LOCAL SYMPLECTIC GROUPOIDS AND THE SGA EQUATION

BY

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DISSERTATION

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ABSTRACT

This thesis discusses several problems related to local symplectic groupoids. In Chapter 1, we prove that if a local symplectic groupoid has uniformly discrete associators, then its associative completion is a symplectic groupoid. It follows that a Poisson manifold is integrable if and only if any of its local integrations has uniformly discrete associators. In Chapter 2, we construct a local symplectic groupoid integrating the Heisenberg-Poisson manifold which is not 6-associative. In Chapter 3, we give the conditions for a function to be the generating function for some local symplectic groupoid structure on the cotangent bundle, both for a coordinate space and for an abstract manifold. We also compare different notions of generating functions and analyze the role of the SGA equation. In Chapter 4, we show that the algebraic equation in the SGA equation is equivalent to a groupoid 2-cocycle condition. Under mild assumptions on the local symplectic groupoid, we find a groupoid 2-cocycle which under the van Est map yields the underlying Poisson bivector.

To my grandmother and my mother.

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INTRODUCTION

In [10], a precise connection between the integrability of a Lie algebroid and the associativity in any of its local integration is established. A notion that plays an important role in bridging the two concepts is the so-called **associative completion**, denoted by $\mathcal{AC}(G)$, of a local Lie groupoid G.

As the name suggests, the associative completion is globally associative and is a genuine groupoid. More precisely, [10] shows that under some connectedness assumptions on G, the associative completion $\mathcal{AC}(G)$ is a topological groupoid, which is smooth if and only if the set of associators is uniformly discrete. As a corollary, one deduces that a Lie algebroid is integrable if and only if it admits a local integration with uniformly discrete associators.

Given any symplectic groupoid $(\mathcal{G}, \omega) \rightrightarrows M$, there is a unique induced Poisson structure π on the space of objects that makes the target map a complete symplectic realization. When this happens, $(\mathcal{G}, \omega)$ is said to be an integration of the Poisson manifold (M, π). Not every Poisson manifold integrates to a symplectic groupoid. In fact, a Poisson manifold is integrable if and only if its cotangent bundle is an integrable Lie algebroid [7].

A local symplectic groupoid is a local Lie groupoid with a multiplicative symplectic form. As for symplectic groupoids, given any local symplectic groupoid, there is a unique Poisson structure on the space of objects for which the target is a symplectic realization. In this case, we say that the local symplectic groupoid is an integration of the Poisson manifold.

In contrast to integration by symplectic groupoids, every Poisson manifold integrates to a local symplectic groupoid [5]. We show in Chapter 1 that if any local integration has uniformly discrete associators, then the Poisson manifold is integrable to a symplectic groupoid:

Theorem 1. *Let (G, Ω) be a local symplectic groupoid whose associative completion is smooth. Then $\mathcal{AC}(G)$ admits a unique multiplicative symplectic form Ω for which the completion map $p : G \to \mathcal{AC}(G)$ is symplectic.*

Given a Lie groupoid $\mathcal{G} \rightrightarrows M$ with Lie algebroid A and anchor ρ, recall (see, e.g., [1]) that a **closed infinitesimal multiplicative (IM) 2-form** on A is a bundle map $\sigma : A \to T^*M$ satisfying

$$\sigma(\alpha)(\rho(\beta)) = -\sigma(\beta)(\rho(\alpha))$$
$$\sigma([\alpha, \beta]) = L_{\rho(\alpha)}\sigma(\beta) - L_{\rho(\beta)}\sigma(\alpha) - \mathrm{d}\sigma(\alpha)(\rho(\beta))$$

When a Lie groupoid has simply-connected source fibers, it is known that every closed IM 2-form on A lifts to a closed multiplicative 2-form on $\mathcal{G}$ whose non-degeneracy is equivalent to that of the former [1]. This implies that when the associative completion $\mathcal{AC}(G)$ is smooth and has simply-connected source fibers, it has a symplectic groupoid structure. However, examples show that, in general, $\mathcal{AC}(G)$ does not have simply-connected source fibers. Therefore, Theorem 1 does not follow from the known integration result for IM forms.

Given a Poisson manifold (S, π), the **Heisenberg-Poisson manifold** associated to S, denoted by $\mathcal{H}S$, is $S \times \mathbb{R}$ equipped with the Poisson structure $y\pi$ where y is the coordinate function on $\mathbb{R}$. If S is symplectic and admits a prequantization, [12] gives a construction of a symplectic groupoid integrating $\mathcal{H}S$ using the explosion construction. We will see in Chapter 2 that when $S = \mathbb{S}^2$, the explosion construction also gives rise to a local integration of $\mathcal{H}S$, which is not globally associative and thus not globalizable.

If M is a manifold and $\omega \in \Omega^2(M)$ is a 2-form, the **prequantum Lie algebroid** A_ω is the bundle $TM \times \mathbb{R}$ with anchor the projection onto the first coordinate and bracket given by

$$[(X, f), (Y, g)] = ([X, Y], Xg - Yf + \omega(X, Y))$$

By the general integrability criteria (see [7], example 3.1), when $M = \mathbb{S}^2$ with the usual area form ω, A_ω is integrable and the A-path construction gives the source 1-connected Lie groupoid integrating A_ω. By shrinking the groupoid structure to the open neighborhood consisting of arrows whose source and target are not antipodal points, one obtains a local Lie groupoid isomorphic to the open

$$G' \subset \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{R}/4\pi\mathbb{Z}$$

consisting of (x_0, x_1, a) such that $x_0 \neq x_1$ with structure maps given by

$$s(x_1, x_0, a) = x_0, \qquad t(x_1, x_0, a) = x_1,$$
$$(x_2, x_1, a_1) \cdot (x_1, x_0, a_0) = (x_2, x_0, a_0 + a_1 + A(\Delta x_0 x_1 x_2)),$$

where $A(\Delta x_0 x_1 x_2)$ is the area of the geodesic triangle with vertices x_0, x_1, x_2. Example 3.5 in [10] shows how one can modify this construction to obtain a local Lie groupoid integrating A_ω that is not 6-associative. Our construction of a non-globalizable local symplectic groupoid integrating $\mathcal{H}\mathbb{S}^2$ is inspired by the similarity between the cotangent algebroid of $\mathcal{H}\mathbb{S}^2$ and A_ω, and this example.

If (M, π) is a Poisson manifold, the **Poisson homotopy groupoid** is the set of cotangent paths modulo cotangent path homotopies:

$$\Pi(M, \pi) = \frac{\text{cotangent paths}}{\text{cotangent path-homotopy}}$$

Theorem 14.5 from [8] says that, when the cotangent algebroid of M is integrable, $\Pi(M, \pi)$ admits a smooth structure and a symplectic form with respect to which it is a symplectic groupoid integrating M. The cotangent algebroid of $\mathcal{H}\mathbb{S}^2$ is integrable by the general integrability criteria [7] or by the construction in [12], so the Poisson homotopy groupoid associated to $\mathcal{H}\mathbb{S}^2$ gives a (global) integration of the Poisson manifold.

In analogy to Example 3.5 in [10], we describe the Poisson homotopy groupoid and find that the part consisting of cotangent paths over the singular part of $\mathcal{H}\mathbb{S}^2$ is identified with $TS \times \mathbb{R}$, with the structure maps given by

$$\mathbf{s}(v_0, F_0) = (x, 0) = \mathbf{t}(v_0, F_0),$$
$$(v_0, F_0) + (v_1, F_1) = (v_0 + v_1, F_0 + F_1 + \frac{1}{2}\omega_S(v_0, v_1)).$$

On the other hand, the part consisting of cotangent paths over the regular part is identified with:

$$\Big\{(x, x', y, F_0) = ((x, y), (x', y), F_0) : x, x' \in S, x \neq -x', F_0 \in \mathbb{R}/\frac{4\pi}{y^2}\mathbb{Z}\Big\}.$$

with structure maps given by

$$\mathbf{s}(x,x',y,F_0) = (x',y), \qquad \mathbf{t}(x,x',y,F_0) = (x,y),$$
$$(x,x',y,F_0) + (x',x'',y,F_1) = (x,x'',y,F_0+F_1+\frac{1}{y^2}A(xx'x'')).$$

We see that the regular part can be viewed as a family of the local groupoid G' as in Example 3.4 in [10] parametrized by y. To construct a non-globalizable integration of the Heisenberg-Poisson manifold $\mathcal{H}\mathbb{S}^2$, we use the same idea as in Example 3.5 in [10], to modify the regular part. That is, in the last coordinate, we will no longer take the quotient by $\frac{4\pi}{y^2}\mathbb{Z}$ and we restrict the multiplication appropriately.

There are two problems we need to solve: 1) we need a smooth structure on the proposed local groupoid that makes it a local Lie groupoid; 2) We need to show that the Lie algebroid of the local Lie groupoid is isomorphic to the cotangent algebroid of the Heisenberg-Poisson manifold.

For the smooth structure, we use the explosion construction. For any Lie groupoid, the explosion along the identity is a Lie groupoid. In particular, the explosion of the pair groupoid $\mathbb{S}^2 \times \mathbb{S}^2$ along the diagonal is the so-called Connes groupoid. We augment it with an $\mathbb{R}$-component to get a smooth structure on the proposed local groupoid.

We will see that the following crucial lemma, concerning the area of geodesic triangles in $\mathbb{S}^2$, allows to show both that this local groupoid is smooth and then that its Lie algebroid is isomorphic to the cotangent algebroid of $\mathcal{H}\mathbb{S}^2$.

Lemma 1. *Let X and Y be vector fields on $\mathbb{S}^2$. Let $x \in \mathbb{S}^2$. Let γ be the integral curve of X starting at x and let $\varphi_Y(t,x) = \varphi_Y^t(x)$ be the flow of Y. Then*

$$\lim_{y\to 0}\frac{1}{y^2}A(x,\gamma(y),\varphi_Y(y,\gamma(y))) = \frac{1}{2}\omega(X(x),Y(x))$$

Although it is easy to write down geodesics on $\mathbb{S}^2$ and compute the area of geodesic triangles, we were unable to give a direct proof of the Lemma. Instead, the proof given in Section 2.7 makes use of the exponential map for the Poisson homotopy groupoid for $\mathcal{H}\mathbb{S}^2$.

Next, in Chapter 3, we turn our attention to generating functions, where our work was inspired by [2]. Recall that the graph of multiplication of (local) symplectic groupoid (G, Ω) is a Lagrangian submanifold of the symplectic manifold $\overline{G} \times \overline{G} \times G$, where $\overline{G}$ denotes the manifold G equipped with the symplectic form $-\Omega$. Now, let us look at the case where $G \subset T^*M$ equipped with the canonical symplectic structure ω_c. The map

$$\phi_0 : T^*\mathbb{R}^n \times T^*\mathbb{R}^n \times T^*\mathbb{R}^n \to T^*(T^*\mathbb{R}^n \times_{\mathbb{R}^n} T^*\mathbb{R}^n),$$
$$((p_1, x_1), (p_2, x_2), (p, x)) \mapsto ((p_1, x_1 - x), (p_2, x_2 - x), (x, p - (p_1 + p_2)),$$

is a symplectomorphism mapping the graph of multiplication for the canonical groupoid structure on the cotangent bundle $T^*\mathbb{R}^n$ to the zero section. If $S : T^*\mathbb{R}^n \times_{\mathbb{R}^n} T^*\mathbb{R}^n \to \mathbb{R}$ is a function, $\phi^{-1}(\mathrm{Graph}(\mathrm{d}S))$ is a Lagrangian. We will show in Theorem 3.3.2 conditions for the Lagrangian $\phi^{-1}(\mathrm{Graph}(\mathrm{d}S))$ to be the graph of multiplication of some local symplectic groupoid structure on $(T^*\mathbb{R}^n, \omega_c)$. Our result is inspired by techniques developed in [4] for the case of formal symplectic groupoids. There the authors introduced the so-called **Symplectic Groupoid Associativity** or **SGA** equation, which also plays crucial role in our result.

We can also ask a more general question. Let (T^*M, ω_c) be the canonical integration of the zero Poisson structure on M. The graph of multiplication $\mathrm{Graph}(m_0)$ is a Lagrangian submanifold of $\overline{T^*M} \times \overline{T^*M} \times T^*M$. Since $\mathrm{Graph}(m_0) \simeq T^*M \times_M T^*M$ via

$$\mathrm{Graph}(m_0) \to T^*M \times_M T^*M, \quad (\alpha_x, \beta_x, \alpha_x + \beta_x) \mapsto (\alpha_x, \beta_x),$$

we can apply the Lagrangian Neighborhood Theorem (see, e.g., [3]) to obtain a symplectomorphism ϕ from a neighborhood of $\mathrm{Graph}(m_0) \subseteq \overline{T^*M} \times \overline{T^*M} \times T^*M$ to a neighborhood of the zero section in $(T^*(T^*M \times_M T^*M), \omega_c)$, which maps $\mathrm{Graph}(m_0)$ to the zero section. If $S : T^*M \times_M T^*M \to \mathbb{R}$ is a function, then as above $\phi^{-1}(\mathrm{Graph}(\mathrm{d}S))$ is a Lagrangian in $\overline{T^*M} \times \overline{T^*M} \times T^*M$, and we ask: when is $\phi^{-1}(\mathrm{Graph}(\mathrm{d}S))$ the graph of a multiplication of a local symplectic groupoid $G \subset T^*M$ with symplectic form ω_c?

An answer to this question is given in Theorem 3.2.1. The system of equations expressing associativity is at the core of the argument. When $M = \mathbb{R}^n$ and $\phi = \phi_0$, we would like to compare this system of equations with the SGA equation. A natural guess is that they are equivalent. However, we are only able to proof this equivalence under additional restrictions on the function S. The details are given in Propositions 3.4.6 and 3.4.7.

There is also a different notion of generating functions introduced in [11]: Let $\pi : Z \to X \times Y$ be a fibration and let S be a function on Z. The set of critical points of S with respect to π is the set of points at which the differential $\mathrm{d}S$ vanishes on vectors in the fiber direction. At each critical point z such that $\pi(z) = (x, y)$, there are unique $\alpha_x \in T^*_x X$, $\alpha_y \in T^*Y$ for which we have $\mathrm{d}_z S = \pi^*(\alpha_x + \alpha_y)$. We denote the map $C_S \to T^*X \times T^*Y$, $z \mapsto -\alpha_x + \alpha_y$ by $\mathrm{d}_{X\times Y}$. We call a Lagrangian submanifold L in $\overline{T^*X} \times T^*Y$ a **canonical relation** and we say that S is a generating function for L with respect to π if $\mathrm{d}_{X\times Y}(C_S) = L$. Given a local symplectic groupoid structure on T^*M, the graph of multiplication $\mathrm{Graph}(m)$ is a Lagrangian relation between $M \times M$ and M and we can study its generating functions in the sense that we just introduced. Since now we have two notions of generating functions for $\mathrm{Graph}(m)$, it is natural to seek a precise relationship between them. However, we were able to do so only in a very special case, but which leads to an interesting interpretation of the SGA equation discussed in Chapter 4.

We show that, for generating functions in the sense of [11], the SGA equation becomes the cocycle condition for a groupoid 2-cocycle. We discuss two examples where we are able to write down this cocycle explicitly. Our final result shows that, under the van Est map, the class of this cocycle in groupoid cohomology is mapped to class of the Poisson bivector in algebroid cohomology, i.e., in Poisson cohomology.

CHAPTER 1

THE ASSOCIATIVE COMPLETION OF LOCAL SYMPLECTIC GROUPOIDS

1.1 Local Lie Groupoids

The following definition of a local Lie groupoid is taken from [10] and it is the one that will be used throughout the rest of this thesis.

Definition 1.1.1. *A* **local Lie groupoid** *over a manifold M is a manifold G with*

- $\mathbf{s}, \mathbf{t} : G \to M$ *(source and target maps) submersions*
- $\mathbf{u} : M \to G$, $x \mapsto 1_x$, *(unit map) a smooth map such that* $\mathbf{s}(u(x)) = x$ *and* $\mathbf{t}(u(x)) = x$ *for all* $x \in M$.
- $\mathbf{m} : \mathcal{U} \to G$, $(g, h) \mapsto gh$, *(multiplication) a submersion where* $\mathcal{U}$ *is an open neighborhood of*

  $$\mathcal{I}_2 := \big(M\ {}_{\mathbf{s}}\!\times_{\mathbf{t}} G\big) \cup \big(G\ {}_{\mathbf{s}}\!\times_{\mathbf{t}} M\big)$$

  in $G^{(2)} := \{(g, h) \in G \times G : \mathbf{s}(g) = \mathbf{t}(h)\}$
- $\mathbf{i} : \mathcal{V} \to \mathcal{V}$, $g \mapsto g^{-1}$, *(inversion map) a smooth map where* $\mathcal{V}$ *is an open neighborhood of* $\mathbf{u}(M)$ *in* G *such that* $\{(g, h) \in V \times V : \mathbf{s}(g) = \mathbf{t}(h)\} \subseteq U$

satisfying the following axioms:

- $\mathbf{s}(gh) = \mathbf{s}(h)$, $\mathbf{t}(gh) = \mathbf{t}(g)$ *for all* $(g, h) \in \mathcal{U}$
- $g1_{\mathbf{s}(g)} = 1_{\mathbf{t}(g)}g = g$ *for all* $g \in G$
- $\mathbf{s}(g^{-1}) = \mathbf{t}(g)$, $\mathbf{t}(g^{-1}) = \mathbf{s}(g)$ *for all* $g \in \mathcal{V}$.
- $g^{-1}g = 1_{\mathbf{s}(g)}$, $gg^{-1} = 1_{\mathbf{t}(g)}$ *for all* $g \in \mathcal{V}$

- *$(gh)l = g(hl)$ for all $(g,h,l) \in \mathcal{W}$*

where $\mathcal{W}$ is an open neighborhood of

$$\mathcal{I}_3 := \big(G\ {}_{\mathbf{s}}\times_{\mathbf{t}} M\ {}_{\mathbf{s}}\times_{\mathbf{t}} M\big) \cup \big(M\ {}_{\mathbf{s}}\times_{\mathbf{t}} G\ {}_{\mathbf{s}}\times_{\mathbf{t}} M\big) \cup \big(M\ {}_{\mathbf{s}}\times_{\mathbf{t}} M\ {}_{\mathbf{s}}\times_{\mathbf{t}} G\big)$$

in $G^{(3)} := \{(g,h,l) : \mathbf{s}(g) = \mathbf{t}(h), \mathbf{s}(h) = \mathbf{t}(l)\}$. The manifold G is allowed to be non-Hausdorff while M, the source and target fibers are required to be Hausdorff.

We will often identify M with its image $\mathbf{u}(M) \subset G$.

We can define a local topological groupoid analogously by requiring the structure maps to be continuous instead of smooth. From here on, we will denote by $G^{(n)}$ the set of all n-tuples of elements in G in which the source of every entry is equal to the target of the next entry, for $n \geq 2$

$$G^{(n)} = \{(g_1, \cdots, g_n) : \mathbf{t}(g_{i+1}) = \mathbf{s}(g_i), 1 \leq i \leq n-1\}$$

Elements in $G^{(n)}$ will be called compatible n-tuples.

Given a local Lie groupoid G, there are two ways of obtaining a smaller one G':

- (Restriction) We say that G' is obtained by restricting G if they have the same space of arrows and objects and the same source and target maps, and if the multiplication and inversion in G' are restrictions of the ones in G to smaller open neighborhoods of $\mathcal{I}_2$ and M, respectively.
- (Shrinking) We say that G' is obtained by shrinking G if G' is an open neighborhood of the unit section in G, the source and target maps are the restrictions of s and t to G', multiplication in G' is the multiplication in G restricted to $U \cap (G')^{(2)} \cap m^{-1}(G')$ and inversion in G' is the inversion in G restricted to $V \cap G' \cap i(G')$.

Below are some examples of a local Lie groupoid.

Examples 1.1.2. *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid.*

1. *Any choice of an open $\mathcal{U} \subseteq \mathcal{G}^{(2)}$ containing $(\mathcal{G}\ {}_{\mathbf{s}}\times_{\mathbf{t}} M) \cup (M\ {}_{\mathbf{s}}\times_{\mathbf{t}} \mathcal{G})$ and an open $\mathcal{V} \subseteq \mathcal{G}$ containg M such that $\mathcal{V} = \mathbf{i}(\mathcal{V})$ determines a local Lie groupoid which is a restriction of $\mathcal{G}$.*

2. *Any open neighborhood in $\mathcal{G}$ containing the unit section determines a local Lie groupoid which is a shrinking of $\mathcal{G}$.*

3. *(One-point compatification) The one-point compactification of $\mathbb{R}$, $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$ becomes a local Lie group (a local Lie groupoid with objects the singleton $\{*\}$) with domain of muliplication $\mathbb{R}_\infty \times \mathbb{R}_\infty \setminus \{(\infty, \infty)\}$ and the domain of inversion $\mathbb{R} \setminus \{\infty\}$ if we set $g + \infty = \infty = \infty + g$ for all $g \in \mathbb{R}$ amd $a + b$ to be the usual addition if $a, b \in \mathbb{R}$ and inversion to be the usual one in $\mathbb{R}$. We cannot extend this multiplication to $\infty + \infty = \infty$, because it would not be smooth at (∞, ∞).*

1.1.1 Lie algebroid of a local Lie groupoid

In the definition of a local Lie groupoid given above, not every element needs to be invertible. One consequence of this is that right multiplication, when defined, may not induce a local diffeomorphism between the source fibers. To define the Lie algebroid of a local Lie groupoid, we need the following assumptions.

Definition 1.1.3. *A local Lie groupoid is said to be:*

- **right-regular** *if for any (g, h) in the domain of multiplication, we have an isomorphism*
$$\mathrm{d}R_h : T^{\mathbf{s}}_g G \to T^{\mathbf{s}}_{gh} G.$$
- **left-regular** *if for any (g, h) in the domain of multiplication, we have an isomorphism*
$$\mathrm{d}R_g : T^{\mathbf{t}}_h G \to T^{\mathbf{t}}_{gh} G.$$
- **bi-regular** *if it is both right regular and left regular.*

Let G be a left-regular local Lie groupoid and let $A := T^{\mathbf{t}}_M G$. For a section $\alpha \in \Gamma(A)$, denote by $\tilde{\alpha}$ the unique left invariant vector field such that $\tilde{\alpha}|_M = \alpha$. We define the **Lie algebroid of** G to be $(A, \rho, [\cdot, \cdot])$ with anchor and bracket given by

$$\rho : A \to TM, \quad v \mapsto \mathrm{d}\mathbf{s}(v)$$
$$[\alpha, \beta] := [\tilde{\alpha}, \tilde{\beta}]|_M$$

We note that any local Lie groupoid G has a shrinking which is left-regular and so one defines the Lie algebroid of G to be the Lie algebroid of any of its left-regular shrinking.

1.1.2 Connectedness in a local Lie groupoid

In this section we list some assumptions on a local Lie groupoid that will be needed when stating results from [10].

Definition 1.1.4. *A local Lie groupoid is*

1. **source-connected** *if all of its source fibers are connected.*
2. **target-connected** *if all of its target fibers are connected.*
3. **inversional** *if every element can be written as a product of invertible elements.*

We also say that the local Lie groupoid has **products connected to the axes** if for any $(g,h) \in \mathcal{U}$, there is a path $\gamma : [0,1] \to \mathbf{s}^{-1}(\mathbf{t}(h))$ such that

$$\begin{cases} (\gamma(\tau), h) \in \mathcal{U}, \ \tau \in [0,1] \\ \gamma(0) = \mathbf{t}(h), \ \gamma(1) = g \end{cases}$$

or there is a path $\gamma : [0,1] \to \mathbf{t}^{-1}(s(g))$ such that

$$\begin{cases} (g, \gamma(\tau)) \in \mathcal{U}, \ \tau \in [0,1] \\ \gamma(0) = \mathbf{s}(g), \ \gamma(1) = h \end{cases}$$

1.1.3 Associativity

Given a triple (g,h,l) such that (g,h), (h,l), (gh,l) and (g,hl) are all elements in $\mathcal{U}$, it may not be true that $(gh)l = g(hl)$, since (g,h,l) may not be in the domain of associativity $\mathcal{W}$. This suggests how one should define 3-associativity on a local groupoid, or more generally, n-associativity.

Definition 1.1.5. *Let $n \geq 3$. A local groupoid is n-**associative** if given any compatible n-tuple, all defined products are equal. A local groupoid is **globally associative** if it is n-associative for all $n \geq 3$.*

As noted above, a local groupoid may fail to be 3-associative. However, a result from [10] shows that for every n, every local groupoid has a restriction which is n-associative. It is clear from the definitions that any restriction *or* shrinking of a (global) Lie groupoid is globally associative. The same is true for any restriction *of* a shrinking of a (global) Lie groupoid, for which we have a name.

Definition 1.1.6. *A local Lie groupoid is **globalizable** if it is a restriction of an open neighborhood of the unit section in a Lie groupoid.*

So the preceding remark says that every globalizable local Lie groupoid is globally associative. Mal'cev's Theorem for local Lie groupoids [10] says that, under some connectedness assumption, the converse statement also holds. The key ingredient in the proof of this theorem is what is called the associative completion of a local Lie groupoid G, which is denoted by $\mathcal{AC}(G)$. The name highlights the fact that the new structure is globally associative and is a genuine groupoid.

1.2 The Associative Completion of a Local Lie Groupoid

If G is a local groupoid we consider the set of well-formed words in G:

$$W(G) := \bigsqcup_{n \geq 1} G^{(n)}.$$

Given any

$$\tilde{g} = (g_1, \dots, g_k, g_{k+1}, \dots, g_n) \in W(G), \quad \text{where} \quad (g_k, g_{k+1}) \in \mathcal{U}$$

if we let

$$\tilde{\tilde{g}} = (g_1, \dots, g_k g_{k+1}, \dots, g_n) \in W(G).$$

we say that $\tilde{g}$ is an *expansion* of $\tilde{\tilde{g}}$ and that $\tilde{\tilde{g}}$ is a *contraction* of $\tilde{g}$. If two words in $W(G)$ are related by an expansion or a contraction, then we say that they are elementarily equivalent. We denote by $\sim$ the equivalence relation generated by these elementary equivalences. We also write $\tilde{g} \leq \tilde{\tilde{g}}$ if $\tilde{g}$ can be obtained from $\tilde{\tilde{g}}$ through a sequence of contractions.

Definition 1.2.1. *If G is a local groupoid, then*

$$\mathcal{AC}(G) = W(G)/\sim$$

is called the ***associative completion*** *of G and the quotient map $p : G \to \mathcal{AC}(G)$, $g \mapsto [(g)]$, is called the* ***completion map****.*

If G is inversional, then $\mathcal{AC}(G)$ is a groupoid with the following structure maps:

- source and target: $\tilde{\mathbf{s}}([(g_1, \dots, g_n)]) = \mathbf{s}(g_n)$, $\tilde{\mathbf{t}}([(g_1, \dots, g_n)]) = \mathbf{t}(g_1)$;
- unit section : $\tilde{\mathbf{u}}(x) = [(1_x)]$;
- multiplication: induced by concatenation of words.

Since G is inversional, every arrow can be written as a product of invertible elements. Then if $[(g_1, \dots, g_n)] \in \mathcal{AC}(G)$, its inverse is defined by

$$[((w^n_{k_n})^{-1}, \dots, (w^n_1)^{-1}, \dots, (w^1_{k_1})^{-1}, \dots, (w^1_1)^{-1})]$$

where $g_i = w^i_1 \dots w^i_{k_i}$ with each w^i_j invertible. This does not depend on the representative of the equivalence class. Moreover, if G is a topological groupoid, then $\mathcal{AC}(G)$, with the quotient topology inherited from $W(G)$, is a topological groupoid.

When G is a local Lie groupoid, $W(G)$ is the disjoint union of smooth manifolds of different dimensions. In this case, we may ask if $\mathcal{AC}(G)$ admits a smooth structure that makes the quotient map $W(G) \to \mathcal{AC}(G)$ a submersion. To answer this question we need the notion of *associator*.

Definition 1.2.2. *An element $g \in \mathbf{s}^{-1}(x) \cap \mathbf{t}^{-1}(x) = G_x$ is an* ***associator*** *if there is $\tilde{g} \in W(G)$ such that $(g) \leq \tilde{g}$ and $(1_x) \leq \tilde{g}$.*

We denote by $\mathrm{Assc}(G)$ the set of associators in G. Clearly, $\mathrm{Assc}(G)$ is contained in the kernel of the completion map $p : G \to \mathcal{AC}(G)$, but the two sets may fail to coincide. However, they are equal under some assumptions on the local Lie groupoid G.

Proposition 1.2.3 ([10])**.** *If G is a biregular local Lie groupoid with product connected to the axes, then $\mathrm{Assc}(G) = Ker(p)$, where $p : G \to \mathcal{AC}(G)$ is the completion map.*

We say that $\mathrm{Assc}(G)$ is **uniformly discrete** if there is an open neighborhood of the unit section $u(M)$ in G that intersects $\mathrm{Assc}(G)$ only at the identities. The following result from [10] gives a sufficient and necessary condition for $\mathcal{AC}(G)$ to be a Lie groupoid.

Theorem 1.2.4 ([10]). *Let G be a biregular local Lie groupoid with product connected to the axes. The associative completion $\mathcal{AC}(G)$ is smooth if and only if $\mathrm{Assc}(G)$ is uniformly discrete. In this case, $p : G \to \mathcal{AC}(G)$ is a local diffeomorphism.*

Remark 1.2.5. *Any local Lie groupoid G has a shrinking which is biregular with product connected to the axes. Note, however, that shrinking a Lie groupoid may change the set of associators.*

Let us recall the construction of the smooth structure on $\mathcal{AC}(G)$ when $\mathrm{Assc}(G)$ is uniformly discrete. Given $\tilde{g} \in \mathcal{AC}(G)$, we construct a chart around $\tilde{g}$ modeled on an open set in G. To do this, we need to make three choices:

- A *representative* $(g_1, \ldots, g_n)$ of $\tilde{g}$;
- A k between 1 and n;
- For each $1 \leq i \leq n$, $i \neq k$, a *local bisection* N_i of G at g_i (that is, a submanifold through g of the same dimension as M that is transverse to both the source and target fibers).

We take N_i small enough so that $\mathbf{s}|_{N_i}, \mathbf{t}|_{N_i}$ are diffeomorphisms. Then a chart is constructed as follows.

Choose an open set O around g_k in G. If O is small enough, each $y \in O$ is such that $\mathbf{s}(y) \in \mathrm{Im}\,\mathbf{t}|_{N_{k+1}}$. In this case, let $y_{k+1} = t|_{N_{k+1}}^{-1}(\mathbf{s}(y))$. Inductively, for $k+1 \leq i \leq n-1$, by choosing O small enough, we can assume $\mathbf{s}(y_i) \in \mathrm{Im}\,\mathbf{t}|_{N_{i+1}}$ and define $y_{i+1} = t|_{N_{i+1}}^{-1}(\mathbf{s}(y_i)$. Similarly, if we take O small enough, then $\mathbf{t}(y) \in \mathrm{Im}\,\mathbf{s}|_{k-1}$. In this case, let $y_{k-1} = \mathbf{s}|_{N_{k-1}}^{-1}(\mathbf{t}(y))$. Inductively, for $1 \leq i \leq k-1$, choose O small enough to be have $\mathbf{t}(y_i) \in \mathrm{Im}\,\mathbf{s}|_{N_{i-1}}$, let $y_{i-1} = \mathbf{s}|_{N_{i-1}}^{-1}(\mathbf{t}(y_i))$. Then the chart corresponding to the three choices $(g_1, \ldots, g_n)$, k and N_i is given by

$$O \ni y \mapsto [(y_1, \ldots, y_{k-1}, y, y_{k+1}, \ldots, y_n)] \in \mathcal{AC}(G).$$

We will denote this chart by $c : O \to \mathcal{AC}(G)$. The proof of the theorem above consists in showing that, if O is small enough, then (i) this map is injective and (ii) for any pair of charts the transition maps are smooth.

1.3 Local Symplectic Groupoids

Let $G \rightrightarrows M$ be a local Lie groupoid. Let $\mathcal{U} \subseteq G^{(2)}$ be the domain of multiplication

$$\mathbf{m} : \mathcal{U} \to G,$$

and let

$$\mathrm{pr}_1, \mathrm{pr}_2 : \mathcal{U} \to G,$$

be the projections onto the first and the second coordinates, respectively.

Definition 1.3.1. *A k-form $\omega \in \Omega^k(G)$ is said to be **multiplicative** if*

$$\mathrm{pr}_1^* \omega - \mathbf{m}^* \omega + \mathrm{pr}_2^* \omega = 0.$$

*A **local symplectic groupoid** is a local Lie groupoid with a multiplicative symplectic form.*

Just like for symplectic groupoids, each local symplectic groupoid induces a unique Poisson structure on the manifold of units. Hence, one also has a (local) integrability problem for Poisson manifolds, which always has an affirmative answer:

Theorem 1.3.2 ([5])**.**

(i) *Given a local symplectic groupoid $(G, \omega) \rightrightarrows M$, there is a unique Poisson structure π on M for which the target map $\mathbf{t} : G \to M$ is a Poisson map.*

(ii) *Given a Poisson manifold (M, π), there is a local symplectic groupoid $(G, \omega) \rightrightarrows M$ such that the target map is a Poisson map.*

1.4 The Associative Completion of a Local Symplectic Groupoid

Theorem 1.2.4 answers the question of the smoothness of the associative completion of a local Lie groupoid. In the symplectic context, we have:

Question 1.4.1. *Given a local symplectic groupoid is the associative completion a symplectic groupoid (assuming that it is smooth)?*

The main result of this chapter answers this question:

Theorem 1.4.2. *Let (G, Ω) be a local symplectic groupoid whose associative completion is smooth, then $\mathcal{AC}(G)$ admits a mutiplicative symplectic form Ω for which the completion map $p : G \to \mathcal{AC}(G)$ is symplectic.*

Corollary 1.4.3. *A Poisson manifold is integrable to a (global) symplectic groupoid if and only if any of its local integrations has uniformly discrete associators.*

Proof of Theorem 1.4.2. To begin the proof, we first define pointwise the 2-form which will be the symplectic form on $\mathcal{AC}(G)$. Let $\tilde{g} \in \mathcal{AC}(G)$ and let $c : O \to \mathcal{AC}(G)$ be a chart around $\tilde{g}$ as constructed at the end of the previous section but where the bisections N_i used in the construction are chosen to be *Lagrangian*:

$$\Omega|_{N_i} = 0.$$

We keep all the notations used there.

Definition 1.4.4. *Let ω be the 2-form on $\mathcal{AC}(G)$ such that*

$$\Omega_g = (\mathrm{d}_g c)^* \omega_{\tilde{g}}.$$

We will prove later that this definition does not depend on the choice of the chart. The proof uses crucially the fact that we only allow Lagrangian bisections. Since for any chart $c : O \to \mathcal{AC}(G)$ we have

$$\Omega|_O = c^*(\omega|_{c(O)}),$$

it is obvious that ω is smooth, closed and non-degenerate.

Multiplicativity of ω is shown as follows. Let

$$\tilde{g} = (g_1, \dots, g_{k-1}, g), \qquad \tilde{g}' = (g', g_{k+2}, \dots, g_n).$$

Let N_i, $1 \le i \le k-1, k+2 \le i \le n$ be Lagrangian bisections at g_i respectively. Let N' be a Lagrangian bisection at g'. Assume $\mathbf{s}(g) = \mathbf{t}(g')$. We make three choices of charts:

1) A chart around $\tilde{g}$: $c : O \to \mathcal{AC}(G)$ determined by an open neighborhood O of g and $N_1, \dots, N_{k-1}$;

2) A chart around $\tilde{g}'$: $c' : O' \to \mathcal{AC}(G)$ determined by an open neighborhood O' of g' and $N_{k+2}, \dots, N_n$;

3) A chart around $\tilde{g}\tilde{g}'$: $c'' : O \to \mathcal{AC}(G)$ determined by O and $N_1, \dots, N_{k-1}$, $N', N_{k+2}, \dots, N_n$.

Let V denote the set of invertible elements in G. Let $\epsilon : O' \to V$ be a map such that $\epsilon(y')y' \in N'$ whenever $y' \in O'$. Such a map exists and is smooth if O' is small enough and, when it exists, it is unique. In the charts above, the multiplication map takes the form

$$O \times O' \to O, \quad (y, y') \mapsto y\epsilon(y')^{-1}.$$

Let $x(\tau)$, $y(\tau)$, $x'(\tau)$, $y'(\tau)$ be curves in O, O' starting at g, g' with $\mathbf{s}(x(\tau)) = \mathbf{t}(x'(\tau))$, $\mathbf{s}(y(\tau)) = \mathbf{t}(y'(\tau))$. Let $\tilde{x}(\tau)$, $\tilde{y}(\tau)$, $\tilde{x}'(\tau)$, $\tilde{y}'(\tau)$ be the corresponding curves in $\mathcal{AC}(G)$. We find:

$$\begin{aligned} m^*\omega_{(\tilde{g},\tilde{g}')}((\dot{\tilde{x}}, \dot{\tilde{x}}'), (\dot{\tilde{y}}, \dot{\tilde{y}}')) &= \omega_{\tilde{g}\tilde{g}'}(\mathrm{d}m(\dot{\tilde{x}}, \dot{\tilde{x}}'), \mathrm{d}m(\dot{\tilde{y}}, \dot{\tilde{y}}')) \\ &= \Omega_g\Big(\frac{\mathrm{d}}{\mathrm{d}\tau}x(\tau)\epsilon(x'(\tau))^{-1}, \frac{\mathrm{d}}{\mathrm{d}\tau}y(\tau)\epsilon(y'(\tau))^{-1}\Big) \\ &= \Omega_g(\dot{x}, \dot{y}) + \Omega_{1_{s(g)}}(\epsilon(\dot{x'})^{-1}, \epsilon(\dot{y'})^{-1}) \\ &= \Omega_g(\dot{x}, \dot{y}) + \Omega'_g(\dot{x}', \dot{y}') \\ &= \omega_{\tilde{g}}(\dot{\tilde{x}}, \dot{\tilde{y}}) + \omega_{\tilde{g}'}(\dot{\tilde{x}}', \dot{\tilde{y}}') \\ &= (pr_1^* + pr_2^*)\omega_{(\tilde{g},\tilde{g}')}((\dot{\tilde{x}}, \dot{\tilde{x}}'), (\dot{\tilde{y}}, \dot{\tilde{y}}')), \end{aligned}$$

where we have used that:

$$\begin{aligned}0 &= \Omega_{g'}(\epsilon(\dot{x'})x', \epsilon(\dot{y'})y') \\ &= \Omega_{1_{s(g')}}(\epsilon(\dot{x'}), \epsilon(\dot{y'})) + \Omega_{g'}(\dot{x}', \dot{y}') \\ &= -\Omega_{1_{s(g')}}(\epsilon(\dot{x'})^{-1}, \epsilon(\dot{y'})^{-1}) + \Omega_{g'}(\dot{x}', \dot{y}'),\end{aligned}$$

which follows from the fact that the bisections are Lagrangian and that inversion is anti-symplectic. This proves multiplicativity.

To see that the completion map $p : (G, \Omega) \to (\mathcal{AC}(G), \omega)$ is a symplectomorphism, we observe that if $O \subset G$ is a small enough open containing g then the restriction $c := p|_O$ is a chart around $\tilde{g} = (g) \in p(G)$. Hence:

$$\Omega_g = c^*\omega_{\tilde{g}} = p^*\omega_{\tilde{g}}$$

To complete the proof it remains to show that the definition of ω is independent of the choice of chart. For that, recall that each chart is determined by three choices: the word representing the element, k and the Lagrangian bisections. We will show that the definition of ω is independent of each.

REPRESENTATIVE: Let $\tilde{g} = (g_1, \dots, g_{k-1}, g, g', g_{k+2}, \dots, g_n) \in \mathcal{AC}(G)$ where $gg' \in G$ is defined. Let U, N_i, N' be as before. Let $y' : U \to N'$ be such that $\mathbf{t}(y'(y)) = \mathbf{s}(y)$ (or, equivalently, $y'(y) = (\mathbf{t}|_{N'})^{-1}(\mathbf{s}(y))$). This map is defined if U is small enough. By making U even smaller, we assume further that $yy'(y)$ is defined in G for all $y \in U$. We choose two charts around $\tilde{g}$, with the same k:

1) A chart with $(g_1, \dots, g_{k-1}, g, g', g_{k+2}, \dots, g_n)$ as representative, N_i and N' as bisections, $U \ni g$ a small enough open.

2) A chart with $(g_1, \dots, g_{k-1}, gg', g_{k+2}, \dots, g_n)$ as representative, N_i as bisections, $U' \ni gg'$ a small enough open.

The transition map between these charts is given by $y \mapsto yy'(y)$. If $x(\tau)$, $y(\tau)$ are curves in U starting at g, we have

$$\Omega_{gg'}(\dot{xy'(x)}, \dot{yy'(y)}) = \Omega_g(\dot{x}, \dot{y}) + \Omega_{g'}(\dot{y'(x)}, \dot{y'(y)}) = \Omega_g(\dot{x}, \dot{y}),$$

where the last equality follows from the fact that N' is lagrangian. Since any two representatives of $\tilde{g}$ are related by a sequence of expansions and contractions, this proves that $\omega_{\tilde{g}}$ does not depend on the choice of representatives.

Choice of k: Let $\tilde{g} = (g_1, \dots, g_{k-1}, g, g', g_{k+2}, \dots, g_n) \in \mathcal{AC}(G)$. Let N_i, N' be as before. Let N be a Lagrangian bisection through g. We choose two charts around $\tilde{g}$ with the same representative word:

1) A chart with $U \ni g$ small enough, N_i, N' as bisections.

2) A chart with $U' \ni g'$ small enough, N_i, N as bisections.

Let $\epsilon : U \to V$ be such that $y\epsilon(y) \in N$ for each $y \in U$ and let $y' : U \to N'$ be such that $\mathbf{t}(y'(y)) = \mathbf{s}(y)$ for each $y \in U$. Such maps exist if U is chosen small enough. The transition map between these charts is given by $y \to \epsilon(y)^{-1}y'(y)$. If $x(\tau)$, $y(\tau)$ are curves in U starting at g, we have

$$\begin{aligned}\Omega_{g'}(\epsilon(x)^{-\dot{1}}y'(x), \epsilon(y)^{-\dot{1}}y'(y)) &= \Omega_{1_{s(g)}}(\epsilon(\dot{x})^{-1}, \epsilon(\dot{y})^{-1}) + \Omega_{g'}(y'(\dot{x}), y'(\dot{y})) \\ &= -\Omega_{1_{s(g)}}(\epsilon(\dot{x}), \epsilon(\dot{y})) \\ &= \Omega_g(\dot{x}, \dot{y}),\end{aligned}$$

where the second to last equality follows from that fact that $\Omega|_M = 0$ and that N' is Lagrangian, while the last equality follows from

$$\begin{aligned}0 &= \Omega_g(x\epsilon(\dot{x}), y\epsilon(\dot{y})) \\ &= \Omega_g(\dot{x}, \dot{y}) + \Omega_{1_{s(g)}}(\epsilon(\dot{x}), \epsilon(\dot{y})),\end{aligned}$$

where the first equality holds since N is Lagrangian. This shows that $\omega_{\tilde{g}}$ does not depend on the choice of k.

Bisections: Let $\tilde{g} \in \mathcal{AC}(G)$ be as before. Let N_i, N_i' be two sets of Lagrangian bisections. This gives rise to two charts around $\tilde{g}$ with the same representative and the same k:

1. A chart with $U \ni g$ small enough, N_i as bisections.

2. A chart with $U' \ni g'$ small enough, N_i' as bisections.

Let $y_i : U \to N_i$ be functions such that for each $y \in U$, $\mathbf{t}(y_{k+1}(y)) = \mathbf{s}(y)$, $\mathbf{t}(y_{i+1}(y)) = \mathbf{s}(y_i(y))$ for all $k+1 \le i \le n-1$ and $\mathbf{s}(y_{k-1}(y) = \mathbf{t}(y)$, $\mathbf{s}(y_i(y) = \mathbf{t}(y_{i+1}(y))$ for all $1 \le i \le k-2$. Let $\epsilon_{n+1} : U \to V$, $\epsilon_0 : U \to V$ be given by $\epsilon_{n+1}(y) = 1_{\mathbf{s}(y_n(y))}$, $\epsilon_0(y) = 1_{\mathbf{t}(y_1(y))}$ respectively. For $k+1 \leq i \leq n$, let $\epsilon_i : U \to V$ be such that $\epsilon_i(y) y_i \epsilon_{i+1}(y)^{-1} \in N_i'$; for $1 \leq i \leq k-1$, let $\epsilon_i : U \to V$ be such that $\epsilon_{i-1}(y)^{-1} y_i \epsilon_i(y) \in N_i'$. These maps exist if U is chosen small enough. In that case, the transition map is given by $y \to \epsilon_{k-1}(y)^{-1} y \epsilon_{k+1}(y)^{-1}$. Let $x(\tau), y(\tau)$ be curves in U starting at g.

We claim that:

$$\Omega(\dot{\epsilon_i(x)}, \dot{\epsilon_i(y)}) = 0, \quad (i = 0, \ldots, n+1).$$

This can be seen by induction: on the one hand, we have

$$\Omega(\dot{\epsilon_{n+1}(x)}, \dot{\epsilon_{n+1}(y)}) = 0,$$

since $\Omega|_M = 0$. For $k+1 \leq i \leq n$, we have

$$\begin{aligned} \Omega(\epsilon_i(x) \dot{y_i(x)} \epsilon_{i+1}^{-1}(x), \epsilon_i(y) \dot{y_i(y)} \epsilon_{i+1}^{-1}(y))) = 0, \\ \Omega(\dot{y_i(x)}, \dot{y_i(y)}) = 0, \end{aligned}$$

since N_i, N_i' are Lagrangian. Hence, $\Omega(\dot{\epsilon_i(x)}, \dot{\epsilon_i(y)}) = 0$.The argument is similar for $1 \leq i \leq k-1$, proving the claim.

Now, since $\Omega|_M = 0$, we also obtain that $\Omega(\dot{\epsilon_i(x)^{-1}}, \dot{\epsilon_i(y)^{-1}}) = 0$ for all i. It follows that

$$\begin{aligned} \Omega(\frac{\mathrm{d}}{\mathrm{d}\tau} \epsilon_{k-1}(x(\tau))^{-1} x(\tau) \epsilon_{k+1}(x(\tau))^{-1}, \frac{\mathrm{d}}{\mathrm{d}\tau} \epsilon_{k-1}(y(\tau))^{-1} y(\tau) \epsilon_{k+1}(y(\tau))^{-1}) = \\ = \Omega(\dot{\epsilon_{k-1}(x)^{-1}}, \dot{\epsilon_{k-1}(y)^{-1}}) + \Omega(\dot{x}, \dot{y}) + \Omega(\dot{\epsilon_{k+1}(x)^{-1}}, \dot{\epsilon_{k+1}(y)^{-1}}) \\ = \Omega(\dot{x}, \dot{y}). \end{aligned}$$

This shows that that $\omega_{\tilde{g}}$ does not depend on the choice of bisections and completes the proof. □

CHAPTER 2

A LOCAL INTEGRATION OF THE HEISENBERG-POISSON MANIFOLD $\mathcal{H}\mathbb{S}^2$

In this chapter we will construct a local symplectic groupoid integrating the Heisenberg-Poisson manifold associated with the symplectic manifold $\mathbb{S}^2$, equipped with its standard symplectic structure. Our construction is inspired by Example 3.5 in [10], which gives a non-globalizable local Lie groupoid integrating the so-called *prequantum Lie algebroid* A_ω associated to a closed 2-form ω. We note the similarity between A_ω and the cotangent algebroid of a Heisenberg-Poisson manifold $\mathcal{H}S$, for an arbitrary symplectic manifold S, and we use it to construct a non-globalizable local symplectic groupoid integrating $\mathcal{H}\mathbb{S}^2$.

2.1 Heisenberg-Poisson manifolds

Definition 2.1.1 ([12])**.** *Let (S, π_S) be a Poisson manifold. The* **Heisenberg-Poisson manifold** $\mathcal{H}S$ *is $M = S \times \mathbb{R}$ equipped with the Poisson structure*

$$\pi_{(x,y)} := y\pi_x.$$

The symplectic leaves of $\mathcal{H}S$ are of two types: (i) $L \times \{y\}$ for $y \neq 0$, where $L \subset S$ runs through the symplectic leaves of π_S, and (ii) the points in $S \times \{0\}$. We will be mostly interested in the case where (S, π_S) is symplectic.

2.2 A non-globalizable local Lie groupoid

In this section, we review the construction in [10, Example 3.5] of a non-globalizable local Lie groupoid. Let M be a manifold and let $\omega \in \Omega^2(M)$ be a closed 2-form. The *prequantum Lie algebroid* A_ω associated to ω is defined as follows. As a bundle, $A_\omega = TM \oplus \mathbb{R}$. The anchor is given by the projection onto the first coordinate and the bracket is given by

$$[(X,f),(Y,g)] = ([X,Y], Xg - Yf + \omega(X,Y)).$$

Example 3.1 in [4] shows that when $M = \mathbb{S}^2$ and ω is the the usual area form, A_ω is integrable and the A-path construction gives the source 1-connected Lie groupoid integrating A_ω. To describe it, consider the map

$$A : \Omega(\mathbb{S}^2) \to \mathbb{R}/4\pi\mathbb{Z}, \quad \gamma \mapsto \int_{H_\gamma} \omega \pmod{4\pi\mathbb{Z}},$$

where $\Omega(\mathbb{S}^2)$ is the space of all loops in $\mathbb{S}^2$ and H_γ is any path-homotopy contracting $\gamma \in \Omega(\mathbb{S}^2)$ to the trivial loop at $\gamma(0) = \gamma(1)$. This map is well-defined since the integrals over two such homotopies differ by $\int_\eta \omega$ for some $[\eta] \in \pi_2(\mathbb{S}^2)$. The source 1-connected Lie groupoid integrating A_ω is

$$\mathcal{G} := \frac{\{\text{piecewise smooth path in } \mathbb{S}^2\} \times \mathbb{R}/4\pi\mathbb{Z}}{\sim},$$

where $(\gamma_0, a_0) \sim (\gamma_1, a_1)$ if

$$\begin{cases} \gamma_0, \gamma_1 \text{ have the same end points} \\ a_1 = a_0 + A(\gamma_1^{-1} \cdot \gamma_0). \end{cases}$$

The source and target maps are

$$s(\gamma, a) = \gamma(0), \quad t(\gamma, a) = \gamma(1),$$

and the multiplication is given by

$$(\gamma_1, a_1) \cdot (\gamma_0, a_0) = (\gamma_1 \cdot \gamma_0, a_1 + a_0).$$

If we restrict to the open set $G' \subset \mathcal{G}$ consisting of all elements $[(\gamma, a)]$ with $\gamma(0) \neq \gamma(1)$, each element has a unique representative whose first entry is a geodesic. This means that G' can be identified with the open set

$$G' \subset \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{R}/4\pi\mathbb{Z}$$

consisting of elements (x_0, x_1, a) such that $x_0 \neq x_1$. We can view G' as a local groupoid (a shrinking of $\mathcal{G}$) whose structure maps are given by

$$s(x_1, x_0, a) = x_0, \qquad t(x_1, x_0, a) = x_1,$$
$$(x_2, x_1, a_1) \cdot (x_1, x_0, a_0) = (x_2, x_0, a_0 + a_1 + A(\Delta x_0 x_1 x_2)),$$

and multiplication is only defined when $x_2 + x_0 \neq 0$. We now let

$$G := \{(x_1, x_0, a) : x_0, x_1 \in \mathbb{S}^2, x_0 \neq x_1, a \in \mathbb{R}\}$$

be equipped with same structure maps as above, except that multiplication is defined only when $x_2 \neq x_0$ and $-\pi < A(\Delta x_0 x_1 x_2) < \pi$. Then G is a local Lie groupoid which is not 6-associative (see [10, Example 3.5]).

In this section, we are going to construct a local symplectic groupoid integrating the Heisenberg-Poisson manifold $\mathcal{H}\mathbb{S}^2$. This local integration restricts along certain orbits to copies of G, and therefore it is also not globalizable.

2.3 Poisson homotopy groupoid

In this section, we recall the definition of the Poisson homotopy groupoid of a Poisson manifold and some of its properties that will be useful later. For more details and proofs we refer to chapters 10 and 14 in [8].

In the following discussion we fix a Poisson manifold (M, π) and we denote its cotangent algebroid by $(T^*M, [\cdot,\cdot]_\pi, \pi^\#)$.

Definition 2.3.1. *A* **cotangent path** *in (M, π) is a path $a : [0,1] \to T^*M$ such that for all $t \in [0,1]$,*

$$\pi^\#(a(t)) = \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{pr} \circ a)(t).$$

If $a : [0,1] \to T^*M$ is a cotangent path and $\tau : [0,1] \to [0,1]$ is smooth with $\tau(0) = 0$ and $\tau(1) = 1$, the path

$$a^\tau [0,1] \to T^*M, \quad t \mapsto \tau'(t) a(\tau(t))$$

is a cotangent path and we call a^τ a reparametrization of a. Observe that one can fix a reparameterization $\tau : [0,1] \to [0,1]$, flat at $t = 0, 1$, so that for any cotangent path $a : [0,1] \to T^*M$ one has a new cotangent path a^τ vanishing at the end points together with all its derivatives.

Definition 2.3.2. *Let Σ be a manifold. A* **cotangent map** *is a bundle map $\Phi : T\Sigma \to T^*M$ with base map $\phi : \Sigma \to M$ such that*

$$\mathrm{d}\Phi^* = \Phi^* \mathrm{d}_\pi,$$

where $\Phi^* : \mathfrak{X}^k(M) \to \Omega^k(\Sigma)$ is given by:

$$(\Phi^* \mathcal{V})_x(v_1, \ldots, v_k) := \mathcal{V}_{\phi(x)}(\Phi(v_1), \ldots, \Phi(v_k)).$$

When $\Sigma = [0,1]$, a bundle map $\Phi : T\Sigma \to T^*M$ takes the form $\Phi = a\mathrm{d}t$, where $a : [0,1] \to T^*M$. It then follows that $\Phi : T[0,1] \to T^*M$ is a cotangent map if and only if a is a cotangent path.

Similarly, a bundle map $\Phi : T([0,1] \times [0,1]) \to T^*M$ takes the form $\Phi := \Phi_1 \mathrm{d}t + \Phi_2 \mathrm{d}\epsilon$ where $\Phi_i : [0,1] \to T^*M$ and one has:

Definition 2.3.3. *We say that two cotangent paths $a_0, a_1 : [0,1] \to T^*M$ are* **cotangent-path homotopic** *if there exist a cotangent map*

$$\Phi : T([0,1] \times [0,1]) \to T^*M, \quad \Phi := \Phi_1 \mathrm{d}t + \Phi_2 \mathrm{d}\epsilon,$$

such that for all $t, \epsilon \in [0,1]$:

$$\begin{aligned} \Phi_1(t,0) &= a_0(t), \qquad & \Phi_2(0,\epsilon) &= 0, \\ \Phi_1(t,1) &= a_1(t), \qquad & \Phi_2(1,\epsilon) &= 0. \end{aligned}$$

Cotangent-path homotopy is an equivalence relation on the space of cotangent paths.

A standard argument shows that a cotangent path is cotangent-path homotopic to any of its reparametrizations. In particular, one sees that:

Lemma 2.3.4. *Every contangent-path homotopy class $[a]$ has a representative $a : [0,1] \to T^*M$ which is flat at $t = 0$ and $t = 1$.*

The definition of cotangent-path homotopy is not very practical. In practice, one uses the following alternative characterization of cotangent-path homotopies.

Proposition 2.3.5. *For a bundle map $\Phi = \Phi_1 \mathrm{d}t + \Phi_2 \mathrm{d}\epsilon : T([0,1]\times[0,1]) \to T^*M$ with base map $\gamma : [0,1]\times[0,1] \to M$, the following are equivalent:*

(i) Φ *is a cotangent map;*

(ii) $\pi^\# \circ \Phi = \mathrm{d}\gamma$ *and for any (t,ϵ)-dependent 1-forms $\alpha_{t,\epsilon}$ and $\beta_{t,\epsilon}$ with*

$$\alpha_{t,\epsilon}(\gamma(t,\epsilon)) = \Phi_1(t,\epsilon), \quad \beta_{t,\epsilon}(\gamma(t,\epsilon)) = \Phi_2(t,\epsilon),$$

one has:

$$\Big(\frac{\mathrm{d}\beta_{t,\epsilon}}{\mathrm{d}t} - \frac{\mathrm{d}\alpha_{t,\epsilon}}{\mathrm{d}\epsilon}\Big)(\gamma(t,\epsilon)) = -[\alpha_{t,\epsilon}, \beta_{t,\epsilon}]_\pi(\gamma(t,\epsilon)).$$

The Poisson homotopy groupoid is

$$\Pi(M,\pi) = \frac{\text{cotangent paths}}{\text{cotangent path-homotopy}}$$

where multiplication is given by concatenation of appropriate reparametrizations, as in Lemma 2.3.4, to keep the concatenated path smooth. Theorem 14.5 from [8] says that if the cotangent algebroid $(T^*M, [\cdot,\cdot]_\pi, \pi^\#)$ is integrable, then $\Pi(M,\pi)$ admits a smooth structure and a symplectic form with respect to which $\Pi(M,\pi)$ is a symplectic groupoid integrating (M,π).

2.4 Poisson Homotopy Groupoid of the Heisenberg-Poisson Manifold

We describe now the Poisson homotopy groupoid of the Heisenberg-Poisson manifold. We start by looking at its Lie algebroid, and then investigate the cotangent paths over the singular and regular parts $\mathcal{H}S$.

2.4.1 Cotangent Algebroid of $\mathcal{H}S$

Let (S,ω_S) be a symplectic manifold and let π_S be the corresponding Poisson structure. Denote by $\mathcal{H}S$ the Heisenberg-Poisson structure on $S\times\mathbb{R}$. We have $T^*\mathcal{H}S = T^*S \times T^*\mathbb{R}$. The anchor and Lie bracket of the cotangent algebroid are given by (for convenience, write $\alpha + f\mathrm{d}y$ as (α, f)):

$$\rho(\alpha, f) = y\pi_S^{\#}(\alpha),$$
$$[(\alpha, f), (\beta, g)] = (y[\alpha, \beta]_{\pi_S}, y\pi_S^{\#}(\alpha)g - y\pi_S^{\#}(\beta)f - \pi_S(\alpha, \beta)).$$

If we identify T^*S with TS using $\pi_S^{\#}$, the cotangent algebroid becomes $T^*\mathcal{H}S \simeq TS \times T^*\mathbb{R}$ with structure maps

$$\rho(X, f) = yX,$$
$$[(X, f), (Y, g)] = (y[X, Y], yXg - yYf + \omega_S(X, Y)).$$

2.4.2 Singular Part

We look at cotangent paths over the symplectic leaves of the form $\{(x, 0)\}$. These are pairs (a_0, f_0), where $a_0 : [0, 1] \to T_x^*S$ and $f_0 : [0, 1] \to \mathbb{R}$. Two such paths (a_0, f_0), (a_1, f_1) are cotangent path-homotopic when a_0, a_1 are cotangent path-homotopic with respect to the zero Poisson structure on S and there is some cotangent path-homotopy $\Phi = a_t\mathrm{d}t + a_\epsilon\mathrm{d}\epsilon$ connecting them such that

$$\int_0^1 f_0 - \int_0^1 f_1 = \int_0^1 \int_0^1 \pi_S(a_t, a_\epsilon)\mathrm{d}t\mathrm{d}\epsilon.$$

Notice that:

- Every cotangent path is homotopic to a constant path (A_0, F_0), where $A_0 \in T_x^*S$ and $F \in \mathbb{R}$.
- Two constant paths (A_0, F_0), (A_1, F_1) are cotangent path-homotopic if and only if $A_0 = A_1$ and for some cotangent homotopy $\Phi = a_t\mathrm{d}t + a_\epsilon\mathrm{d}\epsilon$ from A_0 to A_1, we have

$$F_0 - F_1 = \int_0^1 \int_0^1 \pi_S(a_t, a_\epsilon)\mathrm{d}t\mathrm{d}\epsilon.$$

We claim that this equivalence relation on constant paths is actually trivial. To see this, let $\Phi = a_t\mathrm{d}t + a_\epsilon\mathrm{d}\epsilon$ be a cotangent homotopy from A_0 to $A_1 = A_0$ for which $F_0 - F_1 = \int_0^1 \int_0^1 \pi(a_t, a_\epsilon)\mathrm{d}t\mathrm{d}\epsilon$. If we let $v_0 = \pi_S^{\#}(A_0)$, $v_t = \pi_S^{\#}(a_t)$

and $v_\epsilon = \pi_S^\#(a_\epsilon)$, we have

$$F_0 - F_1 = -\int_0^1 \int_0^1 \omega_S(v_t, v_\epsilon)\mathrm{d}t\mathrm{d}\epsilon.$$

Since π vanishes at $(x, 0)$, using the fact that a_t and a_ϵ are the components of a cotangent path-homotopy, we have that $\frac{\partial v_\epsilon}{\partial t} = \frac{\partial v_t}{\partial \epsilon}$. So there exists a smooth function $F(t,\epsilon) \in T_xS$ such that $v_t = \frac{\partial F}{\partial t}$ and $v_\epsilon = \frac{\partial F}{\partial \epsilon}$. Since $\frac{\partial F}{\partial \epsilon}(0,\epsilon) = 0$, we have $F(0,0) = F(0,1)$. Since $\frac{\partial F}{\partial t}(t,0) = \frac{\partial F}{\partial t}(t,1) = v_0$, we also have $F(t,0) = F(t,1)$. Let $F^0(t,\epsilon) := tv_0$. Then we find

$$\int_0^1 \int_0^1 v_\epsilon \mathrm{d}\epsilon\mathrm{d}t = \int_0^1 F(t,1) - F(t,0)\mathrm{d}t = 0,$$

which gives

$$\begin{aligned}
\int_0^1 \int_0^1 (\omega_S)_x(v_t, v_\epsilon)\mathrm{d}t\mathrm{d}\epsilon &= \int_0^1 \int_0^1 (\omega_S)_x(v_t - v_0, v_\epsilon)\mathrm{d}t\mathrm{d}\epsilon \\
&= \int_0^1 \int_0^1 (\omega_S)_x\Big(\frac{\partial(F - F^0)}{\partial t}, \frac{\partial(F - F^0)}{\partial \epsilon}\Big)\mathrm{d}t\mathrm{d}\epsilon \\
&= \int_{I^2} (F - F^0)^*(\omega_S)_x = \int_{I^2} \mathrm{d}(F - F^0)^*\alpha \\
&= \int_{\partial I^2} (F - F^0)^*\alpha = 0,
\end{aligned}$$

where α is any 1-form on the vector space T_xS such that $(\omega_S)_x = \mathrm{d}\alpha$, and the last equality follows since $(F - F^0)^*\alpha|_{\partial I^2} = 0$. We conclude that $F_0 = F_1$, as claimed.

In conclusion, the arrows in the Poisson homotopy groupoid over the singular part can be identified with the points in $T^*S \times \mathbb{R}$. They form a bundle of abelian groups over S, i.e., the structure maps are given by

$$\begin{gathered}
\mathbf{s}(A_0, F_0) = (x, 0) = \mathbf{t}(A_0, F_0,) \\
(A_0, F_0) + (A_1, F_1) = (A_0 + A_1, F_0 + F_1 - \frac{1}{2}\pi_S(A_0, A_1)).
\end{gathered}$$

Identifying T^*S with TS via $\pi_S^\#$, we obtain an identification of these arrows with $TS \times \mathbb{R}$, with the structure maps given by

$$\mathbf{s}(v_0, F_0) = (x, 0) = \mathbf{t}(v_0, F_0),$$
$$(v_0, F_0) + (v_1, F_1) = (v_0 + v_1, F_0 + F_1 + \frac{1}{2}\omega_S(v_0, v_1)).$$

2.4.3 Regular Part

We now consider the arrows over the regular part. The cotangent paths over a symplectic leaf $S \times \{y\}$, for $y \neq 0$, are pairs (γ, f) where $\gamma : [0,1] \to S \times \{y\}$ and $f : [0,1] \to \mathbb{R}$. Two such paths (γ_0, f_0) and (γ_1, f_1) are cotangent path-homotopic if and only if there is a homotopy γ from γ_0 to γ_1 such that

$$\int_0^1 f_0 - \int_0^1 f_1 = -\frac{1}{y^2} \int_0^1 \int_0^1 \omega_S\Big(\frac{\mathrm{d}\gamma}{\mathrm{d}t}, \frac{\mathrm{d}\gamma}{\mathrm{d}\epsilon}\Big)\mathrm{d}t\mathrm{d}\epsilon.$$

We now notice that:

- Every pair is cotangent path-homotopic to a pair of the form (γ_0, F_0), where $F_0 \in \mathbb{R}$.
- Two such pairs (γ_0, F_0) and (γ_1, F_1) are cotangent path-homotopic if there is a homotopy γ from γ_0 to γ_1 such that

$$F_0 - F_1 = -\frac{1}{y^2} \int_0^1 \int_0^1 \omega_S\Big(\frac{\mathrm{d}\gamma}{\mathrm{d}t}, \frac{\mathrm{d}\gamma}{\mathrm{d}\epsilon}\Big)\mathrm{d}t\mathrm{d}\epsilon.$$

2.4.4 The case of the 2-sphere

When $S = \mathbb{S}^2$ with its usual area form ω_S, we can represent a path connecting non-antipodal points by the shortest geodesic. We conclude that the regular part of the Poisson homotopy groupoid of $\mathcal{H}\mathbb{S}^2$ has an open neighborhood around the identities consisting of the points in $S \times S \times \{y\} \times \mathbb{R}/\frac{4\pi}{y^2}\mathbb{Z}$

$$\Big\{(x, x', y, F_0) = ((x, y), (x', y), F_0) : x, x' \in S, x \neq -x', F_0 \in \mathbb{R}/\frac{4\pi}{y^2}\mathbb{Z}\Big\}.$$

The structure maps are given by

$$\mathbf{s}(x,x',y,F_0) = (x',y), \qquad \mathbf{t}(x,x',y,F_0) = (x,y),$$
$$(x,x',y,F_0) + (x',x'',y,F_1) = (x,x'',y,F_0+F_1+\frac{1}{y^2}A(xx'x'')).$$

where the multiplication is defined only when $x \neq x''$.

2.4.5 Exponential Map

We now describe the exponential map of $\mathcal{H}\mathbb{S}^2$. This will be useful later.

To start, let ∇ be the Levi-Civita connection on $\mathbb{S}^2$ for the round metric. We extend it to a connection ∇' on $\mathbb{S}^2 \times \mathbb{R}$ by setting it to 0 on the $\mathbb{R}$ component. Then ∇' induces a connection on $T^*(\mathbb{S}^2 \times \mathbb{R})$ given by

$$\nabla^*_X \alpha(Y) = X(\alpha(Y)) - \alpha(\nabla'_X Y),$$

for $X, Y \in \mathfrak{X}(\mathbb{S}^2 \times \mathbb{R})$ and $\alpha, \beta \in \Omega^1(\mathbb{S}^2 \times \mathbb{R})$. Finally, this connection determines a $T^*(\mathbb{S}^2 \times \mathbb{R})$-connection on $T^*(\mathbb{S}^2 \times \mathbb{R})$

$$\tilde{\nabla}_\alpha \beta = \nabla^*_{\pi^\#_{\mathbb{S}^2\times\mathbb{R}}(\alpha)}\beta.$$

Let $F \in \mathbb{R}$ and $y_0 \neq 0$. For simplicity, we also denote by F the class $[F]_{\mathbb{R}/\frac{1}{y^2}4\pi\mathbb{Z}}$. An element (x,x',y_0,F) in the cotangent groupoid corresponds to a class of cotangent paths represented by

$$\Big(\frac{1}{y_0}\omega^\flat(\dot{\gamma}), F((\mathrm{d}y)_{y_0})\Big),$$

where $\gamma : [0,1] \to \mathbb{S}^2$ is the geodesic in $\mathbb{S}^2$ such that $\gamma(0) = x$ and $\gamma(1) = x'$.

Lemma 2.4.1. *If $\gamma : [0,1] \to \mathbb{S}^2$ is the geodesic such that $\gamma(0) = x$ and $\gamma(1) = x'$, then the path $\Big(\frac{1}{y_0}\omega^\flat(\dot{\gamma}), F(\mathrm{d}y)_{y_0}\Big)$ is a geodesic with respect to $\tilde{\nabla}$.*

Proof. Let X be a time-dependent vector field on $\mathbb{S}^2$ such that $X(t,\gamma(t)) = \dot{\gamma}(t)$. Since γ is a geodesic, we have $(\nabla_X X + \frac{\mathrm{d}X}{\mathrm{d}t})(t,\gamma(t)) = 0$. Let $\alpha(t,x,y) = (\frac{1}{y_0}\omega^\flat(X(t,x)), F(\mathrm{d}y)_y)$. We have that α is a time-dependent 1-form on $\mathbb{S}^2 \times \mathbb{R}$

extending $(\frac{1}{y_0}\omega^\flat(\dot\gamma), F(\mathrm{d}y)_{y_0})$:

$$\alpha(t,\gamma(t),y_0) = (\frac{1}{y_0}\omega^\flat(\dot\gamma), F(\mathrm{d}y)_{y_0}).$$

Now, notice that:

$$\tilde\nabla_\alpha\alpha = \nabla^*_X(\frac{1}{y_0}\omega^\flat(X), F\mathrm{d}y) = \frac{1}{y_0}\nabla^*_X\omega^\flat(X),$$

where the last 1-form is zero on the $\mathbb{R}$ component. On the other hand, if Y is a vector field on $\mathbb{S}^2$, we find

$$\begin{aligned}\frac{1}{y_0}\nabla^*_X\omega^\flat(X)(Y) &= \frac{1}{y_0}(X(\omega((X,Y)) - \omega(X,\nabla'_XY)))\\ &= \frac{1}{y_0}(X(\omega((X,Y)) - \omega(X,\nabla_XY)))\\ &= -\frac{1}{y_0}\omega(\frac{\mathrm{d}X}{\mathrm{d}t},Y),\end{aligned}$$

where for the last equality we have used that $\nabla_X\omega = 0$ and $\nabla_XX + \frac{\mathrm{d}X}{\mathrm{d}t} = 0$, so that

$$\begin{aligned}0 = \nabla_X\omega(X,Y) &= X(\omega(X,Y)) - \omega(\nabla_XX,Y) - \omega(X,\nabla_XY)\\ &= X(\omega(X,Y)) + \omega(\frac{\mathrm{d}X}{\mathrm{d}t},Y) - \omega(X,\nabla_XY).\end{aligned}$$

It follows that

$$\begin{aligned}\Big(\tilde\nabla_\alpha\alpha + \frac{\mathrm{d}\alpha}{\mathrm{d}t}\Big)(Y) &= \frac{1}{y_0}\nabla^*_X\omega^\flat(X)(Y) + \frac{\mathrm{d}\alpha}{\mathrm{d}t}(Y)\\ &= -\frac{1}{y_0}\omega(\frac{\mathrm{d}X}{\mathrm{d}t},Y) + \frac{\mathrm{d}\alpha}{\mathrm{d}t}(Y)\\ &= -\frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{y_0}\omega^b(X))(Y) + \frac{\mathrm{d}\alpha}{\mathrm{d}t}(Y)\\ &= -\frac{\mathrm{d}\alpha}{\mathrm{d}t}(Y) + \frac{\mathrm{d}\alpha}{\mathrm{d}t}(Y) = 0\end{aligned}$$

□

On the other hand, since $\pi_{\mathbb{S}^2\times\mathbb{R}}(x,0) = 0$, for any $v_x \in T_x\mathbb{S}^2$ and $F \in \mathbb{R}$, the constant path $(\omega^\flat(v_x), F(\mathrm{d}y)_{y=0})$ is a geodesic with respect to $\tilde\nabla$. This and the Lemma imply the following, which we will need later.

Proposition 2.4.2. *Let $\mathcal{G}$ be the cotangent groupoid, let $\gamma : [0,1] \to \mathbb{S}^2$ be the geodesic such that $\gamma(0) = x$ and $\gamma(1) = x'$, and let $\exp_{\tilde{\nabla}} : T^*(\mathbb{S}^2 \times \mathbb{R}) \to \mathcal{G}$ be the exponential map with respect to $\tilde{\nabla}$. We have:*

(i) For $y_0 \neq 0$: $\exp_{\tilde{\nabla}}(\frac{1}{y_0}\omega^\flat(\dot{\gamma}(0)), F(dy)_{y_0}) = [(\frac{1}{y_0}\omega^\flat(\dot{\gamma}), F(\mathrm{d}y)_{y_0})] = (x, x', y_0, F)$;

(ii) For $y_0 = 0$: $\exp_{\tilde{\nabla}}(\omega^\flat(v_x), F(\mathrm{d}y)_{y=0}) = [(\omega^\flat(v_x), F(\mathrm{d}y)_{y=0})] = (v_x, F)$.

In particular:

(i) If $y_0 \neq 0$: $\exp_{\tilde{\nabla}}(\alpha, F(dy)_{y_0}) = (x, \exp_\nabla(v), y_0, F)$, where $v = y_0 \pi_S^\#(\alpha)$.

(ii) If $y_0 = 0$: $\exp_{\tilde{\nabla}}(\alpha, F(\mathrm{d}y)_{y=0}) = (v_x, F)$, where $v = \pi_S^\#(\alpha)$.

2.5 Connes' Groupoid

In the next sections we will construct a local symplectic groupoid integrating the Heisenberg-Poisson manifold, using *explosion*. Our presentation is inspired by the construction in [12] of a global groupoid associated with a Heisenberg-Poisson manifold, which in general is not smooth.

2.5.1 Explosions

Let X be a manifold and let $Y \subset X$ be a submanifold. As a set, the **explosion** of X along Y is

$$E(X,Y) := X \times (\mathbb{R} \setminus \{0\}) \bigcup N(X,Y),$$

where $N(X,Y) = T_Y X / TY$ is the normal bundle of Y in X. One should think of this construction as replacing $X \times \{0\} \subset X \times \mathbb{R}$ by $N(X,Y)$, and this can be made a smooth manifold as follows. The smooth structure on the piece $X \times \mathbb{R} \setminus \{0\}$ is the usual product smooth structure. To construct charts around points in $N(X,Y)$, choose a chart $\mathbb{R}^n \times \mathbb{R}^m \to X$, $(y,z) \mapsto \Phi(y,z)$, with $(y,0) \mapsto \Phi(y,0)$ a chart for Y. The explosion chart

$$E(\Phi) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to E(X,Y)$$

is then given by

$$\epsilon \neq 0 : E(\Phi)(y, z', \epsilon) = (\Phi(y, \epsilon z'), \epsilon),$$
$$\epsilon = 0 : E(\Phi)(y, z', 0) = \mathrm{d}\Phi((0, z)_{(y,0)}),$$

where $(0, z)_{(y,0)}$ is viewed as a tangent vector at $(y, 0)$ and we take its image in the normal bundle.

The explosion along a submanifold is also called the *deformation to the normal cone*. We refer to [9] for more details.

2.5.2 Explosion Along Identity

Let G be a groupoid over M, the explosion $E(G, M)$ has a groupoid structure over $M \times \mathbb{R}$ given as follows.

- On $G \times (\mathbb{R} \setminus \{0\})$, the structure maps are:

$$\tilde{\mathbf{s}}(g, y) = (\mathbf{s}(g), y), \qquad \tilde{\mathbf{t}}(g, y) = (\mathbf{t}(g), y), \qquad (g_0, y)\cdot(g_1, y) = (g_0\cdot g_1, y).$$

- On $N(G, M)$, the structure maps are:

$$\tilde{\mathbf{s}}(v_{1_x}) = (x, 0) = \tilde{\mathbf{t}}(v_{1_x}), \qquad v_{1_x} \cdot w_{1_x} = v_{1_x} + w_{1_x}.$$

2.5.3 The Connes Groupoid

When $G = S \times S$ is the pair groupoid, the groupoid structure on $E(S \times S, S)$ is known as the **Connes groupoid**. Its algebroid has underlying vector bundle isomorphic to $TS \times \mathbb{R} \to S \times \mathbb{R}$ via the identification:

- $y \in \mathbb{R} \setminus \{0\}$: $(v_x, y) \mapsto (0_x, yv_x, 0_y)$;
- $y = 0$: $(v_x, 0) \mapsto [(0_x, v_x)]$;

where we identify $T_{0_x} N_x(S \times S, S) \simeq N_x(S \times S, S)$. The anchor is given by

$$\rho : TS \times \mathbb{R} \to TS \times T\mathbb{R}, \ \rho(v_x, y) = (yv_x, 0_y).$$

For the Lie bracket, given a y-dependent vector field $X = X(x, y)$ on S, we consider the section of the algebroid given by

$$\tilde{X}(x, y) = (X(x, y), y) \in TS \times \mathbb{R},$$

and we set:

$$[\tilde{X}, \tilde{Y}](x, y) := (y[X, Y], y).$$

2.5.4 Central Extension of the Connes Algebroid

The 2-form ω_S induces an algebroid cocycle on $TS \times \mathbb{R}$. The corresponding central extension is the Lie algebroid with supporting vector bundle

$$(TS \times \mathbb{R}) \times \mathbb{R} \to S \times \mathbb{R},\ (v_x, y, F) \mapsto (x, y),$$

and structure maps are given by

$$\rho : (TS \times \mathbb{R}) \times \mathbb{R} \to TS \times T\mathbb{R}, \quad \rho(v_x, y, F) = (yv_x, 0_y),$$
$$[(\tilde{X}, f), (\tilde{Y}, g)](x, y) = ([\tilde{X}, \tilde{Y}](x, y),\ yX(g) - yY(f) + (\omega_S)_x(X, Y)).$$

Using the vector bundle isomorphism

$$(TS \times \mathbb{R}) \times \mathbb{R} \simeq TS \times T^*\mathbb{R},\ (v_x, y, F) \mapsto (v_x, F(\mathrm{d}y)_y),$$

we see that this central extension of the Connes algebroid is isomorphic to the cotangent algebroid of the Heisenberg-Poisson manifold $\mathcal{H}S$ of Section 2.4.1.

2.5.5 Local Groupoid Structure on $E(\mathbb{S}^2 \times \mathbb{S}^2, \mathbb{S}^2) \times \mathbb{R}$

From now on we will set $S = \mathbb{S}^2$. We extend the Conn's groupoid structure on $E(\mathbb{S}^2 \times \mathbb{S}^2, \mathbb{S}^2)$ to a local groupoid structure on $E(\mathbb{S}^2 \times \mathbb{S}^2, \mathbb{S}^2) \times \mathbb{R}$ over the same base $\mathbb{S}^2 \times \mathbb{R} = \mathcal{H}\mathbb{S}^2$:

- On $\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{R} \setminus \{0\} \times \mathbb{R}$, we let

$$\mathbf{s}(x, x', y, F) = (x', y'), \qquad \mathbf{t}(x, x', y, F) = (x, y)$$

and we define multiplication of (x, x', y, F_0) and (x', x'', y, F_1) only when $x \neq -x''$ by:

$$(x, x', y, F_0) \cdot (x', x'', y, F_1) = (x, x'', y, F_0 + F_1 + \frac{1}{y^2} A(xx'x'')).$$

- On $T\mathbb{S}^2 \times \mathbb{R}$,

$$\mathbf{s}(v_x, F) = (x, 0) = \mathbf{t}(v_x, F),$$
$$(v_x, F_0) \cdot (w_x, F_1) = (v_x + w_x, F_0 + F_1 + \frac{1}{2}\omega_{\mathbb{S}^2}(v_x, w_x)).$$

2.6 Smoothness and Algebroid of the Groupoid $E(\mathbb{S}^2 \times \mathbb{S}^2, \mathbb{S}^2) \times \mathbb{R}$

In this section we set $\omega = \omega_{\mathbb{S}^2}$ and $\pi = \pi_{\mathbb{S}^2}$, and we denote the Heisenberg-Poisson structure by $\pi_{\mathbb{S}^2 \times \mathbb{R}}$. We will show that the groupoid $E(\mathbb{S}^2 \times \mathbb{S}^2, \mathbb{S}^2) \times \mathbb{R}$ is smooth and that is algebroid is isomorphic to the cotangent groupoid of $\mathcal{H}\mathbb{S}^2$. For that we will make use of the following auxiliary result:

Lemma 2.6.1. *Let X and Y be vector fields on $\mathbb{S}^2$. Let $x \in \mathbb{S}^2$. Let γ be the integral curve of X starting at x and let $\varphi_Y(t, x) = \varphi_Y^t(x)$ be the flow of Y. Then*

$$\lim_{y \to 0} \frac{1}{y^2} A(x, \gamma(y), \varphi_Y(y, \gamma(y))) = \frac{1}{2}\omega(X(x), Y(x))$$

The proof of this lemma is deferred for later.

2.6.1 Smoothness of $E(\mathbb{S}^2 \times \mathbb{S}^2, \mathbb{S}^2) \times \mathbb{R}$

Proposition 2.6.2. *The groupoid $E(\mathbb{S}^2 \times \mathbb{S}^2, \mathbb{S}^2) \times \mathbb{R}$ is smooth.*

Proof. To show that the groupoid structure defined above is smooth, we need to write down the structure maps in a chart for $E(\mathbb{S}^2 \times \mathbb{S}^2, \mathbb{S}^2) \times \mathbb{R}$. We start by taking an explosion chart for $E(\mathbb{S}^2 \times \mathbb{S}^2, \mathbb{S}^2)$.

Let (θ, ϕ) be a smooth chart on $\mathbb{S}^2$. Denote by $(\theta, \phi, \theta', \phi')$ the induced chart on $\mathbb{S}^2 \times \mathbb{S}^2$. Making the change of coordinates

$$\overline{\theta'} = \theta - \theta',$$
$$\overline{\phi'} = \phi - \phi',$$

then $(\theta, \phi, \overline{\theta'}, \overline{\phi'})$ is a chart adapted to the submanifold $\mathbb{S}^2 = \Delta_{\mathbb{S}^2\times\mathbb{S}^2} \subseteq \mathbb{S}^2\times\mathbb{S}^2$. The induced explosion chart on $E(\mathbb{S}^2 \times \mathbb{S}^2, \mathbb{S}^2)$ is given by

$$(\theta, \phi, \overline{\theta}, \overline{\phi}, y) \mapsto (\theta, \phi, \theta - y\overline{\theta}, \phi - y\overline{\phi}, y) \in \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{R} \setminus \{0\}, \qquad \text{if } y \neq 0,$$
$$(\theta, \phi, \overline{\theta}, \overline{\phi}, 0) \mapsto -\overline{\theta}\, \frac{\partial}{\partial \theta}\bigg|_{(\theta,\phi)} - \overline{\phi}\, \frac{\partial}{\partial \phi}\bigg|_{(\theta,\phi)} \in T\mathbb{S}^2 \simeq N(\mathbb{S}^2 \times \mathbb{S}^2, \mathbb{S}^2), \text{ if } y = 0.$$

On the chart $(\theta, \phi, \overline{\theta}, \overline{\phi}, F)$ for $E(\mathbb{S}^2 \times \mathbb{S}^2, \mathbb{S}^2) \times \mathbb{R}$, where F denotes the coordinate on the $\mathbb{R}$ factor, the structure maps are given by

$$\mathbf{s}(\theta, \phi, \overline{\theta}, \overline{\phi}, y, F) = (\theta - y\overline{\theta}, \phi - y\overline{\phi}, y),$$
$$\mathbf{t}(\theta, \phi, \overline{\theta}, \overline{\phi}, y, F) = (\theta, \phi, y),$$
$$\mathbf{u}(\theta, \phi, y) = (\theta, \phi, 0, 0, y, 0),$$
$$\mathbf{i}(\theta, \phi, \overline{\theta}, \overline{\phi}, y, F) = (\theta - y\overline{\theta}, \phi - y\overline{\phi}, -\overline{\theta}, -\overline{\phi}, y, -F).$$

All these maps are smooth. On the other hand, for the multiplication we find:

- $\underline{y \neq 0,\ \theta - y\overline{\theta} = \theta',\ \phi - y\overline{\phi} = \phi'}$:

  $$(\theta, \phi, \overline{\theta}, \overline{\phi}, y, F_0)\cdot(\theta', \phi', \overline{\theta}', \overline{\phi}', y, F_1) = (\theta, \phi, \overline{\theta}+\overline{\theta}', \overline{\phi}+\overline{\phi}', y, F_0+F_1+L(\theta, \phi, \overline{\theta}, \overline{\phi}, y),$$

  where:

  $$L(\theta, \phi, \overline{\theta}, \overline{\phi}, y) = \frac{1}{y^2} A((\theta, \phi), (\theta - y\overline{\theta}, \phi - y\overline{\phi}), (\theta - y\overline{\theta} - y\overline{\theta}', \phi - y\overline{\phi} - y\overline{\phi}'))).$$

- $\underline{y = 0,\ \theta = \theta',\ \phi = \phi'}$:

  $$(\theta, \phi, \overline{\theta}, \overline{\phi}, 0, F_0)\cdot(\theta', \phi', \overline{\theta}', \overline{\phi}', 0, F_1) = (\theta, \phi, \overline{\theta}+\overline{\theta}', \overline{\phi}+\overline{\phi}', 0, F_0+F_1+L_0(\theta, \phi, \overline{\theta}, \overline{\phi}))$$

where:

$$L_0(\theta,\phi,\overline{\theta},\overline{\phi}) = \frac{1}{2}\omega_{(\theta,\phi)}(-\overline{\theta}\frac{\partial}{\partial\theta} - \overline{\phi}\frac{\partial}{\partial\phi}, -\overline{\theta}'\frac{\partial}{\partial\theta} - \overline{\phi}'\frac{\partial}{\partial\phi})).$$

Applying Lemma 2.6.1 with

$$X = -\overline{\theta}\frac{\partial}{\partial\theta} - \overline{\phi}\frac{\partial}{\partial\phi}, \qquad Y = -\overline{\theta}'\frac{\partial}{\partial\theta} - \overline{\phi}'\frac{\partial}{\partial\phi},$$

we see that:

$$\lim_{y\to 0} L(\theta,\phi,\overline{\theta},\overline{\phi},y) = L_0(\theta,\phi,\overline{\theta},\overline{\phi}),$$

so multiplication is smooth. □

2.6.2 Lie Algebroid of $E(\mathbb{S}^2 \times \mathbb{S}^2, \mathbb{S}^2) \times \mathbb{R}$

We will now show that:

Proposition 2.6.3. *The Lie algebroid of* $E(\mathbb{S}^2 \times \mathbb{S}^2, \mathbb{S}^2) \times \mathbb{R}$ *is isomorphic to the cotangent algebroid of* $\mathbb{S}^2 \times \mathbb{R}$

Proof. For $(x, y) \in \mathbb{S}^2 \times \mathbb{R}$ where $y \neq 0$, the corresponding identity element is $\mathbf{u}(x, y) = (x, x, y, 0) \in E(\mathbb{S}^2 \times \mathbb{S}^2, \mathbb{S}^2) \times \mathbb{R}$. The target fiber and its tangent space at $\mathbf{u}(x, y)$ are given by

$$\begin{aligned} \mathbf{t}^{-1}(x,y) &= \{(x, x', y, F) : x' \in \mathbb{S}^2, F \in \mathbb{R}\}, \\ T_{\mathbf{u}(x,y)}\mathbf{t}^{-1}(x,y) &= \Big\{(0_x, v_x, 0_y, f\tfrac{\partial}{\partial F}\big|_{F=0}) : v_x \in T_x\mathbb{S}^2, f \in \mathbb{R}\Big\}. \end{aligned}$$

We have the identification

$$\begin{aligned} T_x\mathbb{S}^2 \times \{y\} \times \mathbb{R} &\to T_{\mathbf{u}(x,y)}\mathbf{t}^{-1}(x,y), \\ (v_x, y, f) &\mapsto (0_x, yv_x, 0_y, f\tfrac{\partial}{\partial F}\big|_{F=0}). \end{aligned}$$

On the other hand, for $(x, 0) \in \mathbb{S}^2 \times \mathbb{R}$, the identity element is $\mathbf{u}(x, 0) = (0_x, 0) \in T\mathbb{S}^2 \times \mathbb{R} \simeq N(\mathbb{S}^2 \times \mathbb{S}^2, \mathbb{S}^2) \times \mathbb{R} \subseteq E(\mathbb{S}^2 \times \mathbb{S}^2, \mathbb{S}^2) \times \mathbb{R}$. The target fiber and the tangent space at $\mathbf{u}(x, 0)$ are

$$\begin{aligned} \mathbf{t}^{-1}(x,0) &= \{(v_x, F) : v_x \in T_x\mathbb{S}^2, F \in \mathbb{R}\}, \\ T_{\mathbf{u}(x,0)}\mathbf{t}^{-1}(x,0) &= T_x\mathbb{S}^2 \times \mathbb{R}, \end{aligned}$$

where we have identified $T_{0_x}T_x\mathbb{S}^2 \simeq T_x\mathbb{S}^2$ and $T_0\mathbb{R} \simeq \mathbb{R}$. To be precise, we have the isomorphism

$$T_x\mathbb{S}^2 \times \mathbb{R} \to T_{\mathbf{u}(x,0)}\mathbf{t}^{-1}(x,0), \quad (v_x, f) \mapsto \left(\tfrac{\mathrm{d}}{\mathrm{d}t}tv_x, f\tfrac{\partial}{\partial F}\big|_{F=0}\right).$$

We further identify $T_x\mathbb{S}^2 \times \{0\} \times \mathbb{R} \simeq T_x\mathbb{S}^2 \times \mathbb{R}$, so have an isomorphism of vector bundles:

$$T\mathbb{S}^2 \times \mathbb{R} \times \mathbb{R} \simeq \mathrm{LieAlg}(E(\mathbb{S}^2 \times \mathbb{S}^2, \mathbb{S}^2) \times \mathbb{R}). \tag{2.1}$$

We need to find the anchor and bracket of the algebroid under this identification.

- Anchor:

$$\begin{aligned}\rho(v_x, y, f) &= \mathrm{d}\mathbf{s}(0_x, yv_x, 0_y, f\tfrac{\partial}{\partial F}\big|_{F=0}) = (yv_x, 0_y),\\ \rho(v_x, 0, f) &= \mathrm{d}\mathbf{s}(\tfrac{\mathrm{d}}{\mathrm{d}t}tv_x, f\tfrac{\partial}{\partial F}\big|_{F=0}) = (0_x, 0_{y=0}).\end{aligned}$$

This agrees with the anchor for the cotangent algebroid.

- Bracket: Observe that sections $\alpha = (X, f)$ of the form:

$$\begin{aligned}\mathbb{S}^2 \times \mathbb{R} &\to T\mathbb{S}^2 \times \mathbb{R} \times \mathbb{R},\\ (x, y) &\mapsto (X(x), y, f(x, y)),\end{aligned}$$

where $X \in \mathfrak{X}(\mathbb{S}^2)$ and $f \in C^\infty(\mathbb{S}^2 \times \mathbb{R})$ generate the space of all sections of this vector bundle, as a $C^\infty(\mathbb{S}^2 \times \mathbb{R})$-module. Hence, since we have already checked that the anchors are preserved by (2.1), we only need to show that this isomorphism preserves Lie brackets for this type of sections.

For such a section $\alpha = (X, f)$, we find the corresponding left invariant vector field $\tilde{\alpha}$ on the groupoid restricted to the open and dense subset $\mathbb{S}^2 \times \mathbb{S}^2 \times (\mathbb{R} \setminus \{0\}) \times \mathbb{R}$. Along the identity section we have:

$$\tilde{\alpha}(x, x, y, 0) = (0_x, yX(x), 0_y, f(x, y)\tfrac{\partial}{\partial F}\big|_{F=0}).$$

If γ_y and $g(t)$ are curves in $\mathbb{S}^2$ and $\mathbb{R}$, respectively, such that $\dot{\gamma}_y(0) = yX(x)$

and $\dot{g}(0) = f(x,y)(\frac{\partial}{\partial F})_{F=0}$, then at (x', x, y, F) we have for $y \neq 0$:

$$\begin{aligned}
\tilde{\alpha}(x', x, y, F) &= \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}(x', x, y, F)(x, \gamma_y(t), y, g(t)) \\
&= \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\big(x', \gamma_y(t), y, F + g(t) + \tfrac{1}{y^2}A(x', x, \gamma_y(t))\big) \\
&= \big(0_{x'}, yX(x), 0_y, (f(x,y) + \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\tfrac{1}{y^2}A(x', x, \gamma_y(t)))\tfrac{\partial}{\partial F}\big) \\
&= \big(0_{x'}, yX(x), 0_y, (f(x,y) + B_X(x', x, y, F))\tfrac{\partial}{\partial F}\big),
\end{aligned}$$

where we have introduce the notation B_X for the function

$$\begin{aligned}
B_X : \mathbb{S}^2 \times \mathbb{S}^2 \times (\mathbb{R} \setminus \{0\}) \times \mathbb{R} &\to \mathbb{R} \\
(x', x, y, F) &\mapsto \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\tfrac{1}{y^2}A(x', x, \gamma_y(t))).
\end{aligned}$$

Let $\alpha = (X, f), \beta = (Y, g)$ be two sections, corresponding to vector fields $X, Y \in \mathfrak{X}(\mathbb{S}^2)$ and functions $f, g \in C^\infty(\mathbb{S}^2 \times \mathbb{R})$. The associated left invariant vector fields, when restricted to $\mathbb{S}^2 \times \mathbb{S}^2 \times (\mathbb{R} \setminus \{0\}) \times \mathbb{R}$, are then

$$\begin{aligned}
\tilde{\alpha}(x', x, y, F) &= (0_{x'}, yX(x), 0_y, (f(x,y) + B_X(x', x, y, F))\tfrac{\partial}{\partial F}), \\
\tilde{\beta}(x', x, y, F) &= (0_{x'}, yY(x), 0_y, (g(x,y) + B_Y(x', x, y, F))\tfrac{\partial}{\partial F}).
\end{aligned}$$

The Lie bracket of these two sections is given by

$$\begin{aligned}
[\alpha, \beta](x, y) &= [\tilde{\alpha}, \tilde{\beta}](x, x, y, 0) \\
&= (0_x, y^2[X, Y](x), 0_y, (yX(g + B_Y)(x, y) - yY(f + B_X)(x, y))\tfrac{\partial}{\partial F}\big|_{F=0}),
\end{aligned}$$

a section of the algebroid. Under the identification with $T\mathbb{S}^2 \times \mathbb{R} \times \mathbb{R}$, we conclude that:

$$[\alpha, \beta] = (y[X, Y], y, yX(g) - yY(f) + yX(B_Y) - yY(B_X)).$$

Therefore, to show that the bracket is the same as the bracket for the cotangent algebroid, it remains to prove that:

$$yX(B_Y) = \frac{1}{2}\omega(X, Y).$$

Notice that, if γ is the integral curve of X through x and $\varphi_{yY}(t, x) = \varphi_{yY}^t(x)$

is the flow of the vector field yY, then we have

$$yX(B_Y)(x,y) = y\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\tfrac{1}{y^2}A(x,\gamma(t),\varphi_{yY}(s,\gamma(t)))$$

Replacing yY with Y, we see that it is enough to show

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}A(x,\gamma(t),\varphi_Y(s,\gamma(t))) = \frac{1}{2}\omega(X(x),Y(x)). \tag{2.2}$$

We claim that this follows from Lemma 2.6.1. Indeed, if we use the notation $a(s,t) := A(x,\gamma(t),\gamma_y^Y(s,\gamma(t)))$, we have $a(0,t) = 0 = a(s,0)$, so that at $(0,0)$ the following partial derivatives vanish: $a_s, a_t, a_{ss}, a_{tt} = 0$. This implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}a(s,t) = \lim_{y\to 0}\frac{1}{y^2}a(y,y),$$

so (2.2) amounts to:

$$\lim_{y\to 0}\frac{1}{y^2}A(x,\gamma(y),\varphi_Y(y,\gamma(y))) = \frac{1}{2}\omega(X(x),Y(x)),$$

which is precisely the statement of Lemma 2.6.1. □

2.7 Proof of an auxiliary lemma

In this section we prove Lemma 2.6.1. For simplicity, we denote both $F \in \mathbb{R}$ and its class $[F]_{\mathbb{R}/\frac{1}{y^2}4\pi\mathbb{Z}}$ by the same symbol. Also, we let X, Y be vector fields on $\mathbb{S}^2$ and denote by γ the integral curve of X starting at x and by $\varphi_Y(t,x) = \varphi_Y^t(x)$ the flow of Y.

First, we show that:

Lemma 2.7.1. *If $\gamma_y : [0,1] \to \mathbb{S}^2$ is the geodesic with $\gamma_y(0) = \gamma(y)$ and $\gamma_y(1) = \varphi_Y(y,\gamma(y))$ then*

$$\lim_{y\to 0}\frac{1}{y}\dot{\gamma}_y(0) = \frac{\partial}{\partial s}\Big|_{s=0}\varphi_Y(s,x).$$

Proof. Using the standard immersion $\mathbb{S}^2 \subset \mathbb{R}^3$, the geodesic starting at x with tangent vector v is

$$\gamma_{x,v}(t) = \cos(|v|t)x + \sin(|v|t)\tfrac{v}{|v|}.$$

The definition for γ_y implies that $\gamma_y = \gamma_{\gamma(y),v_y}$ where $v_y \in T_{\gamma(y)}\mathbb{S}^2$ is such that $v_0 = 0$ and

$$\varphi_Y(y,\gamma(y)) = \gamma_{\gamma(y),v_y}(1) = \cos(|v_y|)\gamma(y) + \sin(|v_y|)\frac{v_y}{|v_y|}.$$

Differentiating both sides, we have for the left-hand side:

$$\frac{\mathrm{d}}{\mathrm{d}y}\Big|_{y=0} \varphi_Y(y,\gamma(y)) = \frac{\partial}{\partial s}\Big|_{s=0} \varphi_Y(s,x) + \dot\gamma(0),$$

while for the right-hand side:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}y}\Big|_{y=0} \big(\cos(|v_y|)\gamma(y) + \sin(|v_y|)\frac{v_y}{|v_y|}\big) = \\
&\qquad = \dot\gamma(0) + \lim_{y\to 0}\Big(\cos(|v_y|) - \frac{\sin(|v_y|)}{|v_y|}\Big)\frac{v_y|v_y|'}{|v_y|} + \frac{\sin(|v_y|)}{|v_y|}v_y'\Big) \\
&\qquad = \dot\gamma(0) + \lim_{y\to 0} v_y' \\
&\qquad = \dot\gamma(0) + \frac{\mathrm{d}}{\mathrm{d}y}\Big|_{y=0} v_y,
\end{aligned}$$

where the second equality holds since $\frac{v_y|v_y|'}{|v_y|}$ is bounded.

This shows that $\frac{\partial}{\partial s}\big|_{s=0}\varphi_Y(s,x) = \frac{\mathrm{d}}{\mathrm{d}y}\big|_{y=0} v_y$, so the lemma holds.

□

Corollary 2.7.2. *With the same notation as in the Lemma:*

$$\lim_{y\to 0}(x,\gamma(y),y,F) = (X(x),F), \tag{a}$$

$$\lim_{y\to 0}(\gamma(y),\varphi_Y(y,\gamma(y)),y,F) = (Y(x),F), \tag{b}$$

where the left sides are curves in the Poisson homotopy groupoid.

Proof. When $X = 0$, (b) becomes (a), so we only need to prove (b).

By Proposition 2.4.2, (b) amounts to

$$\lim_{y\to 0}(\frac{1}{y}\omega^\flat(\dot\gamma_y(0)), F(\mathrm{d}y)_y) = (Y(x), F(\mathrm{d}y)_{y=0}),$$

or equivalently

$$\lim_{y\to 0}\frac{1}{y}\dot\gamma_y(0) = \frac{\partial}{\partial s}\Big|_{s=0}\varphi_Y(s,x)$$

where γ_y is the geodesic in $\mathbb{S}^2$ such that $\gamma_y(0) = \gamma(y)$ and $\gamma_y(1) = \varphi_Y(y, \gamma(y))$. So the corollary follows from the previous lemma. $\square$

Finally, we can prove Lemma 2.6.1. Recall that the multiplication of the Poisson homotopy groupoid discussed in Section 2.3:

- Regular Part: In $\mathbb{S}^2 \times \mathbb{S}^2 \times (\mathbb{R} \setminus \{0\}) \times (\mathbb{R}/\frac{1}{y^2}4\pi\mathbb{Z})$, the multiplication is given by

$$(x, x', y, F_0) \cdot (x', x'', y, F_1) = (x, x'', y, F_0 + F_1 + \tfrac{1}{y^2}A(x, x', x'')).$$

- Singular Part: In $T\mathbb{S}^2 \times \mathbb{R}$, the multiplication is given by

$$(v_x, F_0) \cdot (v'_x, F_1) = (v_x + v'_x, F_0 + F_1 + \tfrac{1}{2}\omega(v_x, v'_x)).$$

Then, using Corollary 2.7.2, we have:

$$\begin{aligned}
&\lim_{y\to 0}(x, \varphi_Y(y, \gamma(y)), y, \tfrac{1}{y^2}A(x, \gamma y, \varphi_Y(y, \gamma(y)))) = \\
&\qquad = \lim_{y\to 0}(x, \gamma(y), y, 0) \cdot (\gamma(y), \varphi_Y(y, \gamma(y)), y, 0) \\
&\qquad = \lim_{y\to 0}(x, \gamma(y), y, 0) \cdot \lim_{y\to 0}(\gamma(y), \varphi_Y(y, \gamma(y)), y, 0) \\
&\qquad = (X(x), 0) \cdot (Y(x), 0) \\
&\qquad = (X(x) + Y(x), \tfrac{1}{2}\omega(X(x), Y(x))).
\end{aligned}$$

Hence, we have

$$\lim_{y\to 0}(x, \varphi_Y(y, \gamma(y)), y, \tfrac{1}{y^2}A(x, \gamma y, \varphi_Y(y, \gamma(y)))) = (X(x)+Y(x), \tfrac{1}{2}\omega(X(x), Y(x))).$$

Writing $L = \lim_{y\to 0} \frac{1}{y^2}A(x, \gamma(y), \varphi_Y(y, \gamma(y)))$, using Proposition 2.4.2 and an argument as in the proof of Lemma 2.7.1, we find:

$$\begin{aligned}
(X(x) + Y(x), \frac{1}{2}\omega(X(x), Y(x))) &= \lim_{y\to 0}(x, \varphi_Y(y, \gamma(y)), y, +\tfrac{1}{y^2}A(x, \gamma y, \varphi_Y(y, \gamma(y)))) \\
&= \lim_{y\to 0} \exp_\nabla(\frac{1}{y}\omega^\flat(\dot{\gamma}_y(0)) + \tfrac{1}{y^2}A(x, \gamma y, \varphi_Y(y, \gamma(y)))(\mathrm{d}y)_y) \\
&= \exp_\nabla(\omega^\flat(X(x) + Y(x)) + L(\mathrm{d}y)_{y=0}) \\
&= (X(x) + Y(x), L)
\end{aligned}$$

where $\gamma_y : [0,1] \to \mathbb{S}^2$ is the geodesic with $\gamma(0) = x$ and $\gamma(1) = \varphi_Y(y, \gamma(y))$.

We conclude that:

$$\lim_{y \to 0} \frac{1}{y^2} A(x, \gamma y, \varphi_Y(y, \gamma(y))) = L = \frac{1}{2}\omega(X(x), Y(x)),$$

hence Lemma 2.6.1 holds.

2.8 Epilogue

The previous results and [10] allows us to deduce:

Corollary 2.8.1. *The Heisenberg-Poisson manifold $\mathcal{H}\mathbb{S}^2$ admits a local symplectic groupoid integrating it which is not* 6*-associative.*

Indeed, the results in Section 2.6 show that $E(\mathbb{S}^2 \times \mathbb{S}^2, \mathbb{S}^2) \times \mathbb{R}$ is a local Lie groupoid with Lie algebroid isomorphic to the cotangent algebroid of the Heisenberg-Poission manifold $\mathcal{H}\mathbb{S}^2$. Since it has 1-connected target fibers, it admits a multiplicative symplectic form for which it is a local symplectic groupoid integrating $\mathcal{H}\mathbb{S}^2$.

This local groupoid contains a copy of the local Lie groupoid G'' of Example 3.5 in [10], by considering the embedding:

$$G'' \to E(\mathbb{S}^2 \times \mathbb{S}^2, \mathbb{S}^2) \times \mathbb{R}, \quad (x_0, x_1, F) \mapsto (x_0, x_1, 1, F),$$

which is a local groupoid homomorphism over

$$\mathbb{S}^2 \to \mathbb{S}^2 \times \mathbb{R}, \quad x \mapsto (x, 1).$$

It is shown in Example 3.5 of [10] that G'' is not 6-associative. Hence, it follows that the local symplectic groupoid $E(\mathbb{S}^2 \times \mathbb{S}^2, \mathbb{S}^2) \times \mathbb{R}$ is also not 6-associative.

CHAPTER 3

GENERATING FUNCTIONS AND THE SGA EQUATION

3.1 Main Question

Let (T^*M, ω_c) be the canonical integration of the zero Poisson structure on M. This means that ω_c is the canonical symplectic structure, and the graph of multiplication is given by

$$\text{Graph}(m_0) = \{(\alpha_x, \beta_x, \alpha_x + \beta_x) : \alpha_x, \beta_x \in T^*_x M\}.$$

The graph $\text{Graph}(m_0)$ is a Lagrangian submanifold of $\overline{T^*M} \times \overline{T^*M} \times T^*M$, where $\overline{T^*M}$ denotes the symplectic manifold T^*M equipped with the symplectic form $-\omega_c$. Since $\text{Graph}(m_0) \simeq T^*M \times_M T^*M$ via

$$\text{Graph}(m_0) \to T^*M \times_M T^*M, \quad (\alpha_x, \beta_x, \alpha_x + \beta_x) \mapsto (\alpha_x, \beta_x),$$

we can apply the Lagrangian Neighborhood Theorem to obtain a symplectomorphism ϕ from an open neighborhood of $\text{Graph}(m_0) \subseteq \overline{T^*M} \times \overline{T^*M} \times T^*M$ to an open neighborhood of the zero section in $(T^*(T^*M \times_M T^*M), \omega_c)$, such that for all $\alpha_x, \beta_x \in T^*_x M$

$$\phi(\alpha_x, \beta_x, \alpha_x + \beta_x) = 0_{(\alpha_x, \beta_x)}.$$

If $S : T^*M \times_M T^*M \to \mathbb{R}$ is any function, then we obtain a Lagrangian submanifold:

$$\phi^{-1}(\text{Graph}(\mathrm{d}S)) \subset \overline{T^*M} \times \overline{T^*M} \times T^*M.$$

We would like to find under what assumptions on S does $\phi^{-1}(\text{Graph}(\mathrm{d}S))$ arise as the graph of multiplication of a local symplectic groupoid structure on T^*M. More precisely, we have:

Question 3.1.1. *When is there a local symplectic groupoid structure on an open neighborhood of 0_{T^*M} in T^*M equipped with the canonical symplectic form and such that:*

(i) $x \mapsto 0_x$ *is the identity section;*

(ii) $\alpha \mapsto -\alpha$ *is the inversion map;*

(iii) $\mathrm{Graph}(m) \subseteq \phi^{-1}(\mathrm{Graph}(\mathrm{d}S))$.

Definition 3.1.2 ([4])**.** *If G is a local symplectic groupoid satisfying the above conditions, we call S a* **generating function** *for G with respect to ϕ.*

In order to answer the question above, let us assume that there is a local symplectic groupoid $O \subseteq T^*M$ satisfying conditions (i)-(iii) and see what this implies for S. We denote the structure maps of $O \rightrightarrows M$ by

$$\begin{aligned} \mathbf{s}, \mathbf{t} &: O \to M, \\ \mathbf{m} &: U \to O, \\ \mathbf{i} &: V \to V, \\ \mathbf{u} &: M \to O. \end{aligned}$$

Then we can make the following observations:

1. Since $\mathbf{u}(x) = 0_x$ and $(\mathbf{u}(x), \mathbf{u}(x), \mathbf{u}(x)) \in \mathrm{Graph}(\mathbf{m})$ for all $x \in M$, we have

$$\begin{aligned} 0_{(0_x, 0_x)} &= \phi(0_x, 0_x, 0_x) \\ &= \mathrm{d}S(\pi \circ \phi(0_x, 0_x, 0_x)) = \mathrm{d}S(0_x, 0_x); \end{aligned}$$

   where $\pi : T^*(T^*M \times_M T^*M) \to T^*M \times_M T^*M$ is the projection.

2. Since $x = \mathbf{s}(\mathbf{u}(x)) = \mathbf{s}(0_x)$ and $(\alpha, \mathbf{u}(\mathbf{s}(\alpha)), \alpha) \in \mathrm{Graph}(\mathbf{m}) \subseteq \phi^{-1}(\mathrm{Graph}(\mathrm{d}S))$, we see that $x = \mathbf{s}(\alpha)$ is a solution of the equation:

$$\phi(\alpha, 0_x, \alpha) = \mathrm{d}S(\pi \circ \phi(\alpha, 0_x, \alpha)).$$

   Similarly, $y = \mathbf{t}(\alpha)$ is a solution of the equation:

$$\phi(0_y, \alpha, \alpha) = \mathrm{d}S(\pi \circ \phi(0_y, \alpha, \alpha)).$$

3. Since $\mathbf{m}(\alpha, \mathbf{u}(\mathbf{s}(\alpha))) = \mathbf{m}(\mathbf{u}(\mathbf{s}(\alpha)), \alpha) = \alpha$, we have

$$
\begin{aligned}
(\alpha, 0, \overline{\alpha} \in \mathrm{Graph}(\mathbf{m}) \quad &\Longrightarrow \quad \alpha = \overline{\alpha}, \\
(0, \alpha, \overline{\alpha}) \in \mathrm{Graph}(\mathbf{m}) \quad &\Longrightarrow \quad \alpha = \overline{\alpha}.
\end{aligned}
$$

If $\mathrm{Graph}(\mathbf{m}) = \phi^{-1}(\mathrm{Graph}(\mathrm{d}S))$, then

$$
\begin{aligned}
\phi(\alpha, 0, \overline{\alpha}) = \mathrm{d}S(\pi \circ \phi(\alpha, 0, \overline{\alpha})) \quad &\Longrightarrow \quad \alpha = \overline{\alpha}, \\
\phi(0, \alpha, \overline{\alpha}) = \mathrm{d}S(\pi \circ \phi(0, \alpha, \overline{\alpha})) \quad &\Longrightarrow \quad \alpha = \overline{\alpha}.
\end{aligned}
$$

4. Since $\mathbf{i}(\alpha) = -\alpha$ and $\mathbf{m}(\alpha, \mathbf{i}(\alpha)) = 0_{\mathbf{t}(\alpha)}$, we have

$$
\phi(\alpha, -\alpha, 0_{\mathbf{t}(\alpha)}) = \mathrm{d}S(\pi \circ \phi(\alpha, -\alpha, 0_{\mathbf{t}(\alpha)})).
$$

5. Since $\mathbf{t}(\mathbf{i}(\alpha)) = \mathbf{s}(\alpha)$, we have

$$
\phi(0_{\mathbf{s}(\alpha)}, -\alpha, -\alpha) = \mathrm{d}S(\pi \circ \phi(0_{\mathbf{s}(\alpha)}, -\alpha, -\alpha)).
$$

6. For $(\alpha_1, \alpha_2, \alpha_3)$ in the domain of associativity, if $(\alpha_1, \alpha_2), (\alpha_2, \alpha_3) \in U$ and

$$
(\mathbf{m}(\alpha_1, \alpha_2), \alpha_3), (\alpha_1, \mathbf{m}(\alpha_2, \alpha_3)) \in U,
$$

and $\overline{\alpha} = \mathbf{m}(\alpha_1, \alpha_2)$ and $\tilde{\alpha} = \mathbf{m}(\alpha_2, \alpha_3)$, there is α such that

$$
\begin{aligned}
\phi(\alpha_1, \alpha_2, \overline{\alpha}) &= \mathrm{d}S(\pi \circ \phi(\alpha_1, \alpha_2, \overline{\alpha})), \\
\phi(\alpha_2, \alpha_3, \tilde{\alpha}) &= \mathrm{d}S(\pi \circ \phi(\alpha_2, \alpha_3, \tilde{\alpha})), \\
\phi(\alpha_1, \tilde{\alpha}, \alpha) &= \mathrm{d}S(\pi \circ \phi(\alpha_1, \tilde{\alpha}, \alpha)), \\
\phi(\overline{\alpha}, \alpha_3, \alpha) &= \mathrm{d}S(\pi \circ \phi(\overline{\alpha}, \alpha_3, \alpha)).
\end{aligned}
$$

The conditions above give restrictions on a generating function S. We will see in this chapter that they are all it is needed to answer the question posed above.

3.2 Generating function for an abstract manifold

Theorem 3.2.1. *Let $\phi : T^*M \times T^*M \times T^*M \to T^*(T^*M \times_M T^*M)$ be a symplectomorphism around $\{(0_x, 0_x, 0_x) : x \in M\}$ such that $\phi(\alpha_x, \beta_x, \alpha_x + \beta_x) = 0_{(\alpha_x, \beta_x)}$ for any $\alpha_x, \beta_x \in T_x^*M$. Let $S : T^*M \times_M T^*M \to \mathbb{R}$ be a function and assume that there is an open neighborhood O of 0_{T^*M} where the following hold.*

0. $\mathrm{d}S(0_x, 0_x) = 0_{(0_x, 0_x)}$.

1. *(Source and Target Maps) The equations*

$$\phi(\alpha, 0_x, \alpha) = \mathrm{d}S(\pi \circ \phi(\alpha, 0_x, \alpha_x)),$$
$$\phi(0_x, \alpha, \alpha) = \mathrm{d}S(\pi \circ \phi(0_x, \alpha, \alpha_x)).$$

   have unique solutions for x on O. Moreover, there are surjective submersions $\mathbf{s}, \mathbf{t} : O \to \mathbb{R}$ defined by

$$\mathbf{s}(\alpha) = x \quad \text{if} \quad (\phi(\alpha, 0_x, \alpha) = \mathrm{d}S(\pi \circ \phi(\alpha, 0_x, \alpha))),$$
$$\mathbf{t}(\alpha) = x \quad \text{if} \quad (\phi(0_x, \alpha, \alpha) = \mathrm{d}S(\pi \circ \phi(0_x, \alpha, \alpha))).$$

2. *(Naturality)*

   (a) *There is an open neighborhood D of $0_{T^*M} \times T^*M$ in $T^*M \times T^*M$ on which for all $(\alpha, \overline{\alpha})$ in D we have:*

$$\phi(\alpha, 0, \overline{\alpha}) = \mathrm{d}S(\pi \circ \phi(\alpha, 0, \overline{\alpha})) \implies \alpha = \overline{\alpha},$$
$$\phi(0, \alpha, \overline{\alpha}) = \mathrm{d}S(\pi \circ \phi(0, \alpha, \overline{\alpha})) \implies \alpha = \overline{\alpha}.$$

   (b) *We have for all $\alpha \in O$:*

$$\phi(\alpha, -\alpha, 0_{\mathbf{t}(\alpha)}) = \mathrm{d}S(\pi \circ \phi(\alpha, -\alpha, 0_{\mathbf{t}(\alpha)})).$$

3. *(Associativity) There is an open neighborhood $W = \{(\alpha_1, \alpha_2, \alpha_3)\}$ of*

$$\{(\alpha, 0_{\mathbf{s}(\alpha)}, 0_{\mathbf{s}(\alpha)}) : \alpha \in O\} \cup \{(0_{\mathbf{t}(\alpha)}, \alpha, 0_{\mathbf{s}(\alpha)}) : \alpha \in O\} \cup \{(0_{\mathbf{t}(\alpha)}, 0_{\mathbf{t}(\alpha)}, \alpha) : \alpha \in O\}$$

in $O_{\mathbf{s}} \times_{\mathbf{t}} O_{\mathbf{s}} \times_{\mathbf{t}} O$ *on which we have functions*

$$\overline{\alpha} : W \to T^*M, \quad \tilde{\alpha} : W \to T^*M, \quad \alpha : W \to T^*M,$$

such that

(a)
$$\overline{\alpha}(\alpha_0, 0_{\mathbf{s}(\alpha_0)}, 0_{\mathbf{s}(\alpha_0)}) = \overline{\alpha}(0_{\mathbf{t}(\alpha_0)}, \alpha_0, 0_{\mathbf{s}(\alpha_0)}) = \alpha_0,$$
$$\tilde{\alpha}(0_{\mathbf{t}(\alpha_0)}, 0_{\mathbf{t}(\alpha_0)}, \alpha_0) = \tilde{\alpha}(0_{\mathbf{t}(\alpha_0)}, \alpha_0, 0_{\mathbf{s}(\alpha_0)}) = \alpha_0,$$
$$\alpha(0_{\mathbf{t}(\alpha_0)}, 0_{\mathbf{t}(\alpha_0)}, \alpha_0) = \alpha(0_{\mathbf{t}(\alpha_0)}, \alpha_0, 0_{\mathbf{s}(\alpha_0)}) = (\alpha_0, 0_{\mathbf{s}(\alpha_0)}, 0_{\mathbf{s}(\alpha_0)}) = \alpha_0.$$

(b) The following system of equations is satisfied:

$$\begin{aligned}
\phi(\alpha_1, \alpha_2, \overline{\alpha}) &= \mathrm{d}S(\pi \circ \phi(\alpha_1, \alpha_2, \overline{\alpha})), \\
\phi(\alpha_2, \alpha_3, \tilde{\alpha}) &= \mathrm{d}S(\pi \circ \phi(\alpha_2, \alpha_3, \tilde{\alpha})), \\
\phi(\alpha_1, \tilde{\alpha}, \alpha) &= \mathrm{d}S(\pi \circ \phi(\alpha_1, \tilde{\alpha}, \alpha)), \\
\phi(\overline{\alpha}, \alpha_3, \alpha) &= \mathrm{d}S(\pi \circ \phi(\overline{\alpha}, \alpha_3, \alpha)).
\end{aligned}$$

4. *(Multiplication)There is an open neighborhood* $Q \subset (T^*M)^3$ *around*

$$\{(\alpha, 0_{\mathbf{s}(\alpha)}, \alpha) : \alpha \in O\} \cup \{(0_{\mathbf{t}(\alpha)}, \alpha, \alpha) : \alpha \in O\}$$

on which $\phi^{-1}(\mathrm{Graph}(\mathrm{d}S))$ *intersects each* $\{\alpha_1\} \times \{\alpha_2\} \times T^*M$ *at at most one point.*

5. *There is an open neighborhood* $V \subset T^*M$ *of* 0_{T^*M} *on which we have*

$$\phi(0_{\mathbf{s}(\alpha)}, -\alpha, -\alpha) = \mathrm{d}S(\pi \circ \phi(0_{\mathbf{s}(\alpha)}, -\alpha, -\alpha)).$$

Then there is a local symplectic groupoid structure on an open neighborhood of 0_{T^*M} *in* T^*M *compatible with the canonical symplectic form such that*

(i) $x \mapsto 0_x$ *is the identity section;*

(ii) $\alpha \mapsto -\alpha$ *is the inversion map;*

(iii) $\mathrm{Graph}(\mathbf{m}) \subseteq \phi^{-1}(\mathrm{Graph}(\mathrm{d}S))$.

Proof. We first introduce some notations. We set

$$\mathcal{I}_2 = \{(\alpha, 0_{\mathbf{s}(\alpha)}) : \alpha \in O\} \cup \{(0_{\mathbf{t}(\alpha)}, \alpha) : \alpha \in O\},$$
$$\mathcal{I}_3 = \{(\alpha, 0_{\mathbf{s}(\alpha)}, 0_{\mathbf{s}(\alpha)}) : \alpha \in O\} \cup \{(0_{\mathbf{t}(\alpha)}, \alpha, 0_{\mathbf{s}(\alpha)}) : \alpha \in O\} \cup \{(0_{\mathbf{t}(\alpha)}, 0_{\mathbf{t}(\alpha)}, \alpha) : \alpha \in O\}.$$

By assumption on W, there is an open neighborhood $\mathcal{U}' \subset O_s \times_t O$ of $\mathcal{I}_2$ on which we have $(\alpha_1, \alpha_2, 0_{\mathbf{s}(\alpha_2)}) \in W$ and $(0_{\mathbf{t}(\alpha_1)}, \alpha_1, \alpha_2) \in W$. It then follows from assumption 3 that, if $(\alpha_1, \alpha_2) \in \mathcal{U}'$,

$$\phi(\alpha_1, \alpha_2, \overline{\alpha}) = \mathrm{d}S(\pi \circ \phi(\alpha_1, \alpha_2, \overline{\alpha})), \quad \phi(\alpha_1, \alpha_2, \tilde{\alpha}) = \mathrm{d}S(\pi \circ \phi(\alpha_1, \alpha_2, \tilde{\alpha})),$$

where

$$\overline{\alpha} = \overline{\alpha}(\alpha_1, \alpha_2, 0_{\mathbf{s}(\alpha_2)}), \quad \tilde{\alpha} = \tilde{\alpha}(0_{\mathbf{t}(\alpha_1)}, \alpha_1, \alpha_2).$$

If $(\alpha_1, \alpha_2) = (\alpha, 0_{\mathbf{s}(\alpha)})$, by the assumptions on $\overline{\alpha}$ and $\tilde{\alpha}$, we have

$$\overline{\alpha} = \overline{\alpha}(\alpha, 0_{\mathbf{s}(\alpha)}, 0_{\mathbf{s}(\alpha)}) = \alpha = \tilde{\alpha}(0_{\mathbf{t}(\alpha)}, \alpha, 0_{\mathbf{s}(\alpha)}) = \tilde{\alpha},$$

so that $(\alpha_1, \alpha_2, \overline{\alpha}), (\alpha_1, \alpha_2, \tilde{\alpha})$ are contained in Q, as in assumption 4. Similarly, the same equation holds for $(\alpha_1, \alpha_2) = (0_{\mathbf{t}(\alpha)}, \alpha)$. It follows that we can restrict $\mathcal{U}'$ around $\mathcal{I}_2$ and have that, for all $(\alpha_1, \alpha_2) \in \mathcal{U}'$

$$(\alpha_1, \alpha_2, \overline{\alpha}), (\alpha_1, \alpha_2, \tilde{\alpha}) \in Q$$

which implies that $\overline{\alpha} = \tilde{\alpha}$.

This shows that we have a well-defined map

$$\mathbf{m}' : \mathcal{U}' \to O,$$
$$(\alpha_1, \alpha_2) \mapsto \overline{\alpha}(\alpha_1, \alpha_2, 0_{\mathbf{t}(\alpha_2)}) = \tilde{\alpha}(0_{\mathbf{t}(\alpha_1)}, \alpha_1, \alpha_2).$$

Equivalently, we can define $\mathbf{m}'(\alpha_1, \alpha_2) \in O$ to be the unique point such that

1. $\phi(\alpha_1, \alpha_2, \mathbf{m}'(\alpha_1, \alpha_2)) = \mathrm{d}S(\pi \circ \phi(\alpha_1, \alpha_2, \mathbf{m}'(\alpha_1, \alpha_2)))$;
2. $(\alpha_1, \alpha_2, \mathbf{m}'(\alpha_1, \alpha_2)) \in Q$.

If $\alpha \in V$, we have $\mathbf{s}(\alpha) = \mathbf{t}(-\alpha)$ by assumption 5. So let $\mathcal{V}' = V \cap (-V)$

and define

$$\begin{aligned}\mathbf{i}' : \mathcal{V}' &\to \mathcal{V}',\\ \alpha &\mapsto -\alpha.\end{aligned}$$

We now check the groupoid axioms.

1. $\mathbf{s}(0_x) = x$, $\mathbf{t}(0_x) = x$: This follows from the definition of $\mathbf{s}$,$\mathbf{t}$ and that

$$\begin{aligned}\phi(0_x, 0_x, 0_x) &= 0_{(0_x, 0_x)}\\ &= \mathrm{d}S(0_x, 0_x) = \mathrm{d}S(\pi \circ \phi(0_x, 0_x, 0_x)).\end{aligned}$$

2. $\mathbf{m}'(\alpha, 0_{\mathbf{s}(\alpha)}) = \alpha$: For all $\alpha \in O$, we have $(\alpha, 0_{\mathbf{s}(\alpha)}) \in \mathcal{U}'$ and $(\alpha, 0_{\mathbf{s}(\alpha)}, 0_{\mathbf{s}(\alpha)}) \in W$. By the definition of $\mathbf{m}'$

$$\mathbf{m}'(\alpha, 0_{\mathbf{s}(\alpha)}) = \overline{\alpha}(\alpha, 0_{\mathbf{s}(\alpha)}, 0_{\mathbf{s}(\alpha)}) = \alpha.$$

3. $\mathbf{m}'(0_{\mathbf{t}(\alpha)}, \alpha) = \alpha$: For all $\alpha \in O$, we have $(0_{\mathbf{t}(\alpha)}, \alpha) \in \mathcal{U}'$ and $(0_{\mathbf{t}(\alpha)}, 0_{\mathbf{t}(\alpha)}, \alpha) \in W$. By the definition of $\mathbf{m}'$

$$\mathbf{m}'(0_{\mathbf{t}(\alpha)}, \alpha) = \tilde{\alpha}(0_{\mathbf{t}(\alpha)}, 0_{\mathbf{t}(\alpha)}, \alpha) = \alpha.$$

4. $\mathbf{s}(\mathbf{m}'(\alpha_1, \alpha_2)) = \mathbf{s}(\alpha_2)$: If $(\alpha_1, \alpha_2) \in \mathcal{U}'$, $(\alpha_1, \alpha_2, 0_{\mathbf{s}(\alpha_2)}) \in W$. Assumption 3 says that

$$\begin{aligned}\phi(\alpha_1, \alpha_2, \overline{\alpha}) &= \mathrm{d}S(\pi \circ \phi(\alpha_1, \alpha_2, \overline{\alpha})),\\ \phi(\alpha_2, 0_{\mathbf{s}(\alpha_2)}, \tilde{\alpha}) &= \mathrm{d}S(\pi \circ \phi(\alpha_2, 0_{\mathbf{s}(\alpha_2)}, \tilde{\alpha})),\\ \phi(\alpha_1, \tilde{\alpha}, \alpha) &= \mathrm{d}S(\pi \circ \phi(\alpha_1, \tilde{\alpha}, \alpha)),\\ \phi(\overline{\alpha}, 0_{\mathbf{s}(\alpha_2)}, \alpha) &= \mathrm{d}S(\pi \circ \phi(\overline{\alpha}, 0_{\mathbf{s}(\alpha_2)}, \alpha)),\end{aligned}$$

where

$$\begin{aligned}\overline{\alpha} = \overline{\alpha}(\alpha_1, \alpha_2, 0_{\mathbf{s}(\alpha_2)}) = \mathbf{m}'(\alpha_1, \alpha_2), \quad \tilde{\alpha} = \tilde{\alpha}(\alpha_1, \alpha_2, 0_{\mathbf{s}(\alpha_2)}),\\ \alpha = \alpha(\alpha_1, \alpha_2, 0_{\mathbf{s}(\alpha_2)}).\end{aligned}$$

For $(\alpha_1, \alpha_2) = (\alpha, 0_{\mathbf{s}(\alpha)})$, assumption 2(a) and the second equation above imply that $\tilde{\alpha} = 0_{\mathbf{s}(\alpha)}$. For $(\alpha_1, \alpha_2) = (0_{\mathbf{t}(\alpha)}, \alpha)$, we have $\tilde{\alpha} = \alpha$

by assumption 3. This means that we can restrict $\mathcal{U}'$ around $\mathcal{I}_2$ so that $(\alpha_2, 0_{\mathbf{s}(\alpha_2)}, \tilde{\alpha}) \in Q$. The second equation then implies that $\tilde{\alpha} = \alpha_2$ and then the first and third equation imply that $\alpha = \overline{\alpha} = \mathbf{m}'(\alpha_1, \alpha_2)$, where we need to restrict $\mathcal{U}'$ further so that $(\alpha_1, \alpha_2, \alpha) \in Q$. The 4th equation and the definition of $\mathbf{s}$ then imply that $\mathbf{s}(\mathbf{m}'(\alpha_1, \alpha_2)) = \mathbf{s}(\overline{\alpha}) = \mathbf{s}(\alpha_2)$.

5. $\mathbf{t}(\mathbf{m}'(\alpha_1, \alpha_2)) = \mathbf{t}(\alpha_1)$: this is entirely similar to the previous item.

6. $\mathbf{m}'(\alpha, \mathbf{i}'(\alpha)) = 0_{\mathbf{t}(\alpha)}$: By restricting $\mathcal{V}'$ around 0_{T^*M} so that $\mathcal{V}' \subseteq O_b$, then 2(b) implies that

$$\phi(\alpha, -\alpha, 0_{\mathbf{t}(\alpha)}) = \mathrm{d}S(\pi \circ \phi(\alpha, -\alpha, 0_{\mathbf{t}(\alpha)})).$$

We also restrict $\mathcal{V}'$ so that $(\alpha, -\alpha, 0_{\mathbf{t}(\alpha)}) \in Q$. If $(\alpha, -\alpha) \in \mathcal{U}'$, we have

$$\phi(\alpha, -\alpha, \mathbf{m}'(\alpha, -\alpha)) = \mathrm{d}S(\pi \circ \phi(\alpha, -\alpha, \mathbf{m}'(\alpha, -\alpha)))$$

and $(\alpha, -\alpha, \mathbf{m}'(\alpha, -\alpha)) \in Q$. Assumption 4 then gives $\mathbf{m}'(\alpha, -\alpha) = 0_{\mathbf{t}(\alpha)}$.

7. $\mathbf{m}'(\mathbf{i}(\alpha), \alpha) = 0_{\mathbf{s}(\alpha)}$: This follows from (6) and (7) by reversing the sign of α.

The restrictions on the sets $\mathcal{U}'$ and $\mathcal{V}'$ in the proof of the axioms are all done so that we still have $\mathcal{I}_2 \subseteq \mathcal{U}'$ and $0_{T^*M} \subseteq \mathcal{V}'$. We denote the final restricted sets by $\mathcal{U}$ and $\mathcal{V}$, respectively. We define

$$\mathbf{m} : \mathcal{U} \to \mathcal{O}, \quad \mathbf{m} = \mathbf{m}'|_{\mathcal{U}}, \quad \text{and} \quad \mathbf{i} : \mathcal{V} \to \mathcal{V}, \quad \mathbf{i} = \mathbf{i}'|_{\mathcal{V}}.$$

Restrict W to an open neighborhood $\mathcal{W}$ of $\mathcal{I}_3$ in $\mathcal{O}_{\mathbf{s}} \times_{\mathbf{t}} \mathcal{O}_{\mathbf{s}} \times_{\mathbf{t}} \mathcal{O}$ for which we have

$$(\alpha_1, \alpha_2, \overline{\alpha}), \quad (\alpha_2, \alpha_3, \tilde{\alpha}), \quad (\overline{\alpha}, \alpha_3, \alpha), \quad (\alpha_1, \tilde{\alpha}, \alpha),$$

are all elements in Q. This is possible by assumptions 2(a) and 3. Let $(\alpha_1, \alpha_2, \alpha_3) \in \mathcal{W}$ be such that $(\alpha_1, \alpha_2), (\alpha_2, \alpha_3)$ are elements in $\mathcal{U}$ and

$$(\mathbf{m}(\alpha_1, \alpha_2), \alpha_3), \quad (\alpha_1, \mathbf{m}(\alpha_2, \alpha_3)),$$

are also elements in $\mathcal{U}$. If

$$\overline{\alpha} = \overline{\alpha}(\alpha_1, \alpha_2, \alpha_3), \quad \tilde{\alpha} = \tilde{\alpha}(\alpha_1, \alpha_2, \alpha_3), \quad \alpha = \alpha(\alpha_1, \alpha_2, \alpha_3),$$

assumptions 3 and 4 imply that

$$\begin{aligned}
\mathbf{m}(\alpha_1, \alpha_2) &= \overline{\alpha} \\
\mathbf{m}(\alpha_2, \alpha_3) &= \tilde{\alpha} \\
\mathbf{m}(\mathbf{m}(\alpha_1, \alpha_2), \alpha_3) &= \alpha \\
\mathbf{m}(\alpha_1, \mathbf{m}(\alpha_2, \alpha_3)) &= \alpha
\end{aligned}$$

In other words, multiplication m is associative on $\mathcal{W}$. This concludes the proof that

$$(O, M, (\mathcal{U}, \mathbf{m}), (\mathcal{V}, \mathbf{i}), \mathcal{W})$$

is a local symplectic groupoid structure satisfying the required properties. □

3.3 Generating function for $M = \mathbb{R}^n$.

Now we will restrict attention to the special case where $M = \mathbb{R}^n$ and

$$\begin{aligned}
&\phi = \phi_0 : T^*\mathbb{R}^n \times T^*\mathbb{R}^n \times T^*\mathbb{R}^n \to T^*(T^*\mathbb{R}^n \times_{\mathbb{R}^n} T^*\mathbb{R}^n), \\
&((p_1, x_1), (p_2, x_2), (p, x)) \mapsto ((p_1, x_1 - x), (p_2, x_2 - x), (x, p - (p_1 + p_2)),
\end{aligned}$$

where for the range, we use coordinates (p_1, p_2, x) in the base $T^*\mathbb{R}^n \times_{\mathbb{R}^n} T^*\mathbb{R}^n$, and then the identification

$$T^*(T^*\mathbb{R}^n \times_{\mathbb{R}^n} T^*\mathbb{R}^n) \simeq T^*(\mathbb{R}^{*n} \times \mathbb{R}^{*n} \times \mathbb{R}^n).$$

Given a function $S : T^*M \times T^*M \to \mathbb{R}$ elements in $\phi^{-1}(\mathrm{Graph}(\mathrm{d}S))$ take the form

$$((p_1, \nabla_1 S(p_1, p_2, x) + x), (p_2, \nabla_2 S(p_1, p_2, x) + x), (\nabla_x S(p_1, p_2, x) + p_1 + p_2, x))$$

where $\nabla_1 S(p_1, p_2, x)$ means the derivative of S with respect to the first coordinate at the point (p_1, p_2, x), and similarly for $\nabla_2 S(p_1, p_2, x)$ and $\nabla_x S(p_1, p_2, x)$.

Let W be the open set in assumption 3 of Theorem 3.2.1. For any point

$(\alpha_1, \alpha_2, \alpha_3) \in W$, referring to the same assumption, we set

$$\overline{\alpha} = \overline{\alpha}(\alpha_1, \alpha_2, \alpha_3), \qquad \tilde{\alpha} = \tilde{\alpha}(\alpha_1, \alpha_2, \alpha_3), \qquad \alpha = \alpha(\alpha_1, \alpha_2, \alpha_3).$$

Since $M = \mathbb{R}^n$ and $T^*M \simeq \mathbb{R}^{*n} \times \mathbb{R}^n$, we can write these elements in coordinates:

$$\alpha_i = (p_i, x_i), \qquad \overline{\alpha} = (\overline{p}, \overline{x}), \qquad \tilde{\alpha} = (\tilde{p}, \tilde{x}), \qquad \alpha = (p, x).$$

In these coordinates, the equations in items (a) and (b) in assumption 3 of Theorem 3.2.1, which were obtained by writing down the associativity equation

$$m(\alpha_1, m(\alpha_2, \alpha_3)) = m(m(\alpha_1, \alpha_2), \alpha_3)$$

become

$$\begin{aligned}
&((p_1, x_1 - \overline{x}), (p_2, x_2 - \overline{x}), (\overline{x}, \overline{p} - (p_1 + p_2))) = \\
&\qquad = ((p_1, \nabla_1 S(p_1, p_2, \overline{x})), (p_2, \nabla_2 S(p_1, p_2, \overline{x})), (\overline{x}, \nabla_x S(p_1, p_2, \overline{x}))) \\
&((p_2, x_2 - \tilde{x}), (p_3, x_3 - \tilde{x}), (\tilde{x}, \tilde{p} - (p_2 + p_3))) = \\
&\qquad = ((p_2, \nabla_1 S(p_2, p_3, \tilde{x})), (p_2, \nabla_2 S(p_2, p_3, \tilde{x})), (\tilde{x}, \nabla_x S(p_2, p_3, \tilde{x}))) \\
&((\overline{p}, \overline{x} - x), (p_3, x_3 - x), (x, p - (\overline{p} + p_3))) = \\
&\qquad = ((\overline{p}, \nabla_1 S(\overline{p}, p_3, x)), (p_3, \nabla_2 S(\overline{p}, p_3, x)), (x, \nabla_x S(\overline{p}, p_3, x))) \\
&((p_1, x_1 - x), (\tilde{p}, \tilde{x} - x), (x, p - (p_1 + \tilde{p}))) = \\
&\qquad = ((p_1, \nabla_1 S(p_1, \tilde{p}, x)), (\tilde{p}, \nabla_2 S(p_1, \tilde{p}, x)), (x, \nabla_x S(p_1, \tilde{p}, x)))
\end{aligned}$$

or equivalently

$$\begin{aligned}
x_1 - \overline{x} &= \nabla_1 S(p_1, p_2, \overline{x}), & x_2 - \overline{x} &= \nabla_2 S(p_1, p_2, \overline{x}), & \overline{p} - (p_1 + p_2) &= \nabla_x S(p_1, p_2, \overline{x}), \\
x_2 - \tilde{x} &= \nabla_1 S(p_2, p_3, \tilde{x}), & x_3 - \tilde{x} &= \nabla_2 S(p_2, p_3, \tilde{x}), & \tilde{p} - (p_2 + p_3) &= \nabla_x S(p_2, p_3, \tilde{x}), \\
\overline{x} - x &= \nabla_1 S(\overline{p}, p_3, x), & x_3 - x &= \nabla_2 S(\overline{p}, p_3, x), & p - (\overline{p} + p_3) &= \nabla_x S(\overline{p}, p_3, x) \\
x_1 - x &= \nabla_1 S(p_1, \tilde{p}, x), & \tilde{x} - x &= \nabla_2 S(p_1, \tilde{p}, x), & p - (p_1 + \tilde{p}) &= \nabla_x S(p_1, \tilde{p}, x).
\end{aligned}$$

Canceling the variables x_1, x_2, x_3, p, we obtain

$$\begin{aligned}
\nabla_1 S(p_1, p_2, \overline{x}) + \overline{x} &= \nabla_1 S(p_1, \tilde{p}, x) + x, \\
\nabla_2 S(p_1, p_2, \overline{x}) + \overline{x} &= \nabla_1 S(p_2, p_3, \tilde{x}) + \tilde{x}, \\
\nabla_2 S(p_2, p_3, \tilde{x}) + \tilde{x} &= \nabla_2 S(\overline{p}, p_3, x) + x, \\
\nabla_x S(p_1, p_2, \overline{x}) + p_1 + p_2 &= \nabla_x S(\overline{p}, p_3, x) + p_2 + p_3,
\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
\overline{x} &= \nabla_1 S(\overline{p}, p_3, x) + x, \\
\overline{p} &= \nabla_x S(p_1, p_2, \overline{x}) + p_1 + p_2, \\
\tilde{x} &= \nabla_2 S(p_1, \tilde{p}, x) + x, \\
\tilde{p} &= \nabla_x S(p_2, p_3, \tilde{x}) + p_2 + p_3.
\end{aligned}$$

Let us set now:

$$\begin{aligned}
\Delta S(p_1, p_2, p_3, x) &= S(p_1, p_2, \overline{x}) + S(\overline{p}, p_3, x) - (S(p_2, p_3, \tilde{x}) + S(p_1, \tilde{p}, x)) \\
F(p_1, p_2, p_3, x) &= \overline{p}(\overline{x} - x) - \tilde{p}(\tilde{x} - x) + (p_2 + p_3)\tilde{x} - (p_1 + p_2)\overline{x}
\end{aligned}$$

If we view p_1, p_2, p_3, x as independent variables, we find

$$\begin{aligned}
\frac{\partial}{\partial p_1}\Delta S &= \nabla_1 S(p_1, p_2, \overline{x}) + \nabla_x S(p_1, p_2, \overline{x})\frac{\partial \overline{x}}{\partial p_1} + \nabla_1 S(\overline{p}, p_3, x)\frac{\partial \overline{p}}{\partial p_1} + \\
&\quad - (\nabla_x S(p_2, p_3, \tilde{x})\frac{\tilde{x}}{\partial p_1} + \nabla_1 S(p_1, \tilde{p}, x) + \nabla_2 S(p_1, \tilde{p}, x))\frac{\partial \tilde{p}}{\partial p_1} \\
&= x_1 - \overline{x} + (\overline{p} - (p_1 + p_2))\frac{\partial \overline{x}}{\partial p_1} - ((\tilde{p} - (p_2 + p_3))\frac{\partial \tilde{x}}{\partial p_1} + x_1 - x + (\tilde{x} - x)\frac{\partial \tilde{p}}{\partial p_1}) \\
&= \frac{\partial}{\partial p_1}(\overline{p}(\overline{x} - x) - \tilde{p}(\tilde{x} - x) + (p_2 + p_3)\tilde{x} - (p_1 + p_2)\overline{x}) = \frac{\partial F}{\partial p_1}
\end{aligned}$$

and similarly,

$$\frac{\partial}{\partial p_2}\Delta S = \frac{\partial F}{\partial p_2}, \qquad \frac{\partial}{\partial p_3}\Delta S = \frac{\partial F}{\partial p_3}, \qquad \frac{\partial}{\partial x}\Delta S = \frac{\partial F}{\partial x}.$$

If one requires that $S(0, 0, x) = 0$ for all $x \in M$, we then have $\Delta S = F$, which can be rewritten as

$$S(p_1, p_2, \overline{x}) + S(\overline{p}, p_3, x) - \overline{p}(\overline{x} - x) + (p_1 + p_2)\overline{x} = \\ = S(p_2, p_3, \tilde{x}) + S(p_1, \tilde{p}, x) - \tilde{p}(\tilde{x} - x) + (p_2 + p_3)\tilde{x} \quad (3.2)$$

We have obtained the so-called **Symplectic Groupoid Associativity** or **SGA equation** from [4].

Remark 3.3.1. *In [4], the notion of generating functions for a symplectic groupoid structure on $T^*\mathbb{R}^n$ is with respect to the map*

$$\phi_0' : T^*\mathbb{R}^n \times T^*\mathbb{R}^n \times T^*\mathbb{R}^n \to T^*(T^*\mathbb{R}^n \times_{\mathbb{R}^n} T^*\mathbb{R}^n) \\ ((p_1, x_1), (p_2, x_2), (p, x)) \mapsto ((p_1, x_1), (p_2, x_2), (x, p))$$

For a given sympletic groupoid structure on $T^*\mathbb{R}^n$, it is easy to check that S is a generating function with respect to ϕ_0 if and only if $S + S_0$ is a generating function with respect ϕ_0', where $S_0(p_1, p_2, x) = (p_1 + p_2)x$. Note that the latter function is a generating function for the canonical groupoid structure with respect to ϕ_0'.

Having the SGA equation at hand, we can now answer Question 3.1.1 in the case $M = \mathbb{R}^n$. The proof is based on an argument given for formal groupoids in [4].

Theorem 3.3.2. *Assume that there is an open $\mathcal{O} \in T^*\mathbb{R}^n$ around the zero section for which the following hold.*

0. $\mathrm{d}S(0, 0, x) = 0_{(0,0,x)}$ *for all $x \in \mathbb{R}^n$.*

1. (Source and Target Maps) The maps $\mathbf{s}, \mathbf{t} : \mathcal{O} \to \mathbb{R}^n$ given by

$$\mathbf{s}(p, x) = \nabla_2 S(p, 0, x) + x, \qquad \mathbf{t}(p, x) = \nabla_1 S(0, p, x) + x$$

are surjective submersions.

2. (Naturality) For all (p, x) in $\mathcal{O}$:

(a) $S(p, 0, x) = 0, \qquad S(0, p, x) = 0,$

(b) $S(p, -p, x) = 0, \qquad S(-p, p, x) = 0.$

3. (SGA Equation) There is an open neighborhood W *of*

$$\{(p,0,0,x) : (p,x) \in \mathcal{O}\} \cup \{(0,p,0,x) : (p,x) \in \mathcal{O}\} \cup \{(0,0,p,x) : (p,x) \in \mathcal{O}\}$$

on which the SGA equation holds. That is, there are functions

$$\overline{p}, \overline{x}, \tilde{p}, \tilde{x} : W \to T^*\mathbb{R}^n$$

such that, for $(p,x) \in \mathcal{O}$

$$\overline{p}(p,0,0,x) = \overline{p}(0,p,0,x) = p, \quad \tilde{p}(0,p,0,x) = \tilde{p}(0,0,p,x) = p,$$

and we have for all $(p_1,p_2,p_3,x) \in W$*:*

$$\begin{aligned} S(p_1,p_2,\overline{x}) + &S(\overline{p},p_3,x) + (p_1+p_2)\overline{x} + (\overline{p}+p_3)x - \overline{px} \\ &= S(p_2,p_3,\tilde{x}) + S(p_1,\tilde{p},x) + (p_2+p_3)\tilde{x} + (p_1+\tilde{p})x - \tilde{p}\tilde{x}, \\ \overline{p} = \nabla_x S(p_1,p_2,\overline{x}) + p_1 + p_2, &\qquad \overline{x} = \nabla_1 S(\overline{p},p_3,x) + x, \\ \tilde{p} = \nabla_x S(p_2,p_3,\tilde{x}) + p_2 + p_3, &\qquad \tilde{x} = \nabla_2 S(p_1,\tilde{p},x) + x, \end{aligned}$$

4. (Multiplication) The map $(\mathbb{R}^{*n})^2 \times \mathbb{R}^n \to T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ *given by*

$$(p_1,p_2,x) \mapsto ((p_1, \nabla_1 S_1(p_1,p_2,x) + x), (p_2, \nabla_2 S_2(p_1,p_2,x) + x))$$

is an injective immersion on an open neighborhood of

$$\{(p,0,x) : (p,x) \in \mathcal{O}\} \cup \{(0,p,x) : (p,x) \in \mathcal{O}\}.$$

Then there is a local symplectic groupoid structure on $\mathcal{O}$ *compatible with the canonical symplectic form such that* $\mathrm{Graph}(\mathbf{m}) \subseteq \phi^{-1}(\mathrm{Graph}(\mathrm{d}S))$*. The other structure maps are given by*

$$\begin{aligned} \mathbf{s}(p,x) &= \nabla_2 S(p,0,x) + x, &\qquad \mathbf{i}(p,x) &= (-p,x), \\ \mathbf{t}(p,x) &= \nabla_1 S(0,p,x) + x, &\qquad \mathbf{u}(x) &= (0,x). \end{aligned}$$

Proof. We include assumption 0 for convenience as it is actually implied by assumption 2(a). We start by setting up some notations. Let

$$
\begin{aligned}
I_2 &= \{(p,0,x) : (p,x) \in \mathcal{O}\} \cup \{(0,p,x) : (p,x) \in \mathcal{O}\},\\
I_3 &= \{(p,0,0,x) : (p,x) \in \mathcal{O}\} \cup \{(0,p,0,x) : (p,x) \in \mathcal{O}\} \cup \\
&\qquad \cup \{(0,0,p,x) : (p,x) \in \mathcal{O}\},\\
\mathcal{I}_2 &= \{(\alpha, 0_{s(\alpha)}) : \alpha \in \mathcal{O}\} \cup \{(0_{t(\alpha)}, \alpha) : \alpha \in \mathcal{O}\},\\
\mathcal{I}_3 &= \{(\alpha, 0_{s(\alpha)}, 0_{s(\alpha)}) : \alpha \in \mathcal{O}\} \cup \{(0_{t(\alpha)}, \alpha, 0_{s(\alpha)}) : \alpha \in \mathcal{O}\} \cup \\
&\qquad \cup \{(0_{t(\alpha)}, 0_{t(\alpha)}, \alpha) : \alpha \in \mathcal{O}\}.
\end{aligned}
$$

For clarity, all subsets in the coordinate spaces (e.g. $(\mathbb{R}^{*n})^2 \times \mathbb{R}^n$) will be denoted by capital letters and domains of the groupoid structure maps (e.g. domain of multiplication) will be denoted by curly letters. Also, we will refer to items in the above assumptions by their indices.

Note first that, for each $x \in \mathbb{R}^n$

$$
\begin{aligned}
\mathbf{s}(\mathbf{u}(x)) &= \nabla_2 S(0,0,x) + x = x,\\
\mathbf{t}(\mathbf{u}(x)) &= \nabla_1 S(0,0,x) + x = x,
\end{aligned}
$$

by assumption 0 above.

Let

$$
U_1 = \{(p_1, p_3, x) \in \mathbb{R}^{*n} \times \mathbb{R}^{*n} \times \mathbb{R}^n : (p_1, 0, p_3, x) \in W\}.
$$

If $(p_1, p_3, x) \in U_1$, then the SGA equation is satisfied for $(p_1, 0, p_3, x)$. By our assumption on W the set U_1 contains I_2. Consider the map

$$
\begin{aligned}
a : U_1 &\to T^*M,\\
(p_1, p_3, x) &\mapsto (p_1, \overline{x}),
\end{aligned}
$$

where $\overline{x} = \overline{x}(p_1, 0, p_3, x)$ is as in the SGA equation. When $(p,x) \in \mathcal{O}$, we have

$$
\overline{p}(p,0,0,x) = p, \qquad \overline{p}(0,0,p,x) = \nabla_x S(0,0,\overline{x}) = 0,
$$

so that, by 2(a),

$$
\begin{aligned}
\overline{x}(p,0,0,x) &= \nabla_1 S(\overline{p},0,x) + x = \nabla_1 S(p,0,x) + x = x,\\
\overline{x}(0,0,p,x) &= \nabla_1 S(\overline{p},0,x) + x = \nabla_1 S(0,0,x) + x = x.
\end{aligned}
$$

This means that the map a sends I_2 into $\mathcal{O}$. We can then restrict U_1 to $a^{-1}(\mathcal{O})$, keeping $I_2 \subseteq U_1$. In a similar way, we can further restrict U_1 to an open neighborhood around I_2 such that

$$(p_3, \tilde{x}) \in \mathcal{O},$$

for all $(p_1, p_3, x) \in U_1$. With this choice of U_1, assumption 2(a) now implies that, for any $(p_1, p_3, x) \in U_1$, we have

$$\begin{aligned}\overline{p} &= \nabla_x S(p_1, 0, \overline{x}) + p_1 = p_1, \\ \tilde{p} &= \nabla_x S(0, p_3, \tilde{x}) + p_3 = p_3.\end{aligned}$$

It follows that:

$$\begin{aligned}\overline{x} &= \nabla_1 S(p_1, p_3, x) + x, \\ \tilde{x} &= \nabla_2 S(p_1, p_3, x) + x.\end{aligned}$$

Differentiating the SGA equation with respect to p_2 and evaluating at $(p_1, 0, p_3, x)$, we find

$$\begin{aligned}\mathbf{s}(p_1, \nabla_1 S(p_1, p_3, x) + x) &= \mathbf{s}(p_1, \overline{x}) = \nabla_2 S(p_1, 0, \overline{x}) + \overline{x} \\ &= \nabla_1 S(0, p_3, \tilde{x}) + \tilde{x} \\ &= \mathbf{t}(p_3, \tilde{x}) = \mathbf{t}(p_3, \nabla_2 S(p_1, p_3, x) + x).\end{aligned}$$

This means that

$$\mathbf{s}(p_1, \nabla_1 S(p_1, p_3, x) + x) = \mathbf{t}(p_3, \nabla_2 S(p_1, p_3, x) + x).$$

Moreover,

$$((p_1, \nabla_1 S(p_1, p_3, x) + x), (p_3, \nabla_2 S(p_1, p_3, x) + x), (\nabla_x S(p_1, p_3, x) + p_1 + p_3, x))$$

is an element in $\phi^{-1}(\mathrm{Graph}(\mathrm{d}S))$.

Similarly, let $U_2 = \{(p_1, p_3, x) \in \mathbb{R}^{*n} \times \mathbb{R}^{*n} \times \mathbb{R}^n : (p_1, p_3, 0, x) \in W\}$. We restrict U_2 to an open neighborhood of I_2 such that, at each $(p_1, p_3, x) \in U_2$, we have

$$\begin{aligned}
\overline{x}(p_1, p_3, 0, x) &= x,\\
\overline{p}(p_1, p_3, 0, x) &= \nabla_x S(p_1, p_3, x) + p_1 + p_3,\\
\tilde{x}(p_1, p_3, 0, x) &= \nabla_2 S(p_1, p_3, x) + x,\\
\tilde{p}(p_1, p_3, 0, x) &= p_3.
\end{aligned}$$

By differentiating the SGA equation with respect to p_3 and evaluating at $(p_1, p_3, 0, x)$, we have

$$\begin{aligned}
\mathbf{s}(p_3, \nabla_2 S(p_1, p_3, x) + x) &= \mathbf{s}(p_3, \tilde{x}) = \nabla_2 S(p_3, 0, \tilde{x}) + \tilde{x}\\
&= \nabla_2 S(\overline{p}, 0, x) + x\\
&= \mathbf{s}(\overline{p}, x) = \mathbf{s}(\nabla_x S(p_1, p_3, x) + p_1 + p_3, x).
\end{aligned}$$

In the same way, let $U_3 \subseteq \{(p_1, p_3, x) : (0, p_1, p_3, x) \in W\}$ be an open neighborhood of I_2 such that, for any $(p_1, p_3, x) \in U_3$, we have

$$\mathbf{t}(p_1, \nabla_1 S(p_1, p_3, x) + x) = \mathbf{t}(\nabla_x S(p_1, p_3, x) + p_1 + p_3, x).$$

If $U = U_1 \cap U_2 \cap U_3$, since $U \subseteq U_1$, the map

$$\begin{aligned}
U &\to T^*M \times T^*M,\\
(p_1, p_3, x) &\mapsto ((p_1, \nabla_1 S(p_1, p_3, x) + x), (p_2, \nabla_3 S(p_1, p_3, x) + x)),
\end{aligned}$$

has image in $T^*M_s \times_t T^*M$. Restricting U around I_2 if necessary, assumption 4 implies that the map is a diffeomorphism onto an open set. Moreover, for any $\alpha = (p, x) \in \mathcal{O}$, assumption 2 and the definitions of $\mathbf{s}, \mathbf{t}$ imply that

$$\begin{aligned}
\nabla_1 S(p, 0, x) + x &= x,\\
\nabla_2 S(p, 0, x) + x &= \mathbf{s}(\alpha),\\
\nabla_1 S(0, p, x) + x &= \mathbf{t}(\alpha),\\
\nabla_2 S(0, p, x) + x &= x.
\end{aligned}$$

Since $I_2 \subseteq U$, the elements $(\alpha, 0_{s(\alpha)}), (0_{t(\alpha)}, \alpha)$ are in the image of the above map. We denote this image by $\mathcal{U}$:

$$\mathcal{I}_2 \subseteq \mathcal{U} := \{((p_1, \nabla_1 S(p_1,p_3,x)+x), (p_3, \nabla_2 S(p_1,p_3,x)+x)) : (p_1,p_3,x) \in U\},$$

and we define $\mathbf{m} : \mathcal{U} \to T^*\mathbb{R}^n$ by

$$(\alpha_1, \alpha_3) = ((p_1, \nabla_1 S(p_1,p_3,x) + x), (p_3, \nabla_2 S(p_1,p_3,x) + x)) \\ \longmapsto \mathbf{m}(\alpha_1,\alpha_3) = (\nabla_x S(p_1,p_3,x) + p_1 + p_3, x)$$

Summarizing what was shown above, we have $\mathcal{U} \subseteq T^*\mathbb{R}^n \,{}_{\mathbf{s}}\!\times_{\mathbf{t}} T^*\mathbb{R}^n$ and

$$\mathbf{s}(\mathbf{m}(\alpha_1,\alpha_3)) = \mathbf{s}(\alpha_3), \\ \mathbf{t}(\mathbf{m}(\alpha_1,\alpha_3)) = \mathbf{t}(\alpha_1).$$

Moreover, for any $\alpha \in \mathcal{O}$

$$\mathbf{m}(\alpha, 0_{\mathbf{s}(\alpha)}) = \alpha, \\ \mathbf{m}(0_{\mathbf{t}(\alpha)}, \alpha) = \alpha.$$

Now we look at the domain of the inverse map and the remaining groupoid axioms. For $p \in \mathbb{R}^{*n}$, define the following functions $\mathbb{R}^n \to \mathbb{R}^n$

$$F_p(x) = \nabla_2 S(p,0,x) + x, \\ \tilde{F}_p(x) = \nabla_2 S(-p,p,x) + x, \\ G_p(x) = \nabla_1 S(0,p,x) + x, \\ \tilde{G}_p(x) = \nabla_1 S(p,-p,x) + x.$$

Note that for any $(p,x) \in \mathcal{O}$

$$F_p(x) = s(p,x), \qquad G_p(x) = t(p,x).$$

Let

$$\tilde{V} = \{(p,x) \in \mathcal{O} : (p,-p,p,x), (-p,p,0,x), (0,-p,p,x) \in W\}.$$

By definition, for each $(p,x) \in \tilde{V}$ the SGA equation is satisfied with $(p,-p,p,x)$. We restrict $\tilde{V}$ around 0_{T^*M} if necessary to ensure that $(p, \overline{x}(p,-p,p,x)) \in \mathcal{O}$, so that assumption 2(b) implies that

$$\begin{aligned}
\overline{p} &= \nabla_x S(p, -p, \overline{x}) + p - p = 0, \\
\overline{x} &= \nabla_1 S(\overline{p}, p, x) + x = \nabla_1 S(0, p, x) + x.
\end{aligned}$$

Similarly, we restrict $\tilde{V}$ around 0_{T^*M} to have

$$\begin{aligned}
\tilde{p} &= \nabla_x S(-p, p, \tilde{x}) - p + p = 0, \\
\tilde{x} &= \nabla_2 S(p, \tilde{p}, x) = \nabla_2 S(p, 0, x) + x.
\end{aligned}$$

Differentiating the SGA equation with respect to p_2 at $(p, -p, p, x)$ and using assumption 2(a), we have

$$\begin{aligned}
\tilde{F}_p(F_p(x)) &= \nabla_2 S(-p, p, \tilde{x}) + \tilde{x} \\
&= \nabla_2 S(0, p, x) + x = x.
\end{aligned}$$

By definition of $\tilde{V}$, for each $(p, x) \in \tilde{V}$, the SGA equation is satisfied at $(-p, p, 0, x)$. In the same way as the previous paragraph, we restrict $\tilde{V}$ around 0_{T^*M} so that at $(-p, p, 0, x)$:

$$\begin{aligned}
\overline{p} &= \nabla_x S(-p, p, \overline{x}) - p + p = 0, \\
\overline{x} &= \nabla_1 S(\overline{p}, 0, x) + x = x, \\
\tilde{p} &= \nabla_x S(p, 0, \tilde{x}) + p = p, \\
\tilde{x} &= \nabla_2 S(-p, \tilde{p}, x) + x = \nabla_2 S(-p, p, x) + x.
\end{aligned}$$

Differentiating the SGA equation with respect to p_3 at $(-p, p, 0, x)$ and using 2(a) imply that

$$\begin{aligned}
F_p(\tilde{F}_p(x)) &= \nabla_2 S(p, 0, \tilde{x}) + \tilde{x} \\
&= \nabla_2 S(0, 0, x) + x = x.
\end{aligned}$$

A similar argument shows that, for all $(p, x) \in \tilde{V}$,

$$\tilde{G}_p(G_p(x)) = x, \qquad G_p(\tilde{G}_p(x)) = x.$$

On the other hand, assumption 2(b) says that

$$\nabla_1 S(p,-p,x) + x = \nabla_2 S(p,-p,x) + x,$$

or in the notation above, for all $(p,x) \in \tilde{V}$,

$$\tilde{G}_p(x) = \tilde{F}_{-p}(x).$$

By what we have shown above, if $(-p,x) \in \tilde{V}$, then

$$F_{-p}(\tilde{G}_p(x)) = x,$$

and if $(p,x) \in \tilde{V}$, then

$$F_{-p}(y) = G_p(y),$$

where $y = \tilde{G}_p(x)$. It follows that the image of the map $\tilde{V} \cap (-\tilde{V}) \to T^*\mathbb{R}^n$ given by

$$(p,x) \mapsto (p, \tilde{G}_p(x)),$$

contains an open neighborhood V of 0_{T^*M} and for $(p,y) \in V$:

$$\mathbf{s}(-p,y) = F_{-p}(y) = G_p(y) = \mathbf{t}(p,y).$$

Moreover, if $(p,y) \in V \cap (-\tilde{V})$, then

$$y = \tilde{F}_{-p}(G_p(y)) = \nabla_2 S(p,-p,\mathbf{t}(p,y)) + \mathbf{t}(p,y)$$

and if $(p,y) \in (-V) \cap (-\tilde{V})$

$$y = \tilde{G}_{-p}(F_p(y)) = \nabla_1 S(-p,p,\mathbf{s}(p,y)) + \mathbf{s}(p,y)$$

On the other hand, the following two elements belong to $\phi^{-1}(\mathrm{Graph}(\mathrm{d}S))$:

$$((-p, \nabla_1 S(-p,p,s) + s), (p, \nabla_2 S(-p,p,s) + s), (\nabla_x S(-p,p,s) - p + p, s)),$$
$$((p, \nabla_1 S(p,-p,t) + t), (-p, \nabla_2 S(p,-p,t) + t), (\nabla_x S(-p,p,t) - p + p, t)),$$

where $s = \mathbf{s}(p,y)$ and $t = \mathbf{t}(p,y)$.

In summary, we have shown that

$$\begin{aligned}
\mathbf{s}(-p,y) &= \mathbf{t}(p,y), \\
\mathbf{t}(-p,y) &= \mathbf{s}(p,y), \\
((p,y),(-p,y),(0,\mathbf{t}(p,y))) &\in \phi^{-1}(\mathrm{Graph}(\mathrm{d}S)), \\
((-p,y),(p,y),(0,\mathbf{s}(p,y))) &\in \phi^{-1}(\mathrm{Graph}(\mathrm{d}S)).
\end{aligned}$$

for any $(p,y) \in V' := \tilde{V} \cap (-\tilde{V}) \cap V \cap (-V)$.

We restrict V' so that $(p,-p,\mathbf{t}(p,y))$, $(-p,p,\mathbf{s}(p,y))$ are in U and we denote the restriction by $\mathcal{V}$. If $(p,y) \in \mathcal{V}$ is such that $((p,y),(-p,y)) \in \mathcal{U}$, then $(p,-p,\mathbf{t}(p,y)) \in U$ is such that the corresponding point in $\mathcal{U}$ is $((p,y),(-p,y))$. This means that

$$\mathbf{m}((p,y),(-p,y)) = (0,\mathbf{t}(p,y)).$$

Similarly, if $(p,y) \in \mathcal{V}$ is such that $((-p,y),(p,y)) \in \mathcal{U}$, then

$$\mathbf{m}((-p,y),(p,y)) = (0,\mathbf{s}(p,y)).$$

Now we have a domain of multiplication $\mathcal{U}$ and a domain of inverse $\mathcal{V}$ on which the axioms of a groupoid are satisfied. Next we find a domain of associativity. For this, consider the open neighborhood of I_3:

$$\tilde{W} = \{(p_1,p_2,p_3,x) \in W : (p_1,p_2,\overline{x}), (p_2,p_3,\tilde{x}) \in U\}.$$

Also, define $\iota : \tilde{W} \to T^*M \;{}_{\mathbf{s}}\!\times_{\mathbf{t}} T^*M \;{}_{\mathbf{s}}\!\times_{\mathbf{t}} T^*M$ by

$$(p_1,p_2,p_3,x) \mapsto ((p_1,x_1),(p_2,x_2),(p_3,x_3)),$$

where

$$x_1 = \nabla_1 S(p_1,p_2,\overline{x}), \quad x_2 = \nabla_2 S(p_1,p_2,\overline{x}), \quad x_3 = \nabla_2 S(p_2,p_3,\tilde{x}).$$

The definition of $\tilde{W}$ implies that ι is well-defined (that is, the image is indeed in $T^*M \;{}_{\mathbf{s}}\!\times_{\mathbf{t}} T^*M \;{}_{\mathbf{s}}\!\times_{\mathbf{t}} T^*M$).

Let

$$\mathcal{W}_1 = \{(\alpha_1, \alpha_2, \alpha_3) : (\alpha_1, \alpha_2), (\alpha_2, \alpha_3), (\alpha_1, \alpha_2\alpha_3), (\alpha_1\alpha_2, \alpha_3) \in \mathcal{U}\}.$$

Then $\mathcal{W}_1$ is an open neighborhood of $\mathcal{I}_3$ in $T^*M \,{}_{\mathbf{s}}\!\times_{\mathbf{t}} T^*M \,{}_{\mathbf{s}}\!\times_{\mathbf{t}} T^*M$. Around I_3, the map ι has an inverse defined on $\mathcal{W}_1$:

$$\iota^{-1}(\alpha_1, \alpha_2, \alpha_3) = (\pi_1(\alpha_1), \pi_1(\alpha_2), \pi_1(\alpha_3), \pi(\alpha_1(\alpha_2\alpha_3)))$$

This means that there is an open neighborhood around I_3 on which ι is a diffeomorphism onto an open neighborhood $\mathcal{W}$ of $\mathcal{I}_3$ in W_1. We take $\mathcal{W}$ to be the domain of associativity. This finishes the proof.

□

3.4 Comparison of assumptions

In Section 3.1 we deduced *necessary conditions* on a generating function $S : T^*M \times_M T^*M \to \mathbb{R}$ so that $\text{Graph}(\mathrm{d}S) \subset \overline{T^*M \times\ T^*M} \times T^*M$ becomes the graph of multiplication of a local symplectic groupoid. These were the assumptions used in Theorem 3.2.1, which shows that they are also *sufficient*. In contrast, the assumptions in Theorem 3.3.2 are more obscure, and now reason was given for why they should be assumed. In this section, we compare the assumptions made in the two theorems, and we will see that the conditions appearing in Theorem 3.3.2 are in fact necessary conditions too.

3.4.1 Comparison of source/target maps conditions

Since assumptions 0 are identical in both theorems, we start by looking at assumptions 1 which allow to define the source and target maps.

Proposition 3.4.1. *Let $\mathcal{O}$ be an open neighborhood of 0_{T^*M} such that*

$$S(p, 0, x) = 0, \qquad S(0, p, x) = 0, \qquad \textit{for all } (p, x) \in \mathcal{O}.$$

Then the assumptions 1 in Theorems 3.2.1 and 3.3.2 are equivalent.

Proof. Suppose first that there is an open neighborhood O of 0_{T^*M} in T^*M on which the function given by

$$\mathbf{s} : O \to M, \quad (p, x) \mapsto \nabla_2 S(p, 0, x) + x,$$

is a surjective submersion. We have that

$$((p, \nabla_1 S(p, 0, x) + x), (0, \nabla_2 S(p, 0, x) + x), (\nabla_x S(p, 0, x) + p, x))$$

belongs to $\phi^{-1}(\mathrm{Graph}(\mathrm{d}S))$. If $(p, x) \in O \cap \mathcal{O}$ the above element becomes

$$((p, x), (0, \nabla_2 S(p, 0, x) + x), (p, x)).$$

If $y' \in M$ is such that

$$((p, x), (0, y'), (p, x)) \in \phi^{-1}(\mathrm{Graph}(\mathrm{d}S)).$$

We know that there is (p', p'', x') such that

$$\begin{aligned} &((p, x), (0, y'), (p, x)) = \\ &((p', \nabla_1 S(p', p'', x') + x'), (p'', \nabla_2 S(p', p'', x') + x'), (\nabla_x S(p', p'', x') + p' + p'', x')). \end{aligned}$$

It follows that $p' = p$, $p'' = 0$, $x' = x$ and then $y' = \nabla_2 S(p, 0, x) + x$. This shows that assumption 2(a) in Theorem 3.2.1 is satisfied.

Now assume that there is an open neighborhood O of 0_{T^*M} such that for each $\alpha = (p, x) \in O$ there exists a unique $y \in M$ for which we have

$$(\alpha, 0_y, \alpha) \in \phi^{-1}(\mathrm{Graph}(\mathrm{d}S)).$$

This means that

$$\begin{aligned} ((p, x), &(0, y), (p, x)) = \\ &((p, \nabla_1 S(p, 0, x) + x), (0, \nabla_2 S(p, 0, x) + x), (\nabla_x S(p, 0, x) + p, x)). \end{aligned}$$

In particular, we have $y = \nabla_2 S(p, 0, x) + x$ and it follows that the function

$$\mathbf{s} : O \to M, \quad (p, x) \mapsto \nabla_2 (p, 0, x) + x,$$

is a surjective submersion. $\square$

3.4.2 Comparison of Naturality Conditions

In this section we compare assumptions 2(a) and 2(b) in Theorems 3.2.1 and 3.3.2 .

Proposition 3.4.2. *The following statements are equivalent.*

(i) There is an open neighborhood $D_a = \{(p, x)\}$ *around* 0_{T^*M} *in* T^*M *on which we have*

$$S(p, 0, x) = 0.$$

(ii) There is an open neighborhood O_a *around* $0_{T^*M} \times T^*M$ *in* $T^*M \times T^*M$ *on which we have*

$$\phi_0(\alpha, 0, \overline{\alpha}) \in \text{Graph}(\mathrm{d}S) \implies \alpha = \overline{\alpha},$$

for any $(\alpha, \overline{\alpha}) \in O_a$.

Proof. ($\implies$) Indeed, let $\alpha = (p, x)$ and $\overline{\alpha} = (\overline{p}, \overline{x})$. We have $\phi_0(\alpha, 0_y, \overline{\alpha}) \in$ Graph(dS) if and only if

$$\begin{aligned} x - \overline{x} &= \nabla_1 S(p, 0, \overline{x}), \\ y - \overline{x} &= \nabla_2 S(p, 0, \overline{x}), \\ \overline{p} - p &= \nabla_x S(p, 0, \overline{x}). \end{aligned}$$

If $(p, \overline{x}) \in D_a$, then we have $S(p, 0, \overline{x}) = 0$ and the first and third equation imply $x = \overline{x}$ and $p = \overline{p}$, i.e., $\alpha = \overline{\alpha}$. Therefore we can take

$$O_a = \iota^{-1}(D_a)$$

where

$$\iota : T^*M \times T^*M \to T^*M, \quad (\alpha, \overline{\alpha}) \mapsto (\pi^{cot}(\alpha), \pi(\overline{\alpha})),$$

or in coordinates $((p, x), (\overline{p}, \overline{x})) \mapsto (p, \overline{x})$.

($\impliedby$) Conversely, for each (p, x), we have the following element in $\phi^{-1}(\text{Graph}(\mathrm{d}S))$

$$((p, \nabla_1 S(p, 0, x) + x), (0, \nabla_2(p, 0, x) + x), (\nabla_x S(p, 0, x) + p, x)).$$

If $((p, \nabla_1 S(p,0,x)+x), (\nabla_x S(p,0,x)+p, x)) \in O_a$, then we have

$$p = \nabla_x S(p,0,x) + p,$$
$$x = \nabla_1 S(p,0,x) + x.$$

This implies that $\nabla_x S(p,0,x) = 0$ and $\nabla_1 S(p,0,x) = 0$. So if we take

$$D_a = \iota(O_a),$$

where $\iota : T^*M \to T^*M \times T^*M$ is now defined by

$$\iota(p,x) = ((p, \nabla_1 S(p,0,x)+x), (\nabla_x S(p,0,x)+p, x)),$$

we have $\nabla_1 S(p,0,x) = \nabla_x S(p,0,x) = 0$ on D_a. By our normalization condition $S(0,0,x) = 0$, we obtain $S(p,0,x) = 0$ on D_a.

□

Similarly, one shows that:

Proposition 3.4.3. *The following statements are equivalent.*

(i) There is an open neighborhood $D_a = \{(p,x)\}$ around 0_{T^*M} in T^*M on which we have

$$S(0,p,x) = 0$$

(ii) There is an open neighborhood $O_a = \{(\alpha, \overline{\alpha})\}$ around $0_{T^*M} \times T^*M$ in $T^*M \times T^*M$ on which we have

$$\phi_0(0, \alpha, \overline{\alpha}) \in \mathrm{Graph}(\mathrm{d}S) \implies \alpha = \overline{\alpha}$$

This result finishes the proof that 2(a) in Theorem 3.2.1 yields 2(a) in Theorem 3.3.2 when we specialize to $M = \mathbb{R}^n$ and $\phi = \phi_0$.

To show the equivalence of assumptions 2(b) in the special and general case, we need both 2(a) and the SGA equation (as in Theorem 3.3.2).

Proposition 3.4.4. *Assume 2(a) and the SGA equation hold. Then the following statements are equivalent:*

(i) There is an open neighborhood $D_b = \{(p, x)\}$ *around* 0_{T^*M} *in* T^*M *on which we have*

$$\nabla_1 S(p, -p, x) = \nabla_2 S(p, -p, x),$$
$$\nabla_x S(p, -p, x) = 0.$$

(ii) There is an open neighborhood O_b *around* 0_{T^*M} *in* T^*M *on which we have*

$$\phi_0(\alpha, -\alpha, 0_{t(\alpha)}) \in \mathrm{Graph}(\mathrm{d}S).$$

Proof. ($\Longrightarrow$) Notice first that for each (p, x),

$$\phi_0((p, \nabla_1 S(p, -p, x) + x), (-p, \nabla_2 S(p, -p, x) + x), (\nabla_x S(p, -p, x), x))$$

is an element in $\mathrm{Graph}(\mathrm{d}S)$. If $(p, x) \in D_b$, we have

$$\nabla_1 S(p, -p, x) = \nabla_2 S(p, -p, x), \quad \nabla_x S(p, -p, x) = 0.$$

This implies that the above element is of the form $(\alpha, -\alpha, 0_x)$. As in the proof of Theorem 3.3.2, by 2(a) and the SGA equation, there is an open neighborhood of 0_{T^*M} in T^*M on which we have, for each (p_0, x_0),

$$\tilde{F}_{p_0} : x \mapsto \nabla_1 S(p_0, -p_0, x) + x$$

is the inverse of

$$F_{p_0} : x \mapsto \nabla_1 S(0, p_0, x) + x = \mathbf{t}(p_0, x)$$

around x_0. By making D_b smaller, we have, for each $(p, x) \in D_b$,

$$x = F_p \circ \tilde{F}_p(x) = F_p(\nabla_1 S(p, -p, x) + x) = \mathbf{t}(p, \nabla_1 S(p, -p, x) + x).$$

To finish the proof, we need to show that the image of

$$D_b \to T^*M, \quad (p, x) \mapsto (p, \nabla_1 S(p, -p, x) + x),$$

is an open neighborhood around 0_{T^*M} in T^*M. Clearly, 0_{T^*M} is mapped to itself. The above argument with $F_p, \tilde{F}_p$ shows that the map of interest is a local diffeomorphism at each point in D_b. This finishes the proof.

($\Longleftarrow$) If $\alpha = (p, y)$ and $x = \mathbf{t}(\alpha)$, we have $\phi_0(\alpha, -\alpha, 0_x) \in \mathrm{Graph}(\mathrm{d}S)$ if and only if

$$(\alpha, -\alpha, 0_x) = \\ (p, \nabla_1 S(p, -p, x) + x), (-p, \nabla_2 S(p, -p, x) + x), (\nabla_x S(p, -p, x), x).$$

This implies that $\nabla_1 S(p, -p, x) = \nabla_2 S(p, -p, x)$ and $\nabla_x S(p, -p, x) = 0$ if $\alpha \in O_b$. We must show that the image of

$$O_b \to T^*M, \quad (p, y) \mapsto (p, \mathbf{t}(p, y)),$$

is an open neighborhood around 0_{T^*M} in T^*M. Indeed, this follows because what we have shown above implies that this map is a local diffeomorphism and it maps 0_{T^*M} onto itself. □

3.4.3 Different Parameterizations of the SGA Equation

In this section, we compare assumption 3 in Theorems 3.2.1 and 3.3.2 . For convenience, we will reference assumption 3 in Theorem 3.2.1 by (∗1) and assumption 3 in Theorem 3.3.2 by (∗2). We restate the assumptions below for the convenience of the reader.

(∗1). *There is an open neighborhood* $W = \{(\alpha_1, \alpha_2, \alpha_3)\}$ *of*

$$\{(\alpha, 0_{\mathbf{s}(\alpha)}, 0_{\mathbf{s}(\alpha)}) : \alpha \in O\} \cup \{(0_{\mathbf{t}(\alpha)}, \alpha, 0_{\mathbf{s}(\alpha)}) : \alpha \in O\} \cup \{(0_{\mathbf{t}(\alpha)}, 0_{\mathbf{t}(\alpha)}, \alpha) : \alpha \in O\}$$

in $O_{\mathbf{s}} \times_{\mathbf{t}} O_{\mathbf{s}} \times_{\mathbf{t}} O$ *on which we have functions*

$$\overline{\alpha} : W \to T^*M, \quad \tilde{\alpha} : W \to T^*M, \quad \alpha : W \to T^*M,$$

such that

(a)

$$\overline{\alpha}(\alpha_0, 0_{\mathbf{s}(\alpha_0)}, 0_{\mathbf{s}(\alpha_0)}) = \overline{\alpha}(0_{\mathbf{t}(\alpha_0)}, \alpha_0, 0_{\mathbf{s}(\alpha_0)}) = \alpha_0,$$

$$\tilde{\alpha}(0_{\mathbf{t}(\alpha_0)}, 0_{\mathbf{t}(\alpha_0)}, \alpha_0) = \tilde{\alpha}(0_{\mathbf{t}(\alpha_0)}, \alpha_0, 0_{\mathbf{s}(\alpha_0)}) = \alpha_0,$$

$$\alpha(0_{\mathbf{t}(\alpha_0)}, 0_{\mathbf{t}(\alpha_0)}, \alpha_0) = \alpha(0_{\mathbf{t}(\alpha_0)}, \alpha_0, 0_{\mathbf{s}(\alpha_0)}) = (\alpha_0, 0_{\mathbf{s}(\alpha_0)}, 0_{\mathbf{s}(\alpha_0)}) = \alpha_0.$$

(b) The following system of equations is satisfied:

$$\begin{aligned} \phi(\alpha_1, \alpha_2, \overline{\alpha}) &= \mathrm{d}S(\pi \circ \phi(\alpha_1, \alpha_2, \overline{\alpha})), \\ \phi(\alpha_2, \alpha_3, \tilde{\alpha}) &= \mathrm{d}S(\pi \circ \phi(\alpha_2, \alpha_3, \tilde{\alpha})), \\ \phi(\alpha_1, \tilde{\alpha}, \alpha) &= \mathrm{d}S(\pi \circ \phi(\alpha_1, \tilde{\alpha}, \alpha)), \\ \phi(\overline{\alpha}, \alpha_3, \alpha) &= \mathrm{d}S(\pi \circ \phi(\overline{\alpha}, \alpha_3, \alpha)). \end{aligned} \tag{3.3}$$

$(*2)$. *(SGA Equation) There is an open neighborhood W of*

$$\{(p,0,0,x) : (p,x) \in \mathcal{O}\} \cup \{(0,p,0,x) : (p,x) \in \mathcal{O}\} \cup \{(0,0,p,x) : (p,x) \in \mathcal{O}\}$$

on which the SGA equation holds. That is, there are functions

$$\overline{p}, \overline{x}, \tilde{p}, \tilde{x} : W \to T^*\mathbb{R}^n$$

such that, for $(p,x) \in \mathcal{O}$

$$\overline{p}(p,0,0,x) = \overline{p}(0,p,0,x) = p, \quad \tilde{p}(0,p,0,x) = \tilde{p}(0,0,p,x) = p,$$

and we have for all $(p_1, p_2, p_3, x) \in W$:

$$\begin{aligned} S(p_1, p_2, \overline{x}) + &S(\overline{p}, p_3, x) + (p_1 + p_2)\overline{x} + (\overline{p} + p_3)x - \overline{px} \\ &= S(p_2, p_3, \tilde{x}) + S(p_1, \tilde{p}, x) + (p_2 + p_3)\tilde{x} + (p_1 + \tilde{p})x - \tilde{p}\tilde{x}, \quad (3.4) \\ \overline{p} = \nabla_x S(p_1, &p_2, \overline{x}) + p_1 + p_2, \qquad \overline{x} = \nabla_1 S(\overline{p}, p_3, x) + x, \\ \tilde{p} = \nabla_x S(p_2, &p_3, \tilde{x}) + p_2 + p_3, \qquad \tilde{x} = \nabla_2 S(p_1, \tilde{p}, x) + x. \end{aligned}$$

We want to show that, in the setting of Theorem 3.3.2, that is, when $M = \mathbb{R}^n$ and $\phi = \phi_0$, $(*2)$ is equivalent to $(*1)$. We have seen how the SGA equation can be derived from $(*1)$ at the end of section 1, but the computation there only gives rise to $(*2)$ when the differential equations (3.1) are satisfied for (p_1, p_2, p_3, x) on an open subset of $(\mathbb{R}^{*n})^3 \times \mathbb{R}^n$, which is not immediate given the domain of the equations in $(*1)$.

Pointwise, it is clear how equations (3.3) and (3.4) are related. For each $(p_1, p_2, p_3, x) \in W$, take $(\alpha_1, \alpha_2, \alpha_3)$ to be

$$((p_1, \nabla_1 S(p_1, p_2, \overline{x}) + x), (p_2, \nabla_2 S(p_1, p_2, x) + x), (p_3, \nabla_2 S(p_2, p_3, \tilde{x}))).$$

Differentiating the algebraic equation in the SGA equation with respect to p_1, p_2, p_3, x, we recover (3.3) at $(\alpha_1, \alpha_2, \alpha_3)$.

Conversely, if $(\alpha_1, \alpha_2, \alpha_3) \in \mathcal{W}$, take $p_i = \pi^{cot}(\alpha_i)$, $1 \leq i \leq 3$, $x = \pi(\alpha)$ where $\alpha = \alpha(\alpha_1, \alpha_2, \alpha_3)$. Then, as we have seen before, the differential equations (3.1) are satisfied for (p_1, p_2, p_3, x) and if this is true on an open neighborhood in $(\mathbb{R}^{*n})^3 \times \mathbb{R}^n$, equation (3.4) is satisfied at the point (p_1, p_2, p_3, x).

The question we want to answer is the following.

Question 3.4.5. *When are* $(*1)$ *and* $(*2)$ *equivalent?*

For the implication $(*1) \Longleftarrow (*2)$, we need to make two assumptions from Theorem 3.3.2.

Proposition 3.4.6. *Assume that the following statements hold.*

(a) For all $(p, x) \in \mathcal{O}$, *we have*

$$S(p, 0, x) = S(0, p, x) = 0;$$

(b) The map $(\mathbb{R}^{*n})^2 \times \mathbb{R}^n \to T^*M \times T^*M$

$$(p_1, p_2, x) \mapsto ((p_1, \nabla_1 S_1(p_1, p_2, x) + x), (p_2, \nabla_2 S_2(p_1, p_2, x) + x))$$

is an injective immersion on an open neighborhood of

$$\{(p, 0, x) : (p, x) \in \mathcal{O}\} \cup \{(0, p, x) : (p, x) \in \mathcal{O}\}.$$

Then we have $(*1) \Longleftarrow (*2)$.

For the proofs of this section we continue using the notations set up before:

$$\mathcal{I}_3 = \{(\alpha, 0_{s(\alpha)}, 0_{s(\alpha)}) : \alpha \in \mathcal{O}\} \cup \{(0_{\mathbf{t}(\alpha)}, \alpha, 0_{s(\alpha)}) : \alpha \in \mathcal{O}\} \cup \{(0_{\mathbf{t}(\alpha)}, 0_{\mathbf{t}(\alpha)}, \alpha) : \alpha \in \mathcal{O}\},$$
$$I_3 = \{(p, 0, 0, x) : (p, x) \in \mathcal{O}\} \cup \{(0, p, 0, x) : (p, x) \in \mathcal{O}\} \cup \{(0, 0, p, x) : (p, x) \in \mathcal{O}\}.$$

Also, for simplicity of notations, assume that the maps $\mathbf{s}, \mathbf{t}$ are defined on T^*M. Recall that the definitions of these maps in Theorems 3.2.1 and 3.3.2 agree.

Proof of Proposition 3.4.6. Define $\iota : W \to T^*M \times T^*M \times T^*M$ by

$$\iota(p_1, p_2, p_3, x) = ((p_1, \nabla_1 S(p_1, p_2, \overline{x}) + \overline{x}), (p_2, \nabla_2 S(p_1, p_2, \overline{x}) + \overline{x}), (p_3, \nabla_2 S(p_2, p_3, \tilde{x}) + \tilde{x})),$$

where

$$\overline{x} = \overline{x}(p_1, p_2, p_3, x), \quad \tilde{x} = \tilde{x}(p_1, p_2, p_3, x).$$

We have seen in the proof for Theorem 3.3.2 that if (a) holds, then whenever $(p', 0, p'', y) \in W$ one has

$$((p', \nabla_1 S(p', p'', y) + y), (p'', \nabla_2 S(p', p'', y) + y)) \in T^*M \ {}_{\mathbf{s}}\times_{\mathbf{t}} T^*M.$$

One the other hand, it follows from the SGA equation (3.4) that

$$\nabla_2 S(p_1, p_2, \overline{x}) + \overline{x} = \nabla_1 S(p_2, p_3, \tilde{x}) + \tilde{x},$$

so we can restrict W around I_3 and assume that ι has image contained in $T^*M \ {}_{\mathbf{s}}\times_{\mathbf{t}} T^*M \ {}_{\mathbf{s}}\times_{\mathbf{t}} T^*M$.

Let $\mathcal{U} \subseteq T^*M \ {}_{\mathbf{s}}\times_{\mathbf{t}} T^*M$ be the domain of multiplication as in the proof for Theorem 3.3.2. We see there that it is defined whenever (a) and (b) are assumed. The set of compatible triples determined by $\mathcal{U}$ is an open neighborhood $\mathcal{W}'$ of $\mathcal{I}_3$ in $T^*M \ {}_{\mathbf{s}}\times_{\mathbf{t}} T^*M \ {}_{\mathbf{s}}\times_{\mathbf{t}} T^*M$. On $\mathcal{W}'$, we define

$$\kappa : (\alpha_1, \alpha_2, \alpha_3) \mapsto (\pi^{cot}(\alpha_1), \pi^{cot}(\alpha_2), \pi^{cot}(\alpha_3), \pi(\alpha)),$$

where $\alpha = m(\alpha_1, m(\alpha_2, \alpha_3))$. This map is a left inverse to $\iota|_{\iota^{-1}(\mathcal{W}')}$, which implies that $\iota|_{\iota^{-1}(\mathcal{W}')}$ is an injective immersion. Since the two manifolds W and $T^*M \ {}_{\mathbf{s}}\times_{\mathbf{t}} T^*M \ {}_{\mathbf{s}}\times_{\mathbf{t}} T^*M$ have the same dimension, the image of $\iota|_{\iota^{-1}(\mathcal{W}')}$ is an open neighborhood of $\mathcal{I}$ in $T^*M \ {}_{\mathbf{s}}\times_{\mathbf{t}} T^*M \ {}_{\mathbf{s}}\times_{\mathbf{t}} T^*M$. As we have seen above, each element $(\alpha_1, \alpha_2, \alpha_3)$ in the image satisfies the equations in (3.3) with

$$\overline{\alpha} = (\overline{p}, \overline{x}), \quad \tilde{\alpha} = (\tilde{p}, \tilde{x}), \quad \alpha = (p, x),$$

where by assumption, $\overline{p}, \overline{x}, \tilde{p}, \tilde{x}, p$ are smooth functions of $(p_1, p_2, p_3, x) = \kappa(\alpha_1, \alpha_2, \alpha_3)$. This implies that we can take $\mathcal{W} = \iota(W \cap \iota^{-1}(\mathcal{W}'))$ and the restriction of the functions $\overline{\alpha}, \tilde{\alpha}, \alpha$. The normalization properties of $\overline{\alpha}, \tilde{\alpha}, \alpha$ follow from those of $\overline{p}, \tilde{p}$ and (a). ☐

For the implication $(*1) \implies (*2)$, we have the following.

Proposition 3.4.7. *Assume that*

(a) For all $(p, x) \in \mathcal{O}$, we have

$$S(p, 0, x) = S(0, p, x) = 0;$$

(b) The map

$$\begin{aligned} \kappa : \mathcal{W} &\to (\mathbb{R}^{*n})^3 \times \mathbb{R}^n, \\ (\alpha_1, \alpha_2, \alpha_3) &\mapsto (\pi^{cot}(\alpha_1), \pi^{cot}(\alpha_2), \pi^{cot}(\alpha_3), \pi(\alpha)), \end{aligned}$$

where $\alpha = \alpha(\alpha_1, \alpha_2, \alpha_3)$, is injective on an open neighborhood of $\mathcal{I}_3$.

Then we have $(*1) \implies (*2)$.

Proof. We want to show that κ is a local diffeomorphism along $\mathcal{I}_3$ by finding a (local) left inverse. The obvious choice is the map ι defined above. The difficulty here is that the expressions

$$(p_1, \nabla_1 S(p_1, p_2, \overline{x}) + \overline{x}), (p_2, \nabla_2 S(p_1, p_2, \overline{x}) + \overline{x}), (p_3, \nabla_x S(p_2, p_3, \tilde{x}) + \tilde{x})$$

require a solution $\overline{x}$ of

$$\begin{cases} \overline{x} = \nabla_1 S(\overline{p}, p_3, x) + x, \\ \overline{p} = \nabla_x S(p_1, p_2, \overline{x}) + p_1 + p_2, \end{cases} \tag{3.5}$$

and also a solution $\tilde{x}$ of

$$\begin{cases} \tilde{x} = \nabla_2 S(p_1, \tilde{p}, x) + x, \\ \tilde{p} = \nabla_x S(p_2, p_3, \tilde{x}) + p_2 + p_3. \end{cases} \tag{3.6}$$

Without assuming $(*1)$, we do not know if solutions to these equations exist.

To solve this problem, we first rewrite (3.5) as

$$\overline{x} = \nabla_1 S(\nabla_x S(p_1, p_2, \overline{x}) + p_1 + p_2, p_3, x) + x. \tag{3.7}$$

Let $\alpha = (p, x) \in \mathcal{O}$ and $v_0 = (p, 0, 0, x)$. Note that $\kappa(\alpha, 0_x, 0_x) = v_0$. We also let

$$F(\overline{x}, p_1, p_2, p_3, x) = \overline{x} - (\nabla_1 S(\nabla_x S(p_1, p_2, \overline{x}) + p_1 + p_2, p_3, x) + x).$$

At (x, v_0), (a) implies that $F(x, v_0) = 0$ and also that:

$$\begin{aligned} \frac{\partial F}{\partial \overline{x}}\Big|_{(x,v_0)} &= \left(I - \nabla_1 \nabla_1 S(\nabla_x S(p_1, p_2, \overline{x}) + p_1 + p_2, p_3, x) \frac{\partial \nabla_x S(p_1, p_2, \overline{x})}{\partial \overline{x}} \right) \Big|_{(x,v_0)} \\ &= I, \end{aligned}$$

since by (a), $\nabla_x S(p', 0, x') = 0$ for (p', x') in an open set around (p, x).

It follows that the map defined by

$$\tilde{F}(\overline{x}, p_1, p_2, p_3, x) = \big(F(\overline{x}, p_1, p_2, p_3, x), p_1, p_2, p_3, x\big)$$

is a local diffeomorphism around (x, v_0). Let U, U' be open neighborhoods of x and v_0, respectively, such that $\tilde{F}|_{U \times U'}$ is a diffeomorphism onto its image. This image is an open neighborhood of $(0, p, 0, 0, x)$ in $\mathbb{R}^n \times (\mathbb{R}^{*n})^3 \times \mathbb{R}^n$ and denoting by $\tilde{G}$ the inverse of $\tilde{F}|_{U \times U'}$ we have:

$$\tilde{G}(y, p_1, p_2, p_3, x) = (G(y, p_1, p_2, p_3, x), p_1, p_2, p_3, x).$$

Shrinking U' if necessary, we conclude that, for $(p_1, p_2, p_3, x) \in U'$, $\overline{x} = G(0, p_1, p_2, p_3, x)$ is the unique point in U such that $F(\overline{x}, p_1, p_2, p_3, x) = 0$. In particular, $\overline{x}$ is a solution to (3.7). The construction for equation (3.6) is similar.

We conclude that we have an open neighborhood W' of v_0 in $(\mathbb{R}^{*n})^3 \times \mathbb{R}^n$ on which (3.5) (3.6) have solutions. We define

$$\begin{aligned} &\iota : W' \to T^*M \times T^*M \times T^*M, \\ &(p_1, p_2, p_3, x) \mapsto ((p_1, \nabla_1 S(p_1, p_2, \overline{x})), (p_2, \nabla_2 S(p_1, p_2, \overline{x})), (p_3, \nabla_2 S(p_2, p_3, \tilde{x}))). \end{aligned}$$

We claim that, on an open neighborhood of $(\alpha, 0_x, 0_x)$ in $\mathcal{W}$, ι is a left inverse of κ.

Around $(\alpha, 0_x, 0_x)$, to each point $(\alpha_1, \alpha_2, \alpha_3) \in \mathcal{W}$ we associate the points $\overline{\alpha}(\alpha_1, \alpha_2, \alpha_3) = (\overline{p}, \overline{x})$, $\tilde{\alpha}(\alpha_1, \alpha_2, \alpha_3) = (\tilde{p}, \tilde{x})$ as in (3.3). Let us denote the second coordinates of these points by

$$\overline{x}^{\mathcal{W}}(\alpha_1, \alpha_2, \alpha_3) = \overline{x}, \quad \tilde{x}^{\mathcal{W}}(\alpha_1, \alpha_2, \alpha_3) = \tilde{x}.$$

Mapping $(\alpha_1, \alpha_2, \alpha_3)$ via κ, we get a point (p_1, p_2, p_3, x') near $\kappa(\alpha, 0, 0) = (p, 0, 0, x) = v_0$. By the definition of ι, we have

$$\iota(p_1, p_2, p_3, x^{'}) = ((p_1, \nabla_1 S(p_1, p_2, \overline{x})), (p_2, \nabla_2 S(p_1, p_2, \overline{x}), (p_3, \nabla_2 S(p_2, p_3, \tilde{x}))),$$

where $\overline{x}, \tilde{x}$ are obtained as solutions to (3.5)(3.6) as above. We denote them by

$$\overline{x}^{W^{'}}(\kappa(\alpha_1, \alpha_2, \alpha_3)) = \overline{x}, \quad \tilde{x}^{W^{'}}(\kappa(\alpha_1, \alpha_2, \alpha_3)) = \tilde{x}.$$

If we can show that

$$\overline{x}^{\mathcal{W}}(\alpha_1, \alpha_2, \alpha_3) = \overline{x}^{W^{'}}(\kappa(\alpha_1, \alpha_2, \alpha_3)),$$
$$\tilde{x}^{\mathcal{W}}(\alpha_1, \alpha_2, \alpha_3) = \tilde{x}^{W^{'}}(\kappa(\alpha_1, \alpha_2, \alpha_3)),$$

then equations (3.3) imply that ι is a left inverse of κ, as claimed. Although both sides of the equations, together with $(p_1, p_2, p_3, x^{'})$ yield solutions to (3.5) and (3.6), they need not be equal. However, since we know that $\overline{x}^{\mathcal{W}}(\alpha, 0_x, 0_x) = x$, we can choose an open neighborhood of $(\alpha, 0_x, 0_x)$ on which we have

$$\kappa(\alpha_1, \alpha_2, \alpha_3) \in U^{'}, \quad \overline{x}^{\mathcal{W}}(\alpha_1, \alpha_2, \alpha_3) \in U,$$

where $U, U^{'}$ are as in the first part of the proof. Then we have

$$\tilde{F}(\overline{x}^{\mathcal{W}}, p_1, p_2, p_3, x^{'}) = (0, p_1, p_2, p_3, x') = \tilde{F}(\overline{x}^{W^{'}}, p_1, p_2, p_3, x^{'}),$$

and since $\tilde{F}|_{U \times U'}$ is a diffeomorphsim, we must have

$$\overline{x}^{\mathcal{W}}(\alpha_1, \alpha_2, \alpha_3) = \overline{x}^{W^{'}}(\kappa(\alpha_1, \alpha_2, \alpha_3)).$$

The proof for $\tilde{x}$ is similar. This finishes the proof that, at points of the form $(\alpha, 0_x, 0_x)$, κ is an immersion and thus a local diffeomorphism, since $\mathcal{W}$ and $(\mathbb{R}^{*n})^3 \times \mathbb{R}^n$ have the same dimension. The proof around points of the form $(0_x, \alpha, 0_x), (0_x, 0_x, \alpha)$ is similar.

Since κ is injective around $\mathcal{I}_3$, it maps a neighborhood of $\mathcal{I}_3$ to a neighborhood of I_3. By the same argument following the proof of Theorem 3.2.1, on this neighborhood there are smooth functions $\overline{p}, \overline{x}, \tilde{p}, \tilde{x}$ for which the SGA equation (3.4) is satisfied. The normalization properties of $\overline{p}, \tilde{p}$ follow directly from those of $\overline{\alpha}, \tilde{\alpha}$. $\square$

3.5 The SGA equation

In this section we will take a deeper look into the SGA equation (3.2).

3.5.1 The Exponential Map as a Symplectomorphism

In the derivation of the SGA equation, following Theorem 3.2.1, the variables x_1, x_2, x_3, p all cancel so they do no appear in the final equation. In this section, we take a different symplectomorphism ϕ and follow the same procedure and see that now all variables appear in the new form of the equation.

Let M be a manifold. We recall that $X = T^*M \times_M T^*M$ is identified with $\mathrm{Graph}(m_0)$ in $T^*M \times T^*M \times T^*M$. Let g be a Riemannian metric on $T^*M \times T^*M \times T^*M$ so that, around X, the associated exponential map is a diffeomorphism onto its image

$$\exp : NX \to T^*M \times T^*M \times T^*M.$$

On the other hand, we have a diffeomorphism from NX to T^*X

$$\begin{aligned} NX &\to T^*X, \\ v &\mapsto -\omega_0^\flat(v)|_X, \end{aligned}$$

where $\omega_0 = -\operatorname{pr}_1^* \omega_c - \operatorname{pr}_2^* \omega_c + \operatorname{pr}_3^* \omega_c$. Denoting the inverse of this map by ι, we obtain a diffeomorphism

$$\exp \circ \iota : T^*X = T^*(T^*M \times_M T^*M) \to T^*M \times T^*M \times T^*M,$$

which is defined around X in T^*X. When $M = \mathbb{R}^n$ and $T^*M \times T^*M \times T^*M$ has the usual Euclidean metric, the above map is a symplectomorphism. Its inverse is given by

$$\phi : ((p_1, x_1), (p_2, x_2), (p, x)) \mapsto$$
$$((p_1 - \tfrac{1}{3}(p_1 + p_2 - p), x - x_1),(p_2 - \tfrac{1}{3}(p_1 + p_2 - p), x - x_2), (\tfrac{1}{3}(x_1 + x_2 + x), p_1 + p_2 - p)).$$

The SGA equation with respect to ϕ is

$$\begin{aligned} &S_{12-} + S_{23} \ - \tfrac{1}{3}((p_2 + p - \overline{p} - \tilde{p})x_1 + (p_1 - p_3 - \overline{p} + \tilde{p})x_2 + (p_1 + p_2 + p_3 - p)\overline{x}) \\ =&S_{1\sim} + S_{-3} + \tfrac{1}{3}((-p_2 - p + \overline{p} + \tilde{p})x_3 + (-p_1 - p_2 - p_3 + p)\tilde{x} + (-p_1 + \overline{p} - \tilde{p} + p_3)x), \end{aligned}$$

where

$$\begin{aligned} S_{12-} &= S(p_1 - \frac{1}{3}(p_1 + p_2 - \overline{p}), p_2 - \frac{1}{3}(p_1 + p_2 - \overline{p}), \frac{1}{3}(x_1 + x_2 + \overline{x})), \\ S_{23} &= S(p_2 - \frac{1}{3}(p_2 + p_3 - \tilde{p}), p_3 - \frac{1}{3}(p_2 + p_3 - \tilde{p}), \frac{1}{3}(x_2 + x_3 + \tilde{x})), \\ S_{1\sim} &= S(p_1 - \frac{1}{3}(p_1 + \tilde{p} - p), \tilde{p} - \frac{1}{3}(p_1 + \tilde{p} - p), \frac{1}{3}(x_1 + \tilde{x} + x)), \\ S_{-3} &= S(\overline{p} - \frac{1}{3}(\overline{p} + p_3 - p), p_3 - \frac{1}{3}(\overline{p} + p_3 - p), \frac{1}{3}(\overline{x} + x_3 + x)), \end{aligned}$$

and

$$\begin{aligned} p_1 + p_2 - \overline{p} &= \nabla_x S(p_1 - \frac{1}{3}(p_1 + p_2 - \overline{p}), p_2 - \frac{1}{3}(p_1 + p_2 - \overline{p}), \frac{1}{3}(x_1 + x_2 + \overline{x})), \\ p_2 + p_3 - \tilde{p} &= \nabla_x S(p_2 - \frac{1}{3}(p_2 + p_3 - \tilde{p}), p_3 - \frac{1}{3}(p_2 + p_3 - \tilde{p}), \frac{1}{3}(x_2 + x_3 + \tilde{x})), \\ p_1 + \tilde{p} - p &= \nabla_x S(p_1 - \frac{1}{3}(p_1 + \tilde{p} - p), \tilde{p} - \frac{1}{3}(p_1 + \tilde{p} - p), \frac{1}{3}(x_1 + \tilde{x} + x)), \\ \overline{p} + p_3 - p &= \nabla_x S(\overline{p} - \frac{1}{3}(\overline{p} + p_3 - p), p_3 - \frac{1}{3}(\overline{p} + p_3 - p), \frac{1}{3}(\overline{x} + x_3 + x)), \\ \overline{x} - x_1 &= \nabla_1 S(p_1 - \frac{1}{3}(p_1 + p_2 - \overline{p}), p_2 - \frac{1}{3}(p_1 + p_2 - \overline{p}), \frac{1}{3}(x_1 + x_2 + \overline{x})), \\ \tilde{x} - x_2 &= \nabla_1 S(p_2 - \frac{1}{3}(p_2 + p_3 - \tilde{p}), p_3 - \frac{1}{3}(p_2 + p_3 - \tilde{p}), \frac{1}{3}(x_2 + x_3 + \tilde{x})), \\ x - x_1 &= \nabla_1 S(p_1 - \frac{1}{3}(p_1 + \tilde{p} - p), \tilde{p} - \frac{1}{3}(p_1 + \tilde{p} - p), \frac{1}{3}(x_1 + \tilde{x} + x)), \\ x - \overline{x} &= \nabla_1 S(\overline{p} - \frac{1}{3}(\overline{p} + p_3 - p), p_3 - \frac{1}{3}(\overline{p} + p_3 - p), \frac{1}{3}(\overline{x} + x_3 + x)). \end{aligned}$$

The details of these computations are given in appendix A.

3.5.2 An Interpretation of the SGA Equation

From this point on, we will use the SGA equation in its original form (see [4]), which we now recall.

SGA Equation. *There is an open neighborhood W of*

$$I_3 = \{(p,0,0,x) : (p,x) \in \mathcal{O}\} \cup \{(0,p,0,x) : (p,x) \in \mathcal{O}\} \cup \\ \cup \{(0,0,p,x) : (p,x) \in \mathcal{O}\},$$

on which we have functions $\overline{p}, \overline{x}, \tilde{p}, \tilde{x} : W \to \mathbb{R}$ for which the following equations hold:

$$S(p_1,p_2,\overline{x}) + S(\overline{p},p_3,x) - \overline{px} = S(p_2,p_3,\tilde{x}) + S(p_1,\tilde{p},x) - \tilde{p}\tilde{x},$$
$$\overline{p} = \nabla_x S(p_1,p_2,\overline{x}), \quad \overline{x} = \nabla_1 S(\overline{p},p_3,x),$$
$$\tilde{p} = \nabla_x S(p_2,p_3,\tilde{x}), \quad \tilde{x} = \nabla_2 S(p_1,\tilde{p},x).$$

As noted in Remark 3.3.1, we can recover this form of the SGA equation from the equation used in Theorem 3.3.2 by replacing S with $S - (p_1+p_2)x$. Moreover, with a similar argument as in Theorem 3.3.2, we can prove that if the SGA equation and some other assumptions hold, S is a generating function for a local symplectic groupoid structure on some open neighborhood of $0_{T^*\mathbb{R}^n}$ in $T^*\mathbb{R}^n$ with respect to the symplectomorhpism

$$\phi_0' : T^*\mathbb{R}^n \times T^*\mathbb{R}^n \times T^*\mathbb{R}^n \to T^*(T^*\mathbb{R}^n \times_{\mathbb{R}^n} T^*\mathbb{R}^n) \simeq T^*(\mathbb{R}^{*n} \times \mathbb{R}^{*n} \times \mathbb{R}^n),$$
$$((p_1,x_1),(p_2,x_2),(p,x)) \mapsto ((p_1,p_2,x),(x_1,x_2,p)). \tag{3.8}$$

Let X, Y be manifolds. The set $\mathrm{rel}(X,Y)$ of canonical Lagrangian relations between X and Y consists of all Lagrangian submanifolds in $\overline{T^*X} \times T^*Y$, where the overline indicates, as usual, the negative of the canonical symplectic structure.

Let $\pi : Z \to X \times Y$ be a fibration and let $\phi : Z \to \mathbb{R}$ be a smooth function. A point $z \in Z$ is said to be a *critical point* of ϕ with respect to π if $\mathrm{d}_z\phi(v) = 0$ for any vertical tangent vector $v \in \ker(\mathrm{d}_z\pi)$. Denote the set of critical points by C_ϕ. Note that if $z \in C_\phi$ and $\pi(z) = (x,y)$, then there is a unique covector α at (x,y) such that $\mathrm{d}_z\phi = \phi^*\alpha$. We write this covector as

$\alpha = \alpha_x + \alpha_y$, using the identification $T^*(X \times Y) = T^*X \times T^*Y$, and we will denote $\mathrm{d}_{X\times Y}\phi(z) = -\alpha_x + \alpha_y$.

Definition 3.5.1 ([11])**.** *Given a Lagrangian relation $L \in \mathrm{rel}(X, Y)$, we call ϕ a* **generating function** *for L with respect to the fibration π if*

$$L = \{(\mathrm{d}\phi)_{X\times Y}(z) : z \in C_\phi\}.$$

Now, suppose we have a local symplectic groupoid structure on $T^*\mathbb{R}^n$ and $S : \mathbb{R}^{*n} \times \mathbb{R}^{*n} \times \mathbb{R}^n \to \mathbb{R}$ is a generating function with respect to

$$\begin{aligned} \phi_0' : T^*\mathbb{R}^n \times T^*\mathbb{R}^n \times T^*\mathbb{R}^n &\to T^*(\mathbb{R}^{*n} \times \mathbb{R}^{*n} \times \mathbb{R}^n) \\ ((p_1, x_1), (p_2, x_2), (p, x)) &\mapsto ((p_1, p_2, x), (x_1, x_2, p)) \end{aligned}$$

as in [4], that is,

$$\mathrm{Graph}(m) = \{((p_1, \nabla_1 S(p_1, p_2, x)), (p_2, \nabla_2 S(p_1, p_2, x)), (\nabla_x S(p_1, p_2, x), x))\}.$$

Then we also have that the graph of multiplication is a Lagrangian submanifold in $\overline{T^*\mathbb{R}^n} \times \overline{T^*\mathbb{R}^n} \times T^*\mathbb{R}^n = \overline{T^*(\mathbb{R}^n \times \mathbb{R}^n)} \times T^*\mathbb{R}^n$. In other words, $\mathrm{Graph}(m)$ is a Lagrangian relation between $\mathbb{R}^n \times \mathbb{R}^n$ and $\mathbb{R}^n$:

$$\mathrm{Graph}(m) \in \mathrm{rel}(\mathbb{R} \times \mathbb{R}, \mathbb{R}).$$

Then we can find a generating function in the sense of the above definition.

Proposition 3.5.2. *Let $\tilde{S} : (T^*\mathbb{R}^n)^3 \to \mathbb{R}$ be the function*

$$\tilde{S}((p_1, x_1), (p_2, x_2), (p, x)) = -p_1 x_1 - p_2 x_2 + S(p_1, p_2, x), \tag{3.9}$$

and consider the fibration

$$\begin{aligned} \pi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{*n} \times \mathbb{R}^{*n} &\to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n, \\ (x_1, x_2, x_3, p_1, p_2) &\mapsto (x_1, x_2, x_3). \end{aligned}$$

Then $\tilde{S}$ is a generating function for $\phi_0^{-1}(\mathrm{Graph}(\mathrm{d}S))$ with respect to π.

Proof. The set $C_{\tilde{S}}$ of critical points of $\tilde{S}$ is determined by the equations

$$-x_1 + \nabla_1 S(p_1, p_2, \overline{x}) = 0$$
$$-x_2 + \nabla_2 S(p_2, p_3, \tilde{x}) = 0$$

We only need to note that, at each $z \in C_{\tilde{S}}$, we have

$$\mathrm{d}_{(\mathbb{R}^n \times \mathbb{R}^n)\times\mathbb{R}} \tilde{S}(z) = ((p_1, x_1), (p_2, x_2), (\nabla_x S(p_1, p_2, x), x))$$

□

Let us observe now that the graphs of the maps defined on the set of compatible triples by

$$(\alpha_1, \alpha_2, \alpha_3) \mapsto m(\alpha_1, m(\alpha_2, \alpha_3)),$$
$$(\alpha_1, \alpha_2, \alpha_3 \mapsto m(m(\alpha_1, \alpha_2), \alpha_3),$$

are Lagrangian relations between $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ and $\mathbb{R}^n$. Namely, they are the composition of Lagrangian relations:

$$L_1 = \mathrm{Graph}(m) \circ (\mathrm{Graph}(id_{T^*\mathbb{R}^n}) \times \mathrm{Graph}(m)),$$
$$L_2 = \mathrm{Graph}(m) \circ (\mathrm{Graph}(m) \times \mathrm{Graph}(id_{T^*\mathbb{R}^n})).$$

Here, $L_1, L_2 \in \mathrm{rel}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ and for the various Lagrangian relations in these compositions we have:

$$\mathrm{Graph}(m) \in \mathrm{rel}(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n),$$
$$\mathrm{Graph}(id_{T^*\mathbb{R}}) \in \mathrm{rel}(\mathbb{R}^n, \mathbb{R}^n),$$
$$\mathrm{Graph}(id_{T^*\mathbb{R}}) \times \mathrm{Graph}(m) \in \mathrm{rel}(\mathbb{R}^n \times (\mathbb{R}^n \times \mathbb{R}^n), \mathbb{R}^n \times \mathbb{R}^n),$$
$$\mathrm{Graph}(m) \times \mathrm{Graph}(id_{T^*\mathbb{R}^n}) \in \mathrm{rel}((\mathbb{R}^n \times \mathbb{R}^n) \times \mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^n).$$

We can use the function $\tilde{S}$ given by (3.9) to find generating functions for L_1 and L_2. The associativity of the groupoid multiplication is equivalent to the statement that $L_1 = L_2$, as sets. By the definition of a generating function, this equivalent to some condition on the two generating functions, which we would like to determine.

To find the generating functions for L_1, L_2, we use the following result, which can be found in Section 5.7 of [11]:

Lemma 3.5.3. *Let ϕ_1 and ϕ_2 be generating functions for $L_1 \in \mathrm{rel}(X, Y)$ and $L_2 \in \mathrm{rel}(Y, Z)$ with respect to the trivial fibrations $X \times Y \times F \to X \times Y$ and $Y \times Z \times Q \to Y \times Z$, respectively. If $L = L_2 \circ L_1 \in \mathrm{rel}(X, Z)$, then*

$$\phi(x, z, p, q, y) = \phi_1(x, y, p) + \phi_2(y, z, q),$$

is a generating function for L with respect to the trivial fibration $X \times Z \times Y \times P \times Q \to X \times Z$.

Note also that if ϕ and $\phi^{'}$ are generating functions for $L \in \mathrm{rel}(X, Y)$ and $L^{'} \in \mathrm{rel}(X^{'}, Y^{'})$ with respect to the fibrations $\pi : Z \to X \times Y$ and $\pi^{'} : Z^{'} \to X^{'} \times Y^{'}$, respectively, then the *product* relation $L \times L^{'} \in \mathrm{rel}(X \times X^{'}, Y \times Y^{'})$ has a generating function $(z, z^{'}) \mapsto \phi(z) + \phi^{'}(z^{'})$ with respect to the product fibration $Z \times Z^{'} \to X \times Y \times X^{'} \times Y^{'} = X \times X^{'} \times Y \times Y^{'}$.

Coming back to the problem of finding generating functions for L_1 and L_2, observe that $\tilde{T}(x_3, x_3^{'}, q) = q(-x_3 + x_3^{'})$ is a generating function for $\mathrm{Graph}(id_{T^*\mathbb{R}^n}) \in \mathrm{rel}(\mathbb{R}^n, \mathbb{R}^n)$ with respect to the fibration

$$\begin{gathered} \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{*n} \to \mathbb{R}^n \times \mathbb{R}^n, \\ (x_3, x_3^{'}, p_3) \mapsto (x_3, x_3^{'}). \end{gathered}$$

By the above lemma and remark following it, we find that a generating function for L_2 with respect to the fibration

$$\begin{aligned} \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{*n} \times \mathbb{R}^{*n} \times \mathbb{R}^{*n} \times \mathbb{R}^{*n} \times \mathbb{R}^{*n} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \\ (x_1, x_2, x_3, x, p_1, p_2, p_3, \overline{p}, p_3', x_3', \overline{x}) \mapsto (x_1, x_2, x_3, x), \end{aligned}$$

is given by

$$\begin{aligned} \phi_2(x_1, x_2, x_3, x, p_1, p_2, p_3, \overline{p}, p_3', x_3', \overline{x}) = -p_1 x_1 - p_2 x_2 + S(p_1, p_2, \overline{x}) + p_3(-x_3 + x_3') \\ -\overline{p}\overline{x} - p_3' x_3' + S(\overline{p}, p_3', x). \end{aligned}$$

In this expression, we have denoted by $x_3', \overline{x}$ the fiber variables. Similarly, we find that a generating function for L_1 with respect to the fibration

$$\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{*n} \times \mathbb{R}^{*n} \times \mathbb{R}^{*n} \times \mathbb{R}^{*n} \times \mathbb{R}^{*n} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n,$$
$$(x_1, x_2, x_3, x, p_1, p_2, p_3, p_1', \tilde{p}, x_1', \tilde{x}) \mapsto (x_1, x_2, x_3, x),$$

given by

$$\phi_1(x_1, x_2, x_3, x, p_1, p_2, p_3, \tilde{p}, p_1', x_1', \tilde{x}) = -p_2 x_2 - p_3 x_3 + S(p_2, p_3, \tilde{x}) + p_1(-x_1 + x_1')$$
$$- \tilde{p}\tilde{x} - p_1' x_1' + S(p_1', \tilde{p}, x).$$

Now the fiber variables have been denoted by $\tilde{x}, x_1'$.

We now compute the critical points of ϕ_1 with respect to the given fibration. This is done by differentiating ϕ_1 with respect to each of the fiber variables and then set the result equal to 0. We find that $(x_1, x_2, x_3, x, p_1, p_2, q, x_1')$ is a critical point if and only if

$$\begin{aligned}
x_1' &= x_1,\\
p_1' &= p_1\\
x_1 &= \nabla_1 S(p_1', \tilde{p}.x),\\
x_2 &= \nabla_1 S(p_2, p_3, \tilde{x}),\\
x_3 &= \nabla_2 S(p_2, p_3, \tilde{x}),\\
\tilde{x} &= \nabla_2 S(p_1', \tilde{p}, x),\\
\tilde{p} &= \nabla_x S(p_2, p_3, \tilde{x}).
\end{aligned}$$

If this is true, the corresponding covector on the base manifold is given by

$$p_1(\mathrm{d}x_1)_{x_1} + p_2(\mathrm{d}x_2)_{x_2} + p_3(\mathrm{d}x_3)_{x_3} + \nabla_x S(p_1, \tilde{p}, x)(\mathrm{d}x)_x.$$

Similarly, we find that $(x_1, x_2, x_3, x, p_1, p_2, q, x_3')$ is a critical point for ϕ_2 with respect to the given fibration if and only if

$$\begin{aligned}
x_3' &= x_3, \\
p_1' &= p_1, \\
x_1 &= \nabla_1 S(p_1, p_2.\overline{x}), \\
x_2 &= \nabla_2 S(p_1, p_2, \overline{x}), \\
x_3 &= \nabla_2 S(\overline{p}, p_3, x), \\
\overline{x} &= \nabla_1 S(\overline{p}, x_3, x), \\
\tilde{p} &= \nabla_x S(p_1, p_2, \overline{x}),
\end{aligned}$$

and, if this is true, the corresponding covector on the base is

$$p_1(\mathrm{d}x_1)_{x_1} + p_2(\mathrm{d}x_2)_{x_2} + p_3(\mathrm{d}x_3)_{x_3} + \nabla_x S(\overline{p}, p_3, x)(\mathrm{d}x)_x.$$

Finally, we note that, each quadruple (p_1, p_2, p_3, x) determines a unique critical point of ϕ_i, $i = 1, 2$. Denoting the corresponding covectors by $\alpha^i_{(p_1,p_2,p_3,x)}, i = 1, 2$, we conclude that the groupoid multiplication is associative, i.e., that $L_1 = L_2$, if and only if

$$\alpha^1_{(p_1,p_2,p_3,x)} = \alpha^2_{(p_1,p_2,p_3,x)},$$

for every $(p_1, p_2, p_3, x) \in \mathbb{R}^{*n} \times \mathbb{R}^{*n} \times \mathbb{R}^{*n} \times \mathbb{R}^n$. Now this happens if and only if, for all (p_1, p_2, p_3, x), we have:

$$\begin{aligned}
\nabla_1 S(p_1, p_2, \overline{x}) &= \nabla_1 S(p_1, \tilde{p}, x), \\
\nabla_2 S(p_1, p_2, \overline{x}) &= \nabla_1 S(p_2, p_3, \tilde{x}), \\
\nabla_2 S(\overline{p}, p_3, x) &= \nabla_2 S(p_2, p_3, \tilde{x}), \\
\nabla_x S(\overline{p}, p_3, x) &= \nabla_x S(p_1, \tilde{p}, x),
\end{aligned}$$

where

$$\begin{aligned}
\overline{x} &= \nabla_1 S(\overline{p}, p_3, x), \quad \tilde{x} = \nabla_2 S(p_1, \tilde{p}, x), \\
\overline{p} &= \nabla_x S(p_1, p_2, \overline{x}), \quad \tilde{p} = \nabla_x S(p_2, p_3, x).
\end{aligned}$$

If we assume that $S(0, 0, x) = 0$ for all $x \in M$, these equations are equivalent to the SGA equation.

CHAPTER 4

A GROUPOID COCYLE AND THE VAN EST MAP

At the end of the last chapter, we looked at the general notion of generating functions for canonical relations, from [11]. Also, we established a first connection between them and the generating functions appearing in the SGA equation. Namely, if S is a generating function with respect to the symplectomorphism ϕ_0' given by (3.8), then the function defined by $\tilde{S}((p_1, x_1), (p_2, x_2), x) = -p_1 x_1 - p_2 x_2 + S(p_1, p_2, x)$ is a generating function for $\mathrm{Graph}(m)$ viewed as a canonical relation in $\mathrm{rel}(M \times M, M)$, with respect to the projection $T^*M \times T^*M \times M \to M^3$. If we extend $\tilde{S}$ to a function on $(T^*M)^3$ by composing with the projection in the third factor, it is also a generating function with respect to the projection $(T^*M)^3 \to M^3$. By restricting $\tilde{S}$ to the graph of multiplication, the SGA equation on S ammounts to the statement that $\tilde{S}$ is a groupoid cocyle. In this chapter, we will look closer into this phenomenon.

4.1 A Groupoid Cocycle

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. Recall (see, e.g., [6]) that the Bott-Schulman bicomplex consists of differential forms $\Omega^k(\mathcal{G}^{(n)})$, together with:

- the deRham differential $\mathrm{d} : \Omega^k(\mathcal{G}^{(n)}) \to \Omega^{k+1}(\mathcal{G}^{(n)})$, and
- the simplicial differential $\delta : \Omega^k(\mathcal{G}^{(n)}) \to \Omega^k(\mathcal{G}^{(n+1)})$:

$$\delta\omega = \sum_{i=0}^{n+1} (-1)^i \delta_i^* \omega,$$

  where $\delta_i : \mathcal{G}^{(n+1)} \to \mathcal{G}^{(n)}$ denotes the i-th face map which omits the vertex i from a string of $n+1$-composable arrows $(g_1, \ldots, g_{n+1})$.

In the case of a local groupoid $G \rightrightarrows M$, the definition of a cochain should no longer be forms on $G^{(n)}$. For example, two elements g, h could satisfy $s(g) = t(h)$ and not be composable, so δ would not be well-defined. Moreover, since 3-associativity does not hold on all compatible triples, we could have $\delta^2 \neq 0$. For this reason and for our purposes in this section, we will take

$$\begin{aligned} G^{(1)} &= G, \\ G^{(2)} &= U, \\ G^{(3)} &= W, \end{aligned}$$

where U is the domain of multiplication and W is the space of all triples such that both triple products are defined and are equal. The simplicial differentials are defined as in the above formula:

$$\begin{aligned} &\delta : \Omega^k(G) \to \Omega^k(U), \quad \omega \mapsto \mathrm{pr}_1^* \omega - m^* \omega + \mathrm{pr}_2^* \omega, \\ &\delta : \Omega^k(U) \to \Omega^k(W), \quad \omega \mapsto \mathrm{pr}_{2,3}^* \omega - m_{1,2}^* \omega + m_{2,3}^* \omega - \mathrm{pr}_{1,2}^* \omega, \end{aligned}$$

where $m_{1,2}, m_{2,3} : W \to U$ are given by

$$\begin{aligned} m_{1,2}(\alpha_1, \alpha_2, \alpha_3) &= (m(\alpha_1, \alpha_2), \alpha_3), \\ m_{2,3}(\alpha_1, \alpha_2, \alpha_3) &= (\alpha_1, m(\alpha_{2,} \alpha_3)). \end{aligned}$$

We have a commutative diagram

$$\begin{array}{ccc} \Omega^k(G) & \xrightarrow{\mathrm{d}} & \Omega^{k+1}(G) \\ \delta \big\downarrow & & \big\downarrow \delta \\ \Omega^k(U) & \xrightarrow{\mathrm{d}} & \Omega^{k+1}(U) \\ \delta \big\downarrow & & \big\downarrow \delta \\ \Omega^k(W) & \xrightarrow{\mathrm{d}} & \Omega^{k+1}(W) \end{array} \tag{4.1}$$

where d is the De Rham differential. We will call a function $\sigma \in \Omega^0(U)$ a groupoid 2-cocyle if $\delta\phi = 0$.

Now let $(G, \Omega) \rightrightarrows M$ be a local symplectic groupoid. We will first assume that Ω is exact:

$$\text{There is } \alpha \in \Omega^1(G) \text{ such that } \Omega = \mathrm{d}\alpha. \tag{A1}$$

The commutativity of the upper square in (4.1) for $k = 1$ implies that

$$\mathrm{d}\delta\alpha = \delta\mathrm{d}\alpha = \delta\Omega = 0,$$

since Ω is multiplicative.

We make a second assumption:

$$\text{There is } \phi \in \Omega^0(U) \text{ such that } \delta\alpha = \mathrm{d}\phi. \tag{A2}$$

Then, by the commutativity of the upper square for $k = 0$, we have

$$\mathrm{d}\delta\phi = \delta\mathrm{d}\phi = \delta^2\alpha = 0.$$

Here we used that $\delta^2 = 0$, which follows as in the usual case by applying 3-associativity.

Recall that multiplicativity of Ω implies that $\mathbf{u}^*\Omega = 0$. Since Ω is closed, the Tubular Neighborhood Theorem, implies that there is an open neighborhood V of the identity section of G on which Ω is exact:

$$\mathrm{d}\alpha = \Omega|_V.$$

Therefore, by replacing G by V we see that we can assume that (A1) is satisfied. On the other hand, we will see next that if a generating function is given then (A2) is also satisfied.

4.2 Example: $M = \mathbb{R}^n$ with a generating function S.

Let $M = \mathbb{R}^n$ and consider the symplectomorphism (3.8):

$$\begin{gathered}\phi_0' : \overline{T^*M} \times \overline{T^*M} \times T^*M \to T^*(T^*M \times_M T^*M) \simeq T^*(M^* \times M^* \times M),\\ ((p_1, x_1), (p_2, x_2), (p, x)) \mapsto ((x_1, p_1), (x_2, p_2), (p, x)).\end{gathered}$$

Also, let S be a function on $T^*M \times_M T^*M \simeq M^* \times M^* \times M$:

$$S : T^*M \times_M T^*M \to \mathbb{R}, \quad (p_1, p_2, x) \mapsto S(p_1, p_2, x).$$

As we in the previous chapter, elements in $\phi_0'^{-1}(\mathrm{Graph}(\mathrm{d}S)) \subseteq \overline{T^*M} \times \overline{T^*M} \times T^*M$ take the form

$$((p_1, \nabla_1 S(p_1, p_2, x)), (p_2, \nabla_2 S(p_1, p_2, x)), (\nabla_x S(p_1, p_2, x), x))$$

for $(p_1, p_2, x) \in M^* \times M^* \times M$. Assume that the SGA equation holds on some $W \subseteq (M^*)^3 \times M$, that is, there exist functions

$$\overline{p}, \overline{x}, \tilde{p}, \tilde{x} : W \to \mathbb{R}$$

for which we have, for all $(p_1, p_2, p_3, x) \in W$

$$\begin{aligned} S(p_1, p_2, \overline{x}) + S(\overline{p}, p_3, x) - \overline{px} &= S(p_2, p_3, \tilde{x}) + S(p_1, \tilde{p}, x) - \tilde{p}\tilde{x}, \\ \overline{p} = \nabla_x S(p_1, p_2, \overline{x}), &\;\; \overline{x} = \nabla_1 S(\overline{p}, p_3, x), \\ \tilde{p} = \nabla_x S(p_2, p_3, \tilde{x}), &\;\; \tilde{x} = \nabla_2 S(p_1, \tilde{p}, x). \end{aligned}$$

As in the proof of Theorem 3.3.2, define $\iota : W \to (T^*M)^3$ by

$$(p_1, p_2, p_3, x) \mapsto ((p_1, \nabla_1 S(p_1, p_2, \overline{x})), (p_2, \nabla_2 S(p_1, p_2, \overline{x})), (p_3, \nabla_2 S(p_2, p_3, \tilde{x}))).$$

Assume ι is a diffeomorphism onto its image $\mathcal{W}$ (see the previous chapter for assumptions guaranteeing that this holds). For each $(\alpha_1, \alpha_2, \alpha_3) \in \mathcal{W}$, let

$$\begin{aligned} \overline{\alpha}(\alpha_1, \alpha_2, \alpha_3) &= (\overline{p}, \overline{x}), \\ \tilde{\alpha}(\alpha_1, \alpha_2, \alpha_3) &= (\tilde{p}, \tilde{x}), \\ \alpha(\alpha_1, \alpha_2, \alpha_3) &= (\nabla_x S(\overline{p}, p_3, x), x). \end{aligned}$$

Then the SGA equation implies that these define smooth functions on $\mathcal{W}$ for which

$$\left. \begin{array}{c} (\alpha_1, \alpha_2, \overline{\alpha}) \\ (\alpha_2, \alpha_3, \tilde{\alpha}) \\ (\overline{\alpha}, \alpha_3, \alpha) \\ (\alpha_1, \tilde{\alpha}, \alpha) \end{array} \right\} \in \phi_0'^{-1}(\mathrm{Graph}(\mathrm{d}S)).$$

On the other hand, as we saw in the previous section, we can view $\phi_0'^{-1}(\mathrm{Graph}(\mathrm{d}S))$

as a canonical relation in $\mathrm{rel}(M \times M, M)$. The function

$$\begin{aligned} &\tilde{S} : T^*M \times T^*M \times T^*M \to \mathbb{R}, \\ &((p_1, x_1), (p_2, x_2), (p, x)) \mapsto -p_1 x_1 - p_2 x_2 + S(p_1, p_2, x), \end{aligned}$$

is a generating function for $\phi_0'^{-1}(\mathrm{Graph}(\mathrm{d}S))$, in the sense of Definition 3.5.1, with respect to the fibration

$$\begin{aligned} &\pi : T^*M \times T^*M \times T^*M \to M \times M \times M, \\ &((p_1, x_1), (p_2, x_2), (p, x)) \mapsto (x_1, x_2, x). \end{aligned}$$

If we define $\Delta\tilde{S} : \mathcal{W} \to \mathbb{R}$ by

$$\Delta\tilde{S}(\alpha_1, \alpha_2, \alpha_3) = \tilde{S}(\alpha_1, \alpha_2, \overline{\alpha}) + \tilde{S}(\overline{\alpha}, \alpha_3, \alpha) - \tilde{S}(\alpha_2, \alpha_3, \tilde{\alpha}) - \tilde{S}(\alpha_1, \tilde{\alpha}, \alpha),$$

using the definition of $\tilde{S}$, we see that the integral equation in the SGA equation is equivalent to $\Delta\tilde{S} = 0$. So we have $\Delta\tilde{S} = 0$ on $\mathcal{W}$.

Now assume that S is a generating function for a local symplectic groupoid with respect to ϕ_0'. Let U be the domain of multiplication and define

$$\phi : U \to \mathbb{R}, \quad (\alpha_1, \alpha_2) \mapsto \tilde{S}(\alpha_1, \alpha_2, \overline{\alpha}),$$

where $\overline{\alpha} = m(\alpha_1, \alpha_2)$. Then for any $(\alpha_1, \alpha_2, \alpha_3) \in \mathcal{W}$, we have

$$\begin{aligned} \delta\phi(\alpha_1, \alpha_2, \alpha_3) &= \phi(\alpha_1, \alpha_2) + \phi(m(\alpha_1, \alpha_2), \alpha_3) \\ &\quad - \phi(\alpha_2, \alpha_3) - \phi(\alpha_1, m(\alpha_2, \alpha_3)) \\ &= \Delta\tilde{S}(\alpha_1, \alpha_2, \alpha_3) = 0, \end{aligned}$$

which shows that ϕ is a groupoid cocycle.

Notice that (A1) is satisfied since the canonical symplectic form ω has primitive the Liouville 1-formα. On the other hand, $-\phi$ also satisfies (A2):

if $(v_1, v_2, \overline{v}) \in T_{(\alpha_1,\alpha_2,\overline{\alpha})}\,\mathrm{Graph}(m)$, we have

$$\begin{aligned}
\mathrm{d}\phi(\alpha_1,\alpha_2)(v_1,v_2) &= \mathrm{d}\tilde{S}(\alpha_1,\alpha_2,\overline{\alpha})(v_1,v_2,\overline{v}) \\
&= (-\alpha_1,-\alpha_2,\overline{\alpha})\mathrm{d}\pi(v_1,v_2,\overline{v}) \\
&= -\alpha_1(\mathrm{d}p(v_1)) - \alpha_2(\mathrm{d}p(v_2)) + \overline{\alpha}(\mathrm{d}p(\overline{v})) \\
&= -\delta\alpha(\alpha_1,\alpha_2)(v_1,v_2),
\end{aligned}$$

where $p : T^*M \to M$ is the projection. The second equality holds since each $(\alpha_1, \alpha_2, \overline{\alpha}) \in \mathrm{Graph}(m)$ is a critical point of $\tilde{S}$ and

$$\mathrm{d}\tilde{S}(\alpha_1,\alpha_2,\overline{\alpha}) = \pi^*(-\alpha_1,-\alpha_2,\overline{\alpha}).$$

4.3 Example: Manifold M with a generating function

Let M be any manifold and consider a local symplectic groupoid structure on (T^*M, ω_c) over M. The graph of multiplication $\mathrm{Graph}(m)$ is a canonical relation in $\mathrm{rel}(M \times M, M)$. We assume that this canonical relation admits a generating function $\tilde{S}$ with respect to a fibration

$$\pi : Z \to M \times M \times M.$$

Denote the set of critical points of $\tilde{S}$ by $C_{\tilde{S}}$. For each $z \in C_{\tilde{S}}$, the assumption that $\tilde{S}$ is a generating function for $\mathrm{Graph}(m)$ with respect to π means that

$$\mathrm{d}_z\tilde{S} = \pi^*(-\alpha_1,-\alpha_2,\overline{\alpha}),$$

where $(\alpha_1, \alpha_2, \overline{\alpha})$ is an element in $\mathrm{Graph}(m)$. In this notation, we obtain a function $\phi : U \to \mathbb{R}$, defined on the domain of multiplication and such that for all $z \in C_{\tilde{S}}$:

$$\tilde{S}(z) = \phi(\alpha_1,\alpha_2)$$

Let $W \subseteq T^*M_s \times_t T^*M_s \times_t T^*M$ be an open neighborhood of $\mathcal{I}_3$ on which associativity holds. Then $\delta\phi : W \to \mathbb{R}$ is given by

$$\delta\phi(\alpha_1,\alpha_2,\alpha_3) = \phi(\alpha_1,\alpha_2) + \phi(\overline{\alpha},\alpha_3) - \phi(\alpha_2,\alpha_3) - \phi(\alpha_1,\tilde{\alpha}),$$

where

$$\overline{\alpha} = m(\alpha_1, \alpha_2), \quad \tilde{\alpha} = m(\alpha_2, \alpha_3), \quad \alpha = m(\alpha_1, \alpha_2, \alpha_3).$$

If $(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3) \in T_{(\alpha_1,\alpha_2,\alpha_3)}W$ and $v_i = \mathrm{d}p(\tilde{v}_i)$, we find:

$$\begin{aligned}(\mathrm{d}\delta\phi)_{(\alpha_1,\alpha_2,\alpha_3)}(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3) =& (\mathrm{d}\phi)_{(\alpha_1,\alpha_2)}(\tilde{v}_1, \tilde{v}_2) + (\mathrm{d}\phi)_{(\overline{\alpha},\alpha_3)}(\mathrm{d}m(\tilde{v}_1, \tilde{v}_2), \tilde{v}_3) \\ & - (\mathrm{d}\phi)_{(\alpha_2,\alpha_3)}(\tilde{v}_2, \tilde{v}_3) - (\mathrm{d}\phi)_{(\alpha_1,\tilde{\alpha})}(\tilde{v}_1, \mathrm{d}m(\tilde{v}_2, \tilde{v}_3)).\end{aligned}$$

To compute the above expression, let $(\tilde{v}_1, \tilde{v}_2, \mathrm{d}m(\tilde{v}_1, \tilde{v}_2)) \in T_{(\alpha_1,\alpha_2,\overline{\alpha})}\,\mathrm{Graph}(m)$, and let $z \in C_{\tilde{S}}$, $v \in T_z C_{\tilde{S}}$, be such that

$$\begin{cases} \mathrm{d}_z\tilde{S} = \pi^*(-\alpha_1, -\alpha_2, \overline{\alpha}), \\ \mathrm{d}_z l(v) = (\tilde{v}_1, \tilde{v}_2, \mathrm{d}m(\tilde{v}_1, \tilde{v}_2)). \end{cases} \tag{4.2}$$

where $l : C_{\tilde{S}} \to \mathrm{Graph}(m)$ is given by $z \mapsto (\alpha_1, \alpha_2, \overline{\alpha})$. We have

$$\begin{aligned}(\mathrm{d}\phi)_{(\alpha_1,\alpha_2)}(\tilde{v}_1, \tilde{v}_2) &= \mathrm{d}_x\tilde{S}(v) \\ &= \langle \pi^*(-\alpha_1, -\alpha_2, \overline{\alpha}), v\rangle \\ &= \langle (-\alpha_1, \alpha_2, \overline{\alpha}), \mathrm{d}\pi(v)\rangle \\ &= \langle (-\alpha_1, \alpha_2, \overline{\alpha}), \mathrm{d}\pi(\mathrm{d}l(v))\rangle \\ &= \langle (-\alpha_1, \alpha_2, \overline{\alpha}), \mathrm{d}\pi(\tilde{v}_1, \tilde{v}_2, \mathrm{d}m(\tilde{v}_1, \tilde{v}_2))\rangle \\ &= -\alpha_1(\mathrm{d}p(\tilde{v}_1)) - \alpha_2(\mathrm{d}p(\tilde{v}_2)) + \overline{\alpha}(\mathrm{d}p(\mathrm{d}m(\tilde{v}_1, \tilde{v}_2))) \\ &= -\alpha_1(v_1) - \alpha_2(v_2) + \overline{\alpha}(\mathrm{d}p(\mathrm{d}m(\tilde{v}_1, \tilde{v}_2)))\end{aligned}$$

This implies that

$$\begin{aligned}\mathrm{d}\delta\phi(\alpha_1, \alpha_2, \alpha_3)(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3) =& -\alpha_1(v_1) - \alpha_2(v_2) + \overline{\alpha}(\mathrm{d}p(\mathrm{d}m(\tilde{v}_1, \tilde{v}_2))) \\ & - \overline{\alpha}(\mathrm{d}p(\mathrm{d}m(\tilde{v}_1, \tilde{v}_2))) - \alpha_3(v_3) + \alpha(\mathrm{d}p(\mathrm{d}m(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3))) \\ & - (-\alpha_2(v_2) - \alpha_3(v_3) + \overline{\alpha}(\mathrm{d}p(\mathrm{d}m(\tilde{v}_2, \tilde{v}_3))) \\ & - \alpha_3(v_3) - \tilde{\alpha}(\mathrm{d}p(\mathrm{d}m(\tilde{v}_2, \tilde{v}_3))) + \alpha(\mathrm{d}p(\mathrm{d}m(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)))) \\ =& \ 0\end{aligned}$$

If we assume that $\phi(1_x, 1_x) = 0$ for some $x \in M$ and W is connected, we conclude that $\delta\phi = 0$, i.e., ϕ is a groupoid cocycle.

Note also that if α is the Liouville 1-form on T^*M, the above computation shows that

$$\begin{aligned}(\mathrm{d}\phi)_{(\alpha_1,\alpha_2)}(\tilde{v}_1,\tilde{v}_2) &= -\alpha_1(v_1)-\alpha_2(v_2)+\overline{\alpha}(\mathrm{d}p(\mathrm{d}m(\tilde{v}_1,\tilde{v}_2)))\\ &= (-\operatorname{pr}_1^*\alpha-\operatorname{pr}_2^*\alpha+\operatorname{pr}_3^*\alpha)_{(\alpha_1,\alpha_2,\overline{\alpha})}(\tilde{v}_1,\tilde{v}_2,\mathrm{d}m(\tilde{v}_1,\tilde{v}_2))\\ &= -((\operatorname{pr}_1^*-m^*+\operatorname{pr}_2^*)\alpha)_{(\alpha_1,\alpha_2)}(\tilde{v}_1,\tilde{v}_2)\\ &= -(\delta\alpha)_{(\alpha_1,\alpha_2)}(\tilde{v}_1,\tilde{v}_2).\end{aligned}$$

Again, we conclude that (A1) and (A2) are satisfied:

$$\omega=\mathrm{d}\alpha,\quad \mathrm{d}\phi=-\delta\alpha,$$

where α is the Liouville 1-form.

4.4 From groupoid cocycles to Poisson Structures

The following result gives a relationship between the groupoid cocycle associated with a generating function of a local symplectic groupoid and the underlying Poisson structure.

Theorem 4.4.1. *Let (G,Ω) be a local symplectic groupoid with domain of multiplication U, integrating a Poisson manifold (M,π). Assume there exists $\alpha\in\Omega^1(G)$ and $\phi\in\Omega^0(U,\mathbb{R})$ such that*

$$\mathrm{d}\alpha=\Omega,\quad \mathrm{d}\phi=\delta\alpha,\quad \phi|_M=0.$$

Then under the van Est map the class of ϕ is mapped to the class of π:

$$\mathrm{VE}:H^2(U,\mathbb{R})\to H^2(A,\mathbb{R}),\quad [\phi]\mapsto[\pi].$$

Before we prove the theorem, let us recall the definition of the van Est map (see, e.g., [6, 13]). Given a Lie groupoid $\mathcal{G}\rightrightarrows M$ with Lie algebroid A, we denote by $C^n(A,\mathbb{R})$ the space of "A-forms", i.e., all C^∞-multilinear antisymmetric map

$$\omega:(X_1,\cdots,X_n)\mapsto\omega(X_1,\cdots,X_n)\in C^\infty(M),$$

where X_i are sections of A. We also denote by $C^n(\mathcal{G},\mathbb{R})$ the space of groupoid

n-cochains. Then the van Est map, at the level of cochains, is given by

$$\text{VE} : C^n(\mathcal{G}, \mathbb{R}) \to C^n(A, \mathbb{R}),$$
$$\text{VE}(\sigma)(X_1, \cdots, X_n)(x) = \sum_{\tau \in S_n} sgn(\tau)(X_{\tau(1)} \cdots X_{\tau(n)}\sigma)(x),$$

where X_i are sections of A and $x \in M$. The expression $X_{\tau(1)} \cdots X_{\tau(n)}\sigma$ is defined as follows. For a fixed $(g_1, \cdots, g_{n-1}) \in \mathcal{G}^{(n-1)}$. We can view $\sigma(g_1, \cdots, g_{n-1}, -)$ as a function defined on $\mathbf{t}^{-1}(\mathbf{s}(g_{n-1}))$. We differentiate it at $1_{\mathbf{s}(g_{n-1})}$ with respect to $X_{\tau(n)}$ to get a number. This defines $X_{\tau(n)}\sigma$ as a function on $\mathcal{G}^{(n-1)}$. More explicitly, we have

$$X_{\tau(n)}\sigma(g_1, \cdots, g_{n-1}) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \sigma(g_1, \cdots, g_{n-1}, \gamma(t)),$$

where γ is a path in $\mathbf{t}^{-1}(\mathbf{s}(g_{n-1}))$ such that $\gamma(0) = 1_{\mathbf{s}(g_{n-1})}$ and $\dot{\gamma}(0) = X_{\tau(n)}(\mathbf{s}(g_{n-1}))$. We continue this process for $X_{\tau(1)}, \cdots, X_{\tau(n-1)}$ and in the end we are left with a function on $\mathcal{G}^0 = M$.

We are interested in the van Est map at degree 2 for a local symplectic groupoid. There are two things we need to be careful with. The first one concernes the definition of groupoid cochains for local groupoids, which was already discussed at the beginning of this chapter. The second one is the fact that, since A is isomorphic to the cotangent algebroid of M with the induced Poisson structure, we can identify $C^n(A, \mathbb{R})$ with the space of n-multivector fields on M. Hence, in what follows, we will focus on the van Est map in degree 2 and we will view it as a map:

$$\text{VE} : \Omega^0(U, \mathbb{R}) \to \mathfrak{X}^2(M).$$

Proof of Theorem 4.4.1. Under the assumptions of the theorem, notice that ϕ is a groupoid 2-cocycle. As noted above, the Lie algebroid of G is isomorphic to the cotangent algebroid T^*M of the induced Poisson structure on M via the map

$$\sigma_\Omega : A \to T^*M, \quad \alpha \mapsto \Omega^\flat(\alpha)|_M,$$

so the image of an arbitrary $\phi \in \Omega^0(U, \mathbb{R})$ under the van Est map is a bivector on M. We will write down the map more explicitly below.

Let X_1, X_2 be sections of the Lie algebroid of G. The definition of the van

Est map and the identification above mean that we have

$$\mathrm{VE}(\phi)(\Omega^\flat(X_1)|_M, \Omega^\flat(X_2)|_X) = X_1X_2\phi - X_2X_1\phi,$$

where the expressions $X_1X_2\phi$, $X_2X_1\phi$ are computed as follows. For $x \in M$ and $g \in G$

$$X_2\phi(g) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\phi(g, \varphi^s_{X_2}(\mathbf{s}(g))),$$
$$X_1X_2\phi(x) = X_1(X_2\phi)(x) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\phi(\gamma_1(t), \varphi^s_{X_2}(\mathbf{s}(\gamma_1(t)))),$$

where γ_1 is the integral curve of X_1 starting at x and $\varphi^s_{X_2}$ is the flow of X_2. We have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\phi(\gamma_1(t), \varphi^s_{X_2}(\mathbf{s}(\gamma_1(t)))) &= \mathrm{d}\phi(0_{\gamma_1(t)}, X_2(\mathbf{s}(\gamma_1(t)))) \\
&=\delta\alpha(0_{\gamma_1(t)}, X_2(\mathbf{s}(\gamma_1(t)))) \\
&=(-\operatorname{pr}_1^*\alpha - \operatorname{pr}_2^*\alpha + m^*\alpha)(0_{\gamma_1(t)}, X_2(\mathbf{s}(\gamma_1(t)))) \\
&=(-\operatorname{pr}_1^*\alpha - \operatorname{pr}_2^*\alpha)(0_{\gamma_1(t)}, X_2(\mathbf{s}(\gamma_1(t)))) + \alpha(\mathrm{d}L_{\gamma_1(t)}(X_2(\mathbf{s}(\gamma_1(t))))) \\
&= - \alpha(X_2(\mathbf{s}(\gamma_1(t)))) + \alpha(\mathrm{d}L_{\gamma_1(t)}(X_2(\mathbf{s}(\gamma_1(t))))) \\
&= - \alpha(\tilde{X}_2)((\mathbf{s}(\gamma_1(t)))) + \alpha(\tilde{X}_2)(\gamma_1(t)),
\end{aligned}$$

where $\tilde{X}_2$ is the left invariant vector field corresponding to X_2. Therefore, we have

$$\begin{aligned}
X_1X_2\phi(x) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(-\alpha(\tilde{X}_2)((\mathbf{s}(\gamma_1(t)))) + \alpha(\tilde{X}_2)(\gamma_1(t))) \\
&= -\mathrm{d}\mathbf{s}(X_1(x))(\alpha(\tilde{X}_2)) + X_1(x)(\alpha(\tilde{X}_2)),
\end{aligned}$$

and it follows that

$$\begin{aligned}
X_1X_2\phi - X_2X_1\phi = -\mathrm{d}\mathbf{s}(X_1(x))(\alpha(\tilde{X}_2)) + X_1(x)(\alpha(\tilde{X}_2)) \\
+ \mathrm{d}\mathbf{s}(X_2(x))(\alpha(\tilde{X}_1)) - X_2(x)(\alpha(\tilde{X}_1))).
\end{aligned}$$

On the other hand, we also find:

$$\begin{aligned}\pi(\Omega^\flat(X_1)|_M, \Omega^\flat(X_2)|_M) =& \Omega^\flat(X_1)|_M(\pi^\#(\Omega^\flat(X_2)|_M))\\ =& \Omega^\flat(X_1)|_M(\mathrm{d}\mathbf{s}(X_2))\\ =& \Omega(X_1, \mathrm{d}\mathbf{s}(X_2)) = \Omega(X_1, X_2),\end{aligned}$$

where the last equality follows from the fact that $X_2 - \mathrm{d}\mathbf{s}(X_2)$ is tangent to the source fiber, so we have by the multiplicativity of Ω:

$$\Omega(X_1, X_2 - \mathrm{d}\mathbf{s}(X_2)) = 0.$$

Using now the assumption $\Omega = \mathrm{d}\alpha$, we conclude that

$$\begin{aligned}(\mathrm{VE}(\phi) - \pi)(\Omega^\flat(X_1)|_M, &\Omega^\flat(X_2)|_M) = X_1X_2\phi - X_2X_1\phi - \mathrm{d}\alpha(X_1, X_2)\\ &= -\mathrm{d}\mathbf{s}(X_1)(\alpha(\tilde{X}_2)) + \mathrm{d}\mathbf{s}(X_2)(\alpha(\tilde{X}_1)) + \alpha([\tilde{X}_1, \tilde{X}_2])\\ &= -\mathrm{d}\mathbf{s}(X_1)(\alpha(X_2)) + \mathrm{d}\mathbf{s}(X_2)(\alpha(X_1)) + \alpha([X_1, X_2]).\end{aligned}$$

where $\tilde{X}_1, \tilde{X}_2$ are the left invariant vector fields on G corresponding to X_1, X_2. Here, the term $\alpha([X_1, X_2])$ is viewed as a function on M. We claim that the right-hand side coincides with:

$$\mathrm{d}_\pi X(\Omega^\flat(X_1)|_M, \Omega^\flat(X_2)|_M),$$

where X is the vector field corresponding to $-\alpha$ under the isomorphism $\sigma_\Omega : A \to T^*M$:

$$X = -(\sigma_\Omega^*)^{-1}(\alpha).$$

In other words, that we have:

$$\mathrm{VE}(\phi) - \pi = \mathrm{d}_\pi X,$$

and this clearly implies the statement of the theorem.

To prove the claim, notice that from the definition of the Poisson differential one has:

$$
\begin{aligned}
&\mathrm{d}_\pi X(\Omega^\flat(X_1)|_M, \Omega^\flat(X_2)|_M) = \\
&\qquad =\pi^\#(\Omega^\flat(X_1)|_M)(X(\Omega^\flat(X_2)|_M)) - \pi^\#(\Omega^\flat(X_2)|_M)(X(\Omega^\flat(X_1)|_M)) \\
&\qquad\quad - \mathrm{d}(X([\Omega^\flat(X_1)|_M, \Omega^\flat(X_2)|_M)) \\
&\qquad =\mathrm{d}\mathbf{s}(X_1)(X(\Omega^\flat(X_2)|_M) - \mathrm{d}\mathbf{s}(X_2)(X(\Omega^\flat(X_1)|_M) \\
&\qquad\quad - \mathrm{d}(X([\Omega^\flat(X_1)|_M, \Omega^\flat(X_2)|_M)]).
\end{aligned}
$$

By the definition of $\sigma_\Omega : A \to T^*M$, we have

$$
X(\Omega^\flat(Y)|_M) = -\alpha(Y), \tag{4.3}
$$

for every section Y of the Lie algebroid of G. Therefore

$$
\begin{aligned}
\mathrm{d}_\pi X(\Omega^\flat(X_1)|_M, \Omega^\flat(X_2)|_M) &= -\,\mathrm{d}\mathbf{s}(X_1)(\alpha(X_2)) + \mathrm{d}\mathbf{s}(X_2)(\alpha(X_1)) \\
&\qquad - \mathrm{d}X(\Omega^\flat([X_1, X_2])|_M) \\
&= -\,\mathrm{d}\mathbf{s}(X_1)(\alpha(X_2)) + \mathrm{d}\mathbf{s}(X_2)(\alpha(X_1)) \\
&\qquad - \mathrm{d}\alpha([X_1, X_2]).
\end{aligned}
$$

This proves the claim. □

APPENDIX A

DERIVATION OF SGA EQUATION FOR THE EUCLIDEAN EXPONENTIAL MAP

In this appendix, we derive the SGA equation with respect to the Euclidean exponential map:

$$\phi : ((p_1, x_1), (p_2, x_2), (p, x)) \mapsto$$
$$((p_1 - \tfrac{1}{3}(p_1 + p_2 - p), x - x_1),(p_2 - \tfrac{1}{3}(p_1 + p_2 - p), x - x_2), (\tfrac{1}{3}(x_1 + x_2 + x), p_1 + p_2 - p)).$$

As in section 4.2, if S is a generating function for a local symplectic groupoid structure on $T^*\mathbb{R}^n$, associativity implies that

$$\begin{aligned}
&((p_1 - \tfrac{1}{3}(p_1 + p_2 - \overline{p}), \overline{x} - x_1), (p_2 - \tfrac{1}{3}(p_1 + p_2 - \overline{p}), \overline{x} - x_2), (\tfrac{1}{3}(x_1 + x_2 + \overline{x}), p_1 + p_2 - \overline{p})))\\
&=((p_1 - \tfrac{1}{3}(p_1 + p_2 - \overline{p}), \nabla_1 S(p_1 - \tfrac{1}{3}(p_1 + p_2 - \overline{p}), p_2 - \tfrac{1}{3}(p_1 + p_2 - \overline{p}), \tfrac{1}{3}(x_1 + x_2 + \overline{x}))),\\
&\quad (p_2 - \tfrac{1}{3}(p_1 + p_2 - \overline{p}), \nabla_2 S(p_1 - \tfrac{1}{3}(p_1 + p_2 - \overline{p}), p_2 - \tfrac{1}{3}(p_1 + p_{p2} - \overline{p}), \tfrac{1}{3}(x_1 + x_2 + \overline{x}))),\\
&\quad (\tfrac{1}{3}(x_1 + x_2 + \overline{x}), \nabla_x S(p_1 - \tfrac{1}{3}(p_1 + p_2 - \overline{p}), p_2 - \tfrac{1}{3}(p_1 + p_2 - \overline{p}), \tfrac{1}{3}(x_1 + x_2 + \overline{x})))),
\end{aligned}$$

$$\begin{aligned}
&((p_2 - \tfrac{1}{3}(p_2 + p_3 - \tilde{p}), \tilde{x} - x_2), (p_3 - \tfrac{1}{3}(p_2 + p_3 - \tilde{p}), \tilde{x} - x_3), (\tfrac{1}{3}(x_2 + x_3 + \tilde{x}), p_2 + p_3 - \tilde{p})))\\
&=((p_2 - \tfrac{1}{3}(p_2 + p_3 - \tilde{p}), \nabla_1 S(p_2 - \tfrac{1}{3}(p_2 + p_3 - \tilde{p}), p_3 - \tfrac{1}{3}(p_2 + p_3 - \tilde{p}), \tfrac{1}{3}(x_2 + x_3 + \tilde{x}))),\\
&\quad (p_3 - \tfrac{1}{3}(p_2 + p_3 - \tilde{p}), \nabla_2 S(p_2 - \tfrac{1}{3}(p_2 + p_3 - \tilde{p}), p_3 - \tfrac{1}{3}(p_2 + p_3 - \tilde{p}), \tfrac{1}{3}(x_2 + x_3 + \tilde{x}))),\\
&\quad (\tfrac{1}{3}(x_2 + x_3 + \tilde{x}), \nabla_x S(p_2 - \tfrac{1}{3}(p_2 + p_3 - \tilde{p}), p_3 - \tfrac{1}{3}(p_2 + p_3 - \tilde{p}), \tfrac{1}{3}(x_2 + x_3 + \tilde{x})))),
\end{aligned}$$

$$\begin{aligned}
&((p_1 - \tfrac{1}{3}(p_1 + \tilde{p} - p), x - x_1), (\tilde{p} - \tfrac{1}{3}(p_1 + \tilde{p} - p), x - \tilde{x}), (\tfrac{1}{3}(x_1 + \tilde{x} + x), p_1 + \tilde{p} - p)))\\
&=((p_1 - \tfrac{1}{3}(p_1 + \tilde{p} - p), \nabla_1 S(p_1 - \tfrac{1}{3}(p_1 + \tilde{p} - p), \tilde{p} - \tfrac{1}{3}(p_1 + \tilde{p} - p), \tfrac{1}{3}(x_1 + \tilde{x} + x))),\\
&\quad (\tilde{p} - \tfrac{1}{3}(p_1 + \tilde{p} - p), \nabla_2 S(p_1 - \tfrac{1}{3}(p_1 + \tilde{p} - p), \tilde{p} - \tfrac{1}{3}(p_1 + \tilde{p} - p), \tfrac{1}{3}(x_1 + \tilde{x} + x))),\\
&\quad (\tfrac{1}{3}(x_2 + x_3 + \tilde{x}), \nabla_x S(p_1 - \tfrac{1}{3}(p_1 + \tilde{p} - p), \tilde{p} - \tfrac{1}{3}(p_1 + \tilde{p} - p), \tfrac{1}{3}(x_1 + \tilde{x} + x)))),
\end{aligned}$$

$$((\overline{p}-\tfrac{1}{3}(\overline{p}+p_3-p),x-\overline{x}),(p_3-\tfrac{1}{3}(\overline{p}+p_3-p),x-x_3),(\tfrac{1}{3}(\overline{x}+x_3+x),\overline{p}+p_3-p)))$$
$$=((\overline{p}-\tfrac{1}{3}(\overline{p}+p_3-p),\nabla_1 S(\overline{p}-\tfrac{1}{3}(\overline{p}+p_3-p),p_3-\tfrac{1}{3}(\overline{p}+p_3-p),\tfrac{1}{3}(\overline{x}+x_3+x))),$$
$$(p_3-\tfrac{1}{3}(\overline{p}+p_3-p),\nabla_2 S(\overline{p}-\tfrac{1}{3}(\overline{p}+p_3-p),p_3-\tfrac{1}{3}(\overline{p}+p_3-p),\tfrac{1}{3}(\overline{x}+x_3+x))),$$
$$(\tfrac{1}{3}(\overline{x}+x_3+x),\nabla_x S(\overline{p}-\tfrac{1}{3}(\overline{p}+p_3-p),p_3-\tfrac{1}{3}(\overline{p}+p_3-p),\tfrac{1}{3}(\overline{x}+x_3+x)))).$$

These form the same as the following system of equations:

$$\begin{aligned}
\overline{x}-x_1 &= \nabla_1 S(p_1-\tfrac{1}{3}(p_1+p_2-\overline{p}),p_2-\tfrac{1}{3}(p_1+p_2-\overline{p}),\tfrac{1}{3}(x_1+x_2+\overline{x})),\\
\overline{x}-x_2 &= \nabla_2 S(p_1-\tfrac{1}{3}(p_1+p_2-\overline{p}),p_2-\tfrac{1}{3}(p_1+p_2-\overline{p}),\tfrac{1}{3}(x_1+x_2+\overline{x})),\\
p_1+p_2-\overline{p} &= \nabla_x S(p_1-\tfrac{1}{3}(p_1+p_2-\overline{p}),p_2-\tfrac{1}{3}(p_1+p_2-\overline{p}),\tfrac{1}{3}(x_1+x_2+\overline{x})),\\
\tilde{x}-x_2 &= \nabla_1 S(p_2-\tfrac{1}{3}(p_2+p_3-\tilde{p}),p_3-\tfrac{1}{3}(p_2+p_3-\tilde{p}),\tfrac{1}{3}(x_2+x_3+\tilde{x})),\\
\tilde{x}-x_3 &= \nabla_2 S(p_2-\tfrac{1}{3}(p_2+p_3-\tilde{p}),p_3-\tfrac{1}{3}(p_2+p_3-\tilde{p}),\tfrac{1}{3}(x_2+x_3+\tilde{x})),\\
p_2+p_3-\tilde{p} &= \nabla_x S(p_2-\tfrac{1}{3}(p_2+p_3-\tilde{p}),p_3-\tfrac{1}{3}(p_2+p_3-\tilde{p}),\tfrac{1}{3}(x_2+x_3+\tilde{x})),\\
x-x_1 &= \nabla_1 S(p_1-\tfrac{1}{3}(p_1+\tilde{p}-p),\tilde{p}-\tfrac{1}{3}(p_1+\tilde{p}-p),\tfrac{1}{3}(x_1+\tilde{x}+x)),\\
x-\tilde{x} &= \nabla_2 S(p_1-\tfrac{1}{3}(p_1+\tilde{p}-p),\tilde{p}-\tfrac{1}{3}(p_1+\tilde{p}-p),\tfrac{1}{3}(x_1+\tilde{x}+x)),\\
p_1+\tilde{p}-p &= \nabla_x S(p_1-\tfrac{1}{3}(p_1+\tilde{p}-p),\tilde{p}-\tfrac{1}{3}(p_1+\tilde{p}-p),\tfrac{1}{3}(x_1+\tilde{x}+x)),\\
x-\overline{x} &= \nabla_1 S(\overline{p}-\tfrac{1}{3}(\overline{p}+p_3-p),p_3-\tfrac{1}{3}(\overline{p}+p_3-p),\tfrac{1}{3}(\overline{x}+x_3+x)),\\
x-x_3 &= \nabla_2 S(\overline{p}-\tfrac{1}{3}(\overline{p}+p_3-p),p_3-\tfrac{1}{3}(\overline{p}+p_3-p),\tfrac{1}{3}(\overline{x}+x_3+x)),\\
\overline{p}+p_3-p &= \nabla_x S(\overline{p}-\tfrac{1}{3}(\overline{p}+p_3-p),p_3-\tfrac{1}{3}(\overline{p}+p_3-p),\tfrac{1}{3}(\overline{x}+x_3+x)).
\end{aligned}$$

Consider the function:

$$\begin{aligned}
\Delta S(p_1,p_2,p_3,x) =& S(p_1-\tfrac{1}{3}(p_1+p_2-\overline{p}),p_2-\tfrac{1}{3}(p_1+p_2-\overline{p}),\tfrac{1}{3}(x_1+x_2+\overline{x}))\\
&+ S(p_2-\tfrac{1}{3}(p_2+p_3-\tilde{p}),p_3-\tfrac{1}{3}(p_2+p_3-\tilde{p}),\tfrac{1}{3}(x_2+x_3+\tilde{x}))\\
&- S(p_1-\tfrac{1}{3}(p_1+\tilde{p}-p),\tilde{p}-\tfrac{1}{3}(p_1+\tilde{p}-p),\tfrac{1}{3}(x_1+\tilde{x}+x))\\
&- S(\overline{p}-\tfrac{1}{3}(\overline{p}+p_3-p),p_3-\tfrac{1}{3}(\overline{p}+p_3-p),\tfrac{1}{3}(\overline{x}+x_3+x)),
\end{aligned}$$

and define also:

$$\begin{aligned}
F(p_1,p_2,p_3,x) = \tfrac{1}{3}(&(p_2+p-\overline{p}-\tilde{p})x_1+(p_1-p_3-\overline{p}+\tilde{p})x_2\\
&+(-p_2-p+\overline{p}+\tilde{p})x_3+(p_1+p_2+p_3-p)\overline{x}\\
&+(-p_1-p_2-p_3+p)\tilde{x}+(-p_1+\overline{p}-\tilde{p}+p_3)x).
\end{aligned}$$

We have that
$$\frac{\partial \Delta S}{\partial p_i} = \frac{\partial F}{\partial p_i}\ (1 \le i \le 3), \quad \frac{\partial \Delta S}{\partial x} = \frac{\partial F}{\partial x}.$$
By our assumption that $S(0,0,x) = 0$ for all $x \in M$, we have $\Delta S = F$. So we obtain the desired equation:

$$\begin{aligned} &S_{12-} + S_{23} \ - \tfrac{1}{3}((p_2 + p - \overline{p} - \tilde{p})x_1 + (p_1 - p_3 - \overline{p} + \tilde{p})x_2 + (p_1 + p_2 + p_3 - p)\overline{x}) \\ =&S_{1\sim} + S_{-3} + \tfrac{1}{3}((-p_2 - p + \overline{p} + \tilde{p})x_3 + (-p_1 - p_2 - p_3 + p)\tilde{x} + (-p_1 + \overline{p} - \tilde{p} + p_3)x), \end{aligned}$$

where

$$\begin{aligned} S_{12-} &= S(p_1 - \tfrac{1}{3}(p_1 + p_2 - \overline{p}), p_2 - \tfrac{1}{3}(p_1 + p_2 - \overline{p}), \tfrac{1}{3}(x_1 + x_2 + \overline{x})), \\ S_{23} &= S(p_2 - \tfrac{1}{3}(p_2 + p_3 - \tilde{p}), p_3 - \tfrac{1}{3}(p_2 + p_3 - \tilde{p}), \tfrac{1}{3}(x_2 + x_3 + \tilde{x})), \\ S_{1\sim} &= S(p_1 - \tfrac{1}{3}(p_1 + \tilde{p} - p), \tilde{p} - \tfrac{1}{3}(p_1 + \tilde{p} - p), \tfrac{1}{3}(x_1 + \tilde{x} + x)), \\ S_{-3} &= S(\overline{p} - \tfrac{1}{3}(\overline{p} + p_3 - p), p_3 - \tfrac{1}{3}(\overline{p} + p_3 - p), \tfrac{1}{3}(\overline{x} + x_3 + x)), \end{aligned}$$

and

$$\begin{aligned} p_1 + p_2 - \overline{p} &= \nabla_x S(p_1 - \tfrac{1}{3}(p_1 + p_2 - \overline{p}), p_2 - \tfrac{1}{3}(p_1 + p_2 - \overline{p}), \tfrac{1}{3}(x_1 + x_2 + \overline{x})) \\ p_2 + p_3 - \tilde{p} &= \nabla_x S(p_2 - \tfrac{1}{3}(p_2 + p_3 - \tilde{p}), p_3 - \tfrac{1}{3}(p_2 + p_3 - \tilde{p}), \tfrac{1}{3}(x_2 + x_3 + \tilde{x})), \\ p_1 + \tilde{p} - p &= \nabla_x S(p_1 - \tfrac{1}{3}(p_1 + \tilde{p} - p), \tilde{p} - \tfrac{1}{3}(p_1 + \tilde{p} - p), \tfrac{1}{3}(x_1 + \tilde{x} + x)), \\ \overline{p} + p_3 - p &= \nabla_x S(\overline{p} - \tfrac{1}{3}(\overline{p} + p_3 - p), p_3 - \tfrac{1}{3}(\overline{p} + p_3 - p), \tfrac{1}{3}(\overline{x} + x_3 + x)), \\ \overline{x} - x_1 &= \nabla_1 S(p_1 - \tfrac{1}{3}(p_1 + p_2 - \overline{p}), p_2 - \tfrac{1}{3}(p_1 + p_2 - \overline{p}), \tfrac{1}{3}(x_1 + x_2 + \overline{x})), \\ \tilde{x} - x_2 &= \nabla_1 S(p_2 - \tfrac{1}{3}(p_2 + p_3 - \tilde{p}), p_3 - \tfrac{1}{3}(p_2 + p_3 - \tilde{p}), \tfrac{1}{3}(x_2 + x_3 + \tilde{x})), \\ x - x_1 &= \nabla_1 S(p_1 - \tfrac{1}{3}(p_1 + \tilde{p} - p), \tilde{p} - \tfrac{1}{3}(p_1 + \tilde{p} - p), \tfrac{1}{3}(x_1 + \tilde{x} + x)), \\ x - \overline{x} &= \nabla_1 S(\overline{p} - \tfrac{1}{3}(\overline{p} + p_3 - p), p_3 - \tfrac{1}{3}(\overline{p} + p_3 - p), \tfrac{1}{3}(\overline{x} + x_3 + x)). \end{aligned}$$

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