is symplectic if for any choice of local coordinates $(x^1,\ldots,x^m)$ the matrix $\pi^{ij} = \{x^i,x^j\}$ is invertible.*

2.3. **Linear Poisson brackets.** The motion of a rigid body around its center of gravity with moments of inertia $I_1$, $I_2$ and $I_3$ is described by Euler's equations

$$\begin{cases} \dot{x}^1 = \frac{I_2-I_3}{I_2I_3}x^2x^3 \\ \dot{x}^2 = \frac{I_3-I_1}{I_3I_1}x^3x^1, \\ \dot{x}^3 = \frac{I_1-I_2}{I_1I_2}x^1x^2. \end{cases} \tag{7}$$

One can interpret it in the language of Poisson geometry by introducing a Poisson bracket on $\mathbb{R}^3$ defined by

$$\{f,g\}(x) := (\nabla f(x) \times \nabla g(x)) \cdot x.$$

and considering the hamiltonian function

$$h = \sum_{i=1}^{3} \frac{(x^i)^2}{2I_i}.$$

Then Euler's equations (7) can be written in Hamiltonian form

$$\dot{x}^k = \{h,x^k\}, \quad (k = 1,2,3).$$

In general, given a Lie algebra $\mathfrak{g}$ the dual vector space $M = \mathfrak{g}^*$ has a Poisson bracket defined by

$$\{f,g\}(\xi) = \langle [\mathrm{d}_\xi f, \mathrm{d}_\xi g]_{\mathfrak{g}}, \xi \rangle.$$

One recovers the Poisson bracket in the previous example by letting $\mathfrak{g} = \mathfrak{so}(3,\mathbb{R})$, so that

$$M = \mathfrak{so}(3,\mathbb{R})^* \simeq \mathbb{R}^3.$$

These type of Poisson brackets can be characterized as follows.

**Lemma 2.** *A Poisson bracket $\{\cdot,\cdot\}$ on a vector space $V$ is of the form $\mathfrak{g}^*$ if and only for any pair of linear functions $f_1$ and $f_2$, their Poisson bracket $\{f_1, f_2\}$ is a linear function. Equivalently, if in linear coordinates $(x^1,\ldots,x^n)$ the bracket is linear:*

$$\pi^{ij}(x) = \{x^i,x^j\}(x) = \sum_k c_k^{ij} x^k.$$

Notice that for a linear Poisson bracket one has $\pi^{ij}(0) = 0$, so it is never symplectic.

2.4. **Quadratic Poisson structures.** The time evolution of $m$ species in a closed ecological system can often be described by the Lotka-Volterra equations

$$\dot{x}^i = \varepsilon^i x^i + \sum_{j=1}^{m} a_{ij} x^i x^j, \quad (i = 1, \dots, m). \tag{8}$$

for some matrix $(a_{ij})$ and vector $\varepsilon^i$. One calls the system *conservative* if $(a_{ij})$ is skew-symmetric and there exists a solution $q = (q_1, \dots, q_n)$ of the linear system

$$\sum_{j=1}^{n} a_{ji} q^j = \varepsilon^i.$$

A conservative Lotka-Volterra system can be described as a hamiltonian system as follows. On $\mathbb{R}^m$ consider the Poisson bracket

$$\{f_1, f_2\}(x) := \sum_{i<j}^{m} a_{ij} x^i x^j \frac{\partial f_1}{\partial x^i} \frac{\partial f_2}{\partial x^j}.$$

Then the hamiltonian function

$$h = \sum_{i=1}^{m} \left( q^i \log x^i - x^i \right),$$

defines a hamiltonian vector field and the corresponding Hamilton's equations become the Lotka-Volterra equations (8).

The Poisson bracket in this example is *quadratic*. In general, a Poisson bracket on a vector space $V$ is called homogeneous of degree $d$ if for any linear coordinate system $(x^1, \dots, x^n)$ the functions $\pi^{ij}(x) = \{x^i, x^j\}(x)$ are homogeneous of degree $d$.

For example, any quadratic Poisson bracket $\{\cdot,\cdot\}_{\mathbb{R}^{n+1}}$ on $\mathbb{R}^{n+1}$ determines a unique Poisson bracket $\{\cdot,\cdot\}_{\mathbb{P}^n}$ on projective space $\mathbb{P}^n$ for which the projection $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is a **Poisson map**, i.e., satisfies

$$\{f_1, f_2\}_{\mathbb{P}^n} \circ \pi = \{f_1 \circ \pi, f_2 \circ \pi\}_{\mathbb{R}^{n+1}},$$

for any pair of functions $f_1$, $f_2$ on $\mathbb{P}^n$.

2.5. **The symplectic foliation.** The geometry underlying a Poisson bracket on a manifold $M$ has the following three distinct, but intertwined, aspects:

- $M$ inherits an underlying (possibly singular) *foliation* $\mathscr{F}$, i.e., a partition into submanifolds. These submanifolds, called leaves, are the equivalence classes for the equivalence relation on $M$ obtained by declaring two points equivalent if there exists some hamiltonian function $h$ for which the vector field $X_h$ has a trajectory passing through those two points.
- Each leaf $L$ has a unique Poisson structure $\{\cdot,\cdot\}_L$ for which the inclusion $i : L \to M$ is a Poisson map. Moreover, $\{\cdot,\cdot\}_L$ is symplectic, so $\mathscr{F}$ is a foliation by *symplectic leaves*.

- For each $x_0 \in M$ there are coordinates such that $X_{x^i}(x_0) = 0$ and the leaf through $x_0$ is given by $\{x^1 = \cdots = x^d = 0\}$ . Then one has the *isotropy Lie algebra* at $x_0$ with basis $\{e_1, \ldots, e_d\}$, and bracket

$$[e_i, e_j] := \sum_{k=1}^{d} \frac{\partial \pi^{ij}}{\partial x^k} e_k,$$

where the $\pi^{ij}$ are the coefficients of the Poisson bracket, as in (3).

Let us look back at the examples we saw before.

**Example 3.** The (connected) symplectic manifolds are precisely the Poisson manifolds whose symplectic foliation consists of a single leaf. For this reason, the isotropy Lie group/algebra at any point is trivial.

**Example 4.** A linear Poisson structure on the dual of a Lie algebra $\mathfrak{g}^*$ has symplectic foliation consisting of the coadjoint orbits. These are obtained as the orbits of a canonical (linear) action on $\mathfrak{g}^*$ of the 1-connected Lie group $G$ whose Lie algebra is $\mathfrak{g}$ (the coadjoint action/representation). The isotropy/Lie algebras of $\mathfrak{g}^*$ are the ones obtained from this action.

**Example 5.** The symplectic foliation of a quadratic Poisson structure on a vector space $V$ depends heavily on the quadratic homogeneous functions $\pi^{ij}$. Consider for example the quadratic Poisson structure on $\mathbb{R}^3$ with Poisson bracket defined by

$$\{x,y\} = xy, \quad \{x,z\} = 0, \quad \{y,z\} = yz,$$

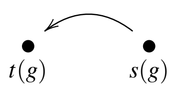
has symplectic leaves the connected components of the surfaces $yz = c$, for $c \neq \{0\}$ and the points (zero dimensional leaves) belonging to the union of the planes $y = 0$ and $z = 0$. On the other hand, the quadratic Poisson bracket defined by

$$\{x,y\} = x^2 + y^2, \quad \{x,z\} = \{y,z\} = 0,$$

has symplectic leaves the cylinders $x^2 + y^2 = c$, with $c > 0$, and the points in the $z$-axis. For a quadratic Poisson bracket the isotropy Lie algebras are all abelian.

2.6. **Symplectic groupoids.** The interaction between Poisson geometry, foliation theory and Lie group theory actually goes much deeper, once Lie algebroids/groupoids are brought into the picture.

Recall that groups typically arise as the symmetries of some given object. The concept of a groupoid allows for more general symmetries, acting on a collection of objects $M$ rather than just a single one. An element in a groupoid $g \in \mathscr{G}$ may be pictured as an arrow from a source object $s(g) \in M$ to a target object $t(g) \in M$



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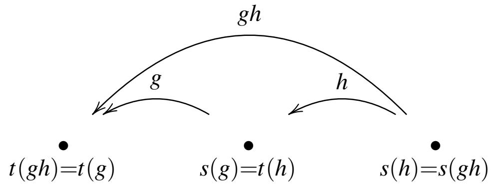
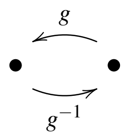
Two such arrows $g, h \in \mathscr{G}$ can be composed to produce a new arrow $gh \in \mathscr{G}$ if and only if the second arrow starts where the first arrow ends, and their product source $s(h)$ and target $t(h)$




For each object one has a unit $1_x$, which is an arrow from $x$ to itself


and for every arrow there is an inverse arrow




These must satisfy the usual axioms

(i) Associativity: $(gh)k = g(hk)$;
(ii) Identity: $1_{t(g)}g = g1_{s(g)} = g$;
(iii) Inverse: $g^{-1}g = 1_{s(g)}$, $gg^{-1} = 1_{t(g)}$.

Using categorical language, one can describe a groupoid simply as a category where every arrow has an inverse.

Just as Lie groups – introduced by Lie in the late XIX century – describe smooth symmetries of an object, Lie groupoids – introduced by Ehresmann in the late 1950's – describe smooth symmetries of a smooth family of objects. That is, the collection of arrows is a manifold $\mathcal{G}$, the set of objects is a manifold $M$, and all the structure maps of the groupoid, namely source, target, multiplication, identity and inverse, are smooth. We denote a groupoid by $\mathcal{G} \rightrightarrows M$.

Given a Lie groupoid $\mathcal{G} \rightrightarrows M$, one has that

- $M$ is partitioned by *orbits* of $\mathcal{G}$, where two objects in $M$ belong to the same orbit if there is an arrow connecting them. The path-connected components of the orbits form a foliation of $M$;
- the arrows with source and target the same object $x \in M$ form a Lie group $\mathcal{G}_x$, called the *isotropy group* at $x$.

The groupoids that arise in Poisson geometry are *symplectic groupoids*, i.e., they possess a symplectic form $\Omega$ on the space of arrows which is compatible with the groupoid multiplication. One says that a symplectic groupoid $(\mathcal{G}, \Omega) \rightrightarrows M$ integrates a Poisson manifold $(M, \{\cdot,\cdot\})$ it the target map $t : (\mathcal{G}, \{\cdot,\cdot\}_\Omega) \to (M, \{\cdot,\cdot\})$ is a Poisson map. When this happens, the symplectic leaves and the isotropy Lie algebras of $(M, \{\cdot,\cdot\})$ coincide, respectively, with the connected components of the orbits and the Lie algebras of the isotropy groups of $\mathscr{G}$. We refer the reader to [13] for details, and look at the examples above.

**Example 6.** Given a manifold $M$ the pair groupoid $M\times M \rightrightarrows M$ has arrows the pairs $(x,y)$. Such a pair has source $x$ and target $x$ and the multiplication of two arrows $(x,y)$ and $(z,w)$ can be performed when $y=z$, the resulting arrow being $(x,z)\cdot(z,w)=(z,w)$. When $(M,\omega)$ is a symplectic manifold, the pair groupoid inherits a symplectic form, namely $\Omega := \mathrm{pr}_1^*\,\omega - \mathrm{pr}_2^*\,\omega$. This form is compatible with multiplication and the resulting symplectic groupoid integrates the Poisson manifold $(M,\{\cdot,\cdot\}_\omega)$.

**Example 7.** Given an action of a Lie group $G$ on a manifold $M$ one can form the action groupoid $G\times M \rightrightarrows M$, where a pair $(g,m)$ is viewed as an arrow from $m$ to $gm$. The multiplication of arrows is given by

$$(h,m')\cdot(g,m) = (hg,m),\ \text{provided}\ m'=gm.$$

The linear Poisson structure on $\mathfrak{g}^*$ can be integrated by the action groupoid $G\times\mathfrak{g}^* \rightrightarrows \mathfrak{g}^*$ associated with the coadjoint action of $G$ on $\mathfrak{g}^*$. The symplectic form is obtained from the canonical identification $T^*G \simeq G\times\mathfrak{g}^*$, since any cotangent bundle has a natural symplectic form.

**Example 8.** Quadratic Poisson brackets also admit integrating symplectic groupoids. The formulas are more complicated, so we consider only the simplest example of the quadratic Poisson bracket on $\mathbb{R}^2$ given by

$$\{x,y\} = xy.$$

Then one finds a symplectic groupoid $(\mathbb{R}^4,\Omega) \rightrightarrows \mathbb{R}^2$, where the source and target maps are

$$s(x,y,u,v) = (x,y), \quad t(x,y,u,v) = (xe^{yv}, ye^{-xu}),$$

and the multiplication is given by the formula

$$(x',y',u',v')\cdot(x,y,u,v) = (x,y,u+e^{yv}u', v+e^{-xu}v').$$

The relevant symplectic form is:

$$\Omega = -\mathrm{d}(xu)\wedge\mathrm{d}(yv) + \mathrm{d}x\wedge\mathrm{d}u + \mathrm{d}y\wedge\mathrm{d}v.$$

Given a Poisson manifold $(M,\{\cdot,\cdot\})$ one may ask if there is some symplectic groupoid integrating $(M,\{\cdot,\cdot\})$. In the three examples above the Poisson manifolds are all integrable, but this does not need to be the case. The obstructions to integrability are well understood – see [10, 11, 12].

In order to study global properties of Poisson manifolds and their symplectic foliations, one must pay attention to their integrating symplectic groupoids. We will see that symplectic groupoids are also central ingredients in quantization schemes for Poisson manifolds, both at formal and non-formal levels [3, 17, 18] and naturally arise in connection with topological field theories (the Poisson sigma model) [8].

## 3. FORMAL DEFORMATION QUANTIZATION

From now it is convenient to work with $\mathbb{C}$-valued functions, so $C^\infty(M)$ will denote the space of $\mathbb{C}$-valued smooth functions and we consider Poisson brackets on such functions.

### 3.1. **Formal star products.**

A **formal star product** for a general Poisson manifold $(M,\{\cdot,\cdot\})$ is a binary operation on the space of formal power series $C^\infty(M)[[\hbar]]$, defined for a pair of functions $f_1, f_2 \in C^\infty(M)$ by an expression of the form

$$f_1 \star_\hbar f_2 := \sum_{k=0}^{\infty} B_k(f_1,f_2)\hbar^k, \tag{9}$$

and extended to the space of formal power series by requiring $\mathbb{C}[[\hbar]]$-linearity. On also imposes the following axioms:

(A1) *Deformation of usual product.* For every $f_1, f_2 \in C^\infty(M)$,

$$f_1 \star_\hbar f_2 = f_1 f_2 + O(\hbar);$$

(A2) *Correspondence principle.* For every $f_1, f_2 \in C^\infty(M)$,

$$\{f_1,f_2\} = \frac{1}{i\hbar}(f_1 \star_\hbar f_2 - f_2 \star_\hbar f_1) + O(h);$$

(A3) *Associativity.* For every $f_1, f_2, f_3 \in C^\infty(M)[[\hbar]]$,

$$(f_1 \star_\hbar f_2) \star_\hbar f_3 = f_1 \star_\hbar (f_2 \star_\hbar f_3);$$

(A4) *Identity.* For every $f \in C^\infty(M)[[\hbar]]$,

$$1 \star_\hbar f = f = f \star_\hbar 1.$$

One calls $\star_\hbar$ *natural* if the $B_k$ are bidifferential operators of order $\leq k$ in each entry.

The formality conjecture stated and proved by Kontsevich [20] yields the following fundamental result.

**Theorem 9** (Kontsevich)**.** *Every Poisson manifold admits a natural formal star product.*

Kontsevich actually gives a classification of star products for a given Poisson manifold and a beautiful explicit formula to construct a star product for any Poisson structure in $\mathbb{R}^n$.

### 3.2. **Examples of formal star products.**

In general, in spite of Kontsevich formula, it is hard to give explicit examples of star products deforming a given Poisson bracket. Still, in some cases, explicit formulas can be given and we list here a few examples.

**Example 10.** On $M = \mathbb{R}^m$ any constant skew-symmetric matrix $(\pi^{ij})$ defines a constant Poisson structure

$$\{x^i, x^j\} := \pi^{ij}.$$

A formal star product deforming this Poisson bracket is given by the Moyal product

$$(10)\qquad \begin{aligned} f_1 \star_\hbar f_2 &:= \exp\left(\frac{i\hbar}{2}\pi^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}\right) f_1(x)f_2(y)\Big|_{y=x} \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{i\hbar}{2}\right)^k \pi^{i_1 j_1}\cdots\pi^{i_k j_k}\frac{\partial^k f_1}{\partial x^{i_1}\cdots\partial x^{i_k}}\frac{\partial^k f_2}{\partial x^{j_1}\cdots\partial x^{j_k}}, \end{aligned}$$

For example, this formula gives a formal star product deforming the symplectic Poisson bracket on $\mathbb{R}^{2d}$ defined by (6).

**Example 11.** Let $M = \mathfrak{g}^*$ with its linear Poisson structure. To obtain a star product deforming it one can use the Poincaré-Birkoff-Witt isomorphism between the symmetric and the universal enveloping algebras of $\mathfrak{g}$

$$q : S(\mathfrak{g}) \to U(\mathfrak{g}).$$

The symmetric algebra $S(\mathfrak{g})$ is naturally isomorphic to the algebra of polynomial functions on $\mathfrak{g}^*$. Then one defines a $\star_\hbar$-product on polynomials by first defining it on homogeneous polynomials $f_1 \in S^k(\mathfrak{g})$ and $f_2 \in S^l(\mathfrak{g})$ by the formula

$$f_1 \star_\hbar f_2 := \sum_{n=0}^{k+l}(i\hbar)^n \operatorname{pr}_{k+l-n}(q^{-1}(q(f_1)\cdot q(f_2))),$$

where $\operatorname{pr}_k : S(\mathfrak{g}) \to S^k(\mathfrak{g})$ denotes the projection onto the homogenous degree $k$ polynomials and $\cdot$ denotes the product in $U(\mathfrak{g})$. The $\mathbb{C}[[\hbar]]$-extension of this map defines a star product on $S(\mathfrak{g})[[\hbar]]$.

This star product is given by a series of the form (9) where the $B_k$ are bidifferential operators, so it also defines a star product on $C^\infty(\mathfrak{g}^*)[[\hbar]]$. Note that if $\{x_1,\ldots,x_m\} \subset \mathfrak{g} = S^1(\mathfrak{g})$ is a basis then the formula above yields

$$x_i \star_\hbar x_j = x_i x_j + \frac{i\hbar}{2}[x_i,x_j].$$

so this star product deforms the linear Poisson structure on $\mathfrak{g}^*$.

**Example 12.** On $M = \mathbb{R}^m$ any skew-symmetric matrix $(a^{ij})$ defines a "diagonal" quadratic Poisson bracket by

$$\{x^i,x^j\} := a^{ij}x^i x^j,$$

A star product deforming this Poisson bracket is also given by a Moyal type formula (10) where now $\pi^{ij} = a^{ij}x^i x^j$ is not constant anymore. One explanation for this coincidence is that the change of coordinates $u^i = \log x^i$ transforms this Poisson structure into a constant Poisson structure (but note this is only valid in the domain $x^i > 0$). If one considers a non-diagonal Poisson structure, such as the second Poisson bracket in Example 5, their star products take a more complicated form.

3.3. **Formal symplectic groupoid.** Given a Poisson manifold, a symplectic groupoid integrating it can be thought of as a first step towards the quantization of the Poisson bracket. In fact, symplectic groupoids were introduced by Karasev [19], Weinstein [28] and Zakrzewski [31, 32] precisely with the aim of quantizing Poisson manifolds. This quantization program was never completed. However, in the formal setting Cattaneo-Dherin-Felder [3] and Karabegov [18] found the following precise connection between groupoids and quantization.

**Theorem 13.** *A natural formal $\star_\hbar$-product deforming a Poisson manifold $(M,\{\cdot,\cdot\})$ determines a formal symplectic groupoid structure on a formal neighborhood of the zero section of $T^*M$ which integrates the Poisson bracket.*

Let us briefly explained the concepts from formal geometry used in the statement of this result. Given a closed submanifold $M \subset N$, denote by $I(M) \subset C^\infty(N)$ the ideal formed by the functions that vanish on $M$. Letting $I^\infty(M) = \bigcap_{k=1}^\infty I(M)^k$, the quotient algebra

$$C^\infty_M(N) := C^\infty(N)/I^\infty(M)$$

is viewed as the algebra of functions on a formal neighborhood of $M$ in $N$.

In order to define a groupoid structure on a formal neighborhood of the zero section of $T^*M$, one reinterprets the structure maps in a groupoid $\mathscr{G} \rightrightarrows M$ as pullback maps. For example, pullback by source and target are algebra maps $s^*,t^* : C^\infty(M) \to C^\infty(\mathscr{G})$, and so in the formal setting they are replaced by algebras maps

$$s^*,t^* : C^\infty(M) \to C^\infty_M(T^*M).$$

Similarly, in the formal setting, the inverse map is replaced by the algebra map

$$i^* : C^\infty_M(T^*M) \to C^\infty_M(T^*M),$$

while the unit map $u : M \to \mathscr{G}$ becomes the evaluation map

$$u^* : C^\infty_M(T^*M) \to C^\infty(M).$$

In order to define the multiplication observe that for a groupoid this is a map $m : \mathscr{G}^{(2)} \to \mathscr{G}$ where the domain is the space of composable arrows

$$\mathscr{G}^{(2)} := \{(g,h) \in \mathscr{G} \times \mathscr{G} : s(g) = t(h)\}.$$

One can identify the formal neighborhood of the diagonal $\Delta_M \subset \mathscr{G}^{(2)}$ using the short exact sequence of algebras

$$0 \longrightarrow I_2 \longrightarrow C^\infty_{M\times M}(\mathscr{G}\times\mathscr{G}) \longrightarrow C^\infty_{\Delta_M}(\mathscr{G}^{(2)}) \longrightarrow 0$$

where $I_2$ is the ideal generated by the image of the map

$$(\mathrm{id}\otimes s^* - t^*\otimes \mathrm{id}) : C^\infty(M) \to C^\infty_{M\times M}(\mathscr{G}\times\mathscr{G}).$$

Hence, in the formal setting, multiplication is defined as an algebra map

$$m^* : C^\infty_M(T^*M)^{(2)} \to C^\infty_M(T^*M),$$

where by definition

$$C^\infty_M(T^*M)^{(2)} := C^\infty_{M\times M}(T^*M\times T^*M)/I_2.$$

Similarly, one translates the axioms that the structure maps must satisfy in terms of pullback diagrams – see [18].

Finally, the ideal $I^\infty(M)$ is also a Poisson ideal for the Poisson bracket $\{\cdot,\cdot\}_{\text{can}}$ associated with the canonical symplectic form $\omega_{\text{can}}$ on $T^*M$. Hence, one obtains a Poisson bracket on the algebra of formal functions $C^\infty_M(T^*M)$. The algebra $C^\infty_M(T^*M)^{(2)}$ also inherits a Poisson bracket, and so one defines a **formal symplectic groupoid** integrating a Poisson manifold $(M,\{\cdot,\cdot\})$, to consist of:

- a formal groupoid structure in the formal neighborhood of the zero section $0\subset T^*M$, such that
- multiplication $m^* : (C^\infty_M(T^*M)^{(2)},\{\cdot,\cdot\}_{\text{can}}) \to (C^\infty(T^*M,M,\{\cdot,\cdot\}_{\text{can}})$ and target $t^* : (C^\infty(M),\{\cdot,\cdot\}) \to (C^\infty_M(T^*M),\{\cdot,\cdot\}_{\text{can}})$ are Poisson maps.

Given a Poisson manifold there always exist a *local* symplectic groupoid integrating it. This is a classical result due to Coste, Dazord and Weinstein [9]. From this it follows also that there exists always a formal symplectic groupoid integrating the Poisson manifold. This can also be seen from quantization via Theorem 13 since, by Kontsevich, any Poisson manifold admits a formal deformation quantization.

## 4. Non-formal deformation quantization

4.1. **Definition and first examples.** We propose to study non-formal star products $\star_\hbar$ given by the sort of semi-classical Fourier integral operators (SCFIO) studied in semi-classical analysis. In this approach a tentative definition go as follows, where we assume for simplicity that $M$ is compact.

A **non-formal deformation quantization** of a Poisson manifold $(M,\{\cdot,\cdot\})$ consists of a bilinear operation $\star_\hbar$ of the form

$$f_1\star_\hbar f_2 := F_\hbar(f_1\otimes f_2),$$

where $F_\hbar$ is a SCFIO of order $k=\frac{\dim M}{4}$, satisfying the following axioms:

(A1) *Deformation of usual product:* For any $f_1,f_2\in C^\infty(M)$:

$$f_1\star_\hbar f_2 - f_1f_2 = O(\hbar).$$

(A2) *Correspondence principle:* For any $f_1,f_2\in C^\infty(M)$:

$$\{f_1,f_2\} - \frac{1}{i\hbar}(f_1\star_\hbar f_2 - f_2\star_\hbar f_1) = O(\hbar).$$

(A3) *Associativity:* For any $h$-dependent families $f_1$, $f_2$ and $f_3$, one has

$$(f_1\star_\hbar f_2)\star_\hbar f_3 - f_1\star_\hbar(f_2\star_\hbar f_3) = O(\hbar^\infty).$$

(A4) *Identity:* For any $h$-dependent family $f$:

$$(1\star_\hbar f - f) = O(\hbar), \quad (f\star_\hbar 1 - f)|_U = O(\hbar).$$

Note that the notation $O(\hbar)$ in the previous axioms has a different meaning from the same notation used in the formal case, in Section 3.1. Here $O(\hbar)$ and $O(\hbar^\infty)$ should be taken for norms in appropriate function spaces. Also, in the last two axioms the equality holds microlocally in a neighborhood of the zero section.

We will say that such non-formal deformation quantization is *natural* when the left and right translation operators, $L_f(g) := f \star_\hbar g$ and $R_f(g) := g \star_\hbar f$, are (semi-classical) pseudo-differential operators of order zero on $M$, microlocally near the zero section. Our assumption about the order of the SCFIO $F_\hbar$ in the definition above is necessary for $L_f$ and $R_f$ to have order 0. We shall only consider non-formal deformation quantizations which are natural.

Underlying any SCFIO $F_\hbar$, and hence also a non-formal deformation quantization, one has canonical relation $\Lambda : M \times M \dashrightarrow M$, i.e., a Lagrangian submanifold

$$\Lambda \subset \overline{T^*M \times T^*M} \times T^*M.$$

Let us give two paradigmatic examples.

**Example 14.** The very first example arises, of course, from the Weyl quantization of the canonical Poisson bracket on $M = \mathbb{R}^{2n}$. The corresponding star products can be expressed as SCFIO by

$$f_1 \star_\hbar f_2(x,p) := \frac{1}{(\pi\hbar)^{2d}} \int_{\alpha_1,\alpha_2 \in \mathbb{R}^{2d}} f_1(\alpha_1) f_2(\alpha_2) e^{\frac{i}{\hbar}S} \mathrm{d}\alpha_1 \mathrm{d}\alpha_2$$

where $\alpha = (x,p)$ and

$$S(\alpha,\alpha_1,\alpha_2) := \int_{\Delta(\alpha,\alpha_1,\alpha_2)} 4\omega_{\mathrm{can}}.$$

with $\Delta(\alpha,\alpha_1,\alpha_2) \subset \mathbb{R}^{2d}$ the euclidean triangle with vertices $\alpha$, $\alpha_1$ and $\alpha_2$.

**Example 15.** A more instructive example, where one already sees the need to work microlocally, is given by the linear Poisson structure on the dual of a Lie algebra $M = \mathfrak{g}^*$. Following an idea of Rieffel [24], one can proceed as follows. Let $0 \in U \subset \mathfrak{g}$ be an open set where the exponential map $\exp : \mathfrak{g} \to G$ restricts to a diffeomorphism onto its image, so $U$ becomes a local Lie group with product denoted $\bullet$. We then choose open balls $D \subsetneq D' \subset U$ centered at 0 such that $D^3 \subset U$, $(D')^2 \subset U$. Also, we choose a Haar density on $G$ and transport it to a density $\mu$ on $U$ using the exponential map. After a possible rescaling, one find that:

$$\mu(p) = \det\left(\tfrac{I - e^{-\mathrm{ad}_p}}{\mathrm{ad}_p}\right) \mathrm{d}p,$$

where $\mathrm{d}p$ denotes the Lebesgue density on $\mathfrak{g}$. Then we set:

$$f_1 \star_\hbar f_2 := \tfrac{1}{(2\pi\hbar)^{n/2}}\, \mathfrak{F}_\hbar^{-1}[(\rho \mathfrak{F}_\hbar(f_1)) \cdot_U (\rho \mathfrak{F}_\hbar(f_2))]$$

where $\mathfrak{F}_\hbar$ denotes the semi-classical Fourier transform, $\rho \in C_c^\infty(U)$ is a cut-off function with $\rho|_D \equiv 1$, and $\cdot_U$ denotes the convolution in the local group $U$ with respect to $\mu$. Explicitly, we find the oscillatory integral expression

$$f_1 \star_\hbar f_2(x) = \frac{1}{(2\pi\hbar)^{n/2}} \int_{x_1,x_2} \int_{p,\tilde{p}\in U} f_1(x_1) f_2(x_2) e^{\frac{i}{\hbar}S} \rho(\tilde{p})\rho(\tilde{p}^{-1} \bullet p)\mu(\tilde{p}) \mathrm{d}x_1 \mathrm{d}x_2,$$

where:

$$S(p,x,x_1,x_2) := px - \tilde{p}x_1 - (\tilde{p}^{-1} \bullet p)x_2.$$

If these expressions did not include a cutoff function $\rho$, we would need the functions $f_i$ to have Fourier transform $\mathfrak{F}_\hbar(f_i)$ with support contained in $D$ so the expressions make sense. This would allow to have the axioms for $\star_\hbar$ be satisfied on the nose, but would greatly limit the domain of $\star_\hbar$. The above choice which includes the cutoff makes the domain bigger while the axioms for $\star_\hbar$ only hold microlocally.

There are other examples of Poisson manifolds that admit these type of star products. In general, the question of which Poisson maniifolds admit such non-formal deformation quantization is widely open.

Our first main (conjectural) result concerns the Lagrangian submanifold $\Lambda \subset T^*(M \times M \times M)$ underlying the SCFIO defining $\star_\hbar$ in the definition above.

**Theorem 16** (conjecture). *If $\star_\hbar$ is a non-formal star product deforming a Poisson manifold $(M,\pi)$ then $\Lambda$ is the graph of multiplication of a local symplectic groupoid integrating $(M,\pi)$.*

This result is the non-formal analogue of Theorem 13. The proof should use the calculus of semi-classical wavefront sets as developed in [1] and the symbol calculus for semi-classical pseudo-differential operators (see, e.g., [15]) applied to the left and right translation operators, $L_f$ and $R_f$.

Our main motivation to propose this semi-classical approach is that it is general enough to properly quantize many Poisson manifolds. Still, we also expect it to be related to formal deformation quantization as follows.

**Theorem 17** (conjecture). *Let $\star_\hbar$ is a non-formal star product deforming a Poisson manifold $(M,\pi)$. For every $x_0 \in M$ and every $f,g \in C_c^\infty(M)$ with small enough support around $x_0$, $f \star_\hbar g$ extends smoothly to $\hbar = 0$ and its Taylor series at $\hbar = 0$*

$$\mathscr{T}_\varepsilon(f \star_\hbar g(x_0)) = \sum_{n\geq 0} \varepsilon^n B_n(f,g)|_{x_0} \tag{11}$$

*defines a natural formal star product $\star_\varepsilon$ on $C^\infty(M)[[\varepsilon]]$ quantizing $(M,\pi)$.*

The proof of this result should follow by using the calculus of wavefront sets and the law of composition of SCFIOs, to conclude that $f \star_\hbar g$ is of degree 0 and hence extends smoothly to $\hbar = 0$. The axioms (A1)-(A4) in definition will then imply that (11) satisfies the axioms of a formal start product.

The definition of non-formal start product $\star_\hbar$ above also yields a *partial algebra* $(\mathscr{A}_W, \star_\hbar)$, where $W \subset T^*M$ is a neighborhood of $0_M$ and

$$\mathscr{A}_W := \{f : \mathrm{WF}_\hbar(f) \subset W\}/O(\hbar^\infty).$$

Here $\mathrm{WF}_\hbar(f)$ denotes the semi-classical wave front set of the $\hbar$-dependent family $f$ (see [33]). We abuse notation and we still use the symbol $\star_\hbar$ for multiplication of equivalence classes in $\mathscr{A}_W$. Appealing to a microlocal continuity property, it is a partially defined product with domain:

$$\mathrm{Dom}(\mathscr{A}_W, \star_\hbar) := \{(f,g) : \mathrm{WF}_\hbar(f \star_\hbar g) \subset W\}.$$

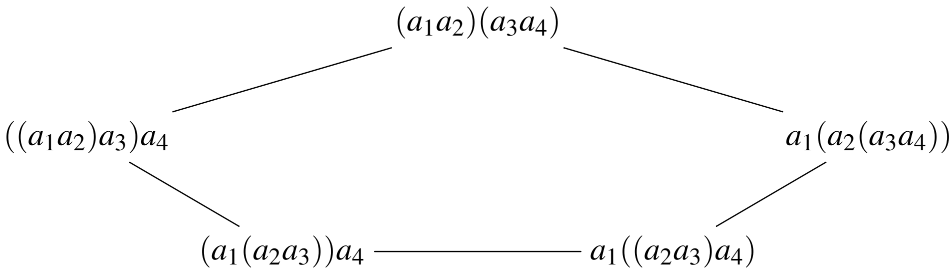
We say that $\star_\hbar$ is a *strict non-formal star product* if there exists $W$ for which $\mathscr{A}_W$ embeds in an algebra. Our third main (conjectural) result is the following theorem relating existence of strict deformation quantizations and integrability of a Poisson structure:

**Theorem 18** (conjectural)**.** *If $(M, \pi)$ admits a strict non-formal deformation quantization then it is integrable by a symplectic groupoid.*

In order to explain why this should hold, recall that in a partial algebra the associativity property reads:

$$a_1 \cdot (a_2 \cdot a_3) = (a_1 \cdot a_2) \cdot a_3, \quad \textit{provided both sides are defined}.$$

This *does not* imply that higher associativities hold. For example, consider all possible products of 4 elements:




Each edge represents a move that uses only the 3-associativity property. In an algebra all vertices of the pentagon are defined and the 3-associativity implies 4-associativity. In a partial algebra it is possible that, for example, $(a_1a_2)(a_3a_4)$ and $(a_1(a_2a_3))a_4$ are defined but none of the other vertices are defined, and then 3-associativity does not allow one to conclude that 4-associativity holds. For a partial algebra to embed in an algebra $n$-associativity must hold for all $n$, in which case one says that the algebra is *globally associative*.

A similar situation happens with local Lie groups and local Lie groupoids. A classical theorem of Mal'cev essentially says that a local Lie group embedds in a global Lie group if and only if the multiplication is globally associative (see [23]). This results was generalized to Lie groupoids in [14] and for the proof of Theorem 18 one needs to show that if $\star_\hbar$ is a strict non-formal star product, then the local symplectic groupoid given by Theorem

16 is globally associative. In [14] the precise relationship between the failure of $n$-associativity and the obstructions to integrability found in [10, 11] is also established. From it, one should also be able to obtain the following answer to Rieffel's question mentioned in the Introduction:

> **Conjecture:** If $(M,\pi)$ admits a *strict deformation quantization*, then it must be integrable by a symplectic groupoid. In particular, the obstruction theory of [10, 11] applies.

Since Rieffel's strict deformation quantization is by $C^*$-algebras, one needs to investigate the relationship with the notion of non-formal deformation quantization above.

Our tentative results also place our theory right at the cross paths of older attempts by Maslov & Karasëv [19], Weinstein [29] and Zakrzewski [31, 32], to construct non-formal deformation quantizations by using symplectic groupoids. In this direction, Hawkins [17] proposed that a Poisson manifold may be quantized by a twisted polarized convolution $C^*$-algebra of a symplectic groupoid. The most important remaining problem is, as already, mentioned

> **Open Problem:** How can one construct a non-formal deformation quantization of an integrable Poisson manifold by a star product of SC-FIO type?

In a series of works, Cattaneo, Dherin and Weinstein [4, 5, 6, 7] also use a semi-classical approach, albeit different from ours (e.g., in [7, Sec. 5.7], the domain of $\star_\hbar$ does not include oscillatory functions and the canonical relations underlying SCFIOs must satisfy extra properties from the outset). Their techniques should be very helpful to make progress in this problem.

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