A NOTE ON PROPER POISSON ACTIONS

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ABSTRACT. We show that the fixed point set of a proper action of a Lie group G on a Poisson manifold M by Poisson automorphisms has a natural induced Poisson structure and we give several applications.

1. INTRODUCTION

In the present work, we consider a *Poisson action* $G \times M \to M$ of a Lie group G on a Poisson manifold M: this means that each element $g \in G$ acts by a Poisson diffeomorphism of M. We recall that the action is called *proper* if the map:

$$G \times M \to M \times M, \quad (g, p) \mapsto (p, g \cdot p),$$

is a proper map[1]. As usual, we will denote by M^G the *fixed point set* of the action:

$$M^G = \{p \in M : g \cdot p = p, \forall g \in G\}.$$

For proper actions, the connected components of the fixed point set M^G are (embedded) submanifolds of M. Notice that these components may have different dimensions.

The main result of this paper is the following:

Theorem 1.1. *Let $G \times M \to M$ be a proper Poisson action. Then the fixed point set M^G has a natural induced Poisson structure.*

This result is a generalization to Poisson geometry of a well-known proposition in symplectic geometry, due to Guillemin and Sternberg (see [6], Theorem 3.5), stating that fixed point sets of symplectic actions are symplectic submanifolds. We stress that the fixed point set *is not* a Poisson submanifold. This happens already in the symplectic case. In the general Poisson case, M^G will be a Poisson-Dirac submanifold in the sense of Crainic and Fernandes (see [1], Section 8) and Xu ([11]).

Proper symplectic/Poisson actions have been study intensively in the last 15 years. For example, the theory of (singular) reduction for Hamiltonian systems has been developed extensively for these kind of actions. We refer the reader to the recent monograph by Ortega and Ratiu [7] for a nice survey of results in this area. Theorem 1.1 should have important applications in symmetry reduction, and this is one of our main motivations for this work. We refer the reader for an upcoming publication ([5]).

This paper is organized as follows. In Section 1, we recall the notion of a Poisson-Dirac submanifold, and some related results which are needed for the proof of Theorem 1.1. In Section 2, we prove our main result. In Section 3, we deduce some consequences and give some applications.

[1]A map $f : X \to Y$ between two topological spaces is called *proper* if for every compact subset $K \subset Y$, the inverse image $f^{-1}(K)$ is compact.

2. Poisson-Dirac submanifolds

Let M be a Poisson manifold. For background in Poisson geometry we refer the reader to Vaisman's book [10]. We will denote by $\pi \in \mathfrak{X}^2(M)$ the Poisson bivector field so that the Poisson bracket is given by:

$$\{f, g\} = \pi(\mathrm{d}f, \mathrm{d}g), \qquad \forall f, g \in C^\infty(M).$$

Recall that a Poisson submanifold $N \subset M$ is a submanifold which has a Poisson bracket and for which the inclusion $i : N \hookrightarrow M$ is a Poisson map:

$$\{f \circ i, g \circ i\}_M = \{f, g\}_N \circ i, \qquad \forall f, g \in C^\infty(N).$$

Such Poisson submanifolds are, in a sense, extremely rare. In fact, they are collections of open subsets of symplectic leaves of M.

Example 2.1. Let M be a symplectic manifold with symplectic form ω. Recall that a *symplectic submanifold* is a submanifold $i : N \hookrightarrow M$ such that the restriction $i^*\omega$ is a symplectic form on N. For every even dimension $0 \le 2i \le \dim M$ there are symplectic submanifolds of dimension $2i$. On the other hand, the only Poisson submanifolds are the open subsets of M.

Crainic and Fernandes in [1] introduce the following natural extension of the notion of a Poisson submanifold:

Definition 2.1. Let M be a Poisson manifold. A submanifold $N \subset M$ is called a **Poisson-Dirac submanifold** if N is a Poisson manifold such that:

(i) the symplectic foliation of N is $N \cap \mathcal{F} = \{L \cap N : L \in \mathcal{F}\}$, and
(ii) for every leaf $L \in \mathcal{F}$, $L \cap N$ is a symplectic submanifold of L.

Note that if $(M, \{\cdot, \cdot\})$ is a Poisson manifold, then the symplectic foliation with the induced symplectic forms on the leaves, gives a smooth (singular) foliation with a smooth family of symplectic forms. Conversely, given a manifold M with a foliation $\mathcal{F}$ furnished with a smooth family of symplectic forms on the leaves, then we have a Poisson bracket on M defined by the formula[2]

$$\{f, g\} \equiv X_f(g),$$

for which the associated symplectic foliation is precisely $\mathcal{F}$. Hence, a Poisson structure can be defined by specifying its symplectic foliation. It follows that a submanifold N of a Poisson manifold M has at most one Poisson structure satisfying conditions (i) and (ii) above, and this Poisson structure is completely determined by the Poisson structure of M.

Example 2.2. If M is a symplectic manifold, then there is only one symplectic leave, and the Poisson-Dirac submanifolds are precisely the symplectic submanifolds of M.

Therefore, we see that the notion of a Poisson-Dirac submanifold generalizes to the Poisson category the notion of a symplectic submanifold.

Example 2.3. Let L be a symplectic leaf of a Poisson manifold, and $N \subset M$ a submanifold which is transverse to L at some x_0:

$$T_{x_0}M = T_{x_0}L \oplus T_{x_0}N.$$

Then one can check that conditions (i) and (ii) in Definition 2.1 are satisfied in some open subset in N containing x_0. In other words, if N is small enough then it is a Poisson-Dirac submanifold. Sometimes one calls the Poisson structure on N

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[2]In a Poisson (or symplectic) manifold, we will denote by X_f the Hamiltonian vector field associated with a function $f : M \to \mathbb{R}$.

the transverse Poisson structure to L at x_0 (up to Poisson diffeomorphisms, this structure does not depend on the transversal N).

The two conditions in Definition 2.1 are not very practical to use. Let us give some alternative criteria to determine if a given submanifold is a Poisson-Dirac submanifold.

Observe that condition (ii) in the definition means that the symplectic forms on a leaf $L \cap N$ are the pull-backs $i^* \omega_L$, where $i : N \cap L \hookrightarrow L$ is the inclusion into a leaf and $\omega_L \in \Omega^2(L)$ is the symplectic form. Denoting by $\# : T^*M \to TM$ the bundle map determined by the Poisson bivector field, we conclude that we must have[3]:

$$(2.1) \qquad TN \cap \#(TN^0) = \{0\},$$

since the left-hand side is the kernel of the pull-back $i^* \omega_L$. If this condition holds, then at each point $x \in N$ we obtain a bivector $\pi_N(x) \in \wedge^2 T_x N$, and one can prove (see [1]):

Proposition 2.1. *Let N be a submanifold of a Poisson manifold M, such that*
   (a) equation (2.1) holds, and
   (b) the induced tensor π_N is smooth.
Then π_N is a Poisson tensor and N is a Poisson-Dirac submanifold.

Notice that, by the remarks above, the converse of the proposition also holds.

Remark 2.1. Equation (2.1) can be interpreted in terms of the Dirac theory of constraints. This is the reason for the use of the term "Poisson-Dirac submanifold". We refer the reader to [1] for more explanations.

On the other hand, from Proposition 2.1, we deduce the following sufficient condition for a submanifold to be a Poisson-Dirac submanifold:

Corollary 2.1. *Let M be a Poisson manifold and $N \subset M$ a submanifold. Assume that there exists a subbundle $E \subset T_N M$ such that:*

$$T_N M = TN \oplus E$$

and $\#(E^0) \subset TN$. Then N is a Poisson-Dirac submanifold.

Proof. Under the assumptions of the corollary, one has a decomposition

$$\pi = \pi_N + \pi_E,$$

where $\pi_N \in \Gamma(\wedge^2 TN)$ and $\pi_E \in \Gamma(\wedge^2 E)$ are both smooth bivector fields. On the other hand, one checks easily that (2.1) holds. By Proposition 2.1, we conclude that N is a Poisson-Dirac submanifold. $\square$

There are Poisson-Dirac submanifolds which do not satisfy the conditions of this corollary. Also, the bundle E may not be unique. For a detailed discussion and examples we refer to [1].

Under the assumptions of the corollary, the Poisson bracket on the Poisson-Dirac submanifold $N \subset M$ is quite simple to describe: Given two smooth functions $f, g \in C^\infty(N)$, to obtain their Poisson bracket we pick extensions $\widetilde{f}, \widetilde{g} \in C^\infty(M)$ such that $\mathrm{d}_x \widetilde{f}, \mathrm{d}_x \widetilde{g} \in E_x^0$. Then the Poisson bracket on N is given by:

$$(2.2) \qquad \{f, g\}_N = \{\widetilde{f}, \widetilde{g}\}|_N.$$

It is not hard to check that this formula does not depend on the choice of extensions.

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[3]For a subspace W of a vector space V, we denote by $W^0 \subset V^*$ its annihilator. Similarly, for a vector subbundle $E \subset F$, we denote by $E^0 \subset F^*$ its annihilator subbundle.

Remark 2.2. Let M be a Poisson manifold and $N \subset M$ a submanifold. Assume that there exists a subbundle $E \subset T_N M$ such that E^0 is a *Lie subalgebroid* of T^*M (equivalently, E is a co-isotropic submanifold of the tangent Poisson manifold TM). Then E satisfies the assumptions of the corollary, so N is a Poisson-Dirac submanifold. This class of Poisson-Dirac submanifolds have very special geometric properties. They where first study by Xu in [11], which calls them **Dirac submanifolds**. They are further discussed by Crainic and Fernandes in [1], where they are called **Lie-Dirac submanifolds**.

3. Fixed point sets of proper Poisson actions

In this section we will give a proof of Theorem 1.1, which we restate now as follows:

Theorem 3.1. *Let $G \times M \to M$ be a proper Poisson action. Then the fixed point set M^G is a Poisson-Dirac submanifold.*

Since the action is proper, the fixed point set M^G is an embedded submanifold of M. Its connected components may have different dimensions, but our argument will be valid for each such component, so we will assume that M^G is a connected submanifold. The proof will consist in showing that there exists a subbundle $E \subset T_{M^G} M$ satisfying the conditions of Corollary 2.1.

First of all, given any action $G \times M \to M$ (proper or not) there exists a lifted action $G \times TM \to TM$. For proper actions we have the following basic property:

Proposition 3.1. *If $G \times M \to M$ is a proper action then there exists a G-invariant metric on TM.*

For a proof of this fact and other elementary properties of proper actions, we refer to [3]. Explicitly, the G-invariance of the metric means that:

$$\langle g \cdot v, g \cdot w \rangle_{g \cdot p} = \langle v, w \rangle_p, \qquad \forall v, w \in T_p M.$$

where $g \in G$ and $p \in M$.

We fix, once and for all, a G-invariant metric $\langle \, , \, \rangle$ for our proper Poisson action $G \times M \to M$. Let us consider the subbundle $E \subset T_{M^G} M$ which is orthogonal to TM^G:

$$E = \{ v \in T_{M^G} M : \langle v, w \rangle = 0, \forall w \in TM^G \}.$$

We have:

Lemma 3.1.

$$T_{M^G} M = TM^G \oplus E \qquad and \qquad \#(E^0) \subset TM^G.$$

Proof. Since $E = (TM^G)^\perp$, the decomposition $T_{M^G} M = TM^G \oplus E$ is obvious. Now for a proper action, we have $(TM)^G = TM^G$ so this decomposition can also be written as:

$$(3.1) \qquad T_{M^G} M = (TM)^G \oplus E,$$

On the other hand, we have the lifted cotangent action $G \times T^*M \to T^*M$, which is related to the lifted tangent action by $g \cdot \xi(v) = \xi(g^{-1} \cdot v)$, $\xi \in T^*M, v \in TM$. We claim that:

$$(3.2) \qquad E^0 \subset (T^*M)^G.$$

In fact, if $v \in TM$ we can use (3.1) to decompose it as $v = v_G + v_E$, where $v_G \in (TM)^G$ and $v_E \in E$. Hence, for $\xi \in E^0$ we find:

$$g \cdot \xi(v_G + v_E) = \xi(g^{-1} \cdot v_G + g^{-1} \cdot v_E)$$
$$= \xi(v_G) + \xi(g^{-1} \cdot v_E)$$

$$= \xi(v_G)$$
$$= \xi(v_G) + \xi(v_E) = \xi(v_G + v_E).$$

We conclude that $g \cdot \xi = \xi$ and (3.2) follows.

Since $G \times M \to M$ is a Poisson action, we see that $\# : T^*M \to TM$ is a G-equivariant bundle map. Hence, if $\xi \in E^0$, we obtain from (3.2) that:

$$g \cdot \#\xi = \#(g \cdot \xi) = \#\xi.$$

This means that $\#\xi \in (TM)^G = TM^G$, so the lemma holds. □

This lemma shows that the conditions of Corollary 2.1 are satisfied, so M^G is a Poisson-Dirac submanifold and the proof of Theorem 3.1 is completed.

Remark 3.1. If one works further with the decomposition (3.1) and its transposed version, it is not hard to show that E^0 is actually a Lie subalgebroid of T^*M. Therefore, the fixed point set M^G of a proper Poisson action is, in fact, a Lie-Dirac submanifold of M (see Remark 2.2).

Remark 3.2. Special cases of Theorem 3.1 where obtained by Damianou and Fernandes in [2] for a compact Lie group G, and by Fernandes and Vanhaecke in [4] for a reductive algebraic group G. Xiang Tang also proves a version of this theorem in his PhD thesis [9].

Notice that the Poisson bracket of functions $f, g \in C^\infty(M^G)$ can be obtained simply by choosing G-invariant extensions $\widetilde{f}, \widetilde{g} \in C^\infty(M)^G$, and setting:

$$\{f, g\}_{M^G} = \{\widetilde{f}, \widetilde{g}\}|_{M^G}.$$

This follows from equation (2.2) and the remark that for any such G-invariant extensions we have $\mathrm{d}_{M^G}\widetilde{f}, \mathrm{d}_{M^G}\widetilde{g} \in E^0$. It is an instructive exercise to prove directly that the bracket on M^G does not depend on the choice of extensions.

4. Applications and further results

Every compact Lie group action is proper. In particular, a finite group action is always a proper. The case $G = \mathbb{Z}_2$ leads to the following result:

Corollary 4.1. *Let $\phi : M \to M$ be an involutive Poisson automorphism of a Poisson manifold M. The fixed point set $\{p \in M : \phi(p) = p\}$ has a natural induced Poisson structure.*

Proof. Apply Theorem 3.1 to the Poisson action of the group $G = \{\mathrm{Id}, \phi\}$. □

This result is known in the literature as the Poisson Involution Theorem (see [2, 4, 11]). It has been applied in [2, 4] to explain the relationship between the geometry of the Toda and Volterra lattices, and there should be similar relations between other known integrable systems. In this respect, it should be interesting to find extensions of our results to infinite dimensional manifolds and actions.

Recall that if an action $G \times M \to M$ is proper and free then the space of orbits M/G is a smooth manifold. For general non-free actions the orbit space can be a very pathological topological space. However, for proper actions the singularities of the orbit space are very much controlled, and M/G is a nicely stratified topological space. For proper symplectic actions there is a beautiful theory of singular symplectic quotients due to Lerman and Sjamaar [8] which describes the geometry of M/G. For proper Poisson actions one should expect that the orbit space still exhibits some nice Poisson geometry. In fact, we will explain in [5] that Theorem 3.1 leads to the following result that generalizes a theorem due to Lerman and Sjamaar:

Theorem 4.1. *Let $G \times M \to M$ be a proper Poisson action. Then the quotient M/G is a Poisson stratified space.*

Note that if a Poisson action is proper *and* free then the orbit space is a smooth Poisson manifold. In this case one can identify the smooth functions on the quotient M/G with the G-invariant functions on M:

$$C^\infty(M/G) \simeq C^\infty(M)^G.$$

In the non-free case, the smooth structure of M/G as a stratified space also leads to such an identification. Rather than explaining in detail the notion of a Poisson stratified space (see the upcoming paper [5]), we will illustrate this result with an example.

Example 4.1. Let $\mathbb{C}^{n+1}$ be the complex $n + 1$-dimensional space with holomorphic coordinates $(z_0, \ldots, z_n)$ and anti-holomorphic coordinates $(\overline{z}_0, \ldots, \overline{z}_n)$. On the (real) manifold $\mathbb{C}^{n+1} - 0$ we will consider a (real) quadratic Poisson bracket of the form:

$$\{z_i, z_j\} = a_{ij} z_i z_j, \qquad \{z_i, \overline{z}_j\} = \{\overline{z}_i, \overline{z}_j\} = 0.$$

where $A = (a_{ij})$ is a skew-symmetric matrix.

The group $\mathbb{C}^*$ of non-zero complex numbers acts on $\mathbb{C}^{n+1} - 0$ by multiplication of complex numbers. This is a free and proper Poisson action, so the quotient $\mathbb{C}P(n) = \mathbb{C}^{n+1} - 0/\mathbb{C}^*$ inherits a Poisson bracket.

Let us consider now the action of the n-torus $\mathbb{T}^n$ on $\mathbb{C}^{n+1} - 0$ defined by:

$$(\theta_1, \ldots, \theta_n) \cdot (z_0, z_1, \cdots, z_n) = (z_0, e^{i\theta_1} z_1, \cdots, e^{i\theta_n} z_n).$$

This is a Poisson action that commutes with the $\mathbb{C}^*$-action. It follows that the $\mathbb{T}^n$-action descends to a Poisson action on $\mathbb{C}P(n)$. Note that the action of $\mathbb{T}^n$ on $\mathbb{C}P(n)$ is proper but not free. The quotient $\mathbb{C}P(n)/\mathbb{T}^n$ is not a manifold but it can be identified with the standard simplex

$$\Delta^n = \{(\mu_0, \ldots, \mu_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^n \mu_i = 1, \mu_i \geq 0\}.$$

This identification is obtained via the map $\mu : \mathbb{C}P(n) \to \Delta^n$ defined by:

$$\mu([z_0 : \cdots : z_n]) = \left(\frac{|z_0|^2}{|z_0|^2 + \cdots + |z_n|^2}, \cdots, \frac{|z_n|^2}{|z_0|^2 + \cdots + |z_n|^2}\right).$$

Let us describe the Poisson stratification of $\Delta^n = \mathbb{C}P(n)/\mathbb{T}^n$. The Poisson bracket on Δ^n is obtained through the identification:

$$C^\infty(\Delta^n) \simeq C^\infty(\mathbb{C}P(n))^{\mathbb{T}^n}.$$

For that, we simply compute the Poisson bracket between the components of the map μ. A more or less straightforward computation will show that:

$$(4.1) \qquad \{\mu_i, \mu_j\} = \left(a_{ij} - \sum_{l=0}^n (a_{il} + a_{lj})\mu_l\right) \mu_i \mu_j, \qquad (i, j = 0, \ldots, n).$$

Now notice that (4.1) actually defines a Poisson bracket on $\mathbb{R}^{n+1}$. For this Poisson bracket, the interior of the simplex and its faces are Poisson submanifolds: a face $\Delta_{i_1, \ldots, i_{n-d}}$ of dimension $0 \leq d \leq n$ is given by equations of the form:

$$\sum_{i=0}^n \mu_i = 1, \quad \mu_{i_1} = \cdots = \mu_{i_{n-d}} = 0, \quad \mu_i > 0 \text{ for } i \notin \{i_1, \ldots, i_{n-d}\}.$$

These equations define Poisson submanifolds since:
  (a) the bracket $\{\mu_i, \mu_l\}$ vanishes whenever $\mu_l = 0$, and
  (b) the bracket $\{\mu_i, \sum_{l=0}^n \mu_l\}$ vanishes whenever $\sum_{l=0}^n \mu_l = 1$.

Therefore, the Poisson stratification of Δ^n consists of strata formed by the faces of dimension $0 \le d \le n$, which are smooth Poisson manifolds.

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