VAN EST'S PROOF OF LIE'S THIRD THEOREM

BY

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THESIS

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1 Introduction

Lie theory originates from the pioneer work of Sophus Lie. Underlying the theory there are three basic results known as Lie's theorems. These theorems indicate that there is a one-to-one correspondence between isomorphism classes of finite-dimensional Lie algebras and isomorphism classes of simply connected Lie groups. Formal statements and a simple proofs of the first two theorems can be found in Warner ([1]). Lie's third theorem, stating that every finite-dimensional Lie algebra integrates to a Lie group, is a much deeper result. There are various proofs of this theorem. An algebraic proof can be given using the result of Ado ([2]), which states that every finite-dimensional Lie algebra has a faithful representation in $\mathfrak{gl}(n, \mathbb{R})$ for some n. A geometric proof was given by Van Est ([3],[4]) in 1962. Another geometric proof can be found in Duistermaat and Kolk ([5]).

This paper will focus on and interpret the proof given by Van Est. We first show that the second Lie algebra cohomology classifies the abelian extension of Lie algebras. Then we introduce the Van Est theorem, which relates Lie group cohomology and Lie algebra cohomology. Finally, we identify any given finite-dimensional Lie algebra with the semi-direct product of its center and its adjoint Lie algebra and use the previous results to prove Lie's third theorem.

2 Lie Algebra Cohomology and Lie Group Cohomology

In this section, we will discuss Lie algebra cohomology and Lie group cohomology. In §2.1, we give the definition of Lie algebra cohomology and Lie group cohomology ([6]). We have a closer look at the second cohomology group of Lie algebras in §2.2 and show how it classifies the abelian extension of Lie algebras in §2.3. Finally we look at a similar assertion for Lie groups in §2.4.

2.1 Lie Algebra Cohomology And Lie Group Cohomology

Given a Lie algebra $\mathfrak{h}$ with a representation $\rho : \mathfrak{h} \to \mathfrak{gl}(V)$. Let

$$\Omega^k(\mathfrak{h}; V) = \{\omega : \mathfrak{h}^n \to V | \text{multilinear and skew-symmetric}\}.$$

We can define $\delta : C^k(\mathfrak{h}; V) \to C^{k+1}(\mathfrak{h}; V)$ by

$$\begin{aligned}(\delta\omega)(h_0, h_1, ..., h_k) = &\sum_{i=0}^{k}(-1)^i \rho(h_i)(\omega(h_0, ..., \widehat{h_i}, ..., h_k)) \\ &+ \sum_{0\leq i<j\leq k}(-1)^{i+j}\omega([h_i, h_j], h_0, ..., \widehat{h_i}, ..., \widehat{h_j}, ..., h_k).\end{aligned}$$

We can check that $\delta^2 = 0$. This allows us define the Lie algebra cohomology.

Definition (Lie Algebra Cohomology)**.** *Given a Lie algebra $\mathfrak{h}$ with a representation $\rho : \mathfrak{h} \to \mathfrak{gl}(V)$, define $\Omega^k(\mathfrak{h}; V)$ and δ as above, its k-th cohomology group with coefficients in V is*

$$H^k(\mathfrak{h}; V) = \frac{\mathrm{Ker}(\delta : \Omega^k(\mathfrak{h}; V) \to \Omega^{k+1}(\mathfrak{h}; V))}{\mathrm{Im}(\delta : \Omega^{k-1}(\mathfrak{h}; V) \to \Omega^k(\mathfrak{h}; V))}.$$

It's easy to check this is a group under addition. When $V = R$ and ρ is the trivial representation, we write $\Omega^k(\mathfrak{g}) = \Omega^k(\mathfrak{g}; \mathbb{R})$ and $H^k(\mathfrak{g}) = H^k(\mathfrak{g}; \mathbb{R})$.

The construction of Lie group cohomology is similar to Lie algebra cohomology. Given a Lie group H with a representation $\rho : H \to Gl(V)$, let

$$C^k(H; V) = \{c : H^n \to V | \text{smooth}\}.$$

Define $\theta : C^k(H; V) \to C^{k+1}(H; V)$ by

$$\begin{aligned}(\theta c)(g_0, g_1, ..., g_k) = &\rho(g_0)c(g_1, ..., g_k) + \sum_{i=0}^{k-1}(-1)^{i+1}c(g_0, ..., g_i g_{i+1}, ..., g_k) \\ &+ (-1)^{k+1}c(g_0, ..., g_{k-1}).\end{aligned}$$

We can also check that $\theta^2 = 0$, thus we can define the Lie group cohomology.

Definition (Lie Group Cohomology)**.** *Given a Lie group H with a representation $\rho : H \to Gl(V)$, define $C^k(H;V)$ and θ as above, its k-th cohomology group with coefficients in V is*

$$H^k(H;V) = \frac{\mathrm{Ker}(\theta : C^k(H;V) \to C^{k+1}(H;V))}{\mathrm{Im}(\theta : C^{k-1}(H;V) \to C^k(H;V))}.$$

It's easy to check this is also a group under addition. When $V = R$ and ρ is the trivial representation, we write $C^k(G) = C^k(G;\mathbb{R})$ and $H^k(G) = H^k(G;\mathbb{R})$.

2.2 The Second Cohomology Group of Lie Algebras

Given a Lie algebra $\mathfrak{h}$ with a representation $\rho : \mathfrak{h} \to \mathfrak{gl}(V)$, for each $\omega \in \Omega^2(\mathfrak{h};V)$ such that $\delta\omega = 0$, we can construct $\mathfrak{g}_\omega = \mathfrak{h} \times V$ with a bracket$[\ ,\ ]_\omega$:

$$[(Y_1, v_1), (Y_2, v_2)]_\omega = ([Y_1, Y_2]_\mathfrak{h}, \rho(Y_1)(v_2) - \rho(Y_2)(v_1) + \omega(Y_1, Y_2)).$$

Claim 1. *$\mathfrak{g}_\omega$ is a Lie algebra.*

Proof. Clearly $[\ ,\ ]_\omega$ is bilinear. And it is also skew-symmetric by construction and noticing that both $[\ ,\ ]_\mathfrak{h}$ and ω are skew-symmetric. We only need to check the Jacobi identity:

$$\begin{aligned}
&[[(Y_1, v_1), (Y_2, v_2)]_\omega, (Y_3, v_3)]_\omega + [[(Y_2, v_2), (Y_3, v_3)]_\omega, (Y_1, v_1)]_\omega + [[(Y_3, v_3), (Y_1, v_1)]_\omega, (Y_2, v_2)]_\omega \\
=&[([Y_1, Y_2]_\mathfrak{h}, \rho(Y_1)(v_2) - \rho(Y_2)(v_1) + \omega(Y_1, Y_2)), (Y_3, v_3)]_\omega \\
&+ [([Y_2, Y_3]_\mathfrak{h}, \rho(Y_2)(v_3) - \rho(Y_3)(v_2) + \omega(Y_2, Y_3)), (Y_1, v_1)]_\omega \\
&+ [([Y_3, Y_1]_\mathfrak{h}, \rho(Y_3)(v_1) - \rho(Y_1)(v_3) + \omega(Y_3, Y_1)), (Y_2, v_2)]_\omega \\
=&([[(Y_1, v_1), (Y_2, v_2)]_\mathfrak{h}, (Y_3, v_3)]_\mathfrak{h} + [[(Y_2, v_2), (Y_3, v_3)]_\mathfrak{h}, (Y_1, v_1)]_\mathfrak{h} + [[(Y_3, v_3), (Y_1, v_1)]_\mathfrak{h}, (Y_2, v_2)]_\mathfrak{h}, \\
&+ (\rho([Y_1, Y_2]) - \rho(Y_1)\rho(Y_2) + (\rho(Y_2)\rho(Y_1))(v_3) + (\rho([Y_2, Y_3]) + \rho(Y_3)\rho(Y_2) - \rho(Y_2)\rho(Y_3))(v_1) \\
&+ (\rho([Y_3, Y_1]) - \rho(Y_3)\rho(Y_1) + (\rho(Y_1)\rho(Y_3))(v_2) - \rho(Y_3)\omega(Y_1, Y_2) - \rho(Y_1)\omega(Y_2, Y_3) - \rho(Y_2)\omega(Y_3, Y_1) \\
&+ \omega(\omega(Y_1, Y_2), Y_3) + \omega(\omega(Y_2, Y_3), Y_1) + \omega(\omega(Y_3, Y_1), Y_2)) \\
=&(0, 0 - (\delta\omega)(Y_1, Y_2, Y_3)) = 0.
\end{aligned}$$

□

Claim 2. *If $[\omega_1]$=$[\omega_2]$ in $H^2(\mathfrak{h};V)$, then $\mathfrak{g}_1 \cong \mathfrak{g}_2$ as Lie algebra.*

Proof. If $[\omega_1] = [\omega_2]$, then $\exists \nu \in \Omega^1(\mathfrak{h}; V)$ such that $\omega_1 - \omega_2 = \delta\nu$. Define:

$$\begin{aligned} \Phi : \mathfrak{g}_{\omega_1} &\to \mathfrak{g}_{\omega_2}, \\ (y, v) &\mapsto (y, v + \nu(y)). \end{aligned}$$

Clearly it's a linear bijection, we only need to check whether it is a homomorphism:

$$\begin{aligned} [\Phi(Y_1, v_1), \Phi(Y_2, v_2)]_{\omega_2} =& [(Y_1, v_1 + \nu(Y_1))), (Y_2, v_2 + \nu(Y_2))]_{\omega_2} \\ =& ([Y_1, Y_2]_{\mathfrak{h}}, \rho(Y_1)(v_2 + \nu(Y_2)) - \rho(Y_2)(v_1 + \nu(Y_1)) + \omega_2(Y_1, Y_2)) \\ =& ([Y_1, Y_2]_{\mathfrak{h}}, \rho(Y_1)(v_2 + \nu(Y_2)) - \rho(Y_2)(v_1 + \nu(Y_1)) + (\omega_1 - (\delta\nu))(Y_1, Y_2)) \\ =& ([Y_1, Y_2]_{\mathfrak{h}}, \rho(Y_1)(v_2) - \rho(Y_2)(v_1) + \omega_1(Y_1, Y_2) - \rho(Y_1) + \nu([Y_1, Y_2])) \\ =& \Phi([(Y_1, v_1), (Y_2, v_2)]_{\omega_1}). \end{aligned}$$

□

2.3 Abelian extension of Lie Algebras

Given any short exact sequence of Lie algebras:

$$0 \to V \xrightarrow{i} \mathfrak{g} \xrightarrow{\phi} \mathfrak{h} \to 0,$$

where $V \in \mathfrak{g}$ is abelian, let $\sigma : \mathfrak{h} \to \mathfrak{g}$ be any linear map such that $\phi \circ \sigma = \text{id}$. We can define:

$$\begin{aligned} \Phi : \mathfrak{g} &\to \mathfrak{h} \times V, \\ X &\mapsto (\phi(X), X - \sigma(\phi(X)). \end{aligned}$$

It's easy to see that Φ is a linear isomorphism with $\Phi^{-1}(Y,v)=\sigma(Y)+v$ as its inverse. To make this a Lie algebra isomorphism, we define:

$$\begin{aligned}\rho_\sigma:\mathfrak{h}&\to\mathfrak{gl}(V),\\ Y&\mapsto\rho_{\sigma,Y},\\ \rho_{\sigma,Y}(v)&=[\sigma(Y),v]_{\mathfrak{g}},\\ \omega_\sigma:\mathfrak{h}\times\mathfrak{h}&\to V,\\ (Y_1,Y_2)&\mapsto[\sigma(Y_1),\sigma(Y_2)]_{\mathfrak{g}}-\sigma([Y_1,Y_2]_{\mathfrak{h}}).\end{aligned}$$

Then define the bracket $[\ ,\ ]_\sigma$ on $\mathfrak{h}\times V$ by:

$$[(Y_1,v_1),(Y_2,v_2)]_\sigma=([Y_1,Y_2]_{\mathfrak{h}},\rho_{\sigma,Y_1}(v_2)-\rho_{\sigma,Y_2}(v_1)+\omega(Y_1,Y_2)).$$

We can check that:

$$\begin{aligned}[\Phi(X_1),\Phi(X_2)]_\sigma=&[(\phi(X_1),X_1-\sigma(\phi(X_1)),(\phi(X_2),X_2-\sigma(\phi(X_2))]_\sigma\\ =&([\phi(X_1),\phi(X_2)]_{\mathfrak{h}},\rho_{\sigma,\phi(X_1)}(X_2-\sigma(\phi(X_2))-\rho_{\sigma,\phi(X_2)}(X_1-\sigma(\phi(X_1))\\ &+\omega_\sigma(\phi(X_1),\phi(X_2)))\\ =&(\phi([X_1,X_2]_{\mathfrak{g}}),[\sigma(\phi(X_1),X_2-\sigma(\phi(X_2)]_{\mathfrak{g}}-[\sigma(\phi(X_2),X_1-\sigma(\phi(X_1)]_{\mathfrak{g}}\\ &+[\sigma(\phi(X_1)),\sigma(\phi(X_2))]_{\mathfrak{g}})-\sigma([\phi(X_1),\phi(X_2)]_{\mathfrak{h}}))\\ =&(\phi([X_1,X_2]_{\mathfrak{g}}),-[X_1-\sigma(\phi(X_1),X_2-\sigma(\phi(X_2)]_{\mathfrak{g}}+[X_1,X_2]_{\mathfrak{g}}-\sigma([\phi(X_1),\phi(X_2)]_{\mathfrak{h}})\\ =&(\phi([X_1,X_2]_{\mathfrak{g}}),[X_1,X_2]_{\mathfrak{g}}-\sigma([\phi(X_1),\phi(X_2)]_{\mathfrak{h}}))\\ =&\Phi([X_1,X_2]_{\mathfrak{g}}).\end{aligned}$$

Thus Φ is a Lie algebra isomorphism.
The following statements show that for fixed ϕ, our choice of σ does not affect the result. First, $\forall\sigma'$ such that $\phi\circ\sigma'=\mathrm{id}$, we have

$$(\rho_\sigma-\rho_{\sigma'})(Y)(v)=(\rho_{\sigma,Y}-\rho_{\sigma',Y})(v)=([\sigma(Y),v]-[\sigma(Y),v])=([(\sigma(Y)-\sigma'(Y)),v]).$$

Recall that $\phi(\sigma-\sigma')=0$, we have $(\sigma-\sigma')(Y)\in\mathrm{Ker}(\phi)=V$, thus $\rho_\sigma=\rho'_\sigma$, we can denote ρ_σ as ρ.

Then can check that:

$$\begin{aligned}\delta(\omega_\sigma))(Y_1,Y_2,Y_3) =&\rho(Y_1)\omega_\sigma([Y_2,Y_3]_\mathfrak{h}) - \rho(Y_2)\omega_\sigma([Y_1,Y_3]_\mathfrak{h}) + \rho(Y_3)\omega_\sigma([Y_1,Y_3]_\mathfrak{h})\\ &- \omega_\sigma([[Y_1,Y_2]_\mathfrak{h},Y_3) + \sigma([[Y_2,Y_3]_\mathfrak{h},Y_1) - \sigma([[Y_1,Y_3]_\mathfrak{h},Y_2)\\ =&2([[\sigma(Y_2),\sigma(Y_3)]_\mathfrak{g},\sigma(Y_1)]_\mathfrak{g} - [[\sigma(Y_1),\sigma(Y_3)]_\mathfrak{g},\sigma(Y_2)]_\mathfrak{g} + [[\sigma(Y_1),\sigma(Y_2)]_\mathfrak{g},\sigma(Y_3)]_\mathfrak{g})\\ =&0.\end{aligned}$$

Thus $\omega_\sigma \in \mathrm{Ker}(\Omega^2(\mathfrak{h};V) \to \Omega^3(\mathfrak{h};V))$.
Finally, $\forall\sigma'$ such that $\phi\circ\sigma' = \mathrm{id}$, we have:

$$\begin{aligned}(\omega_\sigma - \omega_{\sigma'})(Y_1,Y_2) =&[\sigma(Y_1),\sigma(Y_2)]_\mathfrak{g} - \sigma([Y_1,Y_2]_\mathfrak{h}) - ([\sigma'(Y_1),\sigma'(Y_2)]_\mathfrak{g} - \sigma'([Y_1,Y_2]_\mathfrak{h}))\\ =&(\sigma'-\sigma)([Y_1,Y_2]_\mathfrak{h}) + [\sigma(Y_1),\sigma(Y_2)]_\mathfrak{g} - [\sigma'(Y_1),\sigma'(Y_2)]_\mathfrak{g}\\ &- [\sigma(Y_1),\sigma'(Y_2)]_\mathfrak{g} + [\sigma'(Y_2),\sigma(Y_1)]_\mathfrak{g}\\ =&[\sigma(Y_1),(\sigma-\sigma'(Y_2))] - [\sigma'(Y_2),(\sigma-\sigma')(Y_1)] - (\sigma-\sigma')([Y_1,Y_2]_\mathfrak{h})\\ =&\rho(Y_1)(\sigma-\sigma')(Y_2) - \rho(Y_2)(\sigma-\sigma')(Y_1) - (\sigma-\sigma')([Y_1,Y_2]_\mathfrak{h})\\ =&\delta(\sigma-\sigma')(Y_1,Y_2).\end{aligned}$$

Thus $[\omega_\sigma] = [\omega_{\sigma'}]$ in $H^2(\mathfrak{h},V)$.
Then by §2.2, we can conclude that $H^2(\mathfrak{h},V)$ classifies the isomorphism classes of abelian extensions of $\mathfrak{h}$ by V.

2.4 The Second Cohomology Group of Lie Groups

Similarly, for a given Lie group H and its representation $\rho : H \to Gl(V)$, we can construct $G_c = H \times V$ for each $c \in C^2(H;V)$ with a multiplication$\cdot_c$:

$$(h_1,v_1)\cdot_c(h_2,v_2) = (h_1h_2, v_1 + \rho(h_1)(v_2) + c(h_1,h_2)).$$

Claim 3. *G_c is a Lie group.*

Proof. Clearly we have $(\mathbf{1},0)$ as the identity and $(h^{-1},-v)$ as the inverse of

(h, v). It remains for us to check $\cdot_c$ is associative:

$$\begin{aligned}((h_1, v_1) \cdot_c (h_2, v2)) \cdot_c (h_3, v_3) &= (h_1h_2, v_1 + \rho(h_1)(v_2) + c(h_1, h_2)) \cdot_c (h_3, v_3) \\ &= (h_1h_2h_3, v_1 + \rho(h_1)(v_2) + c(h_1, h_2) + \rho(h_1h_2)(v_3) + c(h_1h_2, h_3)).\end{aligned}$$

$$\begin{aligned}(h_1, v_1) \cdot_c ((h_2, v2) \cdot_c (h_3, v_3)) &= (h_1, v_1) \cdot_c (h_2h_3, v_2 + \rho(h_2)(v_3) + c(h_2, h_3)) \\ &= (h_1h_2h_3, v_1 + \rho(h_1)(v_2 + \rho(h_2)(v_3) + c(h_2, h_3)) + c(h_1, h_2h_3)).\end{aligned}$$

$$\begin{aligned}&((h_1, v_1) \cdot_c (h_2, v2)) \cdot_c (h_3, v_3) - (h_1, v_1) \cdot_c ((h_2, v2) \cdot_c (h_3, v_3)) \\ &= (0, c(h_1h_2) + c(h_1h_2, h_3) + \rho(h_1)c(h_1, h_2) - c(h_1, h_2h_3) - \rho(h_1)c(h_2, h_3)) \\ &= (0, -(\delta c)(h_1, h_2, h_3)) = 0.\end{aligned}$$

□

Claim 4. *If $[c_1] = [c_2]$ in $H^2(H; V)$, then $G_{c_1} \cong G_{c_2}$ is a Lie group isomorphism.*

Proof. If $[c_1] = [c_2]$, then $\exists f \in H^2(H; V)$ such that $c_1 - c_2 = \delta f$. Define:

$$\begin{aligned}\Phi : G_{c_1} &\to G_{c_2}, \\ (h, v) &\mapsto (h, v + f(h)).\end{aligned}$$

Clearly this is a diffeomorphism, we only need to check whether it's also a homomorphism:

$$\begin{aligned}\Phi(h_1, v_1) \cdot_{c_2} \Phi(h_2, v_2) &= (h_1, v_1 + f(h_1)) \cdot_{c_2} (h_2, v_2 + f(h_2)) \\ &= (h_1h_2, v_1 + f(h_1) + \rho(h_1)(v_2 + f(h_2)) + c_2(h_1, h_2)) \\ &= (h_1h_2, v_1 + f(h_1) + \rho(h_1)(v_2 + f(h_2)) + (c_1 - (\delta f))(h_1, h_2)) \\ &= (h_1h_2, v_1 + \rho(h_1)(v_2) + c_1(h_1, h_2) + f(h_1h_2) \\ &= \Phi((h_1, v_1) \cdot_{c_1} (h_2, v_2)).\end{aligned}$$

□

A claim similar to what we saw above for abelian extensions of Lie algebras can be made for Lie groups. Its second cohomology group $H^2(H; V)$ classifies the isomorphism classes of abelian extensions of H by V.

3 The Van Est Theorem

In this section, we will introduce the Van Est Theorem and its corollary regarding simply connected Lie groups ([7],[8]). Then we will give some examples to illustrate it.

Theorem (Van Est)**.** *Let G be an m-connected Lie group with Lie algebra $\mathfrak{g}$, $\rho : G \to V$ a representation of G, then $H^i(G;V)$ is isomorphic with $H^i(\mathfrak{g};V)$ for $i = 0, 1, ..., m$*

By a result from Hopf ([9]), every simply connected Lie group is 2-connected, so we have the following corollary.

Corollary 1. *Let G be a simply connected Lie group with Lie algebra $\mathfrak{g}$, then $H^i(G;V)$ is isomorphic with $H^i(\mathfrak{g};V)$ for $i = 0, 1, 2$.*

The following examples illustrate these results.

Example 1. $SU(2)$.

Let $G = SU(2)$, then $\mathfrak{g} = \mathfrak{su}(2)$.
Since $SU(2) \cong S^3$ and the cohomology group of a compact Lie group is always trivial, we have $H^*(SU(2)) \equiv 0$.
Since $\forall \omega \in \Omega^1(\mathfrak{su}(2))$, $(\delta\omega)(X_0, X_1) = 0$ implies $-\omega([X_0, X_1]) = 0$, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ then implies $\omega = 0$. Thus we have $H^1(\mathfrak{su}(2)) = 0$.
To compute the cohomology group of higher degree, we use the following basis for $\mathfrak{su}(2)$:

$$u_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, u_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, u_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Notice that $[u_1, u_2] = 2u_3, [u_2, u_3] = 2u_1, [u_3, u_1] = 2u_2$.
Let

$$\sigma_i(u_j) = \begin{cases} 1, & \text{if } j = i \\ 0, & \text{if } j \neq i \end{cases}.$$

Then the basis for $\Omega^1(\mathfrak{su}(2))$ is $\{\sigma_1, \sigma_2, \sigma_3\}$.
Since

$$(\delta\sigma_1)(u_i, u_j) = -\sigma_1([u_i, u_j]) = \begin{cases} -2, & \text{if } i = 2,\ j = 3 \\ 2, & \text{if } i = 3,\ j = 2 \\ 0, & \text{else} \end{cases},$$

the basis of the image of $\Omega^{k-1}(\mathfrak{su}(2))$ under δ is $\{\sigma_1 \wedge \sigma_2, \sigma_2 \wedge \sigma_3, \sigma_3 \wedge \sigma_1\}$ by symmetricity, which is also a basis of $\Omega^2(\mathfrak{su}(2))$, thus the map δ is surjective. Thus we have $H^2(\mathfrak{su}(2)) = 0$.
For $H^3(\mathfrak{su}(2))$, consider $\omega = \mathrm{Tr}([X_1, X_2]X_3)$, it's easy to check that $\delta\omega = 0$. Now suppose $\exists \omega' \in \Omega^2(\mathfrak{su}(2))$ such that $\delta\omega' = \omega$, then we must have

$$\begin{aligned}\omega(u_1, u_2, u_3) &= -\omega'([u_1, u_2], u_3) + \omega'([u_1, u_3], u_2) - \omega'([u_2, u_3], u_1)\\ &= -2(\omega'(u_3, u_3) + \omega'(u_2, u_2) + \omega'(u_1, u_1))\\ &= 0.\end{aligned}$$

However, $\omega(u_1, u_2, u_3) = \mathrm{Tr}(2u_3^2) = -2 \neq 0$. Thus such ω' does not exist, i.e. $[\omega] \neq 0$ in $H^3(\mathfrak{su})$. So $H^3(\mathfrak{su}(2))$ is not isomorphic with $H^3(SU(2))$, which agrees with the fact that $\pi_3(SU(2)) = \pi_3(S^3) = \mathbb{R} \neq 0$.

Example 2. $\mathbb{T}^n$

Let $G = \mathbb{T}^n$, $\mathfrak{g} = \mathbb{R}^n$. then $\pi_1(\mathbb{T}^n) \neq 0$.
By compactness of $\mathbb{T}^n$, we still have $H^*(\mathbb{T}^n) \equiv 0$.
Since ω is multilinear and $\mathbb{R}^n$ is abelian, $\forall \omega' \in \Omega^{k-1}(\mathbb{T}^n)$:

$$(\delta\omega')(X_0, X_1, ..., X_k) = \sum_{0 \le i < j \le k} (-1)^{i+j} \omega([X_i, X_j], X_0, ..., \widehat{X_i}, ..., \widehat{X_j}, ..., X_k) = 0.$$

However, let $\{e_i\}_{i=1,...,n}$ be the basis of R^n, define σ_i by

$$\sigma_i(e_j) = \begin{cases} 1, & \text{if } j = i \\ 0, & \text{if } j \neq i \end{cases}.$$

We have $\omega = \sigma_1 \wedge \sigma_2 \wedge ... \wedge \sigma_{k+1} \in \Omega^{k+1}(\mathbb{R}^n)$ nontrivial for $(k+1) \le n$. Thus $H^m(\mathbb{R}^n)$ is not isomorphic with $H^m(\mathbb{T}^n)$ for $m \le n$. This does not contradict Van Est since $\pi_1(\mathbb{T}^n) \neq 0$.

4 Proof of Lie's Third Theorem

In this section, we will prove the main theorem. First, for any given finite-dimensional Lie algebra $\mathfrak{g}$, using the results in §2, we can identify it with the semi-direct product of its center and its adjoint Lie algebra, both of which can be integrated into simply connected Lie groups. Then we use the Van Est Theorem to conclude that there is a semi-direct product of these two Lie groups which has a Lie group structure making $\mathfrak{g}$ its Lie algebra.

Theorem (Lie's Third Theorem)**.** *Given any finite-dimensional Lie algebra $\mathfrak{g}$, there is a Lie group G such that $\mathfrak{g}$ is the Lie algebra of G.*

Proof. Let $\mathfrak{g}$ be an arbitary finite-dimensional Lie algebra. Consider the adjoint representation:

$$\begin{aligned} \mathrm{ad} : \mathfrak{g} &\to \mathrm{End}(\mathfrak{g}), \\ X &\mapsto \mathrm{ad}_X, \\ &\quad \mathrm{ad}_X(Y) = [X, Y]. \end{aligned}$$

The kernel of ad is the center of $\mathfrak{g}$, denoted as $\mathrm{Z}(\mathfrak{g})$. Thus we have the following short exact sequence:

$$0 \to \mathrm{Z}(\mathfrak{g}) \xrightarrow{i} \mathfrak{g} \xrightarrow{\mathrm{ad}} \mathrm{ad}(\mathfrak{g}) \to 0.$$

Using the results of §2.3, let $\sigma : \mathrm{ad}(\mathfrak{g}) \to \mathfrak{g}$ be any linear map such that $\mathrm{ad} \circ \sigma = \mathrm{id}$, and define:

$$\begin{aligned} \omega : \mathrm{ad}(\mathfrak{g}) \times \mathrm{ad}(\mathfrak{g}) &\to \mathrm{Z}(\mathfrak{g}), \\ (Y_1, Y_2) &\mapsto \sigma([Y_1, Y_2]_{\mathrm{ad}(\mathfrak{g})}) - [\sigma(Y_1), \sigma(Y_2)]_{\mathrm{ad}(\mathfrak{g})}. \end{aligned}$$

we have the following isomorphism:

$$\begin{aligned} \Phi : \mathfrak{g} &\to \mathrm{ad}(\mathfrak{g}) \times \mathrm{Z}(\mathfrak{g}), \\ X &\mapsto (\mathrm{ad}(X), X - \sigma(\mathrm{ad}\, X)). \end{aligned}$$

The Lie bracket $[\ ,\ ]_{\mathfrak{h}}$ on $\mathrm{ad}(\mathfrak{g}) \times \mathrm{Z}(\mathfrak{g})$ is

$$[(Y_1, z_1), (Y_2, z_2)]_{\mathfrak{h}} = ([Y_1, Y_2]_{\mathrm{ad}(\mathfrak{g})}, \omega(Y_1, Y_2)).$$

There exists a simply connected Lie group $\widehat{\mathrm{Ad}(\mathfrak{g})}$ whose Lie algebra is $\mathrm{ad}(\mathfrak{g})$. Let $\rho : H \to Gl(\mathrm{Z}(\mathfrak{g})) \equiv e$. Then by corollary 1, $\exists c \in C^2(\widehat{\mathrm{Ad}(\mathfrak{g})}; \mathrm{Z}(\mathfrak{g}))$ and

$[dc] = [\omega_\sigma]$. Now let $G = \widehat{\mathrm{Ad}(\mathfrak{g})} \times \mathrm{Z}(\mathfrak{g})$ with the multiplication:

$$(h_1, v_1) \cdot (h_2, v_2) = (h_1 h_2, v_1 + v_2 + c(h_1, h_2)).$$

Then G is a Lie group with Lie algebra $\mathfrak{g}$. $\square$

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