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# Poisson Fibrations and Fibered Symplectic Groupoids

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Abstract. We show that Poisson fibrations integrate to a special kind of symplectic fibrations, called fibered symplectic groupoids.

## 1. Introduction

Our purpose in this paper is to explain that there is a geometric theory of Poisson fibrations that is analogous to the theory of symplectic fibrations. This paper makes no special claims to originality, and improves on previous works of Vorobjev (Poisson case, [**15**]) and Vaisman (Dirac case, [**14**]). Our main contribution is two folded. On one hand, we propose an approach based on gauge theory and Dirac geometry, which gives some natural explanations for some of the mysterious formulas that appear in those works. On the other hand, we look for the first time into the integration of such structures, recovering symplectic fibrations from Poisson fibrations.

A symplectic fibration is a locally trivial fibration with fiber type a symplectic manifold, admitting a collection of trivializations whose transition functions are symplectomorphisms. Symplectic fibrations have a long history going back to the early works of Weinstein *et al.* [**18, 10**] and Guillemin *et al.* ([**12, 11**]). A closed 2-form on the total space of a fibration which restricts to a symplectic form on the fibers, determines a symplectic fibration. Conversely, given a symplectic fibration, one may ask if there exists a closed 2-form which is compatible with the fibration. It is well-known that there are non-trivial obstructions for the existence of such coupling forms and that one can classify all such coupling forms. A very nice exposition of the the theory of symplectic fibrations, where these results are discussed in detail, is given in Chapter 6 of the monograph by McDuff and Salamon [**13**].

We are interested in the more general notion of a *Poisson fibration*, i.e., a locally trivial fibration with fiber type a Poisson manifold, admitting a collection of trivializations whose transition functions are Poisson diffeomorphisms. At first sight, the analogous questions for Poisson geometry are either hopeless or trivial. On one hand, there are simple examples of fibrations with a Poisson structure on

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the total space that restricts to each fiber and which is not a Poisson fibration. On the other hand, every Poisson fibration always admits trivially a compatible Poisson structure: one can just declare the fibers to be Poisson submanifolds.

A closer inspection, however, reveals a very different point of view. Note that for a symplectic fibration one looks for a *presymplectic structure* which intersects each fiber in a symplectic submanifold. Therefore, for a Poisson fibration we should look for a *Dirac structure* for which each presymplectic leaf intersects every fiber in a symplectic leaf of the Poisson structure on the fiber (recall that a Poisson manifold is a (singular) foliation by symplectic manifolds, while a Dirac manifold is a (singular) foliation by presymplectic manifolds). This yields the notion of a *Dirac coupling* for a Poisson fibration. The theory of Poisson fibrations can then be seen as a foliated version of the theory of symplectic fibrations (this is the point of view advocated by Vaisman [**14**]).

Therefore, the questions (and answers) in the theory of Poisson fibrations, are analogous to (and generalize) the theory of symplectic fibrations. For example, given a Poisson fibration one would like to know (i) if it admits coupling Dirac structures and (ii) classify all such couplings. In this direction we have the following generalization of a well-known result in symplectic fibrations (see [**13**, Theorem 6.13]):

THEOREM 1.1. *Let $p : M \to B$ be a Poisson fibration. Then the following statements are equivalent:*

*(i) $p : M \to B$ admits a coupling a Dirac structure.*
*(ii) There exists a Poisson connection on $p : M \to B$ whose holonomy groups act on the fibers in a hamiltonian fashion.*

Its is also possible to classify all such coupling forms. The proof of this result follows the same pattern as in the symplectic case: one builds a Poisson gauge theory where coupling forms are obtained on associated fiber bundles $M = P \times_G F$, starting from a connection on a principal $G$-bundle and a Hamiltonian $G$-action on a Poisson manifold $(F, \pi)$. Note that $G$ is not necessarily a finite dimensional Lie group.

Our second main purpose is the *integration* of Poisson fibrations. Recall that Poisson structures are infinitesimal objects which integrate to global objects called symplectic groupoids. There are obstructions to integrability which were recently understood [**4, 3**], but we will ignore them for the time being. Now, we will see that:

- The global object associated to a Poisson fibration is a *fibered symplectic groupoid.*

Let us explain what we mean by this. By a *fibered groupoid* we mean a groupoid $\mathcal{G} \rightrightarrows M$ which is fibered over $B$:

$$\begin{tikzcd}
\mathcal{G} \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dr] & & M \arrow[dl] \\
& B &
\end{tikzcd}$$

Here, $\mathcal{G}$ and $B$ are fibered as well as all structure maps. So a fibered groupoid maybe thought of as a fiber bundle with fiber type a groupoid where the structure group acts by groupoid automorphism. Then by a *fibered symplectic groupoid* we mean a fibered groupoid $\mathcal{G}$ whose fiber type is a symplectic groupoid. For a fibered symplectic groupoid, $\mathcal{G} \to B$ is a symplectic fibration and the symplectic structure is compatible with the groupoid structure. The base $p : M \to B$ of a fibered symplectic groupoid has a natural structure of a Poisson fibration. Moreover, the fibers of $\mathcal{G} \to B$ are symplectic groupoids over the fibers of $p : M \to B$, inducing the Poisson structures on the fibers. In particular, the fibers $p : M \to B$ are integrable Poisson manifolds. Conversely, we will show the following:

THEOREM 1.2. *Let $p : M \to B$ be a Poisson fibration with fiber type $(F, \pi)$ an integrable Poisson manifold. There exists a unique (up to isomorphism) source 1-connected fibered symplectic groupoid integrating $p : M \to B$. Moreover, this symplectic fibration always admits a coupling 2-form.*

It was proved in [**2**] that the global objects integrating Dirac structures are presymplectic groupoids. Therefore, if $p : M \to B$ is a Poisson fibration which admits a coupling Dirac structure, there are two natural groupoids associated with it:

(i) The source 1-connected fibered symplectic groupoid integrating $p : M \to B$.
(ii) The source 1-connected presymplectic groupoid integrating the coupling Dirac structure of $p : M \to B$.

The precise relationship between these two objects is more involved, and it would take us too far afield, so it will be discussed elsewhere.

## 2. Connections and Dirac structures

One can define a connection on a fibration by specifying on the total space an almost Dirac structure of a special type. In this section we make a preliminary study of connections on fibrations induced by Dirac structures.

**2.1. Connections defined by almost Dirac structures.** In what follows, by a *fibration* we always mean a locally trivial fiber bundle. Given such a fibration $p : M \to B$, we denote by $F_b$ the fiber over $b \in B$, and we let $\mathrm{Vert} \subset TM$ be the *vertical sub-bundle* whose fibers are $\mathrm{Vert}_x \equiv \mathrm{Ker}\, \mathrm{d}_x p = T_x F_{p(x)}$. By a *connection* $\Gamma$ on $p : M \to B$ we mean an Ehresmann connection, i.e., a distribution $\Gamma : x \mapsto \mathrm{Hor}_x$ in $TM$ which splits the tangent bundle as

$$T_x M = \mathrm{Hor}_x \oplus \mathrm{Vert}_x, \tag{2.1}$$

and satisfies the following lifting property:

**Lifting Property:** for each $x_0 \in M$ and each curve $\gamma : [0, 1] \to B$ starting at $b_0 = p(x_0)$ there exists an integral curve $\tilde{\gamma} : [0, 1] \to M$ of $\Gamma$ starting at $x_0$ and covering $\gamma$.

REMARK 2.1. Given a $C^1$-path $\gamma : [0, 1] \to B$ and $x \in F_{\gamma(0)}$ the splitting (2.1) guarantees that there exists a unique *horizontal lift* $\tilde{\gamma}_x : [0, \varepsilon] \to M$ starting at $x$. The Lifting Property says that we can take $\varepsilon = 1$ ([1]).

In the usual way, one obtains the notion of *parallel transport* of fibers: given a piecewise $C^1$-path on the base $\gamma : [0, 1] \to B$ we have the diffeomorphism:

$$\phi_\gamma(t) : F_{\gamma(0)} \to F_{\gamma(t)}, \qquad x \mapsto \tilde{\gamma}_x(t).$$

[1]Note that our curves are always parameterized in the interval $[0, 1]$.

When $\gamma$ is a loop, we call $\phi_\gamma(1)$ the *holonomy* of $\Gamma$ along $\gamma$. For $b \in B$, the *holonomy group* with base point $b$ is the subgroup $\Phi(b) \subset \mathrm{Diff}\,(F_b)$ formed by all holonomy transformations $\phi_\gamma(1)$, where $\gamma : [0,1] \to B$ is any loop based at $b$. Note that to define the concatenation of loops we need to reparameterize our curves, but this causes no problem since "horizontal lift commutes with concatenation" and, hence, two paths differing by a reparameterization determine the same holonomy transformation.

Now our basic observation is that connections can be defined by specifying on the total space of the fibration an almost Dirac structure of a special type. Namely:

DEFINITION 2.2. Let $p : M \to B$ be a fibration. An almost Dirac structure $L$ on the total space of the fibration is called **fiber non-degenerate** if

$$(\mathrm{Vert} \oplus \mathrm{Vert}^0) \cap L = \{0\}. \tag{2.2}$$

In fact, we have:

PROPOSITION 2.3. *Let $p : M \to B$ be a fibration and $L$ a fiber non-degenerate almost Dirac structure. Then $L$ defines a connection $\Gamma_L$ with horizontal space:*

$$\mathrm{Hor} = \left\{X \in TM : \exists \alpha \in (\mathrm{Vert})^0, (X,\alpha) \in L\right\}. \tag{2.3}$$

*Moreover, any connection on $p : M \to B$ can be obtained in this way.*

PROOF. Take a vector space $W$, with a maximal isotropic subspace $L \subset W \times W^*$, and let $V \subset W$ be a subspace such that:

$$(V \times V^0) \cap L = \{0\}.$$

We claim that

$$W = V \oplus H,$$

where $H = \left\{w \in W : \exists \xi \in V^0, (w,\xi) \in L\right\}$. If we take $W = T_xM$, $L = L_x$ and $V = \mathrm{Vert}_x$, this claim yields the first part of the proposition.

To prove the claim, we start by checking that $V \cap H = \{0\}$. In fact, if $v \in H$ there exists $\xi \in V^0$ such that $(v,\xi) \in L$. Hence, if $v \in V \cap H$ we obtain:

$$(v,\xi) \in (V \times V^0) \cap L = \{0\},$$

so we must have $v = 0$. It remains to check that $W = V + H$. First, we observe that since $L$ and $V \times V^0$ intersect trivially and $\dim L = \dim(V \times V^0) = \dim W$, we must have:

$$W \times W^* = (V \times V^0) \oplus L.$$

Therefore, for any $w \in W$ we have a decomposition:

$$(w,0) = (v,\xi) + (h,\eta),$$

where $v \in V$, $\xi \in V^0$ and $(h,\eta) \in L$. It follows that $\eta = -\xi \in V^0$, so that $h \in H$. We conclude that $w = v + h$, with $v \in V$ and $h \in H$, as claimed.

Conversely, if $\Gamma$ is a connection with horizontal distribution $\mathrm{Hor}$, then $L = \mathrm{Hor} \oplus \mathrm{Hor}^0$ defines a fiber non-degenerate almost Dirac structure whose associated connection is $\Gamma$. Hence, every connection arises in this way. $\square$

**2.2. The horizontal 2-form and the vertical bivector field.** Note that two fiber non-degenerate almost Dirac structures $L_1$ and $L_2$ may lead to the same connection $\Gamma_{L_1} = \Gamma_{L_2}$. In fact, as we will see now, there is more structure associated with the specification of a fiber non-degenerate almost Dirac structure.

PROPOSITION 2.4. *Let $p : M \to B$ be a fibration and $L$ a fiber non-degenerate almost Dirac structure, with associated connection $\Gamma_L$. Then the horizontal distribution* $\mathrm{Hor}$ *is contained in the characteristic distribution of $L$, and the pull-back of the natural 2-form yields a smooth 2-form $\omega_L \in \Omega^2(\mathrm{Hor})$.*

We will refer to $\omega_L$ as the **horizontal 2-form of** $L$.

PROOF. Fix a fiber non-degenerate almost Dirac structure $L$ on the total space of a fibration $p : M \to B$. Relations (2.2) and (2.3) together show that, for each $X \in \mathrm{Hor}$, there exists a unique $\alpha \in \mathrm{Vert}^0$ such that $(X, \alpha) \in L$. One can then define a skew-symmetric bilinear form $\omega : \mathrm{Hor} \times \mathrm{Hor} \to \mathbb{R}$ by:

$$\omega_L(X_1, X_2) := \frac{1}{2}\left(\alpha_1(X_2) - \alpha_2(X_1)\right), \tag{2.4}$$

with $\alpha_1, \alpha_2 \in \mathrm{Vert}^0$ the unique elements such that $(X_1, \alpha_1), (X_2, \alpha_2) \in L$. Since $L$ is maximal isotropic we have:

$$0 = 2\langle (X_1, \alpha_1), (X_2, \alpha_2)\rangle_+ = \alpha_1(X_2) - \alpha_2(X_1),$$

so this two form can also be written:

$$\omega_L(X_1, X_2) = \alpha_1(X_2) = -\alpha_2(X_1). \tag{2.5}$$

In this way we obtain a smooth 2-form $\omega_L \in \Omega^2(\mathrm{Hor})$.

From the definition (2.3) of $\mathrm{Hor}$ it is clear that the horizontal distribution $\mathrm{Hor}$ is contained in the characteristic distribution of $L$. From (2.5), it is clear that $\omega_L$ is the pull-back of the natural 2-form on the characteristic distribution of $L$. □

This construction of the horizontal 2-form can be dualized:

PROPOSITION 2.5. *Let $p : M \to B$ be a fibration and let $L$ be a fiber non-degenerate almost Dirac structure, with associated connection $\Gamma_L$. For each fiber $i : F_x = p^{-1}(x) \hookrightarrow M$ the pull-back almost Dirac structure $i^*L$ is well-defined and coincides with the graph of a bivector field $\pi_L \in \mathfrak{X}^2(\mathrm{Vert})$.*

We will refer to $\pi_L$ as the **vertical bivector field of** $L$.

PROOF. First observe that the annihilator of the horizontal space is:

$$\mathrm{Hor}^0 = \{\alpha \in T^*M : \exists X \in \mathrm{Vert}, (X, \alpha) \in L\}. \tag{2.6}$$

Relations (2.2) and (2.6) together show that, for each $\alpha \in \mathrm{Hor}^0$, there exists a unique $X \in \mathrm{Vert}$ such that $(X, \alpha) \in L$. One can then define a skew-symmetric bilinear form $\pi_L : \mathrm{Hor}^0 \times \mathrm{Hor}^0 \to \mathbb{R}$ by:

$$\pi_L(\alpha_1, \alpha_2) := \frac{1}{2}\left(\alpha_1(X_2) - \alpha_2(X_1)\right), \tag{2.7}$$

with $X_1, X_2 \in \mathrm{Vert}$ the unique elements such that $(X_1, \alpha_1), (X_2, \alpha_2) \in L$. Since $L$ is maximal isotropic we have:

$$0 = 2\langle (X_1, \alpha_1), (X_2, \alpha_2)\rangle = \alpha_1(X_2) - \alpha_2(X_1),$$

the form $\pi_L : \mathrm{Hor}^0 \times \mathrm{Hor}^0 \to \mathbb{R}$ can also be written:

$$\pi_L(\alpha_1, \alpha_2) = \alpha_1(X_2) = -\alpha_2(X_1). \tag{2.8}$$

Now we remark that the splitting $TM = \mathrm{Hor} \oplus \mathrm{Vert}$ allows us to identify $\mathrm{Hor}^0 = \mathrm{Vert}^*$, so $\pi_L$ becomes a bivector field on the fibers of $p : M \to B$.

Let us fix a fiber $i : F_x = p^{-1}(x) \hookrightarrow M$. Then $TF_x = \mathrm{Vert}$ and we identify $T^*F_x = \mathrm{Vert}^* \simeq \mathrm{Hor}^0$. The pull-back Dirac structure $i^*L$ is then given by:

$$\begin{aligned} i^*L &= \{(X, \alpha|_{\mathrm{Vert}}) \in \mathrm{Vert} \oplus \mathrm{Vert}^* : (X, \alpha) \in L\} \\ &= \left\{(X, \alpha) \in \mathrm{Vert} \oplus \mathrm{Hor}^0 : X = \pi_L(\alpha, \cdot)\right\} = \mathrm{graph}(\pi_L), \end{aligned}$$

where for the last inequality we have used (2.8). □

Putting together these results, we conclude that:

Corollary 2.6. *To a fiber non-degenerate almost Dirac structure $L$ on a fibration $p : M \to B$ there is associated the following data:*

- *A connection $\Gamma_L$ on $p : M \to B$.*
- *A horizontal 2-form $\omega_L \in \Omega^2(\mathrm{Hor})$.*
- *A vertical bivector field $\pi_L \in \mathfrak{X}^2(\mathrm{Vert})$.*

*Conversely, every such triple $(\Gamma, \omega, \pi)$ on a fibration $p : M \to B$ determines a unique fiber non-degenerate almost Dirac structure $L$, which is given by:*

$$L = \mathrm{graph}\,(\pi_L) \oplus \mathrm{graph}\,(\omega_L). \tag{2.9}$$

**2.3. Fiber non-degenerate Dirac structures.** The next natural question is: Given a fiber non-degenerate almost Dirac structure $L$ on a fibration $p : M \to B$, what are the conditions on the associated triple $(\Gamma_L, \omega_L, \pi_L)$ that guarantee that $L$ is integrable, i.e., is a Dirac structure?

Recall (see [**5**]) that the obstruction to integrability for an almost Dirac structure $L$ is a 3-form $T_L \in \Omega^3(L)$, which is defined on sections $s_1, s_2, s_3 \in \Gamma(L)$ by:

$$T_L(s_1, s_2, s_3) := \langle [\![s_1, s_2]\!], s_3\rangle_+ \tag{2.10}$$

where:

- $[\![\cdot, \cdot]\!]$ denotes the *Courant bracket*, on $\mathfrak{X}(M) \oplus \Omega^1(M)$, given by:

$$[\![(X, \alpha), (Y, \beta)]\!] := ([X, Y], \mathcal{L}_X\beta - \mathcal{L}_Y\alpha + \mathrm{d}\langle(X, \alpha), (Y, \beta)\rangle_-) \tag{2.11}$$

- $\langle\cdot, \cdot\rangle_+$ denotes the *natural pairing* on $\mathfrak{X}(M) \times \Omega^1(M)$, defined by:

$$\langle(X, \alpha), (Y, \beta)\rangle_+ := \frac{1}{2}\left(i_Y\alpha + i_X\beta\right). \tag{2.12}$$

Our next result gives the 3-form $T_L$ of a fiber non-degenerate almost Dirac structure $L$ in terms of the geometric data $(\Gamma_L, \pi_L, \omega_L)$. We need to introduce some notation.

For a vector field $v \in \mathfrak{X}(B)$ we denote by $\widetilde{v} \in \mathfrak{X}(M)$ its horizontal lift, and we let $\pi_L^\# : \mathrm{Vert}^* \to \mathrm{Vert}$ denote the bundle map induced by the vertical bivector field $\pi_L$. Also, we have isomorphisms:

$$\mathrm{Hor} \simeq L \cap (TM \oplus \mathrm{Vert}^0), \quad \mathrm{Hor}^0 \simeq L \cap (\mathrm{Vert} \oplus T^*M).$$

These allow us to identify a horizontal vector field $X \in \Gamma(\mathrm{Hor})$ with a section $s_X = (X, \alpha) \in \Gamma(L)$, where $\alpha \in \Gamma(\mathrm{Vert}^0)$, and a vertical form $\beta \in \Gamma(\mathrm{Hor}^0)$ with a section $s_\beta = (Y, \beta) \in \Gamma(L)$, where $Y \in \Gamma(\mathrm{Vert})$. In this notation we have:

PROPOSITION 2.7. *Let $(\Gamma_L, \pi_L, \omega_L)$ be the geometric data determined by a fiber non-degenerate almost Dirac structure $L$ on a fiber bundle $p : M \to B$. Then:*

*(i) If $\alpha, \beta, \gamma \in \Gamma(\mathrm{Hor}^0)$ then:*

$$T_L(s_\alpha, s_\beta, s_\gamma) = \frac{1}{2}[\pi_L, \pi_L](\alpha, \beta, \gamma).$$

*(ii) If $v \in \mathfrak{X}(B)$ and $\beta, \gamma \in \Gamma(\mathrm{Hor}^0)$, then:*

$$T_L(s_{\widetilde{v}}, s_\beta, s_\gamma) = \frac{1}{2}\mathcal{L}_{\widetilde{v}}\pi_L(\beta, \gamma).$$

*(iii) If $v_1, v_2 \in \mathfrak{X}(B)$ and $\gamma \in \Gamma(\mathrm{Hor}^0)$, then:*

$$T_L(s_{\widetilde{v}_1}, s_{\widetilde{v}_2}, s_\gamma) = \frac{1}{2}(\gamma(\Omega_{\Gamma_L}(\widetilde{v}_1, \widetilde{v}_2)) + \pi_L(\mathrm{d}i_{\widetilde{v}_1} i_{\widetilde{v}_2}\omega_L, \gamma))$$

*where $\Omega_{\Gamma_L}$ is the curvature 2-form of $\Gamma_L$.*

*(iv) If $v_1, v_2, v_3 \in \mathfrak{X}(B)$, then:*

$$T_L(s_{\widetilde{v}_1}, s_{\widetilde{v}_2}, s_{\widetilde{v}_3}) = \frac{1}{2}\mathrm{d}_{\Gamma_L}\omega_L(\widetilde{v}_1, \widetilde{v}_2, \widetilde{v}_3),$$

*where $\mathrm{d}_{\Gamma_L} : \Omega^\bullet(\mathrm{Hor}) \to \Omega^{\bullet+1}(\mathrm{Hor})$ is the differential induced by $\Gamma_L$.*

The proofs are routine calculations so we omit them.

Now observe that sections of the form $s_{\tilde{v}}$ and $s_\alpha$ with $v \in \mathfrak{X}(B)$ and $\alpha \in \Gamma(\mathrm{Hor}^0)$ generate $\Gamma(L)$, as a $C^\infty(M)$-module. Therefore, as a corollary, we obtain the conditions that the triple $(\Gamma_L, \pi_L, \omega_L)$ must satisfy for the associated $L$ to be a Dirac structure:

COROLLARY 2.8. *Let $(\Gamma_L, \pi_L, \omega_L)$ be the geometric data determined by a fiber non-degenerate almost Dirac structure $L$ on a fiber bundle $p : M \to B$. Then $L$ is Dirac iff the following conditions hold:*

*(i) $\pi_L$ is a vertical Poisson structure: $[\pi_L, \pi_L] = 0$.*

*(ii) Parallel transport along $\Gamma_L$ preserves the vertical Poisson structure $\pi_L$:*

$$\mathcal{L}_{\widetilde{v}}\pi_L = 0, \quad \forall v \in \mathfrak{X}(B).$$

*(iii) The horizontal 2-form $\omega_L$ is closed: $\mathrm{d}_\Gamma\omega_L = 0$.*

*(iv) The following* ***curvature identity*** *is satisfied:*

$$\Omega_\Gamma(v_1, v_2) = \pi_L^\#(\mathrm{d}i_{\widetilde{v}_1} i_{\widetilde{v}_2}\omega_L), \quad \forall v_1, v_2 \in \mathfrak{X}(B). \tag{2.13}$$

The curvature identity (2.13) expresses the fact that the curvature 2-form of the connection associated with a fiber non-degenerate Dirac structure $L$ takes values in the vertical Hamiltonian vector fields. We will explore this property later in our study of Poisson fibrations.

**2.4. Presymplectic forms and Poisson structures.** Let us illustrate the previous results with the two extreme cases of Dirac structures determined by Poisson and presymplectic structures.

If $L$ is determined by a presymplectic form, one checks easily:

PROPOSITION 2.9. *Let $\Omega$ be a presymplectic form on the total space of a fibration $p : M \to B$. Then $L = \mathrm{graph}\,(\Omega)$ is a fiber non-degenerate Dirac structure iff the pull-back of $\Omega$ to each fiber is non-degenerate.*

In this case, the vertical Poisson structure $\pi_L$ is non-degenerate on the fibers and coincides with the inverse of the restriction of $\Omega$ to the fibers.

The converse is also true: a fiber non-degenerate Dirac structure $L$ for which the vertical Poisson structure $\pi_L$ is non-degenerate on the fibers, is determined by a presymplectic form $\Omega$. In fact, it follows from (2.9) that:

$$\Omega = \omega_L \oplus (\pi_L)^{-1}.$$

Hence, fiber non-degenerate presymplectic forms are presymplectic forms which restrict to symplectic forms on the fibers.

Dually, for Poisson structures, it is also immediate to check:

PROPOSITION 2.10. *Let* $\Pi$ *denote a Poisson structure on the total space of a fibration* $p: M \to B$. *Then* $L = \mathrm{graph}\,(\Pi)$ *is a fiber non-degenerate Dirac structure iff* $\Pi$ *is horizontal non-degenerate, i.e.,* $\Pi|_{\mathrm{Vert}^0} : \mathrm{Vert}^0 \times \mathrm{Vert}^0 \to \mathbb{R}$ *is a non-degenerate bilinear form.*

In this case, the horizontal 2-form $\omega_L$ is non-degenerate: in fact, $\Pi$ gives an isomorphism $\mathrm{Vert}^0 \to \mathrm{Hor}$, and under this isomorphism $\omega_L$ coincides with the restriction $\Pi|_{\mathrm{Vert}^0}$.

The converse is also true: a fiber non-degenerate Dirac structure $L$ for which the horizontal 2-form $\omega_L$ is non-degenerate, is a Poisson structure $\Pi$. In fact, it follows from (2.9) that:

$$\Pi = (\omega_L)^{-1} \oplus \pi_L.$$

Hence, fiber non-degenerate Poisson structures are the same thing as the horizontal non-degenerate Poisson structures of Vorobjev ([**15**]).

## 3. Poisson fibrations

In this section we will study Poisson fibrations and their relationship to fiber non-degenerate Dirac structures.

**3.1. Poisson and symplectic fibrations.** Let $(F, \pi)$ be a Poisson manifold. We denote by $\mathrm{Diff}_\pi(F)$ the group of Poisson diffeomorphisms of $F$. This is the subgroup of $\mathrm{Diff}\,(F)$ formed by all diffeomorphisms $\phi : F \to F$ such that:

$$(\phi)_* \pi = \pi.$$

We are interested in the following class of fibrations:

DEFINITION 3.1. A **Poisson fibration** $p : M \to B$ is a locally trivial fiber bundle, with fiber type a Poisson manifold $(F, \pi)$ and with structure group a subgroup $G \subset \mathrm{Diff}_\pi(F)$. More precisely, $p$ is a submersion such that there exists a covering $\{U_i\}_i$ of $B$, and trivializations $\phi_i : p^{-1}(U_i) \to U_i \times F$, with transition functions $\phi_j \circ \phi_i^{-1}{}_{|\{x\}\times F}$, $x \in \mathcal{U}_i \cap \mathcal{U}_j$ belonging to $G$. When $\pi$ is symplectic the fibration is called a **symplectic fibration**.

If $p : M \to B$ is a Poisson fibration modeled on a Poisson manifold $(F, \pi)$, each fiber $F_b$ carries a natural Poisson structure $\pi_b$: if $\phi_i : p^{-1}(U_i) \to U_i \times F$ is a local trivialization, $\pi_b$ is defined by:

$$\pi_b = (\phi_i(b)^{-1})_* \pi,$$

for $b \in U_i$. It follows from the definition that this 2-vector field is independent of the choice of trivialization. Note that the Poisson structures $\pi_b$ on the fibers can be glued to a Poisson structure $\pi_V$ on the total space of the fibration:

$$\pi_V(x) = \pi_{p(x)}(x), \quad (x \in M).$$

This 2-vector field is *vertical*: $\pi_V$ takes values in $\wedge^2\mathrm{Vert} \subset \wedge^2 TM$. In this way, the fibers $(F_b, \pi_b)$ become Poisson submanifolds of $(M, \pi_V)$.

Example 3.2. An important class of Poisson fibrations is obtained as follows. Take any Poisson manifold $(P, \Pi)$, fix a *closed* symplectic leaf $B$ of $P$, and let $p : M \to B$ be a tubular neighborhood of $B$ in $P$. Each fiber carries a natural Poisson structure, namely, the transverse Poisson structure ([**17**]). These transverse Poisson structures are all Poisson diffeomorphic and it follows from the Weinstein splitting theorem that this is a Poisson fibration.

More generally, one can take any Dirac manifold $(P, L)$ and fix a closed presymplectic leaf $B$ of $P$. If $p : M \to B$ is a tubular neighborhood of $B$ in $P$, then each fiber carries a natural Poisson structure, called also the transverse Poisson structure ([**8**]). These transverse Poisson structures are all Poisson diffeomorphic and it follows from the generalization of the Weinstein splitting theorem in [**8**] that this is a Poisson fibration.

We saw in the previous section than any fiber non-degenerate Dirac structure $L$ on the total space of a fibration $p : M \to B$ induces Poisson structures on the fibers. In fact, we have the following:

Proposition 3.3. *Let $p : M \to B$ be a fibration with connected base and compact fibers. If $L$ is a fiber non-degenerate Dirac structure, then $p : M \to B$ admits the structure of a Poisson fibration such that $\pi_V = \pi_L$.*

Proof. Corollary 2.8 gives the vertical Poisson structure. While trivializations are obtained by parallel transport along paths in $B$, induced by the connection ($F$ being any chosen fiber). Compactness of the fibers here just ensures completeness of horizontal lifts. □

Note that a Dirac structure on the total space of a fibration $p : M \to B$ for which the fibers are Poisson-Dirac submanifolds may fail to be a Poisson fibration. In other words, Proposition 3.3 becomes false if one omits the assumption of fiber non-degeneracy. This is illustrated by the following simple example.

Example 3.4. Take the fibration $p : \mathbb{R}^3 \to \mathbb{R}$ obtained by projection on the $x$-axis and the Poisson bracket on $\mathbb{R}^3$ defined by:

$$\{x, y\} = \{x, z\} = 0, \quad \{y, z\} = x.$$

Then each fiber is a Poisson-Dirac submanifold: the fiber over $x = 0$ has the zero Poisson structure, while the fibers over $x \neq 0$ are symplectic. Since the fibers are not Poisson diffeomorphic, $p : \mathbb{R}^3 \to \mathbb{R}$ cannot be a Poisson fibration.

**3.2. Coupling Dirac structures.** Motivated by Proposition 3.3 we introduce the following definition:

Definition 3.5. If $p : M \to B$ is a Poisson fibration, we will say that a fiber non-degenerate Dirac structure $L$ is **compatible with the fibration** if $\pi_L = \pi_V$. In this case, we call $L$ a **coupling Dirac structure**.

Note that in the special case of a symplectic fibration, by the results of Section 2.4, a coupling Dirac structure is necessarily given by a presymplectic form $\Omega$. In this case, Proposition 3.3 is well-known (see [**13**], Lemma 6.2). Moreover, given a symplectic fibration $p : M \to B$, there are well-known non-trivial obstructions for the existence of a coupling 2-form $\omega \in \Omega^2(M)$. Our purpose now is to determine the corresponding obstructions for a general Poisson fibration and answer the following question:

- Given a Poisson fibration $p : M \to B$, is there a *coupling* Dirac structure $L$ compatible with the fibration?

Let us recall the notion of a Poisson connection:

Definition 3.6. A connection $\Gamma$ on a Poisson fibration $p : M \to B$ is called a **Poisson connection** if, for every path $\gamma$, parallel transport

$$\phi_\gamma : (F_{\gamma(0)}, \pi_{\gamma(0)}) \to (F_{\gamma(1)}, \pi_{\gamma(1)})$$

is a Poisson diffeomorphism.

Clearly, a connection $\Gamma$ on a Poisson fibration $p : M \to B$ is Poisson iff

$$\mathcal{L}_{\widetilde{v}}\pi_V = 0, \forall v \in \mathfrak{X}(B).$$

Hence, by Corollary 2.8 (ii), an obvious necessary condition for the existence of a coupling Dirac structure is the existence of a Poisson connection. However, one can show that a Poisson fibration always admits such a connection. In fact, we have:

Proposition 3.7. *Let $p : M \to B$ be a Poisson fibration. There exists a fiber non-degenerate almost Dirac structure $L$ on $p : M \to B$ such that:*

*(a) The vertical bivector field $\pi_L$ coincides with $\pi_V$.*
*(b) The connection $\Gamma_L$ is a Poisson connection.*

Proof. Let $p : M \to B$ be a Poisson fibration with fiber $(F, \pi)$ and choose choose local trivializations $\phi_i : p^{-1}(U_i) \to U_i \times F$. Let $L_i$ be the Dirac structure on $U_i \times F$ obtained by pull-back of $L_\pi = \operatorname{graph}(\pi)$ under the projection $U_i \times F \to F$:

$$L_i := \{((v, w), (0, \eta)) \in T(U_i \times F) \oplus T^*(U_i \times F) : w = \pi(\eta, \cdot)\}.$$

Observe that to $L_i$ it is associated the geometric data $(\Gamma_{L_i}, \pi_{L_i}, \omega_{L_i})$ where $\Gamma_{L_i}$ is the canonical flat connection on $U_i \times F \to U_i$, $\pi_{L_i} = \pi$ and $\omega_{L_i} = 0$. Hence $L_i$ is fiber non-degenerate, induces $\pi$ on the fibers, and $\Gamma_{L_i}$ is a Poisson connection.

Next, we choose a partition of unity $\rho_i : B \to \mathbb{R}$ subordinated to the cover $\{U_i\}$, and we define

$$L := \sum_i (\rho_i \circ p)\phi_i^* L_i.$$

By this we mean that the associated geometric data $(\Gamma_L, \pi_L, \omega_L)$ has connection $\Gamma_L = \sum_i (\rho_i \circ p)\phi_i^*\Gamma_{L_i}$, vertical bivector field $\pi_L = \sum_i (\rho_i \circ p)\phi_i^*\pi_{L_i}$ and horizontal 2-form $\omega_L = \sum_i (\rho_i \circ p)\phi_i^*\omega_{L_i}$.

It is clear that $L$ is a fiber non-degenerate almost Dirac structure. Moreover, $\pi_L$ coincides with $\pi_V$. Finally, given $v \in \mathfrak{X}(B)$, if $\widetilde{v}_i$ denotes the horizontal lift relative to $\phi_i^*\Gamma_{L_i}$, we have:

$$\mathcal{L}_{\widetilde{v}_i}\pi_V = 0.$$

The horizontal lift of $v$ relative to $\Gamma_L$ is $\widetilde{v} = \sum_i (\rho_i \circ p)\widetilde{v}_i$. It follows that:

$$\begin{aligned}
\mathcal{L}_{\widetilde{v}}\pi_V &= [\widetilde{v}, \pi_V] \\
&= \sum_i [(\rho_i \circ p)\widetilde{v}_i, \pi_V] \\
&= \sum_i \Big( (\rho_i \circ p)[\widetilde{v}_i, \pi_V] + \pi_V^{\#}(\mathrm{d}(\rho_i \circ p)) \Big) \\
&= \sum_i (\rho_i \circ p)\mathcal{L}_{\widetilde{v}_i}\pi_V = 0,
\end{aligned}$$

where we have used the fact that $\mathrm{d}(\rho_i \circ p) \in \mathrm{Vert}^0$. Hence $L$ is the desired almost Dirac structure. ☐

There is also a procedure to construct coupling Dirac structures due to A. Wade ([**16**]), which is entirely analogous to a construction in symplectic geometry due to A. Weinstein (see [**13**, Theorem 6.17]):

THEOREM 3.8. *Let $G \times F \to F$ be a Hamiltonian action of a compact Lie group $G$ on the Poisson manifold $(F, \pi)$. Every connection on a principal $G$-bundle $P \to B$ determines a coupling Dirac structure $L$ on the associated Poisson fibration $P \times_G F \to B$.*

PROOF. The connection $\Gamma$ on $P$ determines a projection $TP \to \mathrm{Vert}$, along the horizontal distribution $\mathrm{Hor}$. Hence, there is an injection:

$$i_\Gamma : \mathrm{Vert}^* \hookrightarrow T^*P,$$

where $\mathrm{Vert}^*$ is the vertical cotangent bundle with typical fiber $T^*P_b$. Since $\Gamma$ is $G$-invariant, the inclusion is $G$-equivariant. Therefore, the canonical symplectic form $\omega_{\mathrm{can}}$ in $T^*P$ induces a closed 2-form on $\mathrm{Vert}^*$

$$\omega_\Gamma = i_\Gamma^*\omega_{\mathrm{can}},$$

which is $G$-invariant and restricts to the canonical symplectic form on the fibers $T^*P_b$. Also, the action of $G$ on $\mathrm{Vert}^*$ has a moment map $\mu_P \circ i_\Gamma : \mathrm{Vert}^* \to \mathfrak{g}^*$, where $\mu_P$ is the moment map of the lifted cotangent action $G \times T^*P \to T^*P$.

Now let $G \times F \to F$ be a Hamiltonian action with moment map $\mu_F : F \to \mathfrak{g}^*$. This determines a Hamiltonian $G$-action on the Dirac manifold $M := \mathrm{Vert}^* \times F$, with Dirac structure

$$L = \mathrm{graph}\,(\omega_\Gamma) \oplus \mathrm{graph}\,(\pi).$$

and with moment map

$$\mu_M = (\mu_P \circ i_\Gamma) \oplus \mu_F.$$

This action is free and 0 is a regular value of $\mu_M$. Therefore, the reduced space

$$M_{\mathrm{red}} := \mu_M^{-1}(0)/G \simeq P \times_G F,$$

carries a Dirac structure. It is easy to check that this Dirac structure is fiber nondegenerate and restricts to the canonical Poisson structures on the fibers. Hence, $L$ is the desired coupling Dirac structure. ☐

REMARK 3.9. Note that the resulting coupling Dirac structure is presymplectic iff the fiber $(F, \pi)$ is symplectic. Also, it is easy to check that the remaining geometric data associated with the coupling Dirac structure $L$ in the theorem above is the following:

- The connection $\Gamma_L$ is just the connection on the fiber bundle $P \times_G F$ induced from the connection $\Gamma$ on $P$.
- The horizontal 2-form $\omega_L$ is given by the curvature of the connection composed with the moment map

$$\omega_L(\tilde{v}_1, \tilde{v}_2)([u, x]) = \langle \mu_F(x), F_\Gamma(\tilde{v}_1, \tilde{v}_2)_u \rangle, \quad ([u, x] \in P \times_G F).$$

Hence, the resulting coupling Dirac structure is Poisson iff the curvature 2-form of the connection is non-degenerate on the image of $\mu_F$. Such a connection is sometimes called a *fat* connection. The proof of A. Wade in [**16**] consists in proving that this data satifies the conditions of Corollary 2.8 and so defines a coupling.

**3.3. Obstruction to the existence of coupling.** The construction of Theorem 3.8 can be extended for general Poisson fibrations, leading to a characterization of those fibrations which admit a coupling Dirac structure.

Let $L$ be a fiber non-degenerate almost Dirac structure on a Poisson fibration $p : M \to B$. We will say that $L$ is compatible with the fibration if $\pi_L = \pi_V$. By Proposition 3.7, any Poisson fibration admits a compatible $L$ such that $\Gamma_L$ is a Poisson connection. For $L$ to be Dirac this connection must have a much more constrained holonomy:

Theorem 3.10. *Let $p : M \to B$ be a Poisson fibration and let $L$ be a compatible almost Dirac structure such that $\Gamma_L$ is a Poisson connection. Then the following statements are equivalent:*

*(i) $L$ is a Dirac structure: $T_L = 0$.*
*(ii) For every base point $b \in B$, the action of the holonomy group $\Phi(b)$ of $\Gamma_L$ on the fiber $F_b$ is Hamiltonian.*

A proof of this result will be given in the next paragraph, using a Poisson gauge theory. An immediate corollary is the following:

Corollary 3.11. *Let $(F, \pi)$ be a compact Poisson manifold whose first Poisson cohomology group vanishes: $H^1_\pi(F) = 0$. Then any Poisson fibration $p : M \to B$ with fiber $(F, \pi)$, and finite dimensional structure group admits a coupling Dirac structure.*

Proof. Since $H^1_\pi(F) = 0$, the same holds for the fibers $F_b$, and it follows that any Poisson action on the fibers is Hamiltonian. Now apply Proposition 3.7 to construct a fiber non-degenerate almost Dirac structure $L$ compatible with the fibration and such that $\Gamma_L$ is a Poisson connection. Using the implication (ii) $\Rightarrow$ (i) in Theorem 3.10, we conclude that $L$ is a coupling Dirac structure. Compacity of $F$ ensures completeness of the connection. □

Notice that the condition that the holonomy group $\Phi(b)$ of $\Gamma_L$ acts in a Hamiltonian fashion on the fiber $F_b$ is a property of its connected component of the identity $\Phi(b)^0$. This connected component is known as the *restricted holonomy group* and is formed by the holonomy homomorphisms $\phi_\gamma$, where $\gamma$ is a contractible loop based at $b$.

In particular, each $\phi_\gamma$ with $\gamma$ a contractible loop based at $b$, lies in the group of Hamiltonian diffeomorphisms $\mathrm{Ham}\,(F_b, \pi_b)$, which is known to be a normal subgroup of the group of Poisson diffeomorphisms $\mathrm{Diff}\,(F_b, \pi_b)$. The quotient group $\mathrm{Diff}\,(F_b, \pi_b)/\mathrm{Ham}\,(F_b, \pi_b)$ is known as the group of *outer Poisson diffeomorphisms.*

We conclude that a coupling Dirac structure $L$ for a Poisson fibration $p : M \to B$ has an associated *coupling holonomy homomorphism*:

$$\phi : \pi_1(B, b) \to \mathrm{Diff}\,(F_b, \pi_b)/\mathrm{Ham}\,(F_b, \pi_b), \quad [\gamma] \mapsto [\phi_\gamma].$$

EXAMPLE 3.12. A tubular neighborhood $p : M \to B$ of a symplectic leaf $B$ of a Poisson manifold $(P, \Pi)$ (see Example 3.2) admits $L_\Pi$ as a coupling Dirac structure. It follows that the connection $\Gamma_\Pi$ is Poisson and has Hamiltonian holonomy around any contractible loop in $B$. This can also be proved directly using the Weinstein splitting theorem.

In general, the holonomy around a non-contractible loop will not be Hamiltonian and we will have a nontrivial homomorphism

$$\phi : \pi_1(B) \to \mathrm{Diff}\,(F, \pi)/\mathrm{Ham}\,(F, \pi).$$

This is precisely the (reduced) Poisson holonomy of the leaf $B$ introduced in [**6**].

**3.4. Poisson Gauge Theory.** We now turn to the proof of Theorem 3.10. The idea will be to give an analogue of Theorem 3.8, but where the structure group is allowed to be infinite dimensional.

We consider a Poisson fibration $p : M \to B$ with fiber type a Poisson manifold $(F, \pi)$. The structure group of this fibration is the group $G = \mathrm{Diff}\,(F, \pi)$ of Poisson diffeomorphisms. The corresponding principal $G$-bundle is the **Poisson frame bundle**:

$$P \to B$$

whose fiber over a point $b \in B$ is formed by all Poisson diffeomorphisms $u : F \to F_b$. The group $G$ acts on (the right of) $P$ by pre-composition:

$$P \times G \to P : \ (u, g) \mapsto u \circ g.$$

Then our original Poisson fiber bundle is canonically isomorphic to the the associated fiber bundle: $M = P \times_G F$.

Every Poisson connection $\Gamma$ on the Poisson fiber bundle $p : M \to B$ is induced by a principal bundle connection on $P \to B$. To see this, observe that the tangent space $T_uP \subset C^\infty(u^*TM)$ at a point $u \in P$ is formed by the vector fields along $u$, $X(x) \in T_{u(x)}M$ such that:

$$d_{u(x)}p \cdot X(x) = \text{constant}, \quad \mathcal{L}_X \pi_V = 0.$$

The Lie algebra $\mathfrak{g}$ of $G$ is the space of Poisson vector fields: $\mathfrak{g} = \mathfrak{X}(F, \pi)$. The infinitesimal action on $P$ is given by:

$$\rho : \mathfrak{g} \to \mathfrak{X}(P), \quad \rho(X)_u = \mathrm{d}u \cdot X,$$

so the vertical space of $P$ is:

$$\mathrm{Vert}_{\,u} = \{\mathrm{d}u \cdot X : X \in \mathfrak{X}(F, \pi)\}\,.$$

Now a Poisson connection $\Gamma$ on $p : M \to B$ determines a connection in $P \to M$ whose horizontal space is:

$$\mathrm{Hor}_{\,u} = \{\widetilde{v} \circ u : v \in T_bB\}\,,$$

where $u : F \to F_b$ and $\widetilde{v} : F_b \to T_{F_b}M$ denotes the horizontal lift of $v$. Clearly, this defines a principal bundle connection on $P \to B$, whose induced connection on the associated bundle $M = P \times_G F$ is the original Poisson connection $\Gamma$.

Fix a Poisson connection $\Gamma$ on the Poisson fiber bundle $p : M \to B$. Recall that the holonomy group $\Phi(b)$ with base point $b \in B$ is the group of holonomy transformations $\phi_\gamma : F_b \to F_b$, where $\gamma$ is a loop based at $b$. Clearly, we have $\Phi(b) \subset \mathrm{Diff}\,(F_b, \pi_b)$. On the other hand, for $u \in P$ we have the holonomy group $\Phi(u) \subset G = \mathrm{Diff}\,(F, \pi)$ of the corresponding connection in $P$ which induces $\Gamma$: it consist of all elements $g \in G$ such that $u$ and $ug$ can be joined by a horizontal curve in $P$. Obviously, these two groups are isomorphic, for if $u : F \to F_b$ then:

$$\Phi(u) \to \Phi(b),\ g \mapsto u \circ g \circ u^{-1},$$

is an isomorphism.

The curvature of a principal bundle connection is a $\mathfrak{g}$-valued 2-form $F_\Gamma$ on $P$ which transforms as:

$$R_g^* F_\Gamma = \mathrm{Ad}\,(g^{-1}) \cdot F_\Gamma, \quad (g \in G).$$

Therefore, we can also think of the curvature as a 2-form $\Omega_L$ with values in the adjoint bundle $\mathfrak{g}_P := P \times_G \mathfrak{g}$. In the case of the Poisson frame bundle, the adjoint bundle has fiber over $b$ the space $\mathfrak{X}(F_b, \pi_b)$ of Poisson vector fields on the fiber. Hence the curvature of our Poisson connection can be seen as a 2-form $\Omega_\Gamma : T_bB \times T_bB \to \mathfrak{X}(F_b, \pi_b)$. The two curvature connections are related by:

$$\Omega_\Gamma = \mathrm{d}u \circ F_\Gamma \circ u^{-1}. \tag{3.1}$$

Finally, it is easy to check that, in fact, we have:

$$\Omega_\Gamma(v_1, v_2) = [\widetilde{v}_1, \widetilde{v}_2] - \widetilde{[v_1, v_2]},$$

which is the expression we have used before for the curvature.

After these preliminarities, we can now proceed to the proof.

Proof of Theorem 3.10. We will prove the two implications separately.

(i) $\Rightarrow$ (ii). Let us start by observing that given any $u \in P$, a Poisson diffeomorphism $u : F \to F_b$, the curvature identity (2.13) together with (3.1) shows that, for any $v_1, v_2 \in T_bB$, the vector field $F_\Gamma(v_1, v_2)_u \in \mathfrak{g} = \mathfrak{X}(F, \pi)$ is Hamiltonian:

$$F_\Gamma(v_1, v_2)_u = \pi^\# \mathrm{d}(\omega_L(\widetilde{v}_1, \widetilde{v}_2) \circ u).$$

Now fix $u_0 \in P$. The Holonomy Theorem states that the Lie algebra of the holonomy group $\Phi(u_0)$ is generated by all values $F_\Gamma(v_1, v_2)_u$, with $u \in P$ any point that can be connected to $u_0$ by a horizontal curve. Hence, we can define a moment map $\mu_F : F \to (\mathrm{Lie}(\Phi(u_0))^*)$ for the action of $\Phi(u_0)$ on $F$ by:

$$\langle \mu_F(x), F_\Gamma(v_1, v_2)_u \rangle = \omega_L(\widetilde{v}_1, \widetilde{v}_2)_{u(x)}. \tag{3.2}$$

This shows that the action of $\Phi(u_0)$ on $(F, \pi)$ is Hamiltonian, and so (ii) holds (recall the comments above about the relationship between the holonomy groups $\Phi(b)$ and $\Phi(u)$).

(ii) $\Rightarrow$ (i). Again we fix $u_0 \in P$, and we assume now that the action of $\Phi(u_0)$ on $(F, \pi)$ is Hamiltonian with moment map $\mu_F : F \to (\mathrm{Lie}(\Phi(u_0))^*)$. By the Reduction Theorem we can reduce the principal Poisson frame bundle to a principal $\Phi(u_0)$-bundle $P' \to B$. Now we can apply (the infinite dimensional version) of Theorem 3.8 to produce a coupling Dirac structure on the associated Poisson fiber bundle $p : M \to B$. Instead, if the reader does not like an infinite dimensional argument, he can check by himself that the geometric data formed by the connection $\Gamma_L$, the vertical Poisson vector field $\pi_V$ and the 2-form $\omega_L$ defined from (3.2) (we are now given $\mu_F$ and we define $\omega_L$) satisfy the conditions of Corollary 2.8. □

Remark 3.13. Note that our proof really shows that for *any* Poisson fibration the coupling Dirac structure arises as in the construction of Theorem 3.8. Relation (3.2) between the moment map $\mu_F$, the curvature of the connection, and the horizontal 2-form $\omega_L$, was already present there (see Remark 3.9).

## 4. Integration of Poisson fibrations

In this section, we study the integration of Poisson fibrations. Just as Poisson manifolds integrate to symplectic groupoids, we will see that Poisson fibrations integrate to fibered symplectic groupoids.

**4.1. Fibered symplectic groupoids.** Recall that for us a fibration always means a locally trivial fiber bundle. If we fix a base $B$, we have a category **Fib** of fibrations over $B$, where the objects are the fibrations $p : M \to B$ and the morphisms are the fiber preserving maps over the identity:

$$\begin{tikzcd} M_1 \arrow[rr, "\phi"] \arrow[dr, "p_1"'] & & M_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$

A *fibered groupoid* is an internal groupoid in **Fib**, i.e., an internal category where every morphisms is an isomorphism. This means that both the total space $\mathcal{G}$ and the base $M$ of a fibered groupoid are fibrations over $B$ and all structure maps are fibered maps. For example, the source and target maps are fiber preserving maps over the identity:

$$\begin{tikzcd} \mathcal{G} \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dr] & & M \arrow[dl] \\ & B & \end{tikzcd}$$

In particular, each fiber of $\mathcal{G} \to M$ is a groupoid over a fiber of $M \to B$. Moreover, the orbits of $\mathcal{G}$ lie inside the fibers of the base $M$.

A general procedure to construct fibered Lie groupoids is a follows. Let $P \to B$ be a principle $G$-bundle and assume that $G$ acts on a groupoid $\mathcal{F} \rightrightarrows F$ by groupoid automorphisms. Then the associated fiber bundles $\mathcal{G} = P \times_G \mathcal{F}$ and $M = P \times_G F$ are the spaces of arrows and objects of a fibered Lie groupoid. Clearly, every fibered Lie groupoid is of this form provided we allow infinite dimensional structure groups. We will say that the fibered Lie groupoid $\mathcal{G}$ has *fiber type* the Lie groupoid $\mathcal{F}$.

Definition 4.1. A **fibered symplectic groupoid** is a fibered Lie groupoid $\mathcal{G}$ whose fiber type is a symplectic groupoid $(\mathcal{F}, \omega)$.

Therefore, if $\mathcal{G}$ is a fibered symplectic groupoid over $B$, then $\mathcal{G} \to B$ is a symplectic fibration, and each symplectic fiber $\mathcal{F}_b$ is in fact a symplectic groupoid over the corresponding fiber $F_b$ of $M \to B$.

Proposition 4.2. *The base $M \to B$ of a fibered symplectic groupoid $\mathcal{G} \rightrightarrows M$ has a natural structure of a Poisson fibration.*

Proof. Note that (i) the base of any symplectic groupoid has a natural Poisson structure for which the source (respectively, the target) is a Poisson (respectively, anti-Poisson) map, and (ii) any symplectic groupoid isomorphism between two symplectic groupoids covers a Poisson diffeomorphism of the base Poisson manifolds. Hence, each fiber of the base $M \to B$ of a fibered symplectic groupoid carries a natural Poisson structure, and a trivialization of the fibered symplectic groupoid covers a trivialization of $M \to B$ whose transition functions are Poisson diffeomorphisms of the fibers. Therefore the result follows. □

Note that the fibers $p : M \to B$ are integrable Poisson manifolds.

**4.2. Integration of Poisson fibrations.** We just saw that the base of a fibered symplectic groupoid is a Poisson fibration. We will say that a fibered symplectic groupoid $\mathcal{G} \to B$ *integrates* a Poisson fibration $p : M \to B$ whenever this fibration is (Poisson) isomorphic to the Poisson fibration determined by $\mathcal{G} \to B$. If such a a fibered symplectic groupoid exists we say that the Poisson fibration is *integrable*. Note that the fiber type $\mathcal{F}$ of $\mathcal{G}$ is a a symplectic groupoid integrating the fiber type $(F, \pi)$ of $p : M \to B$.

THEOREM 4.3. *A Poisson fibration is integrable iff its fiber type is an integrable Poisson manifold. There exists a 1:1 correspondence between, source 1-connected, fibered symplectic groupoids and integrable Poisson fibrations.*

In one direction, the proof follows from Proposition 4.2. For the other direction, we will offer two proofs. The first proof is an heuristic proof that uses Poisson gauge theory. The second proof uses the approach to integrability through cotangent paths developed in [**3, 4**].

HEURISTIC PROOF VIA GAUGE THEORY. Given a Poisson fibration $p : M \to B$ with fiber type $(F, \pi)$, we start by writing it as an associated fiber bundle:

$$M = P \times_G F$$

where $P$ is the Poisson frame bundle and $G \subset \mathrm{Diff}\,(F, \pi)$ is the structure group of the fibration.

Since $(F, \pi)$ is integrable, there exists a unique source 1-connected symplectic groupoid $\mathcal{F} \rightrightarrows F$ which integrates $(F, \pi)$. The action of $G$ on $F$ lifts to an action of $G$ on $\mathcal{F}$ by symplectic groupoid automorphisms (see [**7**]). Hence, we can form the associated bundle:

$$\mathcal{G} = P \times_G \mathcal{F}.$$

Since the action of $G$ on $\mathcal{F}$ is by groupoid automorphisms, $\mathcal{G} \to B$ becomes a fibered groupoid over $M \to B$. Since this action is by symplectomorphisms, $\mathcal{G} \to B$ becomes a symplectic fibration. Since the groupoid structures and the symplectic structure on the fibers are compatible, $\mathcal{G}$ is a source 1-connected fibered symplectic groupoid with fiber type $\mathcal{F}$. It should be clear that the Poisson fibration determined by $\mathcal{G}$ is isomorphic to the original fibration. □

REMARK 4.4. Note that this heuristic proof becomes a real proof if the structure group $G$ of the fibration is a finite dimensional Lie group. We will illustrate this below in Example 4.3.

PROOF OF THEOREM 4.3. Given a Poisson fibration $p : M \to B$ with fiber type $(F, \pi)$, we denote by $\Sigma(M) \rightrightarrows M$ the symplectic groupoid that integrates the vertical Poisson structure $\pi_V$. Note that since we assume that $(F, \pi)$ is integrable, we have that $(M, \pi_V)$ is integrable, so that $\Sigma(M)$ is a Lie groupoid.

Let us recall (see [**3, 4**] for details and notations) that $\Sigma(M)$ is the space of equivalence classes of cotangent paths:

$$\Sigma(M) = \frac{\{a : [0,1] \to T^*M : \pi_V^\sharp(a(t)) = \frac{\mathrm{d}}{\mathrm{d}t}p(a(t))\}}{\{\text{cotangent homotopies}\}};$$

where $p : T^*M \to M$ is the cotangent bundle projection. Now we observe that $\operatorname{Vert}^0 \subset T^*M$ is a Lie subalgebroid, which is in fact, a bundle of Abelian Lie algebras. This is a direct consequence of $\pi_V$ being vertical. Hence, the equivalence classes of cotangent paths with image in $\operatorname{Vert}^0$ form a closed Lie subgroupoid $\mathcal{K} \subset \Sigma(M)$, which is in fact a bundle of Abelian Lie groups.

Let us consider the quotient Lie groupoid:

$$\mathcal{G} := \Sigma(M)/\mathcal{K}.$$

Notice that $\mathcal{G}$ is fibered over $B$, where the fiber over $b$ is the symplectic groupoid $\Sigma(F_b)$ integrating the fiber $(F_b, \pi_b)$. It is easy to check that $\mathcal{G}$ is, in fact, the desired source 1-connected, fibered symplectic groupoid integrating $p : M \to B$. We leave the details to the reader. $\square$

As we have seen in Proposition 3.7, a Poisson fibration $p : M \to B$ always admits Poisson connections. What does the specification of a Poisson connection on $p : M \to B$ amounts to in the corresponding fibered symplectic groupoid $\mathcal{G}$?

Proposition 4.5. *Let $M \to B$ be a Poisson fibration which integrates to a source 1-connected fibered symplectic groupoid $\mathcal{G} \to B$. The choice of a Poisson connection on the Poisson fibration $M \to B$ determines a coupling form $\Omega$ on the fibered symplectic groupoid $\mathcal{G} \to B$, and conversely.*

Proof. We will give a “gauge theoretical” proof, which is valid at least in the case where the structure group is a finite dimensional Lie group. One can also give a longer proof using paths, which avoids this assumption.

Hence, assume that:

$$M = P \times_G F,$$

where $P$ is a principal $G$-bundle and $F$ is a Poisson $G$-space. As we have mentioned above, the Poisson action $G \times F \to F$ lifts to an action $G \times \mathcal{F} \to \mathcal{F}$ by automorphisms of the symplectic groupoid $\mathcal{F} = \Sigma(F)$, which is Hamiltonian with equivariant moment map $J : \mathcal{F} \to \mathfrak{g}^*$, which is a groupoid cocycle. We have that:

$$\mathcal{G} = P \times_G \mathcal{F}.$$

Now, Poisson connections $\Gamma$ on $M \to B$ are in 1:1 correspondence with principal bundle connections on $P$.

To complete the proof we observe that, since the action $G \times \mathcal{F} \to \mathcal{F}$ is Hamiltonian, a choice of a principal bundle connection on $P$ determines a coupling form on $\mathcal{G}$ and conversely (see Theorem 3.8 or [**13**, Chapter 6] for more details). $\square$

The next natural question is: what does a coupling Dirac structure on the Poisson fibration $p : M \to B$ amounts to in the corresponding fibered symplectic groupoid $\mathcal{G}$? This question is more delicate, and it is intimately related with the pre-symplectic groupoids integrating Dirac structures described in [**2**]. This will be discussed elsewhere.

**4.3. An Example.** Let us denotes by $\mathbb{S}^3 \to \mathbb{S}^2$ the Hopf fibration which we view as a principal $\mathbb{S}^1$-bundle $P \to \mathbb{S}^2$. We will consider as fiber types $(F, \pi)$ the following two Poisson $\mathbb{S}^1$-manifolds:

1) The manifold $F = \mathbb{S}^2$, with the standard area form and the $\mathbb{S}^1$-action by rotations around the north-south poles axis;
2) The manifold $F = \mathfrak{su}(2)^* \simeq \mathbb{R}^3$, with its canonical linear Poisson structure and the $\mathbb{S}^1$-action by rotations around the $z$-axis;

The corresponding Poisson fibrations $M = P \times_{\mathbb{S}^1} F$ are:

1) the non-trivial $\mathbb{S}^2$-bundle $p : M \to \mathbb{S}^2$ (a symplectic fibration), and
2) the non-trivial rank 3 vector bundle $p : E \to \mathbb{S}^2$ (a Poisson fibration which is not symplectic).

The symplectic leaves of $\mathfrak{su}(2)^*$ are the concentric spheres around the origin and the origin itself. Hence the Poisson fibration $p : E \to \mathbb{S}^2$ is foliated by symplectic fibrations isomorphic to $p : M \to \mathbb{S}^2$ and the zero section. Since $\mathbb{S}^2$ is symplectic, we have:

$$H^1_\pi(\mathbb{S}^2) \simeq H^1(\mathbb{S}^2) = \{0\}.$$

Since $\mathfrak{su}(2)$ is semisimple of compact type, we also have:

$$H^1_\pi(\mathfrak{su}(2)^*) = \{0\}.$$

It follows from Corollary 3.11 that both $p : E \to \mathbb{S}^2$ and $p : M \to \mathbb{S}^2$ admit coupling Dirac structures. Of course, since $p : M \to \mathbb{S}^2$ is a symplectic fibration, its Dirac coupling is actually associated with a closed 2-form. We let the reader check that the presymplectic leaves of the Dirac coupling for $p : E \to \mathbb{S}^2$ are the symplectic fibrations isomorphic to $p : M \to \mathbb{S}^2$ (with their coupling forms) and the zero section (with the zero 2-form).

Let us now turn to the fibered symplectic groupoids integrating these fibrations. For that, we use the method in the heuristic proof of Theorem 4.3. Since the structure group is $\mathbb{S}^1$, a finite dimensional Lie group, this is allowed. We need the source 1-connected symplectic groupoid $\mathcal{F} = \Sigma(F)$ integrating the fiber type, and this is well-known in both examples:

1) Since $\mathbb{S}^2$ is symplectic and 1-connected, the associated source 1-connected symplectic groupoid is the pair groupoid $\Sigma(\mathbb{S}^2) = \mathbb{S}^2 \times \overline{\mathbb{S}^2}$, where the bar over the second factor means that we change the sign of symplectic form.
2) From general facts about linear Poisson structures, the symplectic groupoid of $\mathfrak{su}(2)^*$ is $\Sigma(\mathfrak{su}(2)^*) = T^*\mathrm{SU}(2)$, furnished with the canonical cotangent bundle symplectic structure. This groupoid is isomorphic to the action groupoid $\mathrm{SU}(2) \ltimes \mathfrak{su}(2)^*$ for the coadjoint action of $\mathrm{SU}(2)$ on $\mathfrak{su}(2)^*$.

Now we can describe, in both cases, the fibered symplectic groupoid $\mathcal{G} \to \mathbb{S}^2$ given by Theorem 4.3.

For the non-trivial $\mathbb{S}^2$-fibration $M \to \mathbb{S}^2$, the action of $\mathbb{S}^1$ on $\mathbb{S}^2$ lifts to the diagonal $\mathbb{S}^1$-action on $\mathbb{S}^2 \times \overline{\mathbb{S}^2}$, and we have:

$$\mathcal{G}(M) = P \times_{\mathbb{S}^1} (\mathbb{S}^2 \times \overline{\mathbb{S}^2}),$$

which is a non-trivial symplectic $(\mathbb{S}^2 \times \overline{\mathbb{S}^2})$-fibration over $\mathbb{S}^2$ and a groupoid over the non-trivial $\mathbb{S}^2$-fibration.

For the rank 3 vector bundle $E \to \mathbb{S}^2$, the action of $\mathbb{S}^1$ on $\mathfrak{su}(2)^*$ lifts to an action on $\mathrm{SU}(2) \times \mathfrak{su}(2)^*$, which is trivial on the first factor, and we have:

$$\mathcal{G}(E) = P \times_{\mathbb{S}^1} (\mathrm{SU}(2) \times \mathfrak{su}(2)^*) \simeq E \times \mathrm{SU}(2).$$

Note that, contrary to the case of the Poisson fibrations, the symplectic fibered groupoid $\mathcal{G}(M)$ does not sit naturally in $\mathcal{G}(E)$. This is because a Poisson submanifold does not always integrate to a symplectic subgroupoid, and this is exactly the case with the spheres in $\mathfrak{su}(2)^*$.

## References


[1] H. Bursztyn, M. Crainic, Dirac structures, momentum maps, and quasi-Poisson manifolds, in *The breadth of symplectic and Poisson geometry* , 1–40, **Progr. Math.**, 232, Birkhser Boston, Boston, MA, 2005.

[2] H. Bursztyn, M. Crainic, A. Weinstein and C. Zhu, Integration of twisted Dirac brackets, *Duke Math. J.* **123** (2004), no.3, 549–607.

[3] M. Crainic and R. L. Fernandes, Integrability of Lie brackets, *Ann. of Math. (2)* **157** (2003), 575–620.

[4] M. Crainic and R. L. Fernandes, Integrability of Poisson brackets, *J. Differential Geometry* **66** (2004), 71–137.

[5] T. Courant, Dirac manifolds, *Trans. Amer. Math. Soc.* **319** (1990), no. 2, 631–661.

[6] R. L. Fernandes, Connections in Poisson Geometry I: Holonomy and Invariants, *J. Differential Geometry* **54** (2000), 303–166.

[7] R. L. Fernandes, J.P. Ortega and T. Ratiu, Momentum maps in Poisson geometry, *in preparation.*

[8] J.-P. Dufour and A. Wade, On the local structure of Dirac manifolds, *math.SG/0405257.*

[9] C. Ehresmann, Les connexions infinitésimales dans un espace fibré différentiable, *Séminaire Bourbaki*, Vol. 1, Exp. No. 24, 153–168, Soc. Math. France, Paris, 1995.

[10] M. Gotay, R. Lashof, J. Śniatycki and A. Weinstein, Closed forms on symplectic fiber bundles, *Comment. Math. Helv.* **58** (1983), 617–621.

[11] V. Guillemin, E. Lerman and S. Sternberg, *Symplectic fibrations and multiplicity diagrams*, Cambridge University Press, Cambridge, 1996.

[12] V. Guillemin and S. Sternberg, *Symplectic techniques in physics*, 2nd edition. Cambridge University Press, Cambridge, 1990.

[13] D. McDuff and D. Salamon, *Introduction to symplectic topology*, 2nd edition, Oxford Mathematical Monographs, Oxford University Press, New York, 1998.

[14] I. Vaisman, Foliation-coupling Dirac structures, *J. Geom. Phys.* **56** (2006), no. 6, 917–938..

[15] Y. Vorobjev, Coupling tensors and Poisson geometry near a single symplectic leaf,Banach Center Publ. **54** (2001), 249–274.

[16] A. Wade, Poisson fiber bundles and coupling Dirac structures, preprint *math.SG/0507594.*

[17] A. Weinstein, The local structure of Poisson manifolds, *J. Differential Geometry* **18** (1983), 523–557.

[18] A. Weinstein, Fat bundles and symplectic manifolds. *Adv. in Math.* **37** (1980), 239–250.



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