INTEGRABILITY OF THE PERIODIC KM SYSTEM *

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We study the integrability of the periodic Kac-van Moerbeke system. We give a bi-hamiltonian formulation and a Lax pair containing a spectral parameter. Using Griffiths aproach we linearize the system on the Jacobian of the associated spectral curve.

1. The KM system

In two seminal papers [9] for the modern theory of integrable systems Kac and van-Moerbeke introduced the system of o.d.e.'s

$$\dot{u}_i = e^{u_{i+1}} - e^{u_{i-1}}\,, \qquad i = 1, \dots, n, \tag{1}$$

where formally $e^{u_0} = e^{u_{n+1}} = 0$. They showed, for example, that this system arises as a finite-dimensional aproximation of the famous KdV equations. Shortly after, Moser [10] showed that this system can be related to the classical Toda lattice, a rather well studied system. This meant also that interest shifted towards this latter system.

Let us observe that system (1) under the change of variable $u_i \mapsto x_i = e^{u_i}$ is mapped to:

$$\dot{x}_i = x_i(x_{i+1} - x_{i-1})\,, \qquad i = 1, \dots, n, \tag{2}$$

($x_{n+1} = x_0 = 0$). This is a *Lotka-Volterra system*, a classs of systems first studied by Volterra in his famous monograph [14]. There, Volterra introduced general systems of o.d.e's of the form

$$\dot{x}_j = \varepsilon_j x_j + \frac{1}{\beta_j}\sum_{k=1}^{n} a_{jk} x_j x_k\,, \qquad j = 1, \dots, n, \tag{3}$$

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to describe the evolution of n biological species. System (2) corresponds to the case $\varepsilon_j = 0$, $\beta_j = 0$ and (a_{ij}) skew-symmetric. We feel that because of this connection systems of type (2) deserve more attention.

From a dynamical point of view, system (2) is fairly simple: for particles labeled by odd indexes $x_i \to 0$ as $t \to +\infty$, while for particles labeled by even indexes x_i approaches some non-zero limit value as $t \to +\infty$ which depends on the initial conditions. When $t \to -\infty$ the behaviour is reversed, and this led to the integration of the system through the inverse scattering method [10]. A more interesting system, to be discussed in this paper, is obtained by considering the periodic KM system where one identifies indices (mod n). General systems with $\varepsilon_j \neq 0$ have a much more complex behaviour and fall outside the scope of the theory of integrable systems (see [5, 6]).

Let us then consider the system of o.d.e.'s

$$\left\{ \begin{array}{rcl} \dot{x}_i &=& x_i(x_{i+1} - x_{i-1}), \\ x_{i+n} &=& x_i , \end{array} \right. \qquad i = 1, \ldots, n. \tag{4}$$

It is easy to check that the quantity

$$I = \sum_{i=1}^{n} (x_i - \log x_i)$$

is a first integral of the system. Moreover, the point $\boldsymbol{q} = (1, \ldots, 1)$ is an equilibrium and in a neihgborhood of $\boldsymbol{q}$ the level sets of I are $(n-1)$-dimensional spheres. This follows from the relations

$$\begin{array}{rcl} \left(\dfrac{\partial h}{\partial x_i}\right)_{\boldsymbol{q}} &=& \left(1 - \dfrac{1}{x_i}\right)_{\boldsymbol{q}} = 0, \\ \left(\dfrac{\partial^2 h}{\partial {x_i}^2}\right)_{\boldsymbol{q}} &=& \left(\dfrac{1}{x_i^2}\right)_{\boldsymbol{q}} = 1, \\ \left(\dfrac{\partial^2 h}{\partial x_i x_j}\right)_{\boldsymbol{q}} &=& 0, \end{array}$$

and the Morse lemma. The assymptotic behaviour in the periodic case is therefore more complicated than the one in its non-periodic counterpart, and this leads to the use of algebraic-geometric methods to integrate the system rather than the inverse scattering method. In this paper we shall give, for the periodic KM system, a Lax pair depending on a spectral parameter which allows us to linearize the system on the Jacobian of the associated spectral curve. Note that linearizing the system is a task that is not accomplished by showing that the system is completely integrable: it is usually rather difficult to compute explicitely action-angle variables (and this is in fact the case for the KM system).

The paper is organized as follows: in Section 2, we give a bi-hamiltonian formulation which is an extension of the bi-hamiltonian formulation given in [4] for the non-periodic case (but valid only for odd dimension). Also, we introduce a Lax pair depending on a spectral parameter. In Section 3, we study the spectral curve associecited with this Lax pair. In section 4, we use Griffiths method [8] to linearize the system on the Jacobian of the spectral curve.

2. Bi-Hamiltonian formulation and Lax formulation

Some results for the non-periodic case extend without difficulty to the periodic case. For example, system (4) has a Hamiltonian formulation. The Poisson bracket is quadratic and is defined by (recall the identification $x_{i+n} = x_i$)

$$\{x_i, x_{i+1}\}_2 = x_i x_{i+1} \,, \tag{5}$$

with all other brackets $= 0$, while the Hamiltonian function is given by

$$h_1 = \sum_{i=1}^{n} x_i \,. \tag{6}$$

Then the equations for the periodic KM system take the Hamiltonian form

$$\dot{x_i} = \{x_i, h_1\}_2 \,, \qquad i = 1, \ldots, n. \tag{7}$$

This is actually a special case of a general Hamiltonian formulation for system (3) (for details see [11, 5, 6]).

For the non-periodic KM system a bi-Hamiltonian formulation was given in [4] valid only when n is odd. It is interesting to note that this bi-Hamiltonian formulation works for any dimension in the periodic case. It is obtained as follows: one takes as a second Poisson bracket the cubic bracket defined by

$$\{x_i, x_{i+1}\}_3 = x_i x_{i+1}(x_i + x_{i+1}) \,, \qquad \{x_i, x_{i+2}\}_3 = x_i x_{i+1} x_{i+2} \,. \tag{8}$$

It is easy to see that this bracket satisfies the Jacobi identity, and that $\{\ ,\ \}_2 + \{\ ,\ \}_3$ also does (same argument as in [4]). In other words, the quadratic bracket (5) and the cubic bracket (8) are compatible Poisson brackets. Moreover, if we let

$$h_0 = \sum_{i=1}^{n} \log x_i \,, \tag{9}$$

we see that the equations for the peridic KM system are

$$\dot{x_i} = \{x_i, h_1\}_2 = \{x_i, h_0\}_3 \,, \qquad i = 1, \ldots, n. \tag{10}$$

Therefore the system has a bi-Hamiltonian formulation.

Note that both the quadratic and the cubic brackets are degenerate: h_0 is a Casimir for $\{\ ,\ \}_2$ while h_1 is a Casimir for $\{\ ,\ \}_3$, so the bi-Hamiltonian formulation (10) does not give the integrability of the system. One can probably use master symmetries as in [7] to construct a sequence of commuting first integrals, and therefore prove the complete integrability. We shall use instead the Lax pair aproach which has the advantage of leading to the linearization of the system.

Recall that we say that a system admits a Lax pair representation (with spectral parameter) if it can be written in the form

$$\frac{dA(\lambda)}{dt} = [A(\lambda), B(\lambda)] \ , \tag{11}$$

where $A(\lambda)$ and $B(\lambda)$ denote Laurent polynomials with matrix coefficients:

$$A(\lambda) = \sum_{k=-p}^{q} A_k \lambda^k, \tag{12}$$

$$B(\lambda) = \sum_{k=-r}^{s} A_k \lambda^k, \tag{13}$$

(one can consider more general reprensentations where A and B are defined on algebraic curves of genus $g \geq 1$, see [12]). For the system at hand we shall take the following pair:

$$A(\lambda) = \begin{bmatrix} 0 & \sqrt{x_1} & & \cdots & & \lambda^{-1}\sqrt{x_n} \\ \sqrt{x_1} & 0 & \sqrt{x_2} & & & \\ & \sqrt{x_2} & & \ddots & & \vdots \\ \vdots & & \ddots & & \ddots & \\ & & & \ddots & 0 & \sqrt{x_{n-1}} \\ \lambda\sqrt{x_n} & & \cdots & & \sqrt{x_{n-1}} & 0 \end{bmatrix},$$

$$B(\lambda) = \begin{bmatrix} 0 & 0 & \sqrt{x_1x_2} & & \cdots & -\lambda^{-1}\sqrt{x_{n-1}x_n} & 0 \\ 0 & 0 & 0 & \ddots & & & -\lambda^{-1}\sqrt{x_1x_n} \\ -\sqrt{x_1x_2} & 0 & 0 & & \ddots & & \vdots \\ & \ddots & & \ddots & & \ddots & \\ \vdots & & \ddots & & 0 & 0 & \sqrt{x_{n-2}x_{n-1}} \\ \lambda\sqrt{x_{n-1}x_n} & & & \ddots & 0 & 0 & 0 \\ 0 & \lambda\sqrt{x_1x_n} & \cdots & & -\sqrt{x_{n-2}x_{n-1}} & 0 & 0 \end{bmatrix}.$$

This pair is a generalization of the pair given in [4] for the non-periodic KM system (in this case there is no spectral parameter). Note that this pair can also be written in terms of a system of simple roots for the affine Lie algebra $A_{n-1}^{(1)}$ which leads to a Lie-algebraic intrepertation of the Lax pair (this remark deserves more attention and will be explored elsewhere).

Given a system in Lax form (11) we introduce, as usual, the characteristic polynomial defined by

$$Q(\lambda,\mu)=\det\left(A(\lambda)-\mu I\right). \tag{14}$$

It is a standard result that the characteristic polynomial $Q(\lambda,\mu)$ is independent of time, i. e., the flow $t\mapsto A(\lambda)(t)$ is isoespectral. Therefore, the coefficientes of the characteristic polynomial $Q(\lambda,\mu)$ (or the traces $\mathrm{tr}A^k$) give first integrals of the original system. For the Lax pair of the Volterra system the characteristic polynomial takes the form

$$Q(\lambda,\mu)=K(\lambda+\lambda^{-1})+(-1)^n\sum_{j=0}^{n}I_j\mu^j, \tag{15}$$

where

$$K=(-1)^{n+1}\sqrt{\prod_{j=1}^{n}x_i}=(-1)^{n+1}\exp\left(\frac{h_0}{2}\right), \tag{16}$$

and I_j are certain polynomial functions of $x_1,\dots,x_n$, satisfying $I_n=1$ and $I_j=0$ iff $n-j$ is odd.

For a non-degenerate Poisson bracket one says that the system is completely integrable if there are $n/2$ independent integrals in involution. Then the Arnol'd-Liouville theory applies (see [3]). For a degenerate Poisson bracket one has to take into account the Casimirs. Now, for the periodic KM system, we see that the number of functionally independent Casimirs of the quadratic bracket is 1 or 2 depending if n is odd or even. On the other hand, K is a Casimir and the I_j give $[n/2]$ functionally independent integrals in involution with respect to the quadratic Poisson bracket. Theferore, we can say that the Lax pair gives the complete integrability of the system, and in fact we could use, in principle, the Arnol'd-Liouville theorem on each symplectic leave. The problem of course is that the action-angle coordinates are not easy to compute.

3. The spectral curve

While the dependence of the Lax pair in the spectral parameter λ is not relevant to prove the complete integrability, it is crucial for the linearization of the system, a remark that goes back to the fundamental papers of Adler and van Moerbeke [1, 2].

Let us fix a level set of the non-zero first integrals $I_1,\dots,I_m$ appearing as coefficients of the characteristic polynomial

$$I_{\boldsymbol{c}}=\{\boldsymbol{x}\mid I_i(\boldsymbol{x})=c_i,\quad i=1,\dots,m\}. \tag{17}$$

For each value of $\boldsymbol{c}=(c_1,\dots,c_m)$ we have a specific polynomial $Q(\lambda,\mu)$. In the rest of the exposition we consider a generic value[1] of $\boldsymbol{c}$.

For $\lambda\neq 0,\infty$ the set of pairs (λ,μ) satisfying the characteristic equation forms the plane algebraic curve

$$\check{C}_0=\{(\lambda,\mu)\in\mathbb{C}\mid\lambda\neq 0,\quad Q(\lambda,\mu)=0\}.$$

[1] Henceforth, by "generic" we mean on a Zariski open set.

Each pair $(\lambda,\mu) \in \check{C}_0$ gives an eigenvalue μ of the matrix $A(\lambda)$. Generically, μ is a simple eigenvalue, so that for all but a finite number of points the associated eigenspace $L_{(\lambda,\mu)}$ has dimension 1. We would like to consider the line bundle $L \to \check{C}_0$ defined as the sub-bundle of the trivial bundle $\check{C}_0 \times \mathbb{C}^N \to \check{C}_0$ whose fiber over a point (λ,μ) is $L_{(\lambda,\mu)}$. The problem is that $\dim L_{(\lambda,\mu)} = 1$ generically, so there are points were the fibers will have $\dim > 1$. Also, $\check{C}_0$ is not compact, which can cause trouble. To circunvent these problems we take the completion C_0 of $\check{C}_0$ in $\mathbb{P}^2$. This is the projective algebraic curve which coincides with $\check{C}_0$ in $\mathbb{C}^2 - \{\lambda = 0\}$, i.e., the curve

$$C_0 = \left\{ [\zeta_0, \zeta_1, \zeta_2] \in \mathbb{P}^2 \mid {\zeta_0}^n \zeta_1 Q\left(\frac{\zeta_1}{\zeta_0}, \frac{\zeta_1}{\zeta_0}\right) = 0 \right\} . \tag{18}$$

Now we need to get rid of the points with multiplicity greater than one so we consider the normalization C of C_0. Recall that this is a compact Riemann surface C, together with a holomorphic map $\sigma : C \to C_0$, such that if C_{sing} denotes the set of singular points of C_0 the restriction

$$\sigma : C - \sigma^{-1}(C_{\text{sing}}) \to C_0 - C_{\text{sing}}$$

is biholomorphic. It follows from standard results in algebraic geometry that the line bundle $L \to C$, which is only defined in C minus a finite number of points, can be extended to a holomorphic line bundle in all of C. Usually, one calls the normalization C the *spectral curve* associated to the Lax pair (11).

PROPOSITION 3.1 . *C is a hyperelliptic Riemann surface of genus $n-1$.*

Proof: Let $\Sigma = \sum_{j=0}^{n} I_j \mu^j$. If we denote by μ_i $(i = 1, \ldots, 2n)$ the roots of the polynomial $\Sigma^2 - 4K^2$, we see that the curve takes the canonical form

$$\tilde{\lambda}^2 = \sum_{i=1}^{2n} (\mu - \mu_i) ,$$

where $\tilde{\lambda} = 2K\lambda + \Sigma$. Hence the genus of C is $g = n-1$. □

Hence, for each $\boldsymbol{x} \in I_{\boldsymbol{C}}$ we have a line bundle $L \to C$, and therefore a holomorphic map [2]

$$\Phi : I_{\boldsymbol{C}} \to \text{Pic}(C) \tag{19}$$

In fact, the image of this map lies in $\text{Pic}^d(C)$ for some integer d. Now recall that

$$\begin{aligned} \text{Pic}^d(C) &\simeq \text{Pic}^0(C) && \text{(using } L \mapsto L \otimes L_0^{-1}\text{)} \\ &\simeq H^1(\mathcal{O}_C)/H^1(C,\mathbb{Z}) && \text{(using the exponential sheaf sequence),} \end{aligned}$$

where $\mathcal{O}_C$ denotes the sheaf of holomorphic sections on C. Therefore, we see that we can identify the tangent space to $\text{Pic}^d(C)$ at a given point with $H^1(\mathcal{O}_C)$. The problem is now to compute the derivative $\dot{\Phi}$ and verify that it is constant.

[2] As usual, we denote by $\text{Pic}(C)$ (resp. $\text{Pic}^d(C)$) the Picard group of holomorphic line bundles (resp. line bundles with degree d) over the curve C. Also, $J(C) = \text{Pic}^0(C)$ denotes the Jacobi variety.

Before we turn to the computation of $\dot{\Phi}$ we remark that the spectral curve C admits an involution $\sigma : C \to C$ defined by

$$\begin{aligned} \sigma([1,\lambda,\mu]) &= [1,(-1)^n\lambda,-\mu], \\ \sigma(p_1) &= p_1, \\ \sigma(p_2) &= p_2\,, \end{aligned} \tag{20}$$

where $p_1 = [0,1,0]$ and $p_2 = [0,0,1]$. This follows because of the following symmetry of the characteristic polynomial

$$Q(\lambda,\mu) = (-1)^n Q((-1)^n\lambda,-\mu)\,.$$

This involution always has p_1 and p_2 as fixed points. If n is odd there are no other fixed points and if n is even there are two other fixed points. From Proposition 3. and the Riemann-Hurwitz formula we conclude that the quotient curve $C' = C/\sigma$ has genus

$$g' = \left[\frac{n-1}{2}\right]. \tag{21}$$

4. Linearization of the flow

The problem of computing $\dot{\Phi}$ was effectively solved by Griffiths in [8]. To state Griffiths theorem we need to recall the definition of the *residue of* B. Let $D > 0$ be an effective divisor such that $B(\lambda) \in \mathrm{Hom}(\mathrm{V},\mathrm{V(D)})$ and let $\boldsymbol{v} = \boldsymbol{v}(\lambda,\mu)$ be an eingenvector of $A(\lambda)$ with eigenvalue μ holomorphic in a neigbourhood of D. Differentiating $A(\lambda)\boldsymbol{v} = \mu\boldsymbol{v}$ with respect to time we see that $B(\lambda)\boldsymbol{v} + \dot{\boldsymbol{v}}$ is also an eigenvector of $A(\lambda)$ with eigenvalue μ, hence there is a function ψ such that

$$B(\lambda)\boldsymbol{v} = \dot{\boldsymbol{v}} + \psi\boldsymbol{v}\,. \tag{22}$$

The residue of B is defined to be the section $\rho(B) \in H^0(\mathcal{O}_C(D)/\mathcal{O}_C)$ induced by ψ.

Then we have the following result [8]:

THEOREM 4.1. *Let $D > 0$ be an effective divisor such that $B(\lambda) \in \mathrm{Hom}(\mathrm{V},\mathrm{V(D)})$ and consider the short exact sheaf sequence*

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_C(D) \longrightarrow \mathcal{O}_C(D)/\mathcal{O}_C \longrightarrow 0$$

which induces the exact sequence on cohomology

$$H^0(\mathcal{O}_C(D)) \longrightarrow H^0(\mathcal{O}_C(D)/\mathcal{O}_C) \xrightarrow{\partial} H^1(\mathcal{O}_C).$$

If $\boldsymbol{x}(t)$ is a solution of system (11)*, then*

$$\dot{\Phi} = \partial\rho(B)\,. \tag{23}$$

Let $p_1 = [0,1,0]$ and $p_2 = [0,0,1]$ be points in the spectral curve C associated with the periodic KM system. In a neigbourhood of p_1 the following parametrization is valid

$$z \mapsto [-Kz^n, 1, Kz^{n-1} + \mathcal{O}(z^{n-1})]\,, \tag{24}$$

while in a neigbourhood of p_2 the following parametrization is valid

$$z \mapsto [z, \frac{K}{\Sigma} z^{n+1} + \mathcal{O}(z^{3n+1}), 1]\,, \tag{25}$$

and these give the following divisors:

$$\begin{aligned} (\mu) &= -p_1 - p_2 + 2 \text{ zeros}\,, \\ (\lambda) &= -np_1 + np_2\,. \end{aligned}$$

Also, if we denote by Δ_{ij} the (i,j)-minor of the matrix $A(\lambda) - \mu I$, we see that $\boldsymbol{v} = (v_1, \ldots, v_n)$, where

$$v_k = \lambda \frac{\Delta_{nk}}{\Delta_{nn}}\,,$$

is an eigenvector of $A(\lambda)$ with eigenvalue μ. From (24) and (25) we compute the divisors

$$(v_k)_{\{p_1,p_2\}} = -kp_1 + kp_2\,. \tag{26}$$

Proposition 4.1. *Let $D = np_1 + np_2$. There exist local coordinates z_1 and z_2 near p_1 and p_2 such that the residue of B at D is given by*

$$\rho(B) = \frac{1}{{z_1}^2} - \frac{1}{{z_2}^2}\,.$$

Proof: We first look in a neigbourhood of $p_2 \in C$. According to (26) the eigenvector $\boldsymbol{v}$ is holomorphic in a neigbourhood of p_2. The eigenvector equation

$$A(\lambda)\boldsymbol{v} = \lambda \boldsymbol{v}\,,$$

gives the following relations among the components of $\boldsymbol{v}$:

$$\begin{aligned} -\mu v_1 + \sqrt{x_1} v_2 + \sqrt{x_n} &= 0\,, && && (27) \\ \sqrt{x_{k-1}} v_{k-1} - \mu v_k + \sqrt{x_k} v_{k+1} &= 0\,, && k = 2, \ldots, n-1, && (28) \\ \lambda \sqrt{x_n} v_1 + \sqrt{x_{n-1}} v_{n-1} - \mu\lambda &= 0\,, && && (29) \end{aligned}$$

To compute the residue of B, we have to determine ψ satisfying

$$B(\lambda)\boldsymbol{v} = \psi \boldsymbol{v} + \text{holomorphic terms.}$$

It is easy to see that the components of $(B(\lambda)\boldsymbol{v})_i$ are holomorphic except when $i = 1$. For this one we find, using (27) and (29),

$$\begin{aligned} (B(\lambda)\boldsymbol{v})_1 &= -\frac{1}{\lambda}\sqrt{x_{n-1}x_n}\, v_{n-1} + \sqrt{x_1 x_2}\, v_3 \\ &= -\mu\sqrt{x_n} + x_n v_1 + \sqrt{x_1 x_2}\, v_3 \\ &= -\mu^2 v_1 + \sqrt{x_1} v_2 + x_n v_1 + \sqrt{x_1 x_2}\, v_3 \\ &= -\mu^2 v_1 + \text{holomorphic terms.} \end{aligned}$$

Therefore, in a neigbourhood of p_2 we have

$$B(\lambda)\boldsymbol{v} = -\mu^2\boldsymbol{v} + \text{holomorphic terms}\,. \tag{30}$$

Now we look in a neighbourhood of $p_1 \in C$. The eigenvector $\boldsymbol{v}' = \frac{1}{\lambda}\boldsymbol{v}$ is holomorphic in a neighbourhood of p_1:

$$(v'_k)_{\{p_1,p_2\}} = (n-k)p_1 + (k-n)p_2\,. \tag{31}$$

Again, we compute the components of $B(\lambda)\boldsymbol{v}'$ and we see that all execpt $(B(\lambda)\boldsymbol{v})_{n-1}$ and $(B(\lambda)\boldsymbol{v})_n$ are holomorphic. For these two we find using (27), (28) and (29) (similar computations as above)

$$\begin{aligned}(B(\lambda)\boldsymbol{v}')_{n-1} &= (-1)^{n+1}K\frac{\lambda}{\mu^{n-2}}v'_{n-1} + \text{holomorphic terms}\,,\\ (B(\lambda)\boldsymbol{v}')_n &= \mu^2 v'_n + \text{holomorphic terms}\,.\end{aligned}$$

Also, from the equation for the curve, we have in a neigbourhood of p_1

$$\begin{aligned}(-1)^{n+1}K\frac{\lambda}{\mu^{n-2}} &= (-1)^n K\frac{1}{\lambda\mu^{n-2}} + \sum_{j=0}^{n} I_j\mu^{j-n+2}\\ &= \mu^2 + \text{holomorphic terms}\,.\end{aligned}$$

so we conclude that in a neigbourhood of p_1

$$B(\lambda)\boldsymbol{v}' = \mu^2\boldsymbol{v}' + \text{holomorphic terms.} \tag{32}$$

The proposition now follows from the definition of $\rho(B)$ and relations (30) and (32). □

Applying Griffiths theorem with $D = np_1 + np_2$ and using Proposition 4.1 we obtain that $\dot{\Phi}$ is constant, hence:

Corollary 4.1. *The KM periodic system linearizes on the Jacobian $J(C)$ of the spectral curve C associated with the system.*

Now the Jacobi variety $J(C)$ can be identified as the torus

$$J(C) \simeq H^0(\Omega_C)^*/H_1(C,\mathbb{Z}) \qquad \text{(using Abel's theorem)},$$

and so the linear flow is given, up to translation, by

$$(t,\omega) \mapsto t\langle\xi,\omega\rangle, \qquad \xi \in H^0(\Omega_C)^*.$$

According to Corollary 7.10 in [8], this map actually has the explicit form

$$(t,\omega) \mapsto t\sum_i \mathrm{Res_{p_i}}(\rho(\mathrm{B})\omega)\,. \tag{33}$$

The reader might notice that the Jacobi variety $J(C)$ is a torus of dimension $n-1$ while, according to the counting in section 2, the number of functionally independent integrals in involution is $[n/2]$ (discarding the Casimir K). This can be explained by factoring through the involution $\sigma : C \to C$ defined in (20). It is easy to check that

$$\sigma(\rho(B)) = -\rho(B)\,,$$

and by (33) this shows that the linear flow is actually trivial in the $+1$-eigenspace

$$H^0(\Omega_C)^+ = \{\omega \in H^0(\Omega_C) : \sigma^*\omega = \omega\}.$$

If we take the quotient $C' = C/\sigma$ we obtain a double covering $C \to C'$ with associated Prym variety defined by

$$\mathrm{Prym(C/C')} = \mathrm{H^0(\Omega_C)}^{-*}/\mathrm{H_1(C,\mathbb{Z})^-},$$

a torus of dimension $g-g'$. Then the flow induced in $\mathrm{Prym(C/C')}$ is linear and, according to (21), has the right dimension $[n/2]$. In conclusion we have:

Corollary 4.2. *The KM periodic system linearizes on a Prym variety associated with spectral curve C of dimension $[n/2]$.*

As a final note we mention that the flow can also be linearized using the direct aproach of Adler and van Morbeke (see [13] for details).

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