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Hyperelliptic Prym Varieties and Integrable Systems

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Abstract: We introduce two algebraic completely integrable analogues of the Mumford systems which we call hyperelliptic Prym systems, because every hyperelliptic Prym variety appears as a fiber of their momentum map. As an application we show that the general fiber of the momentum map of the periodic Volterra lattice

$$\dot{a}_i = a_i(a_{i-1} - a_{i+1}), \quad i = 1, \dots, n, \quad a_{n+1} = a_1,$$

is an affine part of a hyperelliptic Prym variety, obtained by removing n translates of the theta divisor, and we conclude that this integrable system is algebraic completely integrable.

Contents

1. Introduction

In this paper we introduce two algebraic completely integrable (a.c.i.) systems, similar to the even and odd Mumford systems (see [12] for the odd systems and [15] for the even systems). By a.c.i. we mean that the general level set of the momentum map is

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isomorphic to an affine part of an Abelian variety and that the integrable flows are linearized by this isomorphism ([16]). The phase space of these systems is described by triplets of polynomials $(u(x), v(x), w(x))$, as in the case of the Mumford system, but now we have the extra constraints that u, w are even and v odd for the first system (the "odd" case), and with the opposite parities for the other system (the "even" case). We show that in the odd case the general fiber of the momentum map is an affine part of a Prym variety, obtained by removing three translates of its theta divisor, while in the even case the general fiber has two affine parts of the above form. We call these systems the *odd* and the *even hyperelliptic Prym system* because every hyperelliptic Prym variety (more precisely an affine part of it) appears as the fiber of their momentum map. Thus we find the same universality as in the Mumford system: in the latter every hyperelliptic Jacobian appears as the fiber of its momentum map.

To show that the hyperelliptic Prym systems are a.c.i. we exhibit a family of compatible (linear) Poisson structures, making these systems multi-Hamiltonian. These structures are not just restrictions of the Poisson structures on the Mumford system. Rather they can be identified as follows: the hyperelliptic Prym systems are fixed point varieties of a Poisson involution (with respect to certain Poisson structures of the Mumford system) and we prove a general proposition stating that such a subvariety always inherits a Poisson structure (Prop. 3.4).

As an application we study the algebraic geometry and the Hamiltonian structure of the periodic Volterra lattice

$$\dot{a}_i = a_i(a_{i-1} - a_{i+1}) \quad i = 1, \dots, n; \quad a_{n+1} = a_1. \tag{1}$$

Although systems of this form go back to Volterra's work on population dynamics ([20]), they first appear (in an equivalent form) in the modern theory of integrable system in the pioneer work of Kac and van Moerbeke ([10]), who constructed this system as a discretization of the Korteweg-de Vries equation and who discovered its integrability. Though those authors only considered the non-periodic case, we shall refer to (1) as the *n-body KM system*. In the second part of the paper we give a precise description of the fibers of the momentum map of the KM systems and we prove their algebraic complete integrability.

We can summarize our results as follows:

Theorem. Denote be $\mathcal{M}'$, $\mathcal{P}$, $\mathcal{T}$, $\mathcal{K}$ the phase spaces of the (even) Mumford system, the hyperelliptic Prym system (odd or even), the (periodic) $\mathfrak{sl}$ Toda lattice and the (periodic) KM system. Then there exists a commutative diagram of a.c.i. systems

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\Phi} & \mathcal{M}' \\ \uparrow & & \uparrow \\ \mathcal{K} & \xrightarrow[\Phi]{} & \mathcal{P} \end{array}$$

where the horizontal maps are morphisms of integrable systems, and the vertical maps correspond to a Dirac type reduction.

We stress that the vertical arrows are natural inclusion maps exhibiting for both spaces the subspace as fixed points varieties, but they are *not* Poisson maps. On the other hand, the horizontal arrows are injective maps that map every fiber of the momentum map on the left injectively into (but not onto) a fiber of the momentum map on the right. In order to make these into morphisms of integrable systems, we construct a pencil of quadratic brackets making Toda $\to$ Mumford a Poisson map. For one bracket in this pencil the induced map for the KM systems is also Poisson, so it follows that the diagram is also valid in the Poisson category.

A description of the general fiber of the momentum map of the KM systems as an affine part of a hyperelliptic Prym variety follows. Since the flows of the KM systems are restrictions of *certain* linear flows of the Toda lattices this enables us to show that the KM systems are a.c.i.; moreover the map Φ leads to an explicit linearization of the KM systems.

In order to determine precisely which divisors are missing from the affine varieties that appear in the momentum map we use Painlevé analysis, since it is difficult to read this off from the map Φ. The result is that n ($=$ the number of KM particles) translates of the theta divisor are missing from these affine parts. We also show that each hyperelliptic Prym variety that we get is canonically isomorphic to the Jacobian of a related hyperelliptic Riemann surface, which can be computed explicitly, thereby providing an alternative, simpler description of the geometry of the KM systems.

The plan of this paper is as follows. In Sect. 2 we recall the definition of a Prym variety and specialize it to the case of a hyperelliptic Riemann surface with an involution (different from the hyperelliptic involution). We show that such a Prym variety is canonically isomorphic to a hyperelliptic Jacobian and we use this result to describe the affine parts that show up in Sect. 3, in which the hyperelliptic Prym systems are introduced and in which their algebraic complete integrability is proved. In Sect. 4 we establish the precise relation between the KM systems and the Toda lattices and we construct the injective morphism Φ. We use it to give a first description of the general fiber of the momentum map of the KM systems and we derive its algebraic complete integrability. A more precise description of these fibers is given in Sect. 5 by using Painlevé analysis. We finish the paper with a worked out example ($n = 5$) in which we find a configuration of five genus two curves on an Abelian surface that looks very familiar (Fig. 2).

As a final note we remark that the (periodic) KM systems have received much less attention than the (periodic) Toda lattices, another family of discretizations of the Korteweg–de Vries equation, which besides admitting a Lie algebraic generalization, is also interesting from the point of view of representation theory. It is only recently that the interest in the KM systems has revived (see e.g. [6, 18], and the references therein). We hope that the present work clarifies the connections between these systems and the master systems (Mumford and Prym systems). It was pointed out to us by Vadim Kuznetsov, that an embedding of the KM systems in the Heisenberg magnet was constructed by Volkov in [19].

2. Hyperelliptic Prym Varieties

In this section we recall the definition of a Prym variety and specialize it to the case of a hyperelliptic Riemann surface Γ, equipped with an involution σ. We construct an explicit isomorphism between the Prym variety of (Γ, σ) and the Jacobian of a related hyperelliptic Riemann surface. We use this isomorphism to give a precise description of the affine part of the Prym variety that will appear as the fiber of the momentum map of an integrable system related to KM system.

2.1. The Prym variety of a hyperelliptic Riemann surface. Let Γ be a compact Riemann surface of genus G, equipped with an involution σ with p fixed points. The quotient surface $\Gamma_\sigma = \Gamma/\sigma$ has genus g', with $G = 2g' + p/2 - 1$, and the quotient map $\Gamma \to \Gamma_\sigma$ is a double covering map which is ramified at the p fixed points of σ. We assume that $g' > 0$, i.e., σ is not the hyperelliptic involution on a hyperelliptic Riemann surface Γ. The group of divisors of degree 0 on Γ, denoted by $\mathrm{Div}^0(\Gamma)$, carries a natural equivalence relation, which is compatible with the group structure and which is defined by $\mathcal{D} \sim 0$ iff $\mathcal{D}$ is the divisor of zeros and poles of a meromorphic function on Γ. The quotient group $\mathrm{Div}^0(\Gamma)/\sim$ is a compact complex algebraic torus (Abelian variety) of dimension G, called the *Jacobian* of Γ and denoted by $\mathrm{Jac}(\Gamma)$ ([9], Ch. 2.7), its elements are denoted by $[\mathcal{D}]$, where $\mathcal{D} \in \mathrm{Div}^0(\Gamma)$ and we write $\otimes$ for the group operation in $\mathrm{Jac}(\Gamma)$. Notice that σ induces an involution on $\mathrm{Div}^0(\Gamma)$ and hence on $\mathrm{Jac}(\Gamma)$; we use the same notation σ for these involutions.

Definition 2.1. *The* Prym variety *of* (Γ, σ) *is the* $(G-g')$*-dimensional subtorus of* $\mathrm{Jac}(\Gamma)$ *given by*

$$\mathrm{Prym}(\Gamma/\Gamma_\sigma) = \{[\mathcal{D} - \sigma(\mathcal{D})] \mid \mathcal{D} \in \mathrm{Div}^0(\Gamma)\}.$$

We will be interested in the case in which Γ is the Riemann surface of a hyperelliptic curve $\Gamma^{(0)} : y^2 = f(x)$, where f is a monic even polynomial of degree $2n$ without multiple roots (in particular 0 is not a root of f), so that the curve is non-singular. The Riemann surface Γ has genus $G = n - 1$ and is obtained from $\Gamma^{(0)}$ by adding two points, which are denoted by ∞_1 and ∞_2. The two points of $\Gamma^{(0)}$ for which $x = 0$ are denoted by O_1 and O_2. The $2n$ Weierstrass points of Γ (the points (x, y) of $\Gamma^{(0)}$ for which $y = 0$) come in pairs $(X, 0)$ and $(-X, 0)$; fixing some order we denote them by $W_i = (X_i, 0)$ and $-W_i = (-X_i, 0)$, where $i = 1, \dots, n$. The Riemann surface Γ admits a group of order four of involutions, whose action on $\Gamma^{(0)}$ and on the Weierstrass points $(X_i, 0)$ and whose fixed point set are described in Table 1 for n odd, $n = 2g + 1$ and in Table 2 for n even, $n = 2g + 2$ (ι is the hyperelliptic involution).

Table 1. *n* odd

	(x, y)	O_1	O_2	∞_1	∞_2	W_i	$-W_i$	Fix
ι	$(x, -y)$	O_2	O_1	∞_2	∞_1	W_i	$-W_i$	$W_i, -W_i$
σ	$(-x, y)$	O_1	O_2	∞_2	∞_1	$-W_i$	W_i	O_1, O_2
τ	$(-x, -y)$	O_2	O_1	∞_1	∞_2	$-W_i$	W_i	∞_1, ∞_2

Table 2. *n* even

	(x, y)	O_1	O_2	∞_1	∞_2	W_i	$-W_i$	Fix
ι	$(x, -y)$	O_2	O_1	∞_2	∞_1	W_i	$-W_i$	$W_i, -W_i$
σ	$(-x, -y)$	O_2	O_1	∞_2	∞_1	$-W_i$	W_i	–
τ	$(-x, y)$	O_1	O_2	∞_1	∞_2	$-W_i$	W_i	$O_1, O_2, \infty_1, \infty_2$

For future use we also point out that for points $P \in \Gamma$ which are not indicated on these tables, neither $\sigma(P)$ nor $\tau(P)$ coincide with $\iota(P)$.

The involutions σ and τ lead to two quotient Riemann surfaces $\Gamma_\sigma := \Gamma/\sigma$ and $\Gamma_\tau := \Gamma/\tau$, and to two covering maps $\pi_\sigma : \Gamma \to \Gamma_\sigma$ and $\pi_\tau : \Gamma \to \Gamma_\tau$. It follows from Tables 1 and 2 that the genus of Γ_τ equals g, while the genus g' of Γ_σ is g or $g + 1$ depending on whether n is odd or even. Also, the dimension of $\mathrm{Prym}(\Gamma/\Gamma_\sigma) = g$

(whether n is odd or even). If the equation of $\Gamma^{(0)}$ is written as $y^2 = g(x^2)$ then for n odd, $\Gamma_\sigma^{(0)}$ has an equation $v^2 = g(u)$ while $\Gamma_\tau^{(0)}$ has an equation $v^2 = ug(u)$; for n even the roles of $\Gamma_\sigma^{(0)}$ and $\Gamma_\tau^{(0)}$ are interchanged.

In order to describe $\mathrm{Prym}(\Gamma/\Gamma_\sigma)$, which we will call a *hyperelliptic Prym variety*, we need the following classical results about hyperelliptic Riemann surfaces and their Jacobians (for proofs, see [12], Ch. IIIa).

Lemma 2.2. *Let $\mathcal{D}$ be a divisor of degree $H > G$ on Γ, where G is the genus of Γ, and let P be any point on Γ. There exists an effective divisor $\mathcal{E}$ of degree G on Γ such that*

$$\mathcal{D} \sim \mathcal{E} + (H - G)P.$$

Corollary 2.3. *For any fixed divisor $\mathcal{D}_0$ of degree G,* $\mathrm{Jac}(\Gamma)$ *is given by*

$$\mathrm{Jac}(\Gamma) = \left\{ \left[\sum_{i=1}^{G} P_i - \mathcal{D}_0 \right] \mid P_i \in \Gamma \right\}.$$

Lemma 2.4. *Let $\mathcal{D}$ be a divisor on Γ of the form $\mathcal{D} = \sum_{i=1}^{H} (P_i - Q_j)$, where $H \le G$ and $P_i \neq Q_j$ for all i and j. Then $[\mathcal{D}] = 0$ if and only if H is even and $\mathcal{D}$ is of the form*

$$\mathcal{D} = \sum_{i=1}^{H/2} (R_i + \imath(R_i) - S_i - \imath(S_i)),$$

for some points $R_i,\ S_i \in \Gamma$.

2.2. Hyperelliptic Prym varieties as Jacobians. In the following theorem we show that for any n the Prym variety $\mathrm{Prym}(\Gamma/\Gamma_\sigma)$ associated with the hyperelliptic Riemann surface Γ is canonically isomorphic to the Jacobian of Γ_τ.

This result was first proven by D. Mumford (see [13]) for the case in which $\pi_\sigma : \Gamma \to \Gamma_\sigma$ is unramified (n even) and by S. Dalaljan (see [7]) for the case in which $\pi_\sigma : \Gamma \to \Gamma_\sigma$ has two ramification points (n odd). Our proof, which is valid in both cases, is different and has the advantage of allowing us to describe explicitly the affine parts of the Prym varieties that we will encounter as affine parts of the corresponding Jacobians.

Theorem 2.5. *Let π_τ^* denote the homomorphism* $\mathrm{Div}^0(\Gamma_\tau) \to \mathrm{Div}^0(\Gamma)$ *which sends every point of Γ_τ to the divisor on Γ which consists of its two antecedents (under τ). The induced map*

$$\begin{aligned} \Pi : \mathrm{Jac}(\Gamma_\tau) &\to \mathrm{Prym}(\Gamma/\Gamma_\sigma) \\ [\mathcal{D}] &\mapsto [\pi_\tau^* \mathcal{D}] \end{aligned}$$

is an isomorphism.

Proof. It is clear that the homomorphism Π is a well-defined: if $[\mathcal{D}] = 0$ then $\mathcal{D}$ is the divisor of zeros and poles of a meromorphic function f on Γ_τ, hence $\pi_\tau^* \mathcal{D}$ is the divisor of zeros and poles of $f \circ \tau$ and $[\pi_\tau^* \mathcal{D}] = 0$. To see that the image of Π is contained in $\mathrm{Prym}(\Gamma/\Gamma_\sigma)$, just notice that $\pi_\tau^*(\mathcal{D})$ can be written as $\mathcal{E} + \tau(\mathcal{E})$ for some $\mathcal{E} \in \mathrm{Div}^0(\Gamma)$, so that

$$[\pi_\tau^*(\mathcal{D})] = [\mathcal{E} + \tau(\mathcal{E})] = [\mathcal{E} - \sigma(\mathcal{E})] \in \mathrm{Prym}(\Gamma/\Gamma_\sigma).$$

Since $\mathrm{Jac}(\Gamma_\tau)$ and $\mathrm{Prym}(\Gamma/\Gamma_\sigma)$ both have dimension g it suffices to show that Π is injective. Suppose that $[\pi_\tau^*\mathcal{D}] = 0$ for some $\mathcal{D} \in \mathrm{Div}^0(\Gamma_\tau)$. We need to show that this implies $[\mathcal{D}] = 0$. It follows from Corollary 2.3 that we may assume that $\mathcal{D}$ is of the form $\sum_{i=1}^g p_i - g\pi_\tau(\infty_1)$, where $p_i \in \Gamma_\tau$. Then $\pi_\tau^* D = \sum_{i=1}^g P_i + \tau(P_i) - 2g\infty_1$ $(\pi_\tau(P_i) = p_i)$. Since $2g \leq G$ and $\infty_1 \neq \iota(\infty_1)$ Lemma 2.4 implies that $P_i = \infty_1$, i.e., $p_i = \pi_\tau(\infty_1)$ for all i. $\square$

2.3. The theta divisor. We introduce two divisors on $\mathrm{Jac}(\Gamma)$ by

$$\Theta_1 = \left\{ \left[\sum_{i=1}^{G-1} P_i - (G-1)\infty_1 \right] \mid P_i \in \Gamma \right\}, \tag{2}$$

$$\Theta_2 = \left\{ \left[\sum_{i=1}^{G-1} P_i + \infty_2 - G\infty_1 \right] \mid P_i \in \Gamma \right\}. \tag{3}$$

These two divisors are both translates of the theta divisor and they differ by a shift over $[\infty_2 - \infty_1]$. Since $\infty_2 = \iota(\infty_1)$ they are tangent along their intersection locus, which is given by

$$\Omega = \left\{ \left[\sum_{i=1}^{G-2} P_i + \infty_2 - (G-1)\infty_1 \right] \mid P_i \in \Gamma \right\}.$$

Proposition 2.6. *When n is odd* $\mathrm{Prym}(\Gamma/\Gamma_\sigma) \cap (\Theta_1 \cup \Theta_2)$ *consists of three translates of the theta divisor of* $\mathrm{Jac}(\Gamma_\tau)$*, intersecting as in the following figure.*

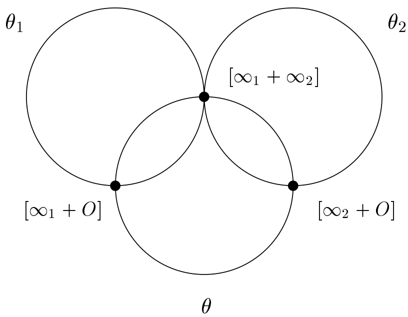

Fig. 1.

Proof. We use the isomorphism Π to determine which divisors of $\mathrm{Jac}(\Gamma_\tau)$ get mapped into Θ_1 and Θ_2. Since O_1 and O_2 are the only points of Γ on which ι and τ coincide, Lemma 2.4 implies that the only divisors $\mathcal{D} = \sum_{i=1}^g p_i - g\pi_\tau(\infty_1) \in \mathrm{Div}(\Gamma_\tau)$ for which $\pi_\tau^*\mathcal{D}$ contains, up to linear equivalence, ∞_1 or ∞_2 are those for which at least one of them contains $\pi_\tau(\infty_1)$ or $\pi_\tau(\infty_2)$ or $\pi_\tau(O_1)$ (=$\pi_\tau(O_2)$). Denoting $O = \pi_\tau(O_1)$ we find that these points constitute the following three divisors on $\mathrm{Jac}(\Gamma_\tau)$:

$$\theta_1 = \left\{ \left[\sum_{i=1}^{g-1} p_i - (g-1)\pi_\tau(\infty_1) \right] \mid p_i \in \Gamma_\tau \right\},$$

$$\theta_2 = \left\{ \left[\sum_{i=1}^{g-1} p_i + \pi_\tau(\infty_2) - g\pi_\tau(\infty_1) \right] \mid p_i \in \Gamma_\tau \right\},$$

$$\theta = \left\{ \left[\sum_{i=1}^{g-1} p_i + O - g\pi_\tau(\infty_1) \right] \mid p_i \in \Gamma_\tau \right\}.$$

They all pass through

$$\omega = \left\{ \left[\sum_{i=1}^{g-2} p_i + \pi_\tau(\infty_2) - (g-1)\pi_\tau(\infty_1) \right] \mid p_i \in \Gamma_\tau \right\},$$

which is the tangency locus of θ_1 and θ_2, and θ_i intersects θ in addition in

$$\omega_i = \left\{ \left[\sum_{i=1}^{g-2} p_i + \pi_\tau(\infty_i) + O - g\pi_\tau(\infty_1) \right] \mid p_i \in \Gamma_\tau \right\},$$

which is a translate of ω. □

When n is even then clearly $\mathrm{Prym}(\Gamma/\Gamma_\sigma)$ is contained in Θ_1, but the following result, similar to Prop. 2.6, holds for an appropriate translate of $\mathrm{Prym}(\Gamma/\Gamma_\sigma)$. The proof is left to the reader.

Proposition 2.7. *When n is even and $i \in \{1, 2\}$ then* $(\mathrm{Prym}(\Gamma/\Gamma_\sigma) \otimes [O_1 - \infty_i]) \cap (\Theta_1 \cup \Theta_2)$ *consists of three translates of the theta divisor of* $\mathrm{Jac}(\Gamma_\tau)$*, intersecting as in Fig. 1 (in which O should now be replaced by O_2).*

We will see in the next section how in both cases (n odd/even) the affine variety obtained by removing these three translates from the theta divisor from $\mathrm{Prym}(\Gamma/\Gamma_\sigma)$ can be described by simple, explicit equations.

3. The Hyperelliptic Prym Systems

In this section we introduce two families of integrable systems, whose members we call the *odd* and the *even hyperelliptic Prym systems*, where the adjective “odd/even” refers to the parity of n, as in the previous section, and where “hyperelliptic Prym” refers to the fact that the fibers of the momentum map of these systems are precisely the affine parts of the hyperelliptic Prym varieties that were considered in the previous section. These systems are intimately related to the even Mumford systems, constructed by the second author (see [15]), as even analogs of the (odd) Mumford systems, constructed by Mumford (see [12]).

3.1. The Mumford systems. We first recall the definition of the g-dimensional odd and even Mumford systems and we describe their geometry. Details, generalizations and applications can be found in [16]. The phase space of each of these systems is an affine space $\mathbb{C}^N$, which is most naturally described as an affine space of triples $(u(x), v(x), w(x))$ of polynomials, often represented as Lax operators

$$L(x) = \begin{pmatrix} v(x) & w(x) \\ u(x) & -v(x) \end{pmatrix},$$

where $u(x),\ v(x)$ and $w(x)$ are subject to certain constraints. Denoting by $\mathcal{M}_g$ (resp. $\mathcal{M}'_g$) the phase space of the g^{th} odd (resp. even) Mumford system these constraints are indicated in the following table:

Table 3.

Phase space	dim	$u(x)$	$v(x)$	$w(x)$
$\mathcal{M}_g$	$3g+1$	monic $\deg = g$	$\deg < g$	monic $\deg = g+1$
$\mathcal{M}'_g$	$3g+2$	monic $\deg = g$	$\deg < g$	monic $\deg = g+2$

It is natural to use the coefficients of the three polynomials $u(x), v(x), w(x)$ as coordinates on $\mathcal{M}_g$ and on $\mathcal{M}'_g$: for $\mathcal{M}'_g$ for example, which will be most important for this paper, we write

$$\begin{aligned} u(x) &= x^g + u_{g-1}x^{g-1} + \cdots + u_0, \\ v(x) &= v_{g-1}x^{g-1} + \cdots + v_0, \\ w(x) &= x^{g+2} + w_{g+1}x^{g+1} + \cdots + w_0, \end{aligned}$$

or, in terms of the Lax operator $L(x)$, as

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x^{g+2} + \begin{pmatrix} 0 & w_{g+1} \\ 0 & 0 \end{pmatrix} x^{g+1} + \begin{pmatrix} 0 & w_g \\ 1 & 0 \end{pmatrix} x^g + \sum_{0 \le i < g} \begin{pmatrix} v_i & w_i \\ u_i & -v_i \end{pmatrix} x^i.$$

We define on $\mathcal{M}_g$ and on $\mathcal{M}'_g$ a map H with values in (a finite-dimensional affine subspace of) $\mathbb{C}[x]$ by

$$H(L(x)) = -\det(L(x)) = u(x)w(x) + v^2(x).$$

Notice that the degree of H, as a polynomial in x, is odd for $\mathcal{M}_g$ and even for $\mathcal{M}'_g$, explaining the terminology odd/even Mumford system. Writing $H(L(x)) = x^{2g+1} + H_{2g}x^{2g} + \cdots + H_0$ in the odd case, and similarly in the even case, we can view the H_i as functions on phase space; clearly they are polynomials in the phase variables u_i, v_i, w_i. These functions are independent and they are in involution with respect to a whole family of compatible Poisson brackets on $\mathcal{M}_g$ (or $\mathcal{M}'_g$). Namely, it is easy to check by direct computation that for any polynomial φ of degree at most g the following formulas:

$$\begin{aligned}
\left\{u(x), u(x')\right\}^{\varphi}_{\mathcal{M}} &= \left\{v(x), v(x')\right\}^{\varphi}_{\mathcal{M}} = 0,\\
\left\{u(x), v(x')\right\}^{\varphi}_{\mathcal{M}} &= \frac{u(x)\varphi(x') - u(x')\varphi(x)}{x - x'},\\
\left\{u(x), w(x')\right\}^{\varphi}_{\mathcal{M}} &= -2\frac{v(x)\varphi(x') - v(x')\varphi(x)}{x - x'},\\
\left\{v(x), w(x')\right\}^{\varphi}_{\mathcal{M}} &= \frac{w(x)\varphi(x') - w(x')\varphi(x)}{x - x'} - \alpha(x + x')u(x)\varphi(x'),\\
\left\{w(x), w(x')\right\}^{\varphi}_{\mathcal{M}} &= 2\alpha(x + x')\left(v(x)\varphi(x') - v(x')\varphi(x)\right),
\end{aligned} \tag{4}$$

define a Poisson structure on $\mathcal{M}_g$ and on $\mathcal{M}'_g$, where $\alpha = 1$ for the odd case (i.e., on $\mathcal{M}_g$) and $\alpha = x + w_{g+1} - u_{g-1}$ for the even case (i.e., on $\mathcal{M}'_g$); we use the same notation $\{\cdot, \cdot\}_{\mathcal{M}}$ for the Poisson brackets on both spaces. When $\varphi \neq 0$ then the rank of this Poisson structure is $2g$ on a Zariski open subset. Since

$$\left\{H(x), H(x')\right\}^{\varphi}_{\mathcal{M}} = \left\{u(x)w(x) + v^2(x), u(x')w(x') + v^2(x')\right\}^{\varphi}_{\mathcal{M}} = 0, \tag{5}$$

the functions H_i are in involution. For any $\varphi \neq 0$ of degree at most g and for any $y \in \mathbb{C}$ which is not a root of φ, let

$$H^{\varphi}(y) = \frac{H(y)}{\varphi(y)} = \frac{u(y)w(y) + v^2(y)}{\varphi(y)}.$$

Then

$$X_y = \{\cdot\,, H^{\varphi}(y)\}^{\varphi}_{\mathcal{M}} = \frac{1}{\varphi(y)}\{\cdot\,, u(y)w(y) + v^2(y)\}^{\varphi}_{\mathcal{M}} \tag{6}$$

is a Hamiltonian vector field, independent of the choice of the Poisson structure $\{\cdot, \cdot\}_{\mathcal{M}}$, i.e., it is a multi-Hamiltonian vector field. Explicitly, X_y can be written as the following Lax equation:

$$X_y L(x) = \frac{1}{x - y}[L(x), L(y) + (x - y)B(x, y)], \tag{7}$$

where

$$B(x, y) = \begin{pmatrix} 0 & \alpha(x + y)u(y) \\ 0 & 0 \end{pmatrix}.$$

In view of (5) the vector fields X_y corresponding to any two values of y commute and a dimension count shows that the Mumford systems are integrable in the sense of Liouville.

In fact, the Mumford systems are algebraic completely integrable systems (a.c.i. systems), meaning that the general level of the momentum map H (equivalently, H^{φ}) is an affine part of an Abelian variety and that the flow of the integrable vector fields X_y is linear on each of these Abelian varieties. A precise description of these fibers is given in the following proposition.

Proposition 3.1. *Let $f(x)$ be a monic polynomial of degree $2g+1$ (resp. $2g+2$) without multiple roots and let Γ denote the Riemann surface corresponding to the (smooth) affine curve (of genus g) defined by $\Gamma^{(0)} : y^2 = f(x)$. Then the fiber over $f(x)$ of $H : \mathcal{M}_g \to \mathbb{C}[x]$ (resp. $H : \mathcal{M}'_g \to \mathbb{C}[x]$) is isomorphic to* $\mathrm{Jac}(\Gamma)$ *minus its theta divisor (resp. minus two translates of its theta divisor which are tangent along their intersection locus).*

Proof. We shortly indicate the idea of the proof, since it will be useful later. To $(u(x), v(x), w(x))$ in the fiber over $f(x)$ of $H : \mathcal{M}_g \to \mathbb{C}[x]$ one associates a divisor $\mathcal{D} = \sum_{i=1}^{g}(x_i, y_i) - g\infty$ on Γ (where $\{\infty\} := \Gamma \setminus \Gamma^{(0)}$) by taking for x_i the roots of $u(x)$ and $y_i = v(x_i)$. This map is injective and its image consists of those divisors $\sum_{i=1}^{g} P_i - g\infty$ for which $P_i \in \Gamma^{(0)}$ and for which $i \neq j \Rightarrow P_i \neq \iota(P_j)$. Mapping $\mathcal{D}$ to its equivalence class $[\mathcal{D}]$ we get an injective map into $\mathrm{Jac}(\Gamma)$ and the complement of its image is the theta divisor $\left\{\left[\sum_{i=1}^{g-1} P_i - (g-1)\infty\right] \mid P_i \in \Gamma\right\}$. For $(u(x), v(x), w(x)) \in \mathcal{M}'_g$ the construction is similar but the complement of the image consists of two translates of the theta divisor because $\Gamma \setminus \Gamma^{(0)}$ consist now of two points. □

Notice that the fact that the general fiber of the momentum map is g-dimensional implies our earlier assertion that the functions H_i are independent, an essential ingredient in the proof of the Liouville integrability of the Mumford systems.

3.2. Phase space and momentum map. In this paragraph we introduce the phase spaces of the hyperelliptic Prym systems, which can be seen as affine subspaces of the even Mumford systems; they are not Poisson subspaces with respect to the Poisson structures (4), as we will discuss in the next paragraph. However, the moment map will just be the restriction of the moment map H to these subspaces.

For $n \geq 1$, let us denote by $\mathcal{P}_n$ and by $\mathcal{P}'_n$ the set of triples of polynomials $(u(x), v(x), w(x))$ which satisfy the constraints indicated in the following table:

Table 4.

Phase space	dim	$u(x)$	$v(x)$	$w(x)$
$\mathcal{P}_n$	$3n+1$	monic even $\deg = 2n$	odd $\deg < 2n$	monic even $\deg = 2n+2$
$\mathcal{P}'_n$	$3n+2$	monic odd $\deg = 2n+1$	even $\deg < 2n+1$	monic odd $\deg = 2n+3$

Comparing this table to Table 3 one sees that $\mathcal{P}_n$ is an affine subspace of $\mathcal{M}'_{2n}$ and that $\mathcal{P}'_n$ is an affine subspace of $\mathcal{M}'_{2n+1}$. For reasons that will become clear shortly, we will say that $\mathcal{P}_n$ (resp. $\mathcal{P}'_n$) is the phase space of the n^{th}*odd* (resp. *even*) *hyperelliptic Prym system*. We denote the restriction of $H : \mathcal{M}'_{2n} \to \mathbb{C}[x]$ to $\mathcal{P}_n$ as well as the restriction of $H : \mathcal{M}'_{2n+1} \to \mathbb{C}[x]$ to $\mathcal{P}'_n$ also by H. Notice that H, which is on these subspaces still given by

$$H(L(x)) = -\det L(x) = u(x)w(x) + v^2(x),$$

takes now values in $\mathbb{C}[x^2]$. It follows that the corresponding Riemann surface Γ is of the type considered in Sect. 2. We show in the following two propositions that the fibers of H are affine parts of hyperelliptic Prym varieties.

Proposition 3.2. *Let $f(x)$ be a monic even polynomial of degree $4g+2$ without multiple roots and let Γ denote the Riemann surface corresponding to the (smooth) affine curve (of genus $2g$) defined by $\Gamma^{(0)} : y^2 = f(x)$. The fiber of $H : \mathcal{P}_g \to \mathbb{C}[x^2]$ over $f(x)$ is isomorphic to $\mathrm{Prym}(\Gamma/\Gamma_\sigma) \cong \mathrm{Jac}(\Gamma_\tau)$ minus three translates of its theta divisor, intersecting as in Fig. 1.*

Proof. Since the fiber over $f(x)$ of $H : \mathcal{P}_g \to \mathbb{C}[x^2]$ is contained in the fiber over $f(x)$ of $H : \mathcal{M}'_{2g} \to \mathbb{C}[x]$ it is a subset of $\mathrm{Jac}(\Gamma)$. In fact it is a subset of $\mathrm{Prym}(\Gamma/\Gamma_\sigma)$. To see this, consider the divisor $\mathcal{D} = \sum_{i=1}^{2g}(x_i, y_i) - 2g\infty_1$ which corresponds to a triple $(u(x), v(x), w(x))$, with u, w even and v odd. The roots of u come in pairs $(x_i, x_j = -x_i)$ and $y_j = v(x_j) = v(-x_i) = -v(x_i) = -y_i$ (recall that 0 can never be a root of u because then f would have 0 as a double root), hence the points in $\mathcal{D}$ come in pairs $P, \tau(P)$ and $[\mathcal{D}]$ belongs to $\mathrm{Prym}(\Gamma/\Gamma_\sigma)$. The points of $\mathrm{Prym}(\Gamma/\Gamma_\sigma)$ which do not belong to the fiber are those $\left[\sum_{i=1}^{2g} P_i - 2g\infty_1\right] \in \mathrm{Prym}(\Gamma/\Gamma_\sigma)$ for which at least one of the P_i equals ∞_1 or ∞_2, i.e., the points on $\Theta_1 \cup \Theta_2$, as defined in (2) and (3). By Prop. 2.6 the fiber is isomorphic to an affine part of $\mathrm{Prym}(\Gamma/\Gamma_\sigma)$ obtained by removing three translates of the theta divisor. □

Proposition 3.3. *Let $f(x)$ be a monic even polynomial of degree $4g+4$ without multiple roots and let Γ denote the Riemann surface corresponding to the (smooth) affine curve (of genus $2g+1$) defined by $\Gamma^{(0)} : y^2 = f(x)$. The fiber over $f(x)$ of $H : \mathcal{P}'_g \to \mathbb{C}[x^2]$ is reducible and each of its two components is isomorphic to $\mathrm{Prym}(\Gamma/\Gamma_\sigma) \cong \mathrm{Jac}(\Gamma_\tau)$ minus three translates of its theta divisor, intersecting as in Fig. 1.*

Proof. Consider the divisor $\mathcal{D} = \sum_{i=1}^{2g+1}(x_i, y_i) - (2g+1)\infty$ which corresponds to a triple $(u(x), v(x), w(x))$, with u, w odd and v even. 0 is a root of u and its other roots come in pairs $x_i, x_j = -x_i$ and $y_j = v(x_j) = v(x_i) = y_i$, hence the points in $\mathcal{D}$ consist of O_1 *or* O_2 and the others come in pairs $P, \tau(P)$. It follows that $[\mathcal{D}]$ belongs to $\mathrm{Prym}(\Gamma/\Gamma_\sigma) \otimes [O_1 - \infty_1]$ or to $\mathrm{Prym}(\Gamma/\Gamma_\sigma) \otimes [O_1 - \infty_2]$. The points of $\mathrm{Prym}(\Gamma/\Gamma_\sigma)$ which do not belong to the fiber are those $\left[\sum_{i=1}^{2g+1} P_i - (2g+1)\infty_1\right]$ for which at least one of the P_i equals ∞_1 or ∞_2, i.e., the points on $\Theta_1 \cup \Theta_2$, as defined in (2) and (3). By Prop. 2.6 the fiber consists of two copies of an affine part of $\mathrm{Prym}(\Gamma/\Gamma_\sigma)$ obtained by removing three translates of the theta divisor. Notice that the fact that the fiber is reducible can also be deduced from the fact that $f(0) = u(0)w(0) + v^2(0) = v_0^2$. □

3.3. Flows and Hamiltonian structure. We now show how the brackets (4) lead to a family of compatible Poisson structures on $\mathcal{P}_n$ and on $\mathcal{P}'_n$, we construct the integrable vector fields for the hyperelliptic Prym systems and we establish their algebraic complete integrability. Consider the natural inclusions $\mathcal{P}_g \hookrightarrow \mathcal{M}'_{2g}$ and $\mathcal{P}'_g \hookrightarrow \mathcal{M}'_{2g+1}$ and consider the involution $\jmath : \mathcal{M}'_n \to \mathcal{M}'_n$ defined by:

$$\jmath : \begin{pmatrix} v(x) & w(x) \\ u(x) & -v(x) \end{pmatrix} \longmapsto \begin{pmatrix} -v(-x) & w(-x) \\ u(-x) & v(-x) \end{pmatrix}.$$

Then we see that the image of $\mathcal{P}_g \hookrightarrow \mathcal{M}'_{2g}$ is the fixed point variety of $\jmath$, while the image of $\mathcal{P}'_g \hookrightarrow \mathcal{M}'_{2g+1}$ is the fixed point variety of $-\jmath$. We claim that $\jmath$ (resp. $-\jmath$) is a Poisson automorphism of $(\mathcal{M}'_{2g}, \{\cdot,\cdot\}^{\varphi}_{\mathcal{M}})$ (resp. $(\mathcal{M}'_{2g+1}, \{\cdot,\cdot\}^{\varphi}_{\mathcal{M}})$), whenever φ is an even (resp. odd) polynomial. In fact, taking φ even we have

$$\left\{u(x)\circ\jmath, v(x')\circ\jmath\right\}^{\varphi}_{\mathcal{M}} = \left\{u(-x), -v(-x')\right\}^{\varphi}_{\mathcal{M}}$$
$$= \frac{u(-x)\varphi(x') - u(-x')\varphi(x)}{x-x'} = \left\{u(x), v(x')\right\}^{\varphi}_{\mathcal{M}} \circ \jmath,$$

showing that for any i and j, $\left\{u_i\circ\jmath, v_j\circ\jmath\right\}^{\varphi}_{\mathcal{M}} = \left\{u_i, v_j\right\}^{\varphi}_{\mathcal{M}}$ and similarly for the Poisson brackets of the other components. Since all brackets are linear in φ the result for φ odd also follows, when $\jmath$ is replaced by $-\jmath$.

The following proposition, which can be seen as a particular case of Dirac reduction, yields a Poisson structure on the fixed point variety of a Poisson involution. For the general theorem on Dirac reduction, see Weinstein ([21], Prop. 1.4) and Courant ([4], Thm. 3.2.1). For our convenience we give a proof in the algebraic category; the proof is easily adapted to smooth manifolds.

Proposition 3.4. *Suppose that $(M, \{\cdot,\cdot\})$ is an affine Poisson variety, equipped with an involution $\jmath$ which is a Poisson map. Let N be the subvariety of M consisting of the fixed points of $\jmath$ and denote the inclusion map $N \hookrightarrow M$ by $\imath$. Then N carries a (unique) Poisson structure $\{\cdot,\cdot\}_N$ such that*

$$\imath^*\{F, G\} = \left\{\imath^*F, \imath^*G\right\}_N \tag{8}$$

for all $F, G \in \mathcal{O}(M)$ that are $\jmath$-invariant.

Proof. For $f, g \in \mathcal{O}(N)$ we choose $F, G \in \mathcal{O}(M)$ such that $f = \imath^*F$ and $g = \imath^*G$. We may assume that F and G are $\jmath$-invariant by replacing F by $(F + \jmath^*(F))/2$ and similarly for G. We define $\{f, g\}_N = \imath^*\{F, G\}$ and show that this definition is independent of the choice of F and G. To do this it is sufficient to show that if G is $\jmath$-invariant and $\imath^*F = 0$, then $\imath^*\{F, G\} = 0$. Since the ideal of functions vanishing on N is generated by $\jmath$-anti-invariant functions ($\jmath^*F = -F$) it suffices to show this for $F \in \mathcal{O}(M)$ such that $\jmath^*F = -F$. By assumption $\jmath$ is a Poisson map, $\jmath\circ\imath = \jmath$ and $\jmath^*G = G$ so that

$$\imath^*\{F, G\} = \imath^*\jmath^*\{F, G\} = \imath^*\left\{\jmath^*F, \jmath^*G\right\} = -\imath^*\{F, G\},$$

showing our claim. Similarly, the bracket of any two $\jmath$-invariant functions is $\jmath$-invariant. In view of this and because the definition of $\{\cdot,\cdot\}_N$ is independent of the choice of F and G we have for any $f, g, h \in \mathcal{O}(N)$ that

$$\left\{\{f, g\}_N, h\right\}_N = \imath^*\{\{F, G\}, H\},$$

leading at once to the Jacobi identity for $\{\cdot,\cdot\}_N$. Similarly the fact that $\{\cdot,\cdot\}_N$ is an anti-symmetric biderivation follows. □

The Hamiltonian structure of the hyperelliptic Prym systems and its algebraic complete integrability is described in the following proposition.

Proposition 3.5. *Let φ be an even (resp. odd) polynomial of degree at most $2g+1$, $\varphi \neq 0$. The Poisson bracket $\{\cdot,\cdot\}^{\varphi}_{\mathcal{M}}$ on $\mathcal{M}'_{2g}$, (resp. on $\mathcal{M}'_{2g+1}$), given by (4) induces a Poisson bracket $\{\cdot,\cdot\}^{\varphi}_{\mathcal{P}}$ on $\mathcal{P}_g$ (resp. on $\mathcal{P}'_g$), the components of $H:\mathcal{P}_g \to \mathbb{C}[x^2]$ (resp. $H:\mathcal{P}'_g \to \mathbb{C}[x^2]$) are in involution and they define an (algebraic completely) integrable system on $\mathcal{P}_g$ (resp. on $\mathcal{P}'_g$).*

Proof. Since for φ even (resp. odd) the image of $\mathcal{P}_g \hookrightarrow \mathcal{M}'_{2g}$ (resp. $\mathcal{P}_g \hookrightarrow \mathcal{M}'_{2g}$) is the fixed point set of the Poisson involution $\jmath$ (resp. -$\jmath$) it follows from Prop. 3.4 that $\mathcal{P}_g$ (resp. $\mathcal{P}'_g$) inherits a Poisson bracket from $\mathcal{M}'_{2g}$, which we denote in both cases by $\{\cdot,\cdot\}^{\varphi}_{\mathcal{P}}$. We exemplify the computation of the reduced brackets by deriving the formula for $\left\{u(x), v(x')\right\}^{\varphi}_{\mathcal{P}}$ on $\mathcal{P}_g$ (φ even). Notice that since u is even and v is odd the polynomial $\{u(x), v(x)\}^{\varphi}_{\mathcal{P}}$, which is a generating function for the Poisson brackets $\left\{u_i, v_j\right\}^{\varphi}_{\mathcal{P}}$, is even in x and odd in x'. Obvious $\jmath$-invariant extensions of the functions u_{2i} and v_{2i+1} are the corresponding functions u_{2i} and v_{2i+1} on $\mathcal{M}'_{2g}$. Therefore $\left\{u(x), v(x')\right\}^{\varphi}_{\mathcal{P}}$ is computed by taking in $\left\{u(x), v(x')\right\}^{\varphi}_{\mathcal{M}}$ the terms that are even in x and odd in x' and restricting the resulting polynomial to the image of $\mathcal{P}_g$, as embedded in $\mathcal{M}_{2g}$. Using the fact that the terms of a bivariate polynomial $F(x,x')$ that are even in x and odd in x' are picked by taking

$$\frac{1}{4}\left(F(x,x') + F(-x,x') - F(x,-x') - F(-x,-x')\right)$$

we find for

$$F(x,x') = \left\{u(x), v(x')\right\}^{\varphi}_{\mathcal{M}} = \frac{u(x)\varphi(x') - u(x')\varphi(x)}{x - x'}$$

that the reduced Poisson bracket, for φ even, is given by

$$\begin{aligned}\left\{u(x), v(x')\right\}^{\varphi}_{\mathcal{P}} &= \frac{x'}{2}\left(\frac{(u(x)+u(-x))\varphi(x') - (u(x')+u(-x'))\varphi(x)}{x^2 - x'^2}\right)\Bigg|_{\mathcal{P}_g}\\ &= x'\frac{u(x)\varphi(x') - u(x')\varphi(x)}{x^2 - x'^2}.\end{aligned}$$

Repeating the same computation for the other coordinates we find the following formulas for $\{\cdot,\cdot\}^{\varphi}_{\mathcal{P}}$,

$$\begin{aligned}\left\{u(x), u(x')\right\}^{\varphi}_{\mathcal{P}} &= \left\{v(x), v(x')\right\}^{\varphi}_{\mathcal{P}} = 0,\\ \left\{u(x), v(x')\right\}^{\varphi}_{\mathcal{P}} &= x'\frac{u(x)\varphi(x') - u(x')\varphi(x)}{x^2 - x'^2},\\ \left\{u(x), w(x')\right\}^{\varphi}_{\mathcal{P}} &= -2\frac{xv(x)\varphi(x') - x'v(x')\varphi(x)}{x^2 - x'^2},\\ \left\{v(x), w(x')\right\}^{\varphi}_{\mathcal{P}} &= x\frac{w(x)\varphi(x') - w(x')\varphi(x)}{x^2 - x'^2} - xu(x)\varphi(x'),\\ \left\{w(x), w(x')\right\}^{\varphi}_{\mathcal{P}} &= 2\left(xv(x)\varphi(x') - x'v(x')\varphi(x)\right).\end{aligned}$$

Using the fact that all Poisson brackets are linear in φ one finds that the formulas for the reduced bracket on $\mathcal{P}'_g$ (with φ odd) are formally identical to the above ones. It is now obvious that the components of the new momentum map H are in involution. Since we know that the fibers of $H : \mathcal{P}_g \to \mathbb{C}[x^2]$ are affine parts of Abelian varieties of dimension g, the components of the new H are independent. The integrable vector fields X_y on $\mathcal{P}_g$ are computed from $\{\cdot\,, H(y)\}^1_{\mathcal{P}}$, to wit

$$X_y L(x) = \frac{1}{x^2 - y^2}\left[L(x), \begin{pmatrix} yv(y) & xw(y) + x(x^2 - y^2)u(y) \\ xu(y) & -yv(y) \end{pmatrix}\right]. \tag{9}$$

Since the formulas for the reduced brackets on $\mathcal{P}_g$ and on $\mathcal{P}'_g$ are formally the same the vector fields X_y on $\mathcal{P}'_g$ are also given by (9). Finally, the flows of all vector fields X_y are linear since they are restrictions of linear flows, showing that the odd hyperelliptic Prym systems are algebraic completely integrable. □

4. The Periodic Toda Lattices and KM Systems

In this section we show that the (periodic) KM systems are related to the (periodic) $\mathfrak{sl}$ Toda lattices in the same way as the hyperelliptic Prym systems are related to the even Mumford systems and we construct a morphism from the Toda lattices to the even Mumford systems, which induces a morphism from the KM systems to the odd or the even hyperelliptic Prym systems. The latter map is then used to describe the level sets of the momentum map of the KM systems.

4.1. From Toda to KM. The phase space $\mathcal{T}_n$ of the periodic $\mathfrak{sl}(n)$ Toda lattice (n-body Toda lattice for short) is the affine variety of all Lax operators in $\mathfrak{sl}(n)[h, h^{-1}]$ of the form

$$L(h) = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & 0 & h^{-1} \\ 1 & b_2 & a_2 & & & 0 \\ 0 & 1 & & & & \vdots \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & & & & b_{n-1} & a_{n-1} \\ ha_n & 0 & \cdots & \cdots & 1 & b_n \end{pmatrix}, \tag{10}$$

with $\prod_{i=1}^n a_i = 1$. It carries a natural $\mathbb{Z}/n$ action, defined by

$$((a_1, a_2 \dots, a_n), (b_1, b_2 \dots, b_n)) \mapsto ((a_2, a_3 \dots, a_1), (b_2, b_3, \dots, b_1)).$$

For this reason it is convenient to view the indices as elements of $\mathbb{Z}/n$ and we put $a_n = a_0,\ b_n = b_0,\ a_{n+1} = a_1,\ b_{n+1} = b_1, \dots$ Define

$$I_i = \frac{1}{1+i}\operatorname{tr}(L(h)^{i+1}), \qquad i = 0, \dots, n-2,$$
$$I_{n-1} = \frac{1}{n}\operatorname{tr}(L(h)^n) - h - \frac{1}{h},$$

and notice that these n functions are independent of h. They are in involution with respect to the linear Poisson bracket $\{\cdot,\cdot\}^1_{\mathcal{T}}$, defined by $\{a_i,a_j\}^1_{\mathcal{T}} = \{b_i,b_j\}^1_{\mathcal{T}} = 0$, $\{a_i,b_j\}^1_{\mathcal{T}} = a_i(\delta_{ij} - \delta_{i+1,j})$, which has I_0 as a Casimir. They are also in involution with respect to the quadratic Poisson bracket $\{\cdot,\cdot\}^x_{\mathcal{T}}$, defined by

$$\begin{aligned}\{a_i,a_j\}^x_{\mathcal{T}} &= a_ia_j(\delta_{i,j+1} - \delta_{i+1,j}), \\ \{b_i,b_j\}^x_{\mathcal{T}} &= a_i(\delta_{i,j+1} - \delta_{i+1,j}),\end{aligned} \qquad \{a_i,b_j\}^x_{\mathcal{T}} = a_ib_j(\delta_{i,j} - \delta_{i+1,j}),$$

which has $\det L$ as a Casimir. Since $\{\cdot,\cdot\}^x_{\mathcal{T}}$ and $\{\cdot,\cdot\}^1_{\mathcal{T}}$ are compatible we may define, for any $\varphi \in \mathbb{C}[x]$ of degree at most 1 a Poisson bracket on $\mathcal{T}_n$ by $\{\cdot,\cdot\}^{\varphi}_{\mathcal{T}} = \varphi_1\{\cdot,\cdot\}^x_{\mathcal{T}} + \varphi_0\{\cdot,\cdot\}^1_{\mathcal{T}}$, where $\varphi(x) = \varphi_1 x + \varphi_0$.

The commuting vector fields $X_i = \{\cdot\,, I_i\}^1_{\mathcal{T}}$ admit the Lax representation

$$X_i L(h) = [L(h), (L(h)^i)_+], \tag{11}$$

where the subscript $+$ denotes projection into the Lie subalgebra of $\mathfrak{sl}(n)[h,h^{-1}]$ generated by the positive roots, i.e.,

$$\left(\sum A_i h^i\right)_+ = \sum_{i>0} A_i h^i + \mathrm{su}(A_0), \tag{12}$$

where $\mathrm{su}(A_0)$ denotes the strictly upper triangular part of A_0. The vector fields X_i are also Hamiltonian with respect to $\{\cdot,\cdot\}^x_{\mathcal{T}}$ and their flows are linear on the general fiber of the momentum map $K : \mathcal{T}_n \to \mathbb{C}[x]$, which is defined by

$$\det(x\,\mathrm{Id} - L(h)) = -h - \frac{1}{h} + K(x)/2;$$

since the general fiber of K is an affine part of a hyperelliptic Jacobian, the n-body Toda lattice is an a.c.i. system (see [3] for details). For higher order brackets for the Toda lattices, see [5].

We now turn to the n-body, periodic, Kac–van Moerbeke system (n-body KM system, for short). Its phase space $\mathcal{K}_n$ is the subspace of $\mathcal{T}_n$ consisting of all Lax operators (10) with zeros on the diagonal. $\mathcal{K}_n$ is not a Poisson subspace of $\mathcal{T}_n$. However, $\mathcal{K}_n$ is the fixed manifold of the involution $\jmath : \mathcal{T}_n \to \mathcal{T}_n$ defined by

$$((a_1,a_2\dots,a_n),(b_1,b_2\dots,b_n)) \mapsto ((a_1,a_2\dots,a_n),(-b_1,-b_2\dots,-b_n)),$$

which is a Poisson automorphism of $(\mathcal{T}_n, \{\cdot,\cdot\}^x_{\mathcal{T}})$. Therefore, by Theorem 3.4, $\mathcal{K}_n$ inherits a Poisson bracket $\{\cdot,\cdot\}_{\mathcal{K}}$ from $\{\cdot,\cdot\}^x_{\mathcal{T}}$, which is given by

$$\{a_i,a_j\}_{\mathcal{K}} = a_ia_j(\delta_{i,j+1} - \delta_{i+1,j}).$$

It follows that the restriction of the momentum map K to $\mathcal{K}_n$ is a momentum map for the n-body KM system. Notice that $I_j = 0$ for even j, while for j odd the Lax equations (11) lead to Lax equations for the n-body KM system, merely by putting $b_1 = \cdots = b_n = 0$. Taking $j = 1$ we find the vector field

$$\dot a_i = a_i(a_{i-1} - a_{i+1}), \quad i = 1,\dots,n, \tag{13}$$

which was already mentioned in the introduction. More generally, taking j odd we find a family of commuting Hamiltonian vector fields on $\mathcal{K}_n$ which are restrictions of the Toda vector fields, while for j even the Toda vector fields X_j are not tangent to $\mathcal{K}_n$. In order to conclude that the KM systems are a.c.i. we need to describe the fibers of the momentum map $K : \mathcal{K}_n \to \mathbb{C}[x]$. This will be done in the next paragraph.

4.2. Algebraic integrability of KM. We first define a map $\Phi : \mathcal{T}_n \to \mathcal{M}'_{n-1}$ which maps the n-body Toda system to the even Mumford system. The following identity, valid for tridiagonal matrices, will be needed.

Lemma 4.1. *Let M be a tridiagonal matrix,*

$$M = \begin{pmatrix} \beta_1 & \alpha_1 & 0 & \cdots & & 0 & 0 \\ \gamma_1 & \beta_2 & \alpha_2 & & & & 0 \\ 0 & \gamma_2 & \beta_3 & & & & \vdots \\ \vdots & & \ddots & \ddots & & & \vdots \\ 0 & & & & & \beta_{n-1} & \alpha_{n-1} \\ 0 & 0 & \cdots & \cdots & & \gamma_{n-1} & \beta_n \end{pmatrix},$$

and denote by $\Delta_{i_1,\dots,i_k}$ the determinant of the minor of M obtained by removing from M the rows $i_1, \dots, i_k$ and the columns $i_1, \dots, i_k$. Then:

$$\Delta_1 \Delta_n - \Delta \Delta_{1,n} = \prod_{i=1}^{n-1} \alpha_i \gamma_i. \tag{14}$$

Proof. For $n = 2$ this is obvious. For $n > 2$ one proceeds by induction, using the following formula for calculating the determinant Δ of M,

$$\Delta = \beta_n \Delta_n - \alpha_{n-1} \gamma_{n-1} \Delta_{n-1,n}. \tag{15}$$

□

In the sequel we use the notation $\Delta_{i_1,\dots,i_k}$ from the above lemma taking as M the tridiagonal matrix obtained from $x \operatorname{Id} - L(h)$ in the obvious way, i.e., by removing the two terms that depend on h. In this notation the characteristic polynomial of $L(h)$ is given by

$$\det(x \operatorname{Id} - L(h)) = -h - h^{-1} + \Delta - a_n \Delta_{1,n}. \tag{16}$$

Proposition 4.2. *For any $m = 1, \dots, n$ the map $\Phi_m : \mathcal{T}_n \to \mathcal{M}'_{n-1}$ defined by*

$$\begin{aligned} u(x) &= \Delta_m, \\ v(x) &= a_{m-1} \Delta_{m-1,m} - a_m \Delta_{m,m+1}, \\ w(x) &= (x - b_m)^2 \Delta_m + 2(x - b_m)(a_{m-1} \Delta_{m-1,m} + a_m \Delta_{m,m+1}) \\ &\quad + 4 a_m a_{m-1} \Delta_{m-1,m,m+1}, \end{aligned} \tag{17}$$

maps each fiber of the momentum map $K : \mathcal{T}_n \to \mathbb{C}[x]$ into a fiber of the momentum map $H : \mathcal{M}'_{n-1} \to \mathbb{C}[x]$. The restriction of Φ_m to $\mathcal{K}_n$ takes values in $\mathcal{P}_{\frac{n-1}{2}}$ when n is odd and in $\mathcal{P}'_{\frac{n}{2}-1}$ when n is even, mapping in both case the fiber of the momentum map $K : \mathcal{K}_n \to \mathbb{C}[x]$ into the fiber of the momentum map $H : \mathcal{P}_{\frac{n-1}{2}} \to \mathbb{C}[x^2]$ (or $H : \mathcal{P}'_{\frac{n}{2}-1} \to \mathbb{C}[x^2]$). As a consequence the general fiber of the momentum map of the KM systems is an affine part of a hyperelliptic Jacobian.

Proof. Since the momentum map is equivariant with respect to the $\mathbb{Z}/n$ action on $\mathcal{T}_n$ it suffices to prove the proposition for $m = n$.

It is easy to see that the triple (u, v, w), defined by (17) satisfies the constraints u, w monic, $\deg w = \deg u + 2 = n + 1$ and $\deg v < n - 1$, so that Φ_n takes values in $\mathcal{M}'_{n-1}$. Moreover, taking $\beta_1 = \cdots = \beta_n = x$ in (15) implies that when all entries on the diagonal of $L(h)$ are zero then $\Delta_{i_1,\ldots,i_p}$ has the same parity as $n - p$, so that the triples (u, v, w) which correspond to points in $\mathcal{K}_n$ have the additional property that v has the same parity as n while u and w have the opposite parity. Therefore the restriction of Φ_n to $\mathcal{K}_n$ takes values in $\mathcal{P}_{\frac{n-1}{2}}$ when n is odd and in $\mathcal{P}'_{\frac{n}{2}-1}$ when n is even.

For $p(x)$ a monic polynomial of degree n, let $L(h) \in K^{-1}(2p(x))$, i.e.,

$$p(x) = (x - b_n)\Delta_n - a_n\Delta_{1n} - a_{n-1}\Delta_{n-1,n}. \tag{18}$$

Proving that $\Phi_n(L(h))$ belongs to $H^{-1}(p^2(x) - 4)$ amounts to showing that $u(x)w(x) + v^2(x) = p^2(x) - 4$, which follows from a direct computation, using (14). The commutativity of the following diagram follows:

$$\begin{array}{ccc} \mathcal{T}_n & \xrightarrow{\Phi} & \mathcal{M}'_{n-1} \\ {\scriptstyle K}\downarrow & & \downarrow{\scriptstyle H} \\ \mathbb{C}[x] & \xrightarrow[\phi]{} & \mathbb{C}[x] \end{array}$$

where ϕ is defined by $\phi(q) = (q/2)^2 - 4$, for $q \in \mathbb{C}[x]$.

To show that the map Φ_n is injective let $(u(x), v(x), w(x)) \in \Phi_n(\mathcal{T}_n)$. We show that the matrix $L(h) \in \mathcal{T}_n$ which is mapped to this point is unique.

First observe that the monic polynomial $p(x) = \Delta - a_n\Delta_{1,n}$ can be recovered from $u(x)w(x) + v(x)^2 = p(x)^2 - 4$. We can then determine b_n from the following two formulas:

$$p(x) = x^n - \left(\sum_{i=1}^{n} b_i\right)x^{n-1} + \cdots,$$

$$u(x) = \Delta_n = x^{n-1} - \left(\sum_{i=1}^{n-1} b_i\right)x^{n-2} + \cdots.$$

Next, the second relation in (17) and (18) lead to the system:

$$\begin{cases} a_{n-1}\Delta_{n-1,n} - a_n\Delta_{1,n} = v(x), \\ a_{n-1}\Delta_{n-1,n} + a_n\Delta_{1,n} = (x - b_n)u(x) - p(x). \end{cases}$$

This linear system completely determines the products $a_n\Delta_{1,n}$ and $a_{n-1}\Delta_{n-1,n}$. Because the determinants of the principal minors of $x\,\mathrm{Id} - L(h)$ are monic polynomials, this means that we know a_n, $\Delta_{1,n}$ and $\Delta_{n-1,n}$ separately. From $\Delta = p(x) + a_n\Delta_{1,n}$ we also obtain Δ.

We have now shown how b_n, a_n, Δ, Δ_n and $\Delta_{n-1,n}$ are determined. We proceed by induction, showing how to determine b_{n-k-1}, a_{n-k-1}, $\Delta_{n-k-1,\ldots,n}$ once we know b_{n-i}, a_{n-i} and $\Delta_{n-i,\ldots,n}$ for $i = 0, \ldots, k$. We use (15) to obtain the recursive relation:

$$\Delta_{n-k+1,\ldots,n} = (x - b_{n-k})\Delta_{n-k,\ldots,n} - a_{n-k-1}\Delta_{n-k-1,\ldots,n}.$$

This determines the product $a_{n-k-1}\Delta_{n-k-1,\dots,n}$, but also a_{n-k-1} and $\Delta_{n-k-1,\dots,n}$ separately, again because $\Delta_{n-k-1,\dots,n}$ is monic. Now from $\Delta_{n-k-1,\dots,n}$ and $\Delta_{n-k,\dots,n}$ we know, as above, the sums $\sum_{i=1}^{n-k-2} b_i$ and $\sum_{i=1}^{n-k-1} b_i$. Hence, b_{n-k-1} is determined. $\square$

We saw in Prop. 3.3 that the fibers of the momentum map of the even Prym system are reducible (two isomorphic pieces), so there remains the question if the same is true for the n-body KM system for even n. To check that this is so, note that the highest degree coefficient of the characteristic polynomial of $L(h)$ gives, for n even, the first integral $I = a_1a_3a_5\cdots a_{n-1} + a_2a_4a_6\cdots a_n$. Since $a_1a_2\cdots a_n = 1$, for generic values of I, the variety defined by

$$a_1a_3a_5\cdots a_{n-1} = \text{constant}, \quad a_2a_4a_6\cdots a_n = \text{constant},$$

is reducible, and the claim follows. Note however that both $a_1a_3a_5\cdots a_{n-1}$ and $a_2a_4a_6\cdots a_n$ are first integrals themselves, so we can construct a momentum map using these integrals (instead of their sum and product) and then the general fiber is irreducible.

The map $\Phi_m : \mathcal{T}_n \to \mathcal{M}'_{n-1}$ not only maps fibers to fibers of the momentum maps, but it maps the whole hierarchy of Toda flows to the Mumford flows defined by (7). To see this, we construct a family of quadratic Poisson brackets $\{\cdot,\cdot\}^{\varphi}_{\mathcal{M},q}$ on $\mathcal{M}'_{n-1}$ which make this map Poisson.

First observe that there exist unique polynomials $p(x)$ and $r(x)$, with $p(x)$ monic of degree n and $r(x)$ of degree less than n, such that

$$u(x)w(x) + v(x)^2 = p(x)^2 + r(x). \tag{19}$$

The coefficients of $p(x)$ and $r(x)$ are regular functions of u_i, v_i and w_i. Hence, we can define a skew-symmetric biderivation on the space of regular functions of $\mathcal{M}'_{n-1}$ by setting, for any $\varphi \in \mathbb{C}[x]$ of degree at most 1,

$$\begin{aligned}
\left\{u(x), u(x')\right\}^{\varphi}_{\mathcal{M},q} &= \left\{v(x), v(x')\right\}^{\varphi}_{\mathcal{M}} = 0,\\
\left\{u(x), v(x')\right\}^{\varphi}_{\mathcal{M},q} &= \left\{u(x), v(x')\right\}^{p\varphi}_{\mathcal{M}} + \alpha^{\varphi}(x+x')u(x)u(x'),\\
\left\{u(x), w(x')\right\}^{\varphi}_{\mathcal{M},q} &= \left\{u(x), w(x')\right\}^{p\varphi}_{\mathcal{M}} - 2\alpha^{\varphi}(x+x')u(x)v(x'),\\
\left\{v(x), w(x')\right\}^{\varphi}_{\mathcal{M},q} &= \left\{v(x), w(x')\right\}^{p\varphi}_{\mathcal{M}} + \alpha^{\varphi}(x+x')u(x)w(x')),\\
\left\{w(x), w(x')\right\}^{\varphi}_{\mathcal{M},q} &= \left\{w(x), w(x')\right\}^{p\varphi}_{\mathcal{M}} + 2\alpha^{\varphi}(x+x')\left(w(x)v(x') - w(x')v(x)\right)),
\end{aligned}$$

where $\alpha^{\varphi}(x) = \varphi(\alpha(2x)/2)$. Notice that the polynomial $p\varphi$, used in the definition of the bracket, depends on the phase variables.

Proposition 4.3. *Let φ be a polynomial of degree at most 1. Then*

(i) $\{\cdot,\cdot\}^{\varphi}_{\mathcal{M},q}$ *is a Poisson bracket on $\mathcal{M}'_{n-1}$ and the maps*

$$\Phi_m : (\mathcal{T}_n, \{\cdot,\cdot\}^{\varphi}_{\mathcal{T}}) \to (\mathcal{M}'_{n-1}, \{\cdot,\cdot\}^{\varphi}_{\mathcal{M},q})$$

are Poisson and map the Toda flows to the Mumford flows;

(ii) *For φ odd, the bracket $\{\cdot,\cdot\}^{\varphi}_{\mathcal{M},q}$ induces a Poisson bracket $\{\cdot,\cdot\}_{\mathcal{P},q}$ on $\mathcal{P}_{(n-1)/2}$ (resp. on $\mathcal{P}'_{n/2-1}$), and the maps*

$$\Phi_m : (\mathcal{K}_n, \{\cdot,\cdot\}_{\mathcal{K}}) \to (\mathcal{P}_{(n-1)/2}, \{\cdot,\cdot\}_{\mathcal{P},q})$$
$$\Phi_m : (\mathcal{K}_n, \{\cdot,\cdot\}_{\mathcal{K}}) \to (\mathcal{P}'_{n/2-1}, \{\cdot,\cdot\}_{\mathcal{P},q})$$

are Poisson and map the flows of the n-body KM system to the flows of the hyperelliptic Prym systems.

Proof. We take the bracket of both sides of (19) with $u(x)$ to obtain

$$2p(y)\varphi(y)\frac{u(x)v(y)-u(y)v(x)}{x-y} = 2p(y)\{u(x),\,p(y)\}^{\varphi}_{\mathcal{M},q} + \{u(x),r(y)\}^{\varphi}_{\mathcal{M},q}\,.$$

It follows that $\{u(x),r(y)\}^{\varphi}_{\mathcal{M},q}$ is divisible by $p(y)$. Since $\{u(x),r(y)\}^{\varphi}_{\mathcal{M},q}$ is of degree less than n in y and since $p(y)$ is monic of degree n we must have $\{u(x),r(y)\}^{\varphi}_{\mathcal{M},q}=0$ and

$$\{u(x),\,p(y)\}^{\varphi}_{\mathcal{M},q} = \frac{u(x)v(y)-u(y)v(x)}{x-y}\varphi(y).$$

Similarly, we find $\{v(x),r(y)\}^{\varphi}_{\mathcal{M},q} = \{w(x),r(y)\}^{\varphi}_{\mathcal{M},q} == 0$ and also that:

$$\{v(x),\,p(y)\}_{\mathcal{M},q} = \frac{\varphi(y)}{2}\left(\frac{w(x)u(y)-u(x)w(y)}{x-y} - \alpha(x+y)u(x)u(y)\right),$$
$$\{w(x),\,p(y)\}_{\mathcal{M},q} = \varphi(y)\left(\frac{v(x)w(y)-w(x)v(y)}{x-y} + \alpha(x+y)v(x)u(y)\right).$$

These expressions also allow one to compute the brackets of $u(x)$, $v(x)$, $w(x)$ and $p(x)$ with $\alpha(y)$, and the check of the Jacobi identity follows easily from it. Therefore, $\{\cdot,\cdot\}^{\varphi}_{\mathcal{M},q}$ is a Poisson bracket for which the coefficients of $r(x)$ are Casimirs.

If we compare the expressions above for the brackets with $p(y)$ with expressions (7) for the Mumford vector fields, we conclude that they are Hamiltonian with respect to $\{\cdot,\cdot\}^{1}_{\mathcal{M},q}$ with Hamiltonian function K. Checking that Φ_m is Poisson can be done by a straightforward (but rather long) computation using the following expressions for the derivatives of $\Delta_{i_1,\dots,i_k}$:

$$\frac{\partial\Delta_{i_1,\dots,i_k}}{\partial a_i} = \begin{cases} -\Delta_{i,i+1,i_1,\dots,i_k}, & i,i+1\notin\{i_1,\dots,i_k\}\,,\\ 0 & \text{otherwise,}\end{cases}$$
$$\frac{\partial\Delta_{i_1,\dots,i_k}}{\partial b_i} = \begin{cases} -\Delta_{i,i_1,\dots,i_k}, & i\notin\{i_1,\dots,i_k\}\,,\\ 0 & \text{otherwise.}\end{cases}$$

For the second statement, one easily checks that when φ is odd then $\jmath$ is a Poisson involution, so that there is an induced bracket on $\mathcal{P}_{(n-1)/2}$ or on $\mathcal{P}'_{n/2-1}$. Explicit formulas for this bracket are computed as in the proof of Proposition 3.5. The other statements in (ii) then follow from (i). □

It is easy to check that the Poisson brackets $\{\cdot,\cdot\}^{\varphi}_{\mathcal{M},q}$ and $\{\cdot,\cdot\}^{\psi}_{\mathcal{M}}$ on $\mathcal{M}'_{n-1}$ are compatible, when φ and ψ have degree at most 1. This is however not true when ψ is of higher degree.

5. Painlevé Analysis

The results in the previous section show that the general fiber of the momentum map of the KM systems is an affine part of a hyperelliptic Prym variety (or two copies of it), which can also be described as a hyperelliptic Jacobian. In order to describe precisely which affine part we determine the divisor which needs to be adjoined to each affine part in order to complete it into an Abelian variety. Since it is difficult to do this by using the maps Φ_m we do this by performing Painlevé analysis of the KM systems.

The method that we use is based on the bijective correspondence between the principal balances of an integrable vector field (Laurent solutions depending on the maximal number of free parameters) and the irreducible components of the divisor which is missing from the fibers of the momentum map (see [1]).

We look for all Laurent solutions

$$a_i(t) = \frac{1}{t^r}\sum_{j=0}^{\infty} a_i^{(j)} t^j, \tag{20}$$

to the vector field (13) of the n-body KM system. The following lemma shows that any such Laurent solution of (13) can have at most simple poles. We may suppose that r in (20) is maximal, i.e., $a_i^{(0)} \neq 0$ for at least one i, and we call r the *order* of the Laurent solution. The order of pole (or zero) of $a_i(t)$ is denoted by r_i, so $r = \max_i r_i$.

Lemma 5.1. *Let the Laurent series $a_i(t)$, $i = 1, \ldots, n$, given by (20) be a solution to the vector field (13) of the n-body KM system. If at least one of the a_i has a pole (for $t = 0$) then it is a Laurent solution of order 1. Moreover the orders of the pole (or zero) of each $a_i(t)$ satisfy*

$$r_i = a_{i+1}^{(0)} - a_{i-1}^{(0)}. \tag{21}$$

Proof. For $s \in \mathbb{N}$ we find from (20):

$$\operatorname*{Res}_{t=0} \frac{\dot a_i(t)}{a_i(t)} t^s = \begin{cases} -r_i, & s = 0 \\ 0, & s > 0. \end{cases}$$

On the other hand, if we use (13) then we find

$$\operatorname*{Res}_{t=0} \frac{\dot a_i(t)}{a_i(t)} t^s = \operatorname*{Res}_{t=0} \left(a_{i-1}(t) - a_{i+1}(t)\right) t^s = a_{i-1}^{(r-s-1)} - a_{i+1}^{(r-s-1)}.$$

We conclude that

$$a_{i-1}^{(k)} - a_{i+1}^{(k)} = \begin{cases} -r_i, & k = r-1 \\ 0, & 0 \le k \le r-2. \end{cases} \tag{22}$$

Now substituting (20) into (13) and comparing the coefficient of $1/t^{r+1}$ the following equation (*the indicial equation*) is obtained:

$$-r a_i^{(0)} = a_i^{(0)}(a_{i-1}^{(0)} - a_{i+1}^{(0)}), \quad i = 1, \ldots, n. \tag{23}$$

If a_i has a pole of order $r > 0$ then $a_i^{(0)} \neq 0$ and (23) implies $a_{i-1}^{(0)} - a_{i+1}^{(0)} = -r$. Comparing with (22) we see that we must have $r = 1$ and that (21) holds. □

Notice that in view of the periodicity of the indices ($a_{i+n} = a_i$) the linear system

$$1 = (a_{i+1}^{(0)} - a_{i-1}^{(0)}), \quad i = 1, \dots, n,$$

has no solutions, so that at least one of the $a_i^{(0)}$ vanishes. If, say, $a_0^{(0)} = a_{k+1}^{(0)} = 0$ while $a_i^{(0)} \neq 0$ for $i = 1, \dots, k$ for some k in the range $1, \dots, n-1$ (this includes the case of a single i for which $a_i^{(0)} = 0$) then the indicial equation specializes to

$$\begin{aligned}
&a_2^{(0)} = 1,\\
&a_{i+1}^{(0)} - a_{i-1}^{(0)} = 1, \quad i = 2, \dots, k-1,\\
&a_{k-1}^{(0)} = -1,
\end{aligned}$$

which has no solution for k odd, and which has a unique solution $(a_1^{(0)}, \dots, a_k^{(0)}) = (-l, 1, 1-l, 2, \dots, -1, l)$ for even k, $k = 2l$. The other variables $a_{k+1}^{(0)} \dots, a_n^{(0)}$ can either be all zero, or they can constitute one or several other solutions of this type (with varying $k = 2l$), separated by zeroes. Using periodicity the other solutions to the indicial equation are obtained by cyclic permutation.

Thus we are led to the following combinatorial description of the solutions to the indicial equation of the n-body KM system. For a subset A of $\mathbb{Z}/n$, and for $p \in \mathbb{Z}/n$ let us denote by $A(p) \subset \mathbb{Z}/p$ the largest subset of A that contains p and that consists of consecutive elements (with the understanding that $A(p) = \emptyset$ when $p \notin A$). If we define

$$\Sigma_n = \{A \subset \mathbb{Z}/n \mid p \in A \Rightarrow \#A(p) \text{ is even}\},$$

then we see that the solutions to the indicial equation are in one to one correspondence with the elements of Σ_n. In the sequel we freely use this bijection. For $A \in \Sigma_n$ we call the integer $\#A/2$ its *order*, denoted by $\operatorname{ord} A$.

For each solution to the indicial equation (i.e., for each $A \in \Sigma_n$) we compute the eigenvalues of the Kowalevski matrix M, whose entries are given by

$$M_{ij} = \frac{\partial F_i}{\partial a_j}(a_1^{(0)}, \dots, a_n^{(0)}) + \delta_{ij},$$

where $F_i = a_i(a_{i-1} - a_{i+1})$, the i^{th}component of (13). The number of non-negative integer eigenvalues of this matrix are precisely the number of free parameters of the family of Laurent solutions whose leading term is given by $(a_1^{(0)}, \dots, a_n^{(0)})$ (see [1]), hence we can deduce from it which strata of the Abelian variety, whose affine part appears as a fiber of the momentum map, are parameterized by it.

Proposition 5.2. *For a solution of the indicial equation corresponding to* $A \in \Sigma_n$ *the Kowalevski matrix* M *has* $n - \operatorname{ord} A$ *non-negative integer eigenvalues.*

Proof. In view of (21) the entries of M can be written in the form

$$M_{ij} = \begin{cases} (1 - r_i)\delta_{i,j}, & \text{if } a_i^{(0)} = 0\\ a_i^{(0)}(\delta_{i,j+1} - \delta_{i,j-1}), & \text{if } a_i^{(0)} \neq 0. \end{cases}$$

Note also that, by using the $\mathbb{Z}/n$ action, we can assume that $1 \in A$, $n \notin A$, and that A is a disjoint union of $A(p_1), \dots, A(p_s)$, with $p_1 < p_2 < \cdots < p_s$. Let $l_i = \operatorname{ord} A(p_i)$. Then M has the following form:

$$M = \begin{pmatrix} C_1 & E_1 & & & & & & -l_1 \\ & D_1 & & & & & 0 & \\ & & C_2 & E_2 & & & & \\ & & & D_2 & & & & \\ & & & & \ddots & & & \\ & 0 & & & & C_s & E_s & \\ & & & & & & D_s & \end{pmatrix}.$$

On the upper right corner the matrix has entry $-l_1$, and the blocks C_i,D_i and E_i, $i = 1, \dots, s$, are matrices as follows:

- C_i is a tridiagonal matrix of size $2l_i$ of the form:

$$C_i = \begin{pmatrix} 0 & l_i & & & & & & \\ 1 & 0 & -1 & & & & & \\ & 1-l_i & 0 & l_i-1 & & & & \\ & & 2 & 0 & -2 & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & l_i-1 & 0 & 1-l_i & \\ & & & & & -1 & 0 & 1 \\ & & & & & & l_i & 0 \end{pmatrix};$$

- D_i is a diagonal matrix of the form $D_i = \operatorname{diag}(1 + l_i, 1, \dots, 1, 1 + l_{i+1})$, with the convention that if D_i is 1×1 then its only entry is $1 + l_i + l_j$;
- E_i is a matrix with only one non-zero entry $-l_i$ in the lower left corner.

It is clear that the set of eigenvalues of M is the union of the set of eigenvalues of the C_i's and D_i's. Now we have:

Lemma 5.3. *The eigenvalues of the matrix C_i are* $\{\pm 1, \pm 2, \dots, \pm l_i\}$.

Assuming the lemma to hold we find that the number of negative eigenvalues of M is equal to $\sum_{i=1}^{s} l_i = \sum_{i=1}^{s} \operatorname{ord} A(p_i) = \operatorname{ord} A$, so the proposition follows.

So we are left with the proof of the lemma. We write l for l_i and we denote by $\mathbf{e}_j$ the j^{th}vector of the standard basis of $\mathbb{C}^{2l}$. In the basis $\mathbf{e}_1, \mathbf{e}_3, \dots, \mathbf{e}_{2l-3}, \mathbf{e}_{2l-1}, \mathbf{e}_{2l}, \mathbf{e}_{2l-2}, \dots, \mathbf{e}_4, \mathbf{e}_2$ the matrix C_i takes form

$$\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix},$$

where A is the transpose of the matrix

$$\Lambda = \begin{pmatrix} 0 & \dots & \dots & 0 & 1 \\ \vdots & & 0 & 2 & -1 \\ \vdots & \iddots & 3 & -2 & 0 \\ 0 & \iddots & \iddots & \iddots & \vdots \\ l & l-1 & 0 & \dots & 0 \end{pmatrix}.$$

We show that this matrix has eigenvalues $1, -2, 3, \dots, (-1)^{l-1}l$. Then the result follows because the eigenvalues of C are $\pm$ the eigenvalues of A.

For $j = 1, \dots, l$, let $\mathbf{f}_j = [1^{j-1}, 2^{j-1}, \dots, l^{j-1}]^T$ and let V_j denote the span of $\mathbf{f}_1, \dots, \mathbf{f}_j$. For $v = [v_1, \dots, v_l]^T \in \mathbb{C}^l$ we have that $v \in V_j$ if and only if there exists a polynomial P of degree less than j such that $v_k = P(k)$ for $k = 1, \dots, l$. Since the k^{th}component of $\Lambda \mathbf{f}_j$ is given by

$$k(l-k+1)^{j-1} + (1-k)(l-k+2)^{j-1} = (-1)^{j-1} j k^{j-1} \left(1 + O\left(\frac{1}{k}\right)\right),$$

we have that $\Lambda \mathbf{f}_j \subset V_j$, more precisely

$$\Lambda \mathbf{f}_j \in (-1)^{j-1} j \mathbf{f}_j + V_{j-1}.$$

This means that in terms of the basis $\{\mathbf{f}_j\}$ the matrix Λ is upper triangular, with the integers $1, -2, 3, \dots, (-1)^{l-1}l$ on the diagonal. □

By the proposition above we can have a Laurent solution depending on $n-1$ free parameters (a principal balance) only for the n choices of A given by $(a_1^{(0)}, \dots, a_n^{(0)}) = (-1, 1, 0, \dots, 0)$ and their cyclic permutations. Let us check that these lead indeed to asymptotic expansions which formally solve (13). By §2 in [1], these solutions are actually convergent and so they define convergent Laurent solutions.

It suffices to do this for the solution $(a_1^{(0)}, \dots, a_n^{(0)}) = (-1, 1, 0, \dots, 0)$ of the indicial equation. By (21) we know that the order of the singularities of this solution are $(r_1, \dots, r_n) = (1, 1, -1, 0, \dots, -1)$ so we have the following ansatz for the formal expansions:

$$\begin{aligned} a_1(t) &= -\frac{1}{t} + \alpha_1 + \beta_1 t + O(t^2), \\ a_2(t) &= \frac{1}{t} + \alpha_2 + \beta_2 t + O(t^2), \\ a_3(t) &= \beta_3 t + O(t^2), \\ a_j(t) &= \alpha_j + \beta_j t + O(t^2), \quad 4 \le j \le n-1, \\ a_n(t) &= \beta_n t + O(t^2). \end{aligned}$$

If we replace these expansions in Eq. (13) defining the n-body KM system we obtain the consistency equations:

$$\begin{aligned}\alpha_1 - \alpha_2 &= 0,\\ 2\beta_1 - \beta_2 &= -\alpha_1\alpha_2 - \beta_n,\\ \beta_1 - 2\beta_2 &= -\alpha_1\alpha_2 + \beta_3,\\ \beta_j &= \alpha_j(\alpha_{j-1} - \alpha_{j+1}), \quad 4 \le j \le n-1.\end{aligned}$$

They give exactly the $n-1$ free parameters $\alpha_1, \alpha_4, \ldots, \alpha_{n-1}, \beta_3, \beta_n$. The coefficients $\mathbf{a}^{(k)} = (a_1^{(k)}, \ldots, a_n^{(k)})$ for $k > 2$ are then completely determined since they satisfy an equation of the form

$$(M - kI) \cdot \mathbf{a}^{(k)} = \text{some polynomial in the } a_i^{(j)} \text{ with } j < k,$$

and the eigenvalues of the Kowalevski matrix M are $-1, 1, 2$, by the proof above.

This leads to the following result.

Theorem 5.4. *When n is odd the general fiber of the momentum map of the n-body KM system is an affine part of a hyperelliptic Prym variety, obtained by removing n translates of its theta divisor. When n is even the general fiber consists of two isomorphic components which admit the same description as in the odd case. In both cases the Prym variety admits an alternative description as a hyperelliptic Jacobian.*

6. Example: $n = 5$

In this section we study the 5-body KM system in more detail. Its phase space is four-dimensional and is given by $\mathcal{K}_5 = \{(a_1, a_2, a_3, a_4, a_5) \mid a_1a_2a_3a_4a_5 = 1\}$, with Lax operator

$$L = \begin{pmatrix} 0 & a_1 & 0 & 0 & h^{-1} \\ 1 & 0 & a_2 & 0 & 0 \\ 0 & 1 & 0 & a_3 & 0 \\ 0 & 0 & 1 & 0 & a_4 \\ ha_5 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The spectral curve $\det(x\,\mathrm{Id} - L) = 0$ is explicitly given by

$$h + \frac{1}{h} = x^5 - Kx^3 + Lx,$$

where

$$\begin{aligned}K &= a_1 + a_2 + a_3 + a_4 + a_5,\\ L &= a_1a_3 + a_2a_4 + a_3a_5 + a_4a_1 + a_5a_2.\end{aligned}$$

These functions are in involution with respect to the quadratic Poisson structure, given by $\{a_i, a_j\} = (\delta_{i,j+1} - \delta_{i+1,j})a_ia_j$. It follows from the previous section that for generic k, l the affine surface $\mathcal{P}_{kl}$ defined by $K = k,\ L = l$ is an affine part of the Jacobian of the genus two Riemann surface Γ_τ minus five translates of its theta divisor, which is isomorphic to Γ_τ. As we have seen, an equation for $\Gamma_\tau^{(0)}$ is given by

$$\Gamma_\tau^{(0)} : y^2 = (u^3 - ku^2 + lu)^2 - 4u. \tag{24}$$

The two commuting Hamiltonian vector fields X_K and X_L are given by

$$\begin{aligned}
\dot a_1 &= a_1(a_5 - a_2), & a_1' &= a_1(a_3a_5 - a_2a_4),\\
\dot a_2 &= a_2(a_1 - a_3), & a_2' &= a_2(a_4a_1 - a_3a_5),\\
\dot a_3 &= a_3(a_2 - a_4), & a_3' &= a_3(a_5a_2 - a_4a_1),\\
\dot a_4 &= a_4(a_3 - a_5), & a_4' &= a_4(a_1a_3 - a_5a_2),\\
\dot a_5 &= a_5(a_4 - a_1), & a_5' &= a_5(a_2a_4 - a_1a_3).
\end{aligned}$$

The principal balance of X_K for which a_1 and a_2 have a pole corresponds, according to Sect. 5, to the following solution of the indicial equations:

$$(a_1^{(0)}, a_2^{(0)}, a_3^{(0)}, a_4^{(0)}, a_5^{(0)}) = (-1, 1, 0, 0, 0),$$

and its first few terms are given by

$$\begin{aligned}
a_1 &= -\frac{1}{t} + \alpha - \frac{1}{3}(\alpha^2 + 2\beta + \gamma)t + O(t^2),\\
a_2 &= \frac{1}{t} + \alpha + \frac{1}{3}(\alpha^2 - \beta - 2\gamma)t + O(t^2),\\
a_3 &= \gamma t + O(t^2),\\
a_4 &= \delta + O(t^2),\\
a_5 &= \beta t + O(t^2).
\end{aligned} \tag{25}$$

Here $\alpha,\ \beta,\ \gamma$ and δ are the free parameters. If we look for Laurent solutions that correspond to the divisor to be added to $\mathcal{P}_{kl}$ we find by substituting the above Laurent solution in $K = k,\ L = l,\ a_1a_2a_3a_4a_5 = 1$,

$$\begin{cases} 2\alpha + \delta = k,\\ 2\alpha\delta + \beta - \gamma = l,\\ \gamma\beta\delta = -1, \end{cases}$$

which means that the Laurent solution depends on two parameters β and δ, bound by the relation

$$(k - \delta)\delta + \beta + \frac{1}{\beta\delta} = l, \tag{26}$$

which is an (affine) equation for the theta divisor, i.e., for Γ_τ; it is easy to see that this curve is birational to the curve (24). The other four principal balances are obtained by cyclic permutation from (25).

$\mathcal{P}_{kl}$ can be embedded explicitly in projective space by using the functions with a pole of order at most 3 along one of the translates of the theta divisor and no other poles. Since the theta divisor defines a principal polarization on its Jacobian, the vector space of such functions has dimension $3^2 = 9$, giving an embedding in $\mathbb{P}^8$. One checks by direct computation that the following functions $z_0, \dots, z_8$ form a basis for the space of functions with a pole of order at most 3 along the divisor associated with the Laurent solution (25) (the first two functions are obvious choices from the expression (25), while the others can be obtained from them by taking the derivative along the two flows):

$$
\begin{aligned}
z_0 &= 1,\\
z_1 &= a_1a_2,\\
z_2 &= a_1a_2a_4,\\
z_3 &= a_1a_2(a_1+a_5),\\
z_4 &= a_1a_2a_4(a_3+a_4+a_5),\\
z_5 &= a_1a_2a_4(a_1-a_2),\\
z_6 &= a_1a_2a_4((a_3+a_4)a_1-(a_4+a_5)a_2),\\
z_7 &= a_1^2a_2^2a_4a_5,\\
z_8 &= a_1a_2^2a_4((a_4+a_5)^2+a_3a_4).
\end{aligned}
$$

The corresponding embedding of the Jacobian in $\mathbb{P}^8$ is then given explicitly on the affine surface $\mathcal{P}_{kl}$ by $(a_1, \dots, a_5) \mapsto (z_0 : \cdots : z_8)$. By substituting the five principal balances in this embedding and letting $t \to 0$ we find an embedding of the five curves $\Gamma_1, \dots, \Gamma_5$ (in that order) which constitute the divisor $\mathrm{Jac}(\Gamma_\tau) \setminus \mathcal{P}_{kl}$:

$$
(\beta,\delta) \mapsto \begin{cases}
(0:0:0:1:0:2\delta:2\delta^2:\beta\delta:-\delta^3)\\
(\beta\delta^2:-\beta^2\delta^2:0:-\beta^2\delta^3:\beta\delta:\beta\delta:\beta\delta^2:0:1-\beta\delta^3)\\
(1:0:\beta\delta:0:\beta\delta(k-\delta):\beta\delta^2:-\beta\delta(\beta+\delta^2-k\delta):0:\beta^2\delta(k-\delta))\\
(\beta^2\delta:0:\beta\delta:-\beta\delta:\beta\delta(k-\delta):-\beta\delta^2:1+\beta\delta^2(\delta-k)\\
\qquad\qquad :-\delta:-\beta\delta^2(\beta-(\delta-k)^2))\\
(\beta\delta^2:-\delta:0:\delta(\delta-k):\beta\delta:-\beta\delta:-\beta\delta^2:-1:1).
\end{cases}
$$

The points on the divisor that correspond to the above Laurent solutions are the ones for which β and δ are finite; notice that all these points in $\mathbb{P}^8$ are different. In order to determine the coordinates of the other points and the incidence relations between these points and the curves Γ_i we choose a local parameter around each of the three points needed to complete (26) into a compact Riemann surface:

(a) $\delta = 1/u,\ \beta = 1/u^2(1+O(t))$;
(b) $\delta = 1/u,\ \beta = u^3(1+O(t))$;
(c) $\beta = 1/u,\ \delta = -u^2(1+O(t))$.

Substituting these in the equations of the five embedded curves we find the following 5 points (each one is found 3 times because it belongs to three of the curves Γ_i)

$$
\begin{aligned}
p_1 &= (0:0:0:1:0:0:0:0:0),\\
p_2 &= (0:0:0:0:0:0:0:0:1),\\
p_3 &= (1:0:0:0:0:0:1:0:-k),\\
p_4 &= (1:0:0:0:0:0:-1:0:0),\\
p_5 &= (0:0:0:0:0:0:0:1:-1).
\end{aligned}
$$

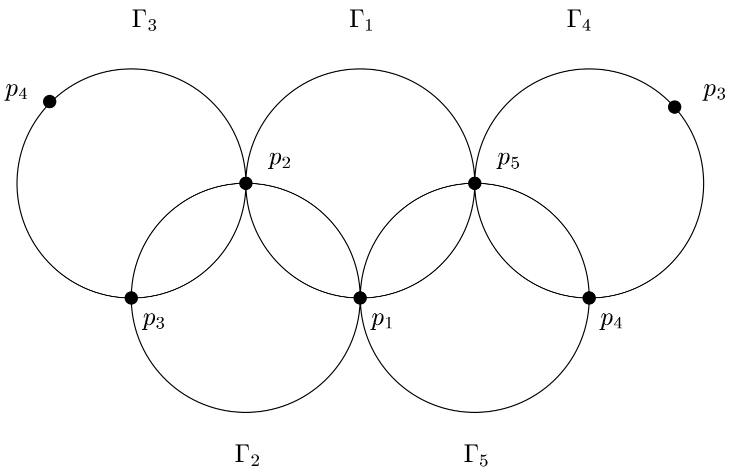

Fig. 2.

With this labeling of the points p_i we have that Γ_i contains the points p_{i-1}, p_i and p_{i+1}. As a corollary we find a 5_3 configuration on the Jacobian, where the incidence pattern of the 5 Painlevé divisors and the 5 points p_i is as in the following picture (to make the picture exact one has to identify the two points labeled p_3, as well as the two points labeled p_4 in such a way that the curves Γ_2 and Γ_4 are tangent, as well as the curves Γ_3 and Γ_5).

Obviously the order 5 automorphism

$$(a_1, a_2, a_3, a_4, a_5) \mapsto (a_2, a_3, a_4, a_5, a_1)$$

preserves the affine surfaces $\mathcal{P}_{kl}$ and maps every curve Γ_i and every point p_i to its neighbor. Since this automorphism does not have any fixed points it is a translation on $\mathrm{Jac}(\Gamma_\tau)$, and since its order is 5 it is a translation over $1/5$ of a period. Notice also that with the above labeling of points and divisors the intersection point between Γ_i and Γ_{i+2} is p_{i+1} (so they are tangent), while the intersection points between Γ_i and Γ_{i+1} are p_i and p_{i+1}. Dually, the divisors that pass through p_i are precisely Γ_{i-1}, Γ_i and Γ_{i+1}. The usual Olympic rings are nothing but an asymmetric projection of this most beautiful Platonic configuration!

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