h-principles around Poisson Geometry

The present thesis is thus organized :

Chapter One: purports to review standard notions and to agree on a notation for them; the reader who is slightly familiar with simplicial objects and Hæfliger structures can freely skip all but the last three sections.

Chapter Two: describes the basic h-theoretic machinery we will use; it is self-contained except for a theorem of Eliashberg's which we are happy to merely quote. The exposition of Section 7 is a solution to exercises of [**32**].

Chapter Three: is the core of this thesis, and presents the applications of the h-Machine to problems revolving around Poisson geometry. After a brief excursion through the features of Poisson Geometry that will later come into play (Sections 1-3), we debate what we wish for a Poisson-geometric h-principle in Section 4, and then proceed to give four applications to specific problems, in Sections 5-8.

I have tried to maintain as standard a notation as possible, and to quote proofs of facts we draw upon when this doesn't take us too far afield. I also hope I haven't forgotten any of the references I used to write this note; please drop me a line if you notice anyone I forgot to quote.

Contents

CHAPTER 1

Setting the Stage

1. Sheaves

Let T be a topological space, $\mathcal{C}$ a category equipped with a faithful forgetful functor

$$E : \mathcal{C} \to \text{Sets}$$

and denote by $\mathcal{O}(T)$ the poset of open sets of T under inclusion, regarded as a discrete category[1]. This defines a cofunctor

$$\mathcal{O} : \text{Top} \to \text{Cat}$$

in the obvious way.

Definition 1. *A* $\mathcal{C}$**-valued presheaf** F *on* T *is a cofunctor*

$$F : \mathcal{O}(T) \to \mathcal{C}$$

A **morphism**

$$\varphi : F_0 \to F_1$$

of two such presheaves is a natural transformation of these cofunctors: for each inclusion

$$i_{U'} : U' \hookrightarrow U$$

we have a commutative $\mathcal{C}$*-diagram*

$$\begin{tikzcd} F_0(U) \arrow[r, "\varphi_U"] \arrow[d, "F_0(\iota_{U'})"'] & F_1(U) \arrow[d, "F_1(\iota_{U'})"] \\ F_0(U') \arrow[r, "\varphi_{U'}"'] & F_1(U') \end{tikzcd}$$

The ensuing category of $\mathcal{C}$*-valued presheaves on* T *and morphisms thereof is denoted* $\text{PSh}(T;\mathcal{C})$*.*

Note that a continuous map

$$f : T_0 \to T_1$$

induces

$$f_* : \text{PSh}(T_0, \mathcal{C}) \to \text{PSh}(T_1, \mathcal{C})$$
$$(f_*F)(U) := F(f^{-1}U)$$

Let also $\text{Cov}(T)$ be the (discrete) category of open coverings of T under refinement; thus an object $\mathfrak{U} = \{U_i\}_{i \in J} \in \text{Cov}(T)$ can be described as a point-set map

$$\mathfrak{U} : J \to \mathcal{O}(T)$$

where $J = J(\mathfrak{U})$ is an arbitrary set, and the only allowed morphisms $\mathfrak{U} \to \mathfrak{U}'$ are cover refinements; i.e., a point-set map $\lambda : J(\mathfrak{U}') \to J(\mathfrak{U})$ such that

$$U'_i \subset U_{\lambda(i)}$$

[1] The reader is encouraged to think of $\mathcal{C}$ = Sets or Top for the moment.

Definition 2. *A presheaf* $F \in \mathrm{PSh}(T, \mathcal{C})$ *is called a* **sheaf** *if for all* $U \in \mathcal{O}(T)$ *and all* $\mathfrak{U} = \{U_i\} \in \mathrm{Cov}(U)$ *we have an equalizer diagram*

$$FU \xrightarrow{e} \prod_i FU_i \overset{p}{\underset{q}{\rightrightarrows}} \prod_{i,j} F(U_i \cap U_j)$$

where

$$e : f \mapsto \{f|U_i\}$$
$$p(\{t_i\}) = \{t_i|U_i \cap U_j\}, \quad q(\{t_i\}) = \{t_j|U_i \cap U_j\}$$

If e *is merely a monomorphism*[2], *we call* F *a* **separated presheaf**.

Definition 3. *Let* $\mathfrak{U} \in \mathrm{Cov}(T)$. *A* **matching family** *for* $\mathfrak{U}$ *with values in* $F \in \mathrm{PSh}(T, \mathcal{C})$ *is a family*

$$\{f_i\}_{i \in J(\mathfrak{U})}$$
$$f_i \in F(U_i)$$
$$f_i|U_i \cap U_j = f_j|U_i \cap U_j$$

and denote by $\mathrm{Match}(\mathfrak{U}, F)$ *the set of all such matching families. Then we can define*

$$F^+T := \operatorname*{colim}_{\mathfrak{U} \in \mathrm{Cov}(T)} \mathrm{Match}(\mathfrak{U}, F)$$

Naturality of the construction defines the **plus construction**

$$^+ : \mathrm{PSh}(T, \mathcal{C}) \to \mathrm{PSh}(T, \mathcal{C})$$

Thus elements of $F^+(U)$ are equivalence classes of matching families for coverings of U, where two such families are regarded as equivalent when they agree on a common refinement.

Note that F^+ is always a separated presheaf – but $F^{++} := (F^+)^+$ is a *sheaf*, called the **sheaffification** of F, as follows from the correspnding statement for $\mathcal{C} = $ Sets and the assumption that $E : \mathcal{C} \to$ Sets is faithful.

Observe that for each $f \in F(T)$ and each $\mathfrak{U} \in \mathrm{Cov}(T)$, we have a well-defined matching family

$$f_{\mathfrak{U}} := \{f_i := f|U_i\}$$

and that under refinement $\mathfrak{U} \to \mathfrak{U}'$, $f_{\mathfrak{U}}$ is sent to $f_{\mathfrak{U}'}$ and thus determines a canonical morphism of $\mathcal{C}$-valued presheaves

$$F \to F^+$$

and one checks that $F^{+++} = F^{++}$.

Note finally that $F \to F^{++}$ can be described alternatively as the universal presheaf morphism through which every morphism of presheaves $F \to F'$ factors when F' is a sheaf :

$$\begin{tikzcd} F \arrow[r] \arrow[d] & F' \\ F^{++} \arrow[ur, dashed] & \end{tikzcd}$$

[2]I.e., with no assumption on it equalizing (p, q).

1.1. Sheaves of sets. Let's devote some attention to the special case $\mathcal{C} = \mathrm{Sets}$.

Denote by $(\mathrm{Etale} \downarrow T)$ the category of étale maps $T' \to T$ with commutative triangles as morphisms. Given $F \in \mathrm{PSh}(T, \mathrm{Sets})$, define its **stalk** F_x at $x \in T$ by

$$F_x := \operatorname*{colim}_{x \in U} F(U)$$

Now set

$$\mathrm{Etale}\, F := \coprod_{x \in T} F_x$$

as a set, and observe the natural point-set maps

$$\mathrm{germ}_x : F(U) \to F_x \text{ for all } x \in U$$

to be compatible with restrictions; furthermore, $\mathrm{Etale}(F)$ comes equipped with a projection $\hat{p}$: $\mathrm{Etale}(F) \to T$ defined by $\mathrm{germ}_x f \mapsto x$.

Now endow $\mathrm{Etale}(F)$ with the topology generated by the subsets

$$\mathcal{O}(U, f) = \{\mathrm{germ}_x f : f \in F(U), x \in U\}$$

One easily sees that $\hat{p}$ becomes an étale map, so $\mathrm{Etale}(F) \to T$ is an object of $(\mathrm{Etale} \downarrow T)$, and that for a presheaf morphism

$$\varphi : F_0 \to F_1$$

the map

$$\mathrm{Etale}\,\varphi : \mathrm{Etale}(F_0) \to \mathrm{Etale}(F_1)$$
$$\mathrm{Etale}\,\varphi(\mathrm{germ}_x f) := \mathrm{germ}_x(\varphi_{\mathrm{dom}\, f} f)$$

is continuous, thus automatically étale, and that the assignment

$$\varphi \mapsto \mathrm{Etale}\,\varphi$$

is functorial.

Wrapping up, we have a functor

$$\mathrm{Etale} : \mathrm{PSh}(T, \mathrm{Sets}) \to (\mathrm{Etale} \downarrow T)$$

In the other direction, consider the functor

$$\Gamma : (\mathrm{Etale} \downarrow T) \to \mathrm{PSh}(T, \mathrm{Sets})$$
$$\big(T' \to T\big) \mapsto \Gamma(\cdot, T')$$

where $\Gamma(\cdot, T')$ denotes the sheaf of local sections of $T' \to T$. For obvious reasons, this functor factors by the inclusion $\mathrm{Sh}(T, \mathrm{Sets}) \hookrightarrow \mathrm{PSh}(T, \mathrm{Sets})$, and it is not difficult to see that

$$\begin{tikzcd}
\mathrm{PSh}(T, \mathrm{Sets}) \arrow[rr, "++"] \arrow[dr, "\mathrm{Etale}"'] & & \mathrm{PSh}(T, \mathrm{Sets}) \\
& (\mathrm{Etale} \downarrow T) \arrow[ur, "\Gamma"'] &
\end{tikzcd}$$

commutes and is a projection. Thus, for a sheaf F, we have $F(U) = \Gamma(U, \mathrm{Etale}\, F)$. Moreover, for a continuous map

$$f : T_0 \to T_1$$

we consider the pullback diagram

$$\begin{tikzcd} T_0 \times_{T_1} \mathrm{Etale}(F) \arrow[r] \arrow[d] & \mathrm{Etale}(F) \arrow[d] \\ T_0 \arrow[r] & T_1 \end{tikzcd}$$

and define

$$f^* : \mathrm{Sh}(T_1, \mathrm{Sets}) \to \mathrm{Sh}(T_0, \mathrm{Sets})$$

by

$$(f^*F)(U_0) = \Gamma(U_0, T_0 \times_{T_1} \mathrm{Etale}(F))$$

It is easy to see that

$$f^*F = G^{++}$$

where G is the presheaf

$$U_0 \mapsto \operatorname*{colim}_{U_1 \supset f(U_0)} F(U_1)$$

and that f_*, f^* are adjoint functors – i.e., for any two sheaves $F_i \in \mathrm{Sh}(T_i, \mathrm{Sets})$, there is a natural bijection

$$\mathrm{Hom}(f^*F_1, F_0) \simeq \mathrm{Hom}(F_1, f_*F_0)$$

2. Topological groupoids and Hæfliger structures

A small category $\mathcal{C}$ is called a **topological category** if $\mathrm{Ar}\,\mathcal{C}$ and $\mathrm{Ob}\,\mathcal{C}$ are endowed with topologies such that all five structural maps

$$\begin{aligned} s, t &: \mathrm{Ar}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{C} \\ \circ &: \mathrm{Ar}\,\mathcal{C}_t \times_s \mathrm{Ar}\,\mathcal{C} \to \mathrm{Ar}\,\mathcal{C} \\ 1_\bullet &: \mathrm{Ob}\,\mathcal{C} \to \mathrm{Ar}\,\mathcal{C} \\ \bullet^{-1} &: \mathrm{Iso}\,\mathcal{C} \to \mathrm{Iso}\,\mathcal{C} \end{aligned}$$

are continuous[3], where $\mathrm{Iso}\,\mathcal{C} \subset \mathrm{Ar}\,\mathcal{C}$ denotes the subspace of invertible arrows. Such a category $G = (G_1 \rightrightarrows G_0)$ is called a **topological groupoid** if $\mathrm{Iso}\,G = \mathrm{Ar}\,G$. A topological groupoid G is called **étale** if s is an étale map, i.e., is a local homeomorphism.

A **continuous functor** $F : \mathcal{C} \to \mathcal{C}'$ between topological categories $\mathcal{C}, \mathcal{C}'$ is a functor for which $F(\varphi) : F(c_0) \to F(c_1)$ varies continuously with φ.

Example 1. *Of course, every small category can be regarded as a* discrete *topological category. Another quite trivial example is that of a topological space T, which can be seen as topological groupoid, all of whose arrows are identities.*

Example 2. *A slightly more interesting one is that of the* **Čech groupoid of a covering**. *Namely, let $\mathfrak{U} = \{U_i\}_{i \in I}$ be an open covering of a topological space T, and define*

$$\begin{aligned} \mathrm{Ob}\,T_{\mathfrak{U}} &:= \{(x, i) : x \in U_i\} \\ \mathrm{Ar}\,T_{\mathfrak{U}} &:= \{(x, j, i) : x \in U_i \cap U_j\} \end{aligned}$$

[3] As Gustavo Granja points out, it is not quite standard to require explicitly continuity of the inverse map.

with structure maps

$$s(x,j,i) = (x,i), \quad t(x,j,i) = (x,j), \quad 1_{(x,i)} = (x,i,i)$$
$$(x,k,j) \circ (x,j,i) = (x,k,i), \quad (x,j,i)^{-1} = (x,i,j)$$

which are continuous in the natural topology of $T_{\mathfrak{U}}$. This will quite obviously serve as a model for $\mathfrak{U}$-trivializable objects on T; note that it is equivalent, as a category, to T regarded as a groupoid as in the previous example.

Let now V be a smooth manifold, and consider

$$\mathrm{diff}(V) = \{U_0 \underset{\simeq}{\xrightarrow{d}}, U_1 : U_i \text{ open in } V \text{ and } d \text{ a diffeomorphism}\}$$

For notational convenience, we will omit U_i from the notation, writing $U_0 = \operatorname{dom} d$ and $U_1 = \operatorname{codom} d$. The subset with $\operatorname{dom} d = V = \operatorname{codom} d$ is the group of automorphisms of V, denoted by $\mathrm{Diff}(V)$, which we topologize under the C^∞-compact-open topology.

DEFINITION 4. *A* **pseudogroup of diffeomorphisms** *of V is a subset $\mathfrak{D} \subset \mathrm{diff}(V)$ such that :*

- *If $d \in \mathfrak{D}$ and U is an open subset of* $\operatorname{dom} d$, *then $d|U \in \mathfrak{D}$;*
- *A* homeomorphism $d : U \to U'$ *lies in $\mathfrak{D}$ iff there exists an open cover $\{U_i\}$ of U such that $d|U_i \in \mathfrak{D}$;*
- $\mathrm{id}_U \in \mathfrak{D}$ *for every open $U \subset V$;*
- *$d \in \mathfrak{D}$ iff $d^{-1} \in \mathfrak{D}$;*
- *$d, d' \in \mathfrak{D}$ and* $\operatorname{dom} d' \subset \operatorname{codom} d$ *implies $d' \circ d \in \mathfrak{D}$*

EXAMPLE 3. *Let $\mathfrak{D}$ be a pseudogroup of diffeomorphisms of V. We can construct an associated étale groupoid $\Gamma \rightrightarrows V$ by letting*

$$\Gamma = \{\mathrm{germ}_x\, d : x \in \mathrm{dom}(d), d \in \mathfrak{D}\}$$

endowed with the sheaf topology*, which is generated by the subsets of the form*

$$\mathcal{O}(d) = \{\mathrm{germ}_x\, d : x \in \mathrm{dom}(d)\}$$

where d ranges over all elements in $\mathfrak{D}$.

Then we set

$$s, t : \Gamma \to V$$
$$s(\mathrm{germ}_x\, d) := x, \quad t(\mathrm{germ}_x\, d) := d(x)$$
$$1 : V \to \Gamma$$
$$1_x := \mathrm{germ}_x(\mathrm{id}_V)$$
$$(\mathrm{germ}_x\, d)^{-1} := \mathrm{germ}_{d(x)}\, d^{-1}$$
$$(\mathrm{germ}_{x'}\, d') \circ (\mathrm{germ}_x\, d) := \mathrm{germ}_x(d' \circ d)$$

thus obtaining an étale (effective) groupoid as claimed.

When $V = \mathbb{R}^n$, $\mathfrak{D} = \mathrm{diff}(\mathbb{R}^n)$, we will employ Hæfliger's notation Γ_n for Γ.

Hæfliger structures. Let again $T_{\mathfrak{U}}$ denote the Čech groupoid associated to a covering $\mathfrak{U}$ of a topological space T.

Suppose $G = (G_1 \rightrightarrows G_0)$ is any topological groupoid. A **Hæfliger G-cocycle** on $\mathfrak{U}$ is just a continuous functor

$$F : T_{\mathfrak{U}} \to G$$

Two such cocycles

$$F_1 : T_{\mathfrak{U}_1} \to G$$
$$F_2 : T_{\mathfrak{U}_2} \to G$$

are said to *differ by a coboundary* if one can find a G-cocycle for $\mathfrak{U}_1 \coprod \mathfrak{U}_2$ rendering the following diagram commutative

$$\begin{tikzcd}
T_{\mathfrak{U}_1} \arrow[dr] \arrow[drr, "F_1"] & & \\
& T_{\mathfrak{U}_1 \coprod \mathfrak{U}_2} \arrow[r, dashed] & G \\
T_{\mathfrak{U}_2} \arrow[ur] \arrow[urr, "F_2"'] & &
\end{tikzcd}$$

where

$$T_{\mathfrak{U}_1} \hookrightarrow T_{\mathfrak{U}_1 \coprod \mathfrak{U}_2} \hookleftarrow T_{\mathfrak{U}_2}$$

are the natural continuous functors induced by the natural inclusions

$$\mathfrak{U}_1 \hookrightarrow \mathfrak{U}_1 \coprod \mathfrak{U}_2 \hookleftarrow \mathfrak{U}_2$$

This is clearly an equivalence relation, and we denote by $H^1(T;G)$ the set of equivalence classes of such structures, nicknamed **Hæfliger G-structures**; an equivalent description would be to define $H^1(\mathfrak{U}, G)$ as the equivalence classes of cocycles $T_{\mathfrak{U}} \to G$, and define

$$H^1(T,G) := \operatorname*{colim}_{\mathrm{Cov}(T)} H^1(\mathfrak{U}, G)$$

We observe that

$$H^1(\cdot,\cdot) : \mathrm{Top}^{\mathrm{op}} \times \mathrm{TopGrpd} \to \mathrm{Sets}$$

is bifunctorial. When G is a group, it factors through

$$\mathrm{Top}^{\mathrm{op}} \times \mathrm{TopGrpd} \to \mathrm{hTop}^{\mathrm{op}} \times \mathrm{TopGrpd}$$

as follows from the classical lemma that homotopic principal group-bundles are isomorphic. For general groupoids G this need not be the case[4]. If we insist on homotopy invariance, we are led to the following notion :

Definition 5. *Two Hæfliger G-structures F_0, F_1 are called* **concordant** *if there is*

$$F \in H^1(T \times I; G)$$

restricting to F_i on $T \times \{i\}$. The set of concordance classes of such structures will be denoted by $h^1(T;G)$.

[4]It is easy to construct two non-isomorphic Hæfliger G-structures on contractible spaces; say,

$$T = \mathbb{R}, \quad \mathfrak{U}_0 = \{\mathbb{R}\} = \mathfrak{U}_1$$

and

$$F_0, F_1 : \mathbb{R} \to \Gamma_1$$

where

$$f_i : \mathbb{R} \to \mathbb{R}, \quad x \mapsto ix$$

By its very construction, $h^1(\cdot,\cdot)$ is homotopy-invariant, i.e., factors through

$$\mathrm{Top} \times \mathrm{TopGrpd} \to \mathrm{hTop} \times \mathrm{TopGrpd}$$

There is another point of view about such structures that is perhaps more transparent for the more bundle-inclined, with the modification resulting from the fact that groupoids act on *maps*, rather than spaces.

Indeed, let $G_1 \rightrightarrows G_0$ be a topological groupoid, E a space and

$$\pi : E \to G_0$$

an arrow.

DEFINITION 6. *A* **left action** *of G on $\pi : E \to G_0$ is a map*

$$\mu : G_1 \times_{G_0} E \to E$$

satisfying

Fibredness: $\pi(g \cdot e) = \pi(e)$*;*
Associativity: $g_1 \cdot (g_2 \cdot e) = (g_1 g_2) \cdot (e)$*;*
Identity: $1_g \cdot e = e$

DEFINITION 7. *A* **left G-bundle** *on a topological space T consists then of a* bundle[5]

$$p : E \to T$$

with a left G-action on $\pi : E \to G_0$, such that p is G-invariant.

A **morphism** *between two left G-bundles $E_i \to T_i$ is a G-equivariant arrow $E_1 \to E_2$, commuting with projections to G_0, and giving rise to a commutative diagram of*

$$\begin{tikzcd} E_1 \arrow[r] \arrow[d] & E_2 \arrow[d] \\ T_1 \arrow[r] & T_2 \end{tikzcd}$$

A left G-bundle $p : E \to T$ is called **principal** *if p is an open surjection and*

$$G_1 \times_{G_0} E \to E \times_M E$$
$$(g, e) \mapsto (e, g \cdot e)$$

is a homeomorphism (hence G acts freely and transitvely on the fibres of p).

We thus have a category PBun_G of isomorphism types of principal left G-bundles $E \to T$. Observe that when G is étale, p is étale.

A distinguished example of left principal G-bundle is obtained by taking $T := G_0$, $E := G_1$, $p := s$ and $\pi := t$. This is the so-called *unit bundle* of G. A left principal G-bundle will be called **trivial** if it is isomorphic to the unit bundle. Moreover, note that the operation of pullback through continuous maps of the bases is well-defined for left G-bundles, and preserves principality of a such bundle.

Suppose now we are given a cocycle

$$F : T_{\mathfrak{U}} \to G.$$

We can construct a principal G-bundle $E_F \to T$ by defining

$$\widetilde{E_F} := G_1 \times_{G_0} \coprod_i U_i$$

$$E_F := \widetilde{E_F} / \sim$$

$$\text{where } (g, (x, j)) \sim (g \circ F(x, j, i), (x, i))$$

[5]Mind that, encoded in the word *bundle*, we assume *local triviality.*

Let now

$$p : E_F \to T$$
$$p([g, x, i]) := x$$
$$\pi : E_F \to G_0$$
$$\pi([g, x, i]) := s(g)$$
$$G_1 \times_{G_0} E_F \to E_F$$
$$g' \cdot [g, x, i] = [g'g, x, i]$$

and one easily checks that this indeed defines a left principal G-bundle $E_F \to T$.

Note that if $\mathfrak{U}'$ refines $\mathfrak{U}$,

$$U'_i \subset U_{\tau(i)}$$

then this naturally defines a $\mathrm{ref}_{\mathfrak{U}'} F : T_{\mathfrak{U}'} \to G$, and also a

$$E_{\mathrm{ref}_{\mathfrak{U}'} F} \to E_F$$
$$[g, x, i] \mapsto [g, x, \tau(i)]$$

which is G-equivariant and therefore a homeomorphism. Therefore, the assignment of a cocycle F to the principal $E_F \to T$ maps cohomologous cocycles to isomorphic principal G-bundles, i.e.,

$$H^1(T, G) \to \mathrm{PBun}_G(T)$$
$$[F] \to [E_F]$$

is well-defined.

That is is also a bijection is seen by constructing an inverse : let $E \to T$ be a left principal G-bundle. In fact, $p : E \to T$ a bundle means that we can find an open cover $\mathfrak{U} = \{U_i\}$ of T and sections

$$t_i : U_i \to E|U_i$$

By principality of E, there is a homeomorphism

$$E \times_{G_0} E \to G_1 \times_{G_0} E$$
$$(e, e') \mapsto (\theta(e, e'), e')$$

so we let

$$F : T_{\mathfrak{U}} \to G$$
$$F(x, i, j) = \theta(t_i(x), t_j(x))$$

Thus there is a well-defined natural bijection

$$\{\text{isomorphism classes of left principal } G\text{-bundles}\} \leftrightarrow \{G\text{-structures}\}$$

3. Classifying spaces for Hæfliger structures

References are [**6**], [**18**], [**26**], [**51**], [**63**], [**64**], [**65**].

Simplicial primer. For each natural number n, let $[n]$ denote the discrete category with object-set $\{0, 1, ..., n\}$ and exactly one arrow $i \to j$ whenever $i \leqslant j$, and by $[\mathbb{N}]$ the category obtained from the union of all such $[n]$. The **simplicial category** Δ is defined as the discrete category with object-set consisting of all $[n]$, and having as arrows $[n] \to [m]$ all functors $[n] \to [m]$; its nth truncation if the full subcategory $\Delta_{\leqslant n}$ spanned by the objects $[0], ..., [n]$.

Note that a functor

$$\mu : [n] \to [m]$$

is tantamount to a monotone map $\widetilde{\mu} : \mathrm{Ob}[n] \to \mathrm{Ob}[m]$, i.e.,

$$i < j \Longrightarrow \widetilde{\mu}(i) \leqslant \widetilde{\mu}(j)$$

REMARK 8. *Observe the distinguished*

$$d^i : [n-1] \to [n], \quad 0 \leqslant i \leqslant n$$
$$s^j : [n+1] \to [n], \quad 0 \leqslant j \leqslant n$$

called **cofaces** *and* **codegeneracies***, respectively, and defined by*

$$d^i : (0 \to 1 \to \cdots \to n-1) \mapsto (0 \to 1 \to \cdots i-1 \to i+1 \to \cdots \to n)$$
$$s^j : (0 \to 1 \to \cdots \to n+1) \mapsto (0 \to 1 \to \cdots j-1 \to j = j \to j+1 \to \cdots \to n)$$

These satisfy the obvious cosimplicial relations

$$\begin{cases} d^j d^i = d^i d^j & i < j \\ s^j d^i = d^i s^{j-1} & i < j \\ s^j d^j = 1 = s^j d^{j+1} & \\ s^j d^i = d^{i-1} s^j & i > j+1 \\ s^j s^i = s^i s^{j+1} & i \leqslant j \end{cases}$$

Observe that every arrow $[n] \to [m]$ in Δ factors uniquely as the composition of an epimorphism $[n] \to [k]$ with a monomorphism $[k] \to [m]$; clearly, monos are composites of coface maps, whereas epis are composites of codegeneracy maps.

A **simplicial object** in a category $\mathcal{C}$ is a contravariant functor

$$S : \Delta \to \mathcal{C}$$

i.e., an object of $\mathcal{C}^{\Delta^{\mathrm{op}}}$; when $\mathcal{C} = \mathrm{Sets}$ we call S a **simplicial set**, and when $\mathcal{C} = \mathrm{Top}$, a **simplicial space**; their respective categories are denoted SSets and SSpaces.

A morphism of simplicial objects in $\mathcal{C}$ is a natural transformation between two such functors.

Note that, according to the previous remark, an equivalent description of the simplicial object S would be :

- A collection of objects S_n in $\mathcal{C}$;
- **Face** maps

$$d_i := S(d^i) : S_n \to S_{n-1}$$

  and **degeneracy** maps

$$s_j := S(s^j) : S_n \to S_{n+1}$$

  satisfying the **simplicial relations** dual to those of Remark 8.

EXAMPLE 4. *If T is a space, we define the simplicial set of* **singular simplices**

$$S(T) : \Delta^{\mathrm{op}} \to \mathrm{Sets}$$
$$S(T)[n] = \mathrm{Top}(\Delta^n, T)$$

Recall that the singular homology $H^{\bullet}_{sing}(T; A)$ of T is defined as the homology of the complex $(\mathbb{Z}S(T)_{\bullet} \otimes A, d)$ where A is an Abelian group and

$$d : \mathbb{Z}S(T)_n \otimes A \to \mathbb{Z}S(T)_{n+1} \otimes A$$
$$d := \sum (-1)^i d_i \otimes 1_A$$

Definition 9. *Suppose S, S' are simplicial spaces. Their product $S \times S'$ is the simplicial space*

$$(S \times S')_n := S_n \times S'_n$$
$$d_i^{S\times S'} = d_i^S \times d_i^{S'}$$
$$s_j^{S\times S'} = s_j^S \times s_j^{S'}$$

In fact, SSets possesses all limits and colimits, which are constructed levelwise as we exemplified with $\times$, i.e., given any J-diagram of simplicial sets, $F : J \to$ SSets, its colimit is constructed as

$$\left(\lim_{\to} F\right)[n] := \lim_{\to} F[n]$$

where $F[n]$ is the induced J-diagram of sets, and

$$\left(\lim_{\to} F\right)(\mu) := \lim_{\to} F[\mu] : \lim_{\to} F[m] \to \lim_{\to} F[n]$$

for all $\mu : [n] \to [m]$; it is evident that this indeed has the categorical properties defining the limit of the J-diagram F.

The analogous construction can be performed, *mutatis mutandis*, for colimits in SSets.

Homotopy notions.

Definition 10. *The* **standard n-simplex** Δ^n *in the category of simplicial sets is the simplicial set represented by* $[n]$*:*

$$\Delta^n = \Delta(\cdot, [n])$$

Observe that by Yoneda's lemma[6]

$$S_n \simeq \mathrm{SSpaces}(\Delta^n, S)$$

and let ι_n for the simplex corresponding to $\mathrm{id}_{[n]} \in \Delta([n],[n])$. *Its* **boundary** $\partial\Delta^n$ *is the smallest subsimplicial set of Δ^n containing all faces $d_j(\iota_n)$, where $0 \leqslant j \leqslant n$. The* **$k$th horn** Λ^n_k *is the subsimplicial set of Δ^n generated by all faces $d_j(\iota_n)$ except for the kth.*

Definition 11. *A map of simplicial spaces $X \to Y$ is called a* **(Kan) fibration** *if all diagrams of the form*

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

are solvable, $n \geqslant 0$.

It is called a **trivial (Kan) fibration** *if all diagrams of the form*

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

are solvable, $n \geqslant 0$.

A simplicial space X is called **fibrant** *or* **Kan** *is $X \to$ pt is a fibration.*

Definition 12. *Let $f_0, f_1 : X \to Y$ be morphisms of simplicial spaces. We call a morphism*

$$f : X \times \Delta^1 \to Y$$

[6]Here one identifies SSets with level-wise discrete simplicial spaces.

rendering

$$\begin{tikzcd} X \times \Delta^0 \arrow[d, "\mathrm{id} \times d^0"'] \arrow[dr, "f_0"] & \\ X \times \Delta^1 \arrow[r, "f"] & Y \\ X \times \Delta^0 \arrow[u, "\mathrm{id} \times d^1"] \arrow[ur, "f_1"'] & \end{tikzcd}$$

commutative a **homotopy** *between* f_0 *and* f_1.

Such a homotopy is called **relative** *to a subcomplex* $A \subset X$ *iff* $f_0|A = f_1|A$ *and*

$$\begin{tikzcd} X \times \Delta^1 \arrow[r] & Y \\ A \times \Delta^1 \arrow[u] \arrow[r, "\mathrm{pr}_A"'] & A \arrow[u, "f_0|A=f_1|A"'] \end{tikzcd}$$

commutes.

Definition 13. *Let* X *be a fibrant simplicial set, and* $x \in X_0$. *Define* $\pi_n(X, x)$ *as the set of rel* $\partial\Delta^n$ *homotopy types of*

$$\alpha : \Delta^n \to X$$

fitting into

$$\begin{tikzcd} \Delta^n \arrow[r, "\alpha"] & X \\ \partial\Delta^n \arrow[u] \arrow[r] & \Delta^0 \arrow[u, "x"'] \end{tikzcd}$$

The fibrant hypothesis on X guarantees that *having the same rel* $\partial\Delta^n$ *homotopy type* indeed defines an equivalence relation. As usual, these are pointed sets for $n \geqslant 0$, groups for $n \geqslant 1$ and Abelian groups for $n \geqslant 2$.

Observe the following relation between simplicial sets and spaces :

Proposition 14. *If* Y *is a simplicial space, there exists a simplicial set* X *and a trivial Kan fibration*

$$f : X \to Y$$

Proof. Suppose we have built $\mathrm{Sk}^n X$ and a map $\mathrm{Sk}^n X \to Y$ such that all diagrams of the form

$$\begin{tikzcd} \partial\Delta^m \arrow[r] \arrow[d] & \mathrm{Sk}^n X \arrow[d] \\ \Delta^m \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

are solvable for $m \leqslant n$.

Now, for each (isomorphism type of) diagram

$$\begin{tikzcd} \partial\Delta^{n+1} \arrow[r] \arrow[d] & \mathrm{Sk}^n X \arrow[d] \\ \Delta^{n+1} \arrow[r] & Y \end{tikzcd}$$

one attaches a non-degenerate Δ^{n+1} to $\mathrm{Sk}^n X$ through the obvious attaching map $\partial\Delta^{n+1} \to X$, thus defining Sk^{n+1} in which all

$$\begin{tikzcd} \partial\Delta^m \arrow[r] \arrow[d] & \mathrm{Sk}^{n+1} X \arrow[d] \\ \Delta^m \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

are solvable for $m \leqslant n+1$, see [**49**]. □

Geometric realization. Every small category $\mathcal{C}$ gives rise to a simplicial set, called the **nerve** $\mathfrak{N}\mathcal{C}$ of $\mathcal{C}$, defined by setting $\mathfrak{N}\mathcal{C}[n]$ to be the space of all functors

$$[n] \to \mathcal{C}$$

and the obvious

$$\mathfrak{N}\mathcal{C}(\mu) : \mathfrak{N}\mathcal{C}[m] \to \mathfrak{N}\mathcal{C}[n]$$

for each $\mu \in \Delta([n],[m])$.

Observe that this defines a functor

$$\mathfrak{N} : \mathrm{TopCat} \to \mathrm{SSpaces}$$

Let now Δ^n denote the standard n-simplex in $\mathbb{R}^{n+1}$:

$$\Delta^n = \left\{ \sum_0^n t_i e_i : t_i \geqslant 0 \text{ and } \sum_0^n t_i = 1 \right\}$$

Given $\mu \in \Delta([n],[m])$, define

$$\mu_* : \Delta^m \to \Delta^n$$

by the rule

$$\mu_* \left(\sum_0^m t_i e_i \right) = \sum_0^n t_{\mu(i)} e_i$$

Then given a simplicial space S, we define its **geometric realization** $\|S\|$ to be the quotient of

$$\|S\| = \coprod_n S([n]) \times \Delta^n$$

by the relations

$$(S(\mu)\, x, v) \sim (x, \mu_* v)$$

for all $\mu \in \Delta([n],[m])$ and all n, m.

Observe that any morphism $\varphi : S \to S'$ between simplicial spaces gives rise to a continuous map $\|\varphi\|$ between the corresponding geometric realizations, and the assignment

$$\varphi \mapsto \|\varphi\|$$

is functorial.

Definition 15. *The* **Segal functor**

$$\mathfrak{S} : \mathrm{TopCat} \to \mathrm{TopCat}$$

assigns to a top-cat $\mathcal{C}$ the top-cat $\mathfrak{S}\mathcal{C}$ which has the same object-set as $\mathbb{N} \times \mathcal{C}$, and for morphisms all identities and all $(n, c) \to (n', c')$ with $n < n'$.

The **(Segal) classifying space** *$B\mathcal{C}$ of $\mathcal{C}$ is then defined as the geometric realization of the nerve of $\mathfrak{S}\mathcal{C}$.*

Remark 16. *The geometric realization of the product $X \times X'$ of two simplicial spaces is naturally homeomorphic to $\|X\| \times \|X'\|$ if $\|X\|$ is either locally compact or compactly generated, see* [**6**].

Consider now a simplicial space S and denote by S^δ the discretization of S, i.e., its composition with the adjoint pair Top $\to$ Sets (forgetful functor) and Sets $\to$ Top (endowing a set with the discrete topology).

Let T be any topological space. Given a homotopy

$$H : T \times I \to \|S\|$$

we can always find a set-theoretic diagram

$$\begin{tikzcd} T \times I \arrow[r, "H"] & \|S\| \\ T^\delta \times I \arrow[u] \arrow[r, "H^\delta"'] & \|S^\delta\| \arrow[u] \end{tikzcd}$$

with both vertical arrows continuous, since $\|S\| = \|S^\delta\|$ as *sets*.

This means essentially that, for each $x \in T$, the induced path $H_x : I \to \|S\|$ factors through $\|S^\delta\| \to \|S\|$, i.e., H_x moves only in the "linear" direction of the simplices Δ^n appearing in the construction of $\|S\|$.

Definition 17. *We call H a* **linear homotopy** *if H^δ is also continuous. We denote by* $\mathrm{Lin}(T, \|S\|)$ *the set of equivalence classes of continuous maps $T \to \|S\|$ under the equivalence relation of being linearly homotopic.*

Numerable and numerated structures.

Definition 18. *A* **partition of unity** *on a space T is the data of continuous maps*

$$t_n : T \to [0, 1], \quad n \in \mathbb{N}$$

which is locally finite[7] *and*

$$\sum_n t_n(x) = 1, \quad \forall x \in T$$

A partition of unity is called **subordinated** *to a covering $\mathfrak{U} \in \mathrm{Cov}(T)$ if the support of each t_n is contained in some member of the covering.*

If a covering $\mathfrak{U}$ admits a partition of unity subordinated to it, we call it **numerable**.

Definition 19. *A G-structure on T is said to be* **numerable** *if is represented by some cocycle $F : T_{\mathfrak{U}} \to G$, where $\mathfrak{U}$ is a numerable covering of T.*

Let Δ^∞ denote the infinite simplex with countably many vertices $\{e_n\}_{n\in\mathbb{N}}$. Observe that a partition of unity gives rise to a continuous map

$$p : T \to \Delta^\infty$$
$$x \mapsto \sum_n t_n(x) e_n$$

The notion of continuous map into Δ^∞ is thus closely related to that of partition of unity, and will be used in what follows.

Recall now that the **carrier** of a point $x \in \|S\|$ in the geometric realization of a simplicial space S, $\mathrm{car}(x)$, is the unique simplex of S to whose interior x belongs.

Now observe that given a continuous map

$$p : T \to \|\mathbb{N}\| = \Delta^\infty$$

not necessarily arising from a partition of unity on T, we can consider the topological subcategory (called **numerated by** p)

$$T_p \subset \mathbb{N} \times T$$

[7]That is, each $x \in T$ has an open neighborhood U where all but finitely many t_n's vanish identically.

spanned by all objects of the form (n, x), where $n \in \mathrm{car}(p(x))$, with the subspace topology. The unique morphism $(n, x) \longrightarrow (m, x)$ will be denoted by (m, n, x).

Definition 20. *A p-**numerated cocycle** is a continuous homomorpism $T_p \to G$, and we call a cocycle **numerated** if it is a p-numerated cocycle for some p.*

Now given partitions of unity

$$p_0, p_1 : T \to \Delta^\infty$$

define

$$\begin{aligned} p_0 \sharp p_1 &: T \to \Delta^\infty \\ x &\mapsto \frac{1}{2}\left(\|h_0\| \circ p_0 + \|h_1\| \circ p_1\right) \end{aligned}$$

where

$$h_i : \mathbb{N} \to \mathbb{N}$$
$$h_i : n \mapsto 2n + i$$

and declare two numerated cocycles $T_{p_0} \to G$, $T_{p_1} \to G$ cohomologous if there is a numerated cocycle $F_{p_0 \sharp p_1} \to G$ such that the following diagram commutes :

$$\begin{tikzcd} T_{p_0} \arrow[dr] \arrow[drr, "F_{p_0}"] & & \\ & T_{p_0 \sharp p_1} \arrow[r, "F_{p_0 \sharp p_1}"] & G \\ T_{p_1} \arrow[ur] \arrow[urr, "F_{p_1}"'] & & \end{tikzcd}$$

where $T_{p_i} \to T_{p_0 \sharp p_1}$ are the canonical functors $(m, n, x) \mapsto (2m + i, 2n + i, x)$. The equivalence classes under this relation are called **numerated G-structures**. The collection of all numerated G-structures on T will be denoted by $\mathrm{Num}(T, G)$.

Remark 21. *It should be pointed out that the realization of the canonical functors*

$$T_{p_i} \to T_{p_0 \sharp p_1}$$

are linear homotopy equivalences, as is easy to verify.

Definition 22. *Two numerated cocycles $F_{p_0} : T_{p_0} \to G, F_{p_1} : T_{p_1} \to G$ are called* **concordant** *if there is a homotopy*

$$p : T \times I \to \Delta^\infty$$

between p_0 and p_1, and a numerated cocycle

$$F_p : (T \times I)_p \to G$$

such that the induced

$$\begin{tikzcd} T_{p_0} \arrow[dr] \arrow[drr, "F_{p_0}"] & & \\ & (T \times I)_p \arrow[r, "F_p"] & G \\ T_{p_1} \arrow[ur] \arrow[urr, "F_{p_1}"'] & & \end{tikzcd}$$

commutes.

The collection of all concordance classes of numerated G-structures will be written $\mathrm{num}(T,G)$.

Observe in the above definition that the realizations of $T_{p_i} \to (T\times I)_p$ are homotopy equivalences, as follows from the commutativity of

$$\begin{tikzcd} \|T_{p_i}\| \arrow[r] & \|(T\times I)_p\| \\ T \arrow[u, "s_{p_i}"] \arrow[r, "\mathrm{id}_{T\times\{i\}}"'] & T\times I \arrow[u, "s_p"'] \end{tikzcd}$$

and the fact that s_{p_i}, s_p and $\mathrm{id}_{T\times\{i\}}$ are homotopy equivalences.

One readily checks that, given two partitions of unity $p_0, p_1 : T \to \Delta^\infty$, for which two cocycles $F_{p_0} : T_{p_0} \to G, F_{p_1} : T_{p_1} \to G$ are cohomologous, then they are also concordant through a homotopy $p : p_0 \to p_1$ and numerated $F_p : (T\times I)_p \to G$.

Observe that part of the data of a numerated cocycle is a partition of unity p, so such a continuous $T_p \to G$ defines canonically a numerable cocycle by the assignment

$$U_n := \{x : n \in \mathrm{car}(p(x))\}$$
$$F_{U_n}(x) := F_p(n,x)$$
$$F_{U_m,U_n}(x) := \begin{cases} F_p(m,n,x) & \text{if } m \geqslant n \\ F_p(n,m,x)^{-1} & \text{otherwise.} \end{cases}$$

and two numerated cocycles representing the same numerated G-structure obviously give rise to numerable cocycles in the same equivalence class.

Conversely, given a numerable cocycle $F : T_{\mathfrak{U}} \to G$, equipped with a partition of unity $\{t_n\}$ subordinated to $\mathfrak{U} = \{U_i\}$, one can construct, in a *canonical* fashion, a new partition of unity $\{\widetilde{t_n}\}$ (call it its Husemoller refinement) with the property that

$$\widetilde{t}_n^{-1}(0,1] = \coprod_i \widetilde{U}_{ni}, \quad \widetilde{U}_{ni} \subset U_i.$$

With this refinement, F determines a continuous homomorphism $T_{\widetilde{p}} \to G$, and any two such numerated cocycles must represent the same numerated G-structure.

The functor $\mathrm{Top}(\cdot, BG)$**.** Notice now that by construction BG comes equipped with a canonical

$$q : BG \to \Delta^\infty$$

induced by the restriction of

$$\mathrm{pr}_{\mathbb{N}} : \mathfrak{S}G \to \mathbb{N}$$

Hence for each continuous map

$$f : T \to BG$$

we can assign a partition of unity

$$p := q \circ f : T \to \Delta^\infty$$

If we denote by T_f the topological subcategory of $T \times G_{\mathbb{N}}$ spanned by the objects $(x,(n,g))$ where (n,g) belongs to the carrier of $f(x)$, it is then obvious that the restriction of

$$\mathrm{pr}_{\mathbb{N}} \times \mathrm{pr}_T$$

to T_f induces a continuous functor

$$T_f \to T_p$$

and that of pr_G induces a continuous

$$T_f \to G$$

LEMMA 23. *$T_f \to T_p$ is an isomorphism of topological categories.*

PROOF. This is a mere consequence of the following observation : by the construction of T_f, $(x,(n,g))$ is an object in this category iff $f(x)$ lies in the interior of the simplex

$$n_0 \xrightarrow{g_0} n_1 \xrightarrow{g_1} \cdots \xrightarrow{g_{k-1}} n_k$$

with $(n,g) = (n_i, g_i)$ for some $0 \leqslant i \leqslant k$.

Since $n_0 < n_1 < \ldots < n_k$, this means that for each $n \in \mathrm{car}(p(x))$, there is one, and only one, $g \in G$ such that $(n,g) \in \mathrm{car}(f(x))$ hence $T_f \to T_p$ is bijective.

Since it is obviously continuous (for it is the restriction of a continuous functor), all that remains to check is that the functor is also open. But this is a direct consequence of the commutativity of the diagram

$$\begin{tikzcd} T_f \arrow[rr] \arrow[dr, "\mathrm{pr}_{\mathbb{N}}"'] & & T_p \arrow[dl, "\mathrm{pr}_{\mathbb{N}}"] \\ & \mathbb{N} & \end{tikzcd}$$

openness of the induced functors

$$T_f \cap \mathrm{pr}_{\mathbb{N}}^{-1}(\{n\}) \to T_p \cap \mathrm{pr}_{\mathbb{N}}^{-1}(\{n\})$$

for each n, and discreteness of $\mathbb{N}$. □

By means of the above lemma, we have defined a correspondence

$$\mathrm{Top}(T, BG) \to \mathrm{Num}(T, G)$$

assigning to each $f : T \to BG$ the numerated cocycle $F_{q\circ f} : T_{q\circ f} \to G$ constructed above. Observe that, by construction, $F_p(n,x)$ is the unique g for which $(x,(n,g))$ is an object of T_f.

LEMMA 24. *The restriction of*

$$\mathrm{Top}(\cdot, BG) \to \mathrm{Num}(\cdot, G)$$

to locally compact, or compactly generated spaces, gives a natural isomorphism.

PROOF. Given a numerated cocycle $F_p : T_p \to G$ there is a canonical map

$$T_p \to \Delta^\infty \times T$$

induced by the inclusion $T_p \subset \mathbb{N} \times T$, since our hypothesis on T is such that

$$\|\mathbb{N} \times T\| \simeq \Delta^\infty \times T$$

It is now straightforward to check that we obtain a commutative diagram

$$\begin{tikzcd} T \arrow[rr, "x \mapsto (p(x), x)"] \arrow[dr, "s_p"'] & & \Delta^\infty \times T \\ & |T_p| \arrow[ur] & \end{tikzcd}$$

and that $\mathrm{pr}_{\mathbb{N}} \times F_p$ factors through the canonical inclusion $G_{\mathbb{N}} \to \mathbb{N} \times G$:

$$\begin{tikzcd} T_p \arrow[rr, "\mathrm{pr}_{\mathbb{N}} \times F_p"] \arrow[dr, "\Phi_p"'] & & \mathbb{N} \times G \\ & G_{\mathbb{N}} \arrow[ur] & \end{tikzcd}$$

This allows us to consider the map

$$\begin{aligned} \mathrm{Num}(T;G) &\to \mathrm{Top}(T,BG) \\ F_p &\mapsto \|\Phi_p\| \circ s_p \end{aligned}$$

which is easily seen to be an inverse to the previously defined $\mathrm{Top}(T,BG) \to \mathrm{Num}(T,G)$. $\square$

Finally, observe that the data of a concordance $(T \times I)_p \to G$ between two numerated cocycles F_{p_0}, F_{p_1} gives rise to a commutative

$$\begin{tikzcd}
T_{p_0} \arrow[dr] \arrow[drr, "\Phi_{p_0}"] & & \\
& (T \times I)_h \arrow[r, "F_p"] & G_{\mathbb{N}} \\
T_{p_1} \arrow[ur] \arrow[urr, "\Phi_{p_1}"'] & &
\end{tikzcd}$$

Consider now the diagram

$$\begin{tikzcd}
T \arrow[r, "s_{p_0}"] \arrow[d, "\mathrm{id}_{T\times\{0\}}"'] & \|T_{p_0}\| \arrow[d] \arrow[dr, "\|\Phi_{p_0}\|"] & \\
T \times I \arrow[r, "s_p"] & \|(T \times I)_p\| \arrow[r, "\|\Phi_p\|"] & BG \\
T \arrow[r, "s_{p_1}"'] \arrow[u, "\mathrm{id}_{T\times\{1\}}"] & \|T_{p_1}\| \arrow[u] \arrow[ur, "\|\Phi_{p_1}\|"'] &
\end{tikzcd}$$

The right triangles are commutative by functoriality of realization; on the other hand, the left squares are homotopy-commutative (where all maps involved are homotopy equivalences). Thus

$$\begin{tikzcd}
& \|T_{p_0}\| \arrow[dr, "\|\Phi_{p_0}\|"] & \\
T \arrow[ur, "s_{p_0}"] \arrow[dr, "s_{p_1}"'] & & BG \\
& \|T_{p_1}\| \arrow[ur, "\|\Phi_{p_1}\|"] &
\end{tikzcd}$$

is homotopy-commutative, which amounts to saying that $\|\Phi_0\| \circ s_{p_0}$ and $\|\Phi_1\| \circ s_{p_1}$ are homotopic. Hence there is a well-defined

$$\mathrm{num}(T,G) \to [T,BG]$$

This is seen to be bijective (for locally compact or compactly generated T) by functoriality of $\mathrm{Num}(\cdot,G) \simeq \mathrm{Top}(\cdot,BG)$ applied to the diagram

$$\begin{tikzcd}
T \times \{0\} \arrow[d] \arrow[dr, "\|\Phi_0\| \circ s_{p_0}"] & \\
T \times I \arrow[r, "H"] & BG \\
T \times \{1\} \arrow[u] \arrow[ur, "\|\Phi_1\| \circ s_{p_1}"'] &
\end{tikzcd}$$

Finally, we observe that, over paracompact T, all G-cocycles can be numerably refined.
We have thus proved :

THEOREM 25. *For all T paracompact which is also locally compact or compactly generated :*
1. $\mathrm{Top}(T, BG) \to H^1(T, G)$ *is a surjection;*
2. *The induced* $[T, BG] \to h^1(T, G)$ *is bijective.*

REMARK 26. *In* [**6**], *Bracho shows that there is an intermediate step in the correspondence above, namely, that*

$$H^1(T, G) \simeq \mathrm{Lin}(T, BG)$$

Hence control of the cohomology *class of a cocycle F, and not merely its concordance type, is controlled at the level of maps under linear homotopy; however, our poor understanding of this latter equivalence makes it difficult to use ths fact significantly in applications.*

4. Quasi-topological spaces

Quasi-topological spaces were first devised to provide a "convenient" category for the algebraic topology of function spaces. In it, several categorical niceties of Sets carry over.

DEFINITION 27. *A* **quasi-topology** *on a set X is an assignment, for all topological spaces T, of a subset* $\mathrm{QTop}(T, X) \subset \mathrm{Sets}(T, X)$, *elements of which are called* **quasi-continuous** *maps of T into X, such that the following properties are verified :*
1. *All constant maps* $\mathrm{pt} \to X$ *are quasi-continuous;*
2. *If $f : T \to X$ is quasi-continuous and $g : T' \to T$ is continuous, then $f \circ g : T' \to X$ is quasi-continuous;*
3. *A map $f \in \mathrm{Sets}(T, X)$ is quasi-continuous iff it is locally quasi-continuous, i.e., every point $t \in T$ has a neighbourhood U such that $f|U : U \to X$ is quasi-continuous;*
4. *If T_1, T_2 are closed subsets of T and $f \in \mathrm{Sets}(T_1 \cup T_2, X)$, then f is quasi-continuous iff $f|T_1$ and $f|T_2$ are quasi-continuous.*

A **morphism**

$$X \to X'$$

of quasi-topological spaces is a set-theoretic map which sends quasi-continuous maps into X to quasi-continuous maps into X', i.e., induces a natural transformation

$$f_* : \mathrm{QTop}(\cdot, X) \to \mathrm{QTop}(\cdot, X')$$

We thus have a category, denoted QTop, *of quasi-topological spaces and morphisms between them.*

Observe that a quasi-topological space is the same as specifying a contravariant functor :

$$X : \mathrm{Top} \to \mathrm{Sets}$$

which abides by the sheaf properties for open and finite closed covers on each space T.

Of course, if T is a topological space, it is a quasi-topological space in a natural way, by defining quasi-continuous maps into T to be exactly the continuous maps into T. This defines a **Yoneda embedding**

$$\mathfrak{y} : \mathrm{Top} \to \mathrm{QTop}$$

which allows us to regard the category of topological spaces as a full subcategory of quasi-topological spaces.

Let us also point out the notion of **subspaces**; if X is a set endowed with a quasi-topology, and A is a subset of X, a quasi-topology on A is determined by declaring quasi-continuous those maps $T \to A$ whose composition with the (set-theoretic) inclusion $A \hookrightarrow X$ is quasi-continuous; thus this inclusion is automatically a morphism of quasi-topological spaces.

We also have a product $\times$ in QTop; on the underlying sets, it coincides with the product in Sets, and we define a set-theoretic map

$$T \to X \times X'$$

to be quasi-continuous iff its composition with each (set-theoretic) projection is quasi-continuous. Moreover, if we topologize the hom-sets $\mathrm{QTop}(X, X')$ by defining as quasi-continuous those (set-theoretic) maps

$$T \to \mathrm{QTop}(X, X')$$

for which

$$T \times X \to X'$$

is a morphism of quasi-topological spaces, then it is clear that we have natural isomorphisms

$$\mathrm{QTop}(X, \mathrm{QTop}(X', X'')) \simeq \mathrm{QTop}(X \times X', X'')$$

As for more general limits : suppose

$$F : J \to \mathrm{QTop}$$

is a functor from a (small) index category J, and let

$$E : \mathrm{QTop} \to \mathrm{Sets}$$

be the functor that forgets the quasi-topological structure. We define the limit of $F : J \to \mathrm{QTop}$, as a set, to be the limit of $EF : J \to \mathrm{Sets}$, with the following quasi-topology : a set-theoretic

$$f : T \to \varprojlim F$$

is quasi-continuous iff its composition with all natural arrows

$$\varprojlim F \to F(j)$$

is quasi-continuous.

Similarly for colimits, we first consider the colimit construction of EF, and then declare a

$$f : T \to \varinjlim F$$

quasi-continuous iff each

$$T \times_{\varinjlim EF} F(j) \to F(j)$$

is quasi-continuous.

For mental-sanity preservation, let us refer to quasi-topological spaces as *quasi-spaces*, and morphisms between them as *maps*. Also, set $X(T) := \mathrm{QTop}(T, X)$.

Observe furthermore that the notion of homotopy type is simplicially imposed on the category of quasi-spaces :

Definition 28. *A* **homotopy** *f between two quasi-continuous maps $f_0, f_1 \in X(T)$ is an element $f \in X(T \times I)$ such that*

$$f_i = f | T \times \{i\}$$

in which case f_0 and f_1 will be called homotopic*.*

This defines an equivalence relation on $X(T)$, the quotient by which we denote $X[T]$.

Definition 29. *A map*

$$\varphi : X \to X'$$

*of quasi-spaces is called an n-***equivalence** *if it the induced*

$$X[S^q] \to X'[S^q]$$

are surjective for $q \leqslant n$ *and injective for* $q < n$*. It is called a* **weak equivalence** *if it is an* n*-equivalence for all* n*. Finally,* φ *is said to be a* **homotopy equivalence** *if*

$$X[T] \to X'[T]$$

are bijections for all spaces T*.*

It should also be noted that, for X a quasi-space, composition with the standard cosimplicial object

$$\Delta \to \mathrm{Top}$$
$$[n] \mapsto \Delta^n$$

defines a simplicial set

$$BX : \Delta^{\mathrm{op}} \to \mathrm{Sets}$$

and, upon realization

$$\| \cdot \| : \mathrm{Sets}^{\Delta^{\mathrm{op}}} \to \mathrm{Top}$$

a true space.

The price we pay for insisting in realizing X by an *actual* space $\|BX\|$ is that we only maintain control on its weak homotopy type[8].

Another notion of homotopy theory of spaces that will be dear to us in the context of quasi-spaces is recalled in the sequel.

Definition 30. *A map*

$$\varphi : X \to Y$$

of quasi-spaces is called a **Serre fibration** *when for each finite polyhedron* P*, every diagram*

$$\begin{tikzcd} P \times \{0\} \arrow[r] \arrow[d] & X \arrow[d, "\varphi"] \\ P \times I \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

is solvable. It is called a **Serre microfibration** *if given such a diagram there is* $\varepsilon > 0$ *such that the following has a solution :*

$$\begin{tikzcd} & P \times \{0\} \arrow[r] \arrow[d] \arrow[dl] & X \arrow[d, "\varphi"] \\ P \times I & P \times [0,\varepsilon] \arrow[l] \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Finally, let

$$\Phi : \mathcal{O}(T)^{\mathrm{op}} \to \mathrm{Top}$$

be a sheaf of spaces on the space T. We have already dscribed a way of constructing a sheaf of quasi-spaces out of Φ, namely, by composition with the Yoneda map $\mathfrak{y}$:

$$\mathfrak{y}\Phi : \mathcal{O}(T)^{\mathrm{op}} \to \mathrm{QTop}$$
$$\mathfrak{y}\Phi(U) : T' \mapsto \mathrm{Top}(T', \Phi(U))$$

But (co-)completeness of QTop allows another natural construction : for each subset $S \subset T$, let

$$S \downarrow \mathcal{O}(T) = \{U \hookrightarrow U' : U, U' \text{ open in } T \text{ and containing } S\}$$

[8]By this we mean that the composite

$$\mathrm{Top} \xhookrightarrow{\mathfrak{y}} \mathrm{QTop} \xrightarrow{\|B\cdot\|} \mathrm{Top}$$

is a levelwise weak equivalence of spaces.

Then consider the $S \downarrow \mathcal{O}(T)^{\mathrm{op}}$ diagram of quasi-spaces induced by $\mathfrak{y}\Phi$, and let

$$\Phi^{\text{✡}}(S) := \lim_{S\downarrow\mathcal{O}(T)^{\mathrm{op}}} \mathfrak{y}\Phi$$

Observe that this actually defines a

$$\Phi^{\text{✡}} : \wp(T)^{\mathrm{op}} \to \mathrm{QTop}$$

where $\wp(T) \supset \mathcal{O}(T)$ denotes the poset of all subsets of T under inclusion, called the **sheaf of germs of sections of** Φ. Note in passing that $\Phi^{\text{✡}}(T) = \mathfrak{y}(\Phi(T))$, that $\Phi^{\text{✡}}(x) = \mathfrak{y}(\Phi_x)$ for all $x \in T$ and also that the construction of $\Phi^{\text{✡}}$ out of Φ makes sense (deleting the $\mathfrak{y}$'s throughout) if Φ were a sheaf of quasi-spaces from the beginning.

In dealing with germs in so pervasive a manner, it is useful to agree on the following figure of speech :

Convention 31. *Let $S \subset T$ be an arbitrary subset. Let us convene that* $\mathrm{Op}\, S$ *will stand for an* **opening** *of S in T; that is : any open set U containing S, treated* as a variable.

Thus when we say that a statement Ⓢ holds true on $\mathrm{Op}\, S$*, what we actually mean, in full-fledged form, is that*

"There exists an open neighborhood $U_{Ⓢ}$ of S in T where Ⓢ holds true".

Observe the dependence of U on both S and *Ⓢ.*

Under this convention, it is natural to think of the "value" of a sheaf $F \in \mathrm{Sh}(T, \mathcal{C})$ at $\mathrm{Op}\, S$ *as the cocone*

$$F : (S \downarrow \mathcal{O}(T))^{\mathrm{op}} \to \mathcal{C}$$

In particular, when $\mathcal{C} = \mathrm{QTop}$, the cocone $F(\mathrm{Op}\, S)$ has a coliimit, namely, $F^{\text{✡}}(S)$.

¡ Caveat !. *Except for this section, we will abuse notation and drop the symbol ✡ from $\Phi^{\text{✡}}$. It should be clear from the context which of $\Phi, \Phi^{\text{✡}}$ is being discussed.*

(As a rule of thumb, if $S \subset T$ is not *open, $\Phi(S)$ stands for $\Phi^{\text{✡}}(S)$. The only ambiguity occurrs when S is open.)*

We wish to point out an important fact : sheaves of spaces

$$\Psi : \mathcal{O}(T)^{\mathrm{op}} \to \mathrm{Top}$$

are stable under push-forwards, i.e., if

$$f : T \to T'$$

is continuous, then $f_*\Psi$ naturally inherits the structure of a sheaf of spaces on T'.

On the other hand, sheaves of spaces are *not* stable under base change : there is *no natural structure* of a sheaf of spaces on $f^*\Psi$ for a continuous

$$f : T' \to T$$

This is another inconvenience that is overcome by the quasi-space formalism : if

$$\Psi : \mathcal{O}(T)^{\mathrm{op}} \to \mathrm{QTop}$$

and $f : T' \to T$ is continuous, then we define $f^*\Psi$ first as a sheaf of sets in the usual way, i.e., $f^*\Psi(U') = \Gamma(U', T' \times_T \mathrm{Etale}\,\Psi)$:

$$\begin{tikzcd} T' \times_T \mathrm{Etale}\,\Psi \arrow[r] \arrow[d] & \mathrm{Etale}\,\Psi \arrow[d] \\ T' \arrow[r, "f"'] & T \end{tikzcd}$$

Now a quasi-topology is constructed on $f^*\Psi(U')$ by singling out those set-theoretic maps

$$T'' \to f^*\Psi(U')$$

whose corresponding

$$T'' \times U' \to \text{Etale}\,\Psi$$

are continuous. This is clearly consistent with the restrictions on $f^*\Psi$.

5. Differential Relations and the h-zoo

Assume given a smooth locally trivial fibre bundle $p : E \longrightarrow V$.

Associated to it we have the the jet fibrations $p^r : J^r E \to V$ and its associated fibrations :

$$p^r_s : J^r E \longrightarrow J^s E \text{ for } 0 \leqslant s \leqslant r, \quad J^0 E = E,$$
$$p^r_t = p^r_s \circ p^s_t \text{ for } t \leqslant s \leqslant r$$

together with a sequence of jet maps at the level of sections:

$$j^r : \Gamma(V, E) \longrightarrow \Gamma(V, J^r E), \quad p^r_s \circ j^r = j^s.$$

Definition 32. *Sections in the image of j^r are called* **holonomic**.

Observe these form a very “thin” subspace inside $\Gamma(V, J^r E)$.

Definition 33. *A* **differential relation** $\mathcal{R}$ **of order** r **in** E *is defined to be an arbitrary subspace of $J^r E$.*

Let Φ denote the sheaf of *holonomic* solutions to $\mathcal{R}$, i.e.,

$$\Phi : \mathcal{O}(V)^{\text{op}} \to \text{Top}$$
$$\Phi(U) := \{f \in \Gamma(U, E) : j^r f(U) \subset \mathcal{R}\}$$

equipped with the compact-open C^r topology. We will refer to sections of Φ as **solutions** to $\mathcal{R}$.

On the other hand, there is the sheaf of **sections** of $\mathcal{R}$:

$$\Gamma(\cdot, \mathcal{R}) : \mathcal{O}(V)^{\text{op}} \to \text{Top}$$
$$\Gamma(U, \mathcal{R}) := \{F \in \Gamma(U, J^r E) : F(U) \subset \mathcal{R}\}$$

which we again endow with the compact-open topology.

There is then defined a natural continuous map

$$j^r : \Phi \to \Gamma(\cdot, \mathcal{R})$$

and it is clear that a necessary condition for a solution to $\mathcal{R}$ to exist is that there be *sections* of $\mathcal{R}$.

The meta-h-principle can be summarized as follows :

Meta-Principle. *If $\mathcal{R}$ is “soft”, then the above obstruction is complete up to homotopy. That is :*

> *any continuous family of sections of $\mathcal{R}$ can be deformed to a continuous family of* solutions *to $\mathcal{R}$.*

This meta-principle introduces a notion of “softness” for problems of differential nature, which turns out to be (in a sense not to be made precise in this note) quite *generic*.

It turns out to be very useful then to know whether a given problem (usually of differential geometric nature) is soft or not; a criterion for “softness” developed by Gromov will be discussed later on.

This very vague notion of “softness” can be given more concreteness through the following taxonomy :

DEFINITION 34. *The h-**principle** is said to hold for a differential relation $\mathcal{R}$ if the natural map*

$$j^r : \Phi(V) \to \Gamma(V, \mathcal{R})$$

is a weak equivalence.

*The e-**principle** is said to hold if this map induces a bijection between connected components.*

*The e- or h-principle is said to hold in C^0-**dense fashion** if the homotopies implied in the above definitions can be chosen to be arbitrarily C^0-small; i.e., if the image of the natural map*

$$\Phi(V) \to C^0_{Whit}(V, E)$$

is dense in the image of the also natural

$$\Gamma(V, \mathcal{R}) \to C^0_{Whit}(V, E)$$

There also exist *local* and *relative* formulations :

DEFINITION 35. *Let $V_0 \subset V$. Then we say that $\mathcal{R}$ abides by the* **local h-principle at V_0** *if*

$$j^r : \Phi(V_0) \to \Gamma(V_0, \mathcal{R})$$

is a weak equivalence; here we regard Φ and $\Gamma(\cdot, \mathcal{R})$ as the associated sheaves of germs of the original $\Phi, \Gamma(\cdot, \mathcal{R})$.

The **local e-principle** and the **C^0-dense local** h and **e-principles** can be defined through the obvious modifications of the above.

Finally, suppose $V_1 \subset V$, and that $f_1 \in \Phi(V_1)$. Let

$$\Phi(V, f_1) := \{f \in \Phi(V) : \mathrm{germ}_{V_1} f = f_1\}$$
$$\Gamma(V, j^r f_1, \mathcal{R}) := \{F \in \Gamma(V, \mathcal{R}) : \mathrm{germ}_{V_1} F = j^r f_1\}$$

with the quasi-topolgy induced from $\Phi, \Gamma(\cdot, \mathcal{R})$.

DEFINITION 36. *The* **relative h-principle** *holds for the pair (V, V_1) if*

$$j^r : \Phi(V, f_1) \to \Gamma(V, j^r f_1 \mathcal{R})$$

is a weak equivalence for all $f_1 \in \Phi(V_1)$.

The relative e-principle, C^0-dense h- and e-principles can again be defined. The local *version of the relative h-principle on a triple (V, V_0, V_1), $V_1 \subset V_0$, claims a weak equivalence for*

$$j^r : \Phi(V_0, f_1) \to \Gamma(V_0, j^r f_1, \mathcal{R})$$

where (obviously)

$$\Phi(V_0, f_1) := \{f \in \Phi(V_0) : \mathrm{germ}_{V_1} f = f_1\}$$
$$\Gamma(V_0, j^r f_1, \mathcal{R}) := \{F \in \Gamma(V_0, \mathcal{R}) : \mathrm{germ}_{V_1} F = j^r f_1\}$$

6. Sheaves of parametric germs and the sheaf-theoretic h-principle

Let

$$\Phi : \mathcal{O}(V)^{\mathrm{op}} \to \mathrm{Top}$$

be a sheaf of spaces on V.

Given a space T, the bifunctor

$$\Phi^T : \mathcal{O}(T)^{\mathrm{op}} \times \mathcal{O}(V)^{\mathrm{op}} \to \mathrm{QTop}$$
$$\Phi^T(T' \times V') := \mathrm{Top}(T', \Phi(V'))$$

with the quasi-topology described by

$$\mathrm{QTop}(T'', \Phi^T(T' \times V')) \simeq \mathrm{Top}(T'' \times T', \Phi(V')) \quad \text{for all spaces } T''$$

extends uniquely to a sheaf of quasi-spaces

$$\Phi^T : \mathcal{O}(T \times V)^{\mathrm{op}} \to \mathrm{QTop}$$

Concretely speaking, a section

$$f \in \Phi^T(W)$$

can be represented, on an open product cover

$$\mathfrak{U} = \{T_i \times V_i\}, \quad \bigcup_i T_i \times V_i = W$$

by continuous functions

$$f_i : T_i \to \Phi(V_i)$$

abiding by

$$\begin{tikzcd} T_i \arrow[r, "f_i"] & \Phi(V_i) \arrow[d] \\ T_i \cap T_j \arrow[u] \arrow[d] & \Phi(V_i \cap V_j) \\ T_j \arrow[r, "f_j"'] & \Phi(V_j) \arrow[u] \end{tikzcd}$$

Let now

$$\Delta_V : V \to V \times V$$

and set $\Phi^\flat := \Delta_V^* \Phi^V$.

Definition 37. *We will call $\Phi^\flat$ the* **sheaf of parametric germs** *of Φ.*

Observe the natural map

$$\Phi \to \Phi^\flat$$

of (the associated sheaf of quasi-spaces of) Φ to its sheaf of parametric germs, defined by assigning to each section of Φ its family of germs at points :

$$\begin{aligned} \Phi(U) &\to \Phi^\flat(U) \\ f &\mapsto \mathrm{germ}_{\Delta_U}(f \circ \mathrm{pr}_2) \end{aligned}$$

where $f \circ \mathrm{pr}_2$ is thought of as a section of $\Phi^U(U \times U)$.

Remark 38. *Briefly ressucitating the notational distinction between Φ and $\Phi^{\text{✡}}$, we see that $\Phi^\flat$ is described as*

$$\Phi^\flat = \left(\Phi^V\right)^{\text{✡}}(\Delta_V)$$

Suppose now that Φ is the sheaf of solutions to some differential relation $\mathcal{R} \subset J^r E$. Then there is also a natural map

$$\begin{aligned} \Phi^\flat &\to \Gamma(\cdot, \mathcal{R}) \\ F &\mapsto [v \mapsto j^r_v F(\cdot, v)] \end{aligned}$$

which assigns to each family of parametric germs of Φ the corresponding family of parametric r-jets. rendering the diagram below commutative :

$$\begin{tikzcd} \Phi \arrow[r, "j^r"] \arrow[d, "\flat"'] & \Gamma(\cdot, \mathcal{R}) \\ \Phi^\flat \arrow[ur] & \end{tikzcd}$$

Definition 39. *A differential relation* $\mathcal{R}$ *is* **germifiable** *if for all compact* $C' \supset C$*,*

$$\begin{tikzcd} \Phi^\flat(C') \arrow[r] \arrow[d] & \Gamma(C', \mathcal{R}) \arrow[d] \\ \Phi^\flat(C) \arrow[r] & \Gamma(C, \mathcal{R}) \end{tikzcd}$$

are such that the induced

$$\Phi^\flat(C') \to \Gamma(C', \mathcal{R}) \times_{\Gamma(C,\mathcal{R})} \Phi^\flat(C)$$

is a weak equivalence over $\Gamma(C, \mathcal{R})$*.*

Of course, all *open* relations are germifiable.

For *closed* relations this need not be the case; recall that there exist smooth linear differential operators which admit no local solution, see e.g. [**47**]. On a more geometric note, there exist Riemannian manifolds V_0, V_1 for which the isometric immersion relation $V_0 \to V_1$ is *not* germifiable, see [**23**].

In any case, the **sheaf-theoretic version** of the h-principle is defined by replacing $\Phi^\flat$ for $\Gamma(\cdot, \mathcal{R})$ in the string of definitions of Section 5 – and thus clearly makes sense, except for the dense incarnation, for every sheaf of quasi-spaces on V, whether there is an underlying differential relation or not.

And, of course, for germifiable relations, the sheaf-theoretic h-principle implies the (usual) h-principle.

CHAPTER 2

Towards the h-principle

The goal of this Chapter is to show how to cook up solutions (or extensions to partial solutions) to invariant differential relations from solutions to old-fashioned obstruction-theoretic problems involving the pertinent classifying spaces. This is an old trick of Hæfliger's [**40**] as seen and generalized by Gromov in his book [**32**].

1. Gromov groupoids of diff-invariant relations

Let $\mathfrak{D}$ be a pseudogroup of diffeomorphisms of V (see Chapter 1, Section 2), and suppose $p : E \to V$ is a smooth fibre bundle.

Definition 40. *An* **extension** *of $\mathfrak{D}$ to E is a pseudogroup of diffeomorphisms $\tilde{\mathfrak{D}}$ of E and a map*

$$\begin{aligned} \mathfrak{D} &\to \tilde{\mathfrak{D}} \\ h &\mapsto h_* \end{aligned}$$

satisfying :

- $\operatorname{dom} h_* = p^{-1} \operatorname{dom} h$ *and* $\operatorname{codom} h_* = p^{-1} \operatorname{codom} h$*;*
- $(\mathrm{id}_U)_* = \mathrm{id}_{p^{-1}U}$*;*
- $(h' \circ h)_* = h'_* \circ h_*$

Such an extension is called **continuous** *if for all open $U \subset V$, the induced*

$$\mathrm{Diff}(U) \to \mathrm{Diff}(p^{-1}U)$$

is continuous.

Example 5. *Tensor bundles, Grassmann bundles, and such, all admit natural continuous extensions of* $\mathrm{diff}(V)$*.*

Observe that such a continuous extension induces continuous extensions to *all* jet bundles of E.

Assume then given a continuous extension of some $\mathfrak{D}$ to E. This provides an action of $\mathfrak{D}$ on sections of E, hence of $J^r E$.

Definition 41. *The sheaf $\Phi \subset \Gamma(\cdot, E)$ is called $\mathfrak{D}$-invariant if it is left invariant by $\tilde{\mathfrak{D}}$.*

Observe that an action of some $\mathfrak{D}$ on Φ has a natural prolongation to an action of $\mathfrak{D}$ on $\Phi^\flat$; concretely,

$$\begin{gathered} d \in \mathfrak{D}, \quad F \in \Phi^\flat(\operatorname{dom} d) \\ (d_\flat F) : (v_1, v_2) \mapsto d_* F(d^{-1}v_1, d^{-1}v_2) \end{gathered}$$

Bear in mind that in the case of *natural* bundles there is a canonical extension of $\mathfrak{D}$. We also refer to "invariance" *tout court* to imply $\mathrm{diff}(V)$-invariance.

Moreover, if $\Gamma_V \rightrightarrows V$ denotes the (étale) topological groupoid of germs of local diffeomorphisms,

$$\Gamma_V = \{\mathrm{germ}_v\, d : d \in \mathrm{diff}(V), v \in \operatorname{dom} d\}$$

with the sheaf topology, and Γ is the subgroupoid consisting of germs of those local diffeomorphisms in $\mathfrak{D}$, the invariance hypothesis gives continuous actions

$$\mathrm{Etale}(\Phi) \times_V \Gamma \to \mathrm{Etale}(\Phi)$$
$$\mathrm{Etale}(\Phi^\flat) \times_V \Gamma \to \mathrm{Etale}(\Phi^\flat)$$

We will denote the associated action groupoids by Σ_Φ and $\Sigma_\Phi^\flat$, respectively, and call them the **Gromov groupoids** of $\Phi, \Phi^\flat$.

(Note that another description of these groupoids is obtained by considering the étale spaces associated to $\Phi, \Phi^\flat$ as non-Hausdorff, smooth n-dimensional manifolds, obtained by glueing of local sections; then $\mathfrak{D}$ can naturally be regarded as a pseudogroup of local diffeomorphisms of these étale spaces, and thus give rise to groupoids by taking germs at points.)

Observe that the natural

$$\Phi \to \Phi^\flat$$

induces a continuous homomorphism

$$\Sigma_\Phi \to \Sigma_\Phi^\flat$$

as well as forgetful maps

$$\Sigma_\Phi, \Sigma_\Phi^\flat \to \Gamma$$

Let us now turn to case of $\mathrm{diff}(V)$ invariant sheaves $\Phi, \Phi^\flat$. Such sheaves are equivariantly modelled on sheaves on $\mathbb{R}^n$,by which we mean that there is a $\mathrm{diff}(\mathbb{R}^n)$-invariant sheaf Ψ on $\mathbb{R}^n$, such that for every point $x \in V$ there is an open neighbourhood U and compatible equivariant isomorphisms

$$\begin{tikzcd}
\Phi|U \arrow[r] \arrow[d, "\simeq"] & \Phi^\flat|U \arrow[d, "\simeq"'] \\
\Psi|U \arrow[r] & \Psi^\flat|U
\end{tikzcd}$$

Definition 42. *A* **Gromov Φ-structure** *on a topological space T is a Hæfliger Σ_Ψ-structure on T; the collection of all such is denoted by $H^1(T, \Phi)$. Similarly, a* **Gromov $\Phi^\flat$-structure** *on T is a Hæfliger $\Sigma_\Psi^\flat$-structure on T; we denote their reunion by $H^1(T, \Phi^\flat)$.*

Remark 43. *Observe that the above construction/definition depends only on the local equivariant model Ψ of Φ, and not on Φ itself.*

Let us unwind the above definition. If a Gromov Φ-structure $\varphi \in H^1(T; \Phi)$ is represented by a cocycle

$$F : T_{\mathfrak{U}} \to \Sigma_\Psi$$

then it actually consists of the following data :

(1) Continuous maps
$$\varphi_i : U_i \to \mathrm{Etale}(\Psi)$$
(2) For each i, j, a continuous
$$\gamma_{ji} : U_i \cap U_j \to \Gamma_n$$
such that
$$\gamma_{ji*}^x \varphi_i^x = \varphi_j^x$$

On the other hand, a cocycle

$$F : T_{\mathfrak{U}} \to \Sigma_{\Psi}^{\flat}$$

consists of :

(1) For each i, there is a continuous map

$$f_i : U_i \to \mathrm{Etale}(\Psi)^{\flat}$$

(2) For each i, j, a continuous map

$$\gamma_{ji} : U_i \cap U_j \to \Gamma_n$$

such that

$$\gamma_{ji*}^{x} \varphi_i^x = \varphi_j^x$$

But observe that to give a point in $\mathrm{Etale}(\Psi^{\flat})$ above a $v \in V$ is the same as giving a germ at v of a continuous map

$$\mathrm{Op}(v) \to \Psi(\mathrm{Op}(v))$$

so that the data defining a Gromov $\Phi^{\flat}$-structure consists of

- Quasi-continuous

$$f_i : U_i \to \mathrm{Etale}(\Psi)^{\mathrm{top}}$$

and quasi-continuous

$$\gamma_{ij} : U_i \cap U_j \to \Gamma_n$$
$$\gamma_{ij*}^{v} f_i(v) = f_j(v)$$

where $\mathrm{Etale}(\Psi)^{\mathrm{top}}$ is defined as having the same point-set as $\mathrm{Etale}(\Psi)$ and a map

$$f : T \to \mathrm{Etale}(\Psi)^{\mathrm{top}}$$

is distinguished if the composition $pf : T \to \mathbb{R}^n$ is continuous and for each $t_0 \in T$ there is $t_0 \in T' \subset T$ and $pf(t_0) \in U \subset V$ and a continuous map

$$F : T' \to \Psi(U)$$

with

$$f(t) = \mathrm{germ}_{pf(t)} F(t)$$

for all $t \in T'$.

In this parlance, those quasi-continuous maps $f : T \to \mathrm{Etale}(\Psi)^{\mathrm{top}}$ which factor through the natural quasi-continuous map

$$\mathrm{Etale}(\Psi) \to \mathrm{Etale}(\Psi)^{\mathrm{top}}$$

are those for which the F in the description above is *locally constant*, i.e., T' being chosen small enough, F should factor through $\mathrm{pt} \to \Psi(U)$.

Recall also that we have natural (forgetful) transformations

$$\mathfrak{H} : H^1(\cdot; \Phi), H^1(\cdot; \Phi^{\flat}) \to H^1(\cdot; \Gamma_n)$$

so we are entitled to speak of the Gromov $\Phi^{\flat}$-structure **underlying** a given Gromov Φ-structure, and of the Γ_n-structure **underlying** a given Φ- or $\Phi^{\flat}$-structure.

Note that if V is an n-manifold there is a well-defined

$$\mathfrak{G}_{\bullet} : \Phi(V) \to \mathfrak{H}^{-1}(\tau_V) =: \mathfrak{H}(V)$$
$$\varphi \mapsto \mathfrak{G}_{\varphi}$$

where $\{(U_i, \gamma_{ii*}(\varphi|U_i), \gamma_{ji})\}$ represents $\mathfrak{G}_{\varphi}$ and $\{(U_i, \gamma_{ii}, \gamma_{ji})\}$ represents the differential structure τ_V of V.

Conversely, if $F = [\{(U_i, \psi_i, \gamma_{ji})\}]$ represents a Gromov Φ-structure on V, and

$$\tau_V = \mathfrak{H}F \in H^1(V; \Gamma_n)$$

then

$$\varphi_F(x) := (\gamma_{ii}^x)_*^{-1}\, \psi_i^x$$

is a solution to Φ, $\varphi_F \in \Phi(V)$, and clearly

$$\mathfrak{G}_{\varphi_F} = F, \quad \varphi_{\mathfrak{G}_\varphi} = \varphi$$

Thus

$$\mathfrak{G}_\bullet : \Phi(V) \leftrightarrows \mathfrak{H}(V) : \varphi_\bullet$$

are mutually inverse bijections :

Proposition 44. *Solutions to Φ can be identified with Gromov Φ-structures lying above τ_V.*

Definition 45. *Let*

$$F \in H^1(T; \Phi)$$

be a Gromov Φ-structure.

A **graph** *of F consists of :*

1. *a bundle $p : E(F) \to T$ with fibres diffeomorphic to $\mathbb{R}^n$;*
2. *a section $s : T \to E(F)$;*
3. *a Gromov Φ-structure $\mathfrak{G}(F) \in H^1(E(F); \Phi)$, whose underlying Γ_n-structure defines a foliation transverse to the fibres of $E(F) \to T$, and such that the following* **tautological equation** *holds :*

$$s^*\mathfrak{G}(F) = F$$

Two such data $(E_0, p_0, s_0, \mathfrak{G}_0), (E_1, p_1, s_1, \mathfrak{G}_1)$ are identified if they they have isomorphic germs along the sections :

$$\mathrm{germ}_{s_0T}(E_0, p_0, \mathfrak{G}_0) \simeq \mathrm{germ}_{s_1T}(E_1, p_1, \mathfrak{G}_1)$$

.

We proceed to construct a graph for Gromov Φ-structures over a (paracompact) T:

Proposition 46. *Let T be paracompact. Then every Gromov Φ-structure on T has a graph.*

Proof. Let a cocycle for Gromov Φ-structure

$$F : T_{\mathfrak{U}} \to \Sigma_\Phi$$

be given, and set $s_i := p \circ \varphi_i$.

By the sheaf topology in Etale(Ψ), we can assume (at the expense of refining $\mathfrak{U}$), that the φ_i are *globally defined*, i.e., that there exist open sets

$$s_i(U_i) \subset W_i \subset \mathbb{R}^n$$

and sections $\phi_i \in \Psi(W_i)$ with

$$U_i \ni x \Longrightarrow \varphi_i^x = \mathrm{germ}_{s_i(x)}\, \phi_i$$

If T is paracompact, we assume further that $\mathfrak{U}$ is *locally finite*, i.e., that the sets

$$J(x) := \{i : U_i \cap \mathrm{Op}(x) \neq \varnothing\}$$

are finite for each $x \in T$.

Fix now $x_0 \in T$. Given i, j distinct indices in $J(x_0)$, the equation

$$\gamma_{ji*}^x \varphi_i^x = \varphi_j^x$$

implies the existence of :

(1) an open subset $U_{i,j}^{x_0} \subset U_i \cap U_j$;
(2) open subsets

$$s_i(x_0) \in O_{ij}^{x_0} \subset W_i$$
$$s_j(x_0) \in O_{ji}^{x_0} \subset W_j$$

(3) a diffeomorphism

$$\sigma_{ji}^{x_0} : O_{ij}^{x_0} \xrightarrow{\simeq} O_{ji}^{x_0}$$

satisfying

$$\sigma_{ji*}^{x_0}(\phi_i | O_{ij}^{x_0}) = \phi_j | O_{ji}^{x_0}$$
$$\gamma_{ji}^{x} = \mathrm{germ}_{s_i(x)}\, \sigma_{ji}^{x_0}$$

for all $x \in U_{i,j}^{x_0}$.

Set then

$$U^{x_0} := \bigcap_{j \neq i} U_{i,j}^{x_0}$$

and note that the equation

$$\gamma_{kj}^{x_0} \gamma_{ji}^{x_0} = \gamma_{ki}^{x_0}$$

for each $i, j, k \in J(x_0)$ implies that for some smaller $O_{ijk}^{x_0} \subset O_{ij}^{x_0} \cap O_{ik}^{x_0}$, this equality hold at the level of representatives :

$$\sigma_{kj}^{x_0} \circ \sigma_{ji}^{x_0} = \sigma_{ki}^{x_0}$$

so that we can choose open sets

$$O_i^{x_0} \subset \bigcap_{j,k} O_{ijk}^{x_0}$$

so that, for $x \in U^{x_0}$ and all i, j, k, we have

$$\varphi_i^{x} = \mathrm{germ}_{s_i(x)}\, \phi_i$$
$$\gamma_{ji}^{x_0} = \mathrm{germ}_{s_i(x)}\, \sigma_{ji}^{x_0}$$

Construction of $E(F)$: For each i, we consider a neighbourhood $\mathcal{O}_i$ of the graph of

$$s_i : U_i \to \mathbb{R}^n$$

of the form

$$\mathcal{O}_i = \bigcup_{x \in U_i} U^x \times O_i^x$$

After possibly shrinking the $\mathcal{O}_i$, we can identify $\mathcal{O}_i$ with $\mathcal{O}_j$ over $U_i \cap U_j$ by the maps

$$(x, y) \mapsto (x, \sigma_{ji}^{x}(y))$$

and define a fibre bundle with fibres *diffeomorphic* to $\mathbb{R}^n$,

$$E(F) \to T$$

obtained from

$$\mathrm{pr}_1 : \mathcal{O}_i \to \mathrm{pr}_1(\mathcal{O}_i) \subset T :$$

by glueing, after having possibly shrunk the $\mathcal{O}_i$'s once more.

This bundle comes equipped with some extra structure arising tautologically from F, namely :

A section $s : T \to E(F)$: Indeed, the local sections s_i of $\mathrm{pr}_1 : \mathcal{O}_i \to \mathrm{pr}_1(\mathcal{O}_i)$ glue into a global section $s : T \to E(F)$ according to

$$\sigma_{ji}^{x} \circ s_i = s_j \quad \text{on } U^x$$

A Gromov Φ-structure $\mathfrak{G}(F)$: Observe that

$$\bigcup_{x \in U_i} U^x \times O_i^x$$

carries a Gromov Φ-structure $\mathfrak{G}_i(F)$ induced from $\phi_i \in \Psi(W_i)$ by pr_2. Thus the underlying Γ_n-structure $P\mathfrak{G}_i(F)$ is that induced from pr_2 by the foliation of $\mathbb{R}^n$ by points – and is thus transverse to the pr_1-fibres.

Now, the equation

$$\sigma^x_{ji*}\phi_i = \phi_j \quad \text{on } U^x$$

implies that such partial Gromov structures $\mathfrak{G}_i(F)$ agree on overlapping domains, thus giving rise to

$$\mathfrak{G}(F) \in H^1(E(F); \Phi)$$

with $P\mathfrak{G}(F)$ transverse to the fibres of $E(F) \to T$.

Finally, observe that

$$s^*\mathfrak{G}(F) = F$$

holds tautologically. ☐

We highlight a few points that follow from the proof of the above Proposition :

(1) First, observe that, for any smooth manifold (of arbitrary finite dimension), the correspondence

$$H^1(W, \Gamma_n) \ni F \mapsto \mathfrak{G}(F) \in H^1(E(F), \Gamma_n)$$

factors through the inclusion

$$\mathrm{Fol}^n_{\pitchfork}(E(F) \to W) \to H^1(E(F), \Gamma_n)$$

(2) Moreover, observe the commutativity of

$$\begin{array}{ccc} H^1(V;\Phi) & \xrightarrow{\mathfrak{G}} & H^1(E(\cdot), \Phi) \\ \downarrow{\scriptstyle \mathfrak{H}} & & \downarrow{\scriptstyle \mathfrak{H}} \\ H^1(V;\Gamma_n) & \xrightarrow{\mathfrak{G}} & H^1(E(\cdot), \Gamma_n) \end{array}$$

and that, in fact,

$$\mathfrak{G}(\mathfrak{H}F) = \mathfrak{H}\mathfrak{G}(F) \quad \text{and} \quad E(F) = E(\mathfrak{H}F)$$

2. Flexibility

In the next three sections we follow [**32**]'s exposition closely.

This section deals with the homotopy-theoretic condition on a sheaf of quasi-spaces Φ to ensure that its weak homotopy type is recovered from its *local* weak homotopy type.

Suppose Φ is a sheaf of quasi-spaces on V, $\Phi : \mathcal{O}(V)^{\mathrm{op}} \to \mathrm{QTop}$.

Observe that, for any *subset* S of V, the sheaf of Φ-germs along S *naturally inherits* the structure of a sheaf of quasi-spaces on S, and inclusions of subsets $S \subset S'$ give rise to morphisms of such sheaves.

DEFINITION 47. *We say that a sheaf of quasi-spaces Φ is* **flexible** *over a nested pair of compact sets $C' \supset C$ if the induced*

$$\Phi(C') \to \Phi(C)$$

is a Serre fibration.

A sheaf Φ is called **flexible** *if it is flexible over all pair of nested compact sets (C', C).*

EXAMPLE 6. $\mathrm{Top}(\cdot, T)$ *with the compact-open topology, where* T *is a fixed topological space, is flexible.*

The sheaf $\mathrm{Hol}(\cdot; J^r E)$ *of holonomic sections of* $J^r E \to V$,

$$\mathrm{Hol}(U, J^r E) = \{F : U \to J^r E : F = j^r f, f \in \Gamma(U, E)\}$$

is flexible.

The sheaf $Z^p_{dR}(\cdot)$ *of closed* p*-forms is* not *flexible over* $(D^{p+1}, \partial D^{p+1})$*, having Stokes' theorem as obstruction; thus it is not flexible. But the sheaf* $B^p_{dR}(\cdot)$ *of* exact p*-forms* is *flexible.*

LEMMA 48. *A sheaf* Φ *of quasi-spaces on* V *is flexible iff the restriction of* Φ *to every simplex of a triangulation of* V *is flexible.*

PROOF. The "only if" is obvious, and the "if" part can be dealt with by this inductive argument:

Choose a triangulation of V and denote by Σ the resulting simplicial set. Define $\mathrm{Sk}^k \Sigma$ to be the subcomplex generated by the k-simplices of Σ. Suppose now that $\Phi || \mathrm{Sk}^k \Sigma|$ was proven to be flexible. Let $C \subset C' \subset |\mathrm{Sk}^{k+1} \Sigma|$ be a pair of nested compact sets, and let there be given

$$\begin{tikzcd} P \times \{0\} \arrow[r] \arrow[d] & \Phi(C') \arrow[d] \\ P \times I \arrow[r, "h"'] \arrow[ur, dashed, "h'"] & \Phi(C) \end{tikzcd}$$

Observe that

$$\begin{tikzcd} \Phi(|\mathrm{Sk}^{k+1} \Sigma|) \arrow[r] \arrow[d] & \prod_\sigma \Phi(\sigma) \arrow[d] \\ \Phi(|\mathrm{Sk}^k \Sigma|) \arrow[r] & \prod_\sigma \Phi(\partial\sigma) \end{tikzcd}$$

is a pullback diagram, where σ ranges over non-degenerate $(k+1)$-simplices of Σ, and that by hypothesis

$$\prod_\sigma \Phi(\sigma) \to \prod_\sigma \Phi(\partial\sigma)$$

$$\Phi(|\mathrm{Sk}^k \Sigma|) \to \prod_\sigma \Phi(\partial\sigma)$$

are Serre fibrations.

Hence a solution to the induced

$$\begin{tikzcd} P \times \{0\} \arrow[r] \arrow[d] & \prod_\sigma \Phi(\sigma \cap C') \arrow[d] \\ P \times I \arrow[r] \arrow[ur, dashed] & \prod_\sigma \Phi(\partial\sigma \cap C) \end{tikzcd}$$

can be lifted to a solution of the original diagram. ☐

By refining triangulations, we conclude that :

COROLLARY 49. Φ *is flexible iff it is locally flexible, i.e., iff* Φ *restricts to flexible sheaves over the open sets of a covering of* V.

Being flexible means that deformations of section over compact sets C

$$P \times I \to \Phi(C)$$

can be represented by deformations

$$P \times I \to \Phi(U), \quad U \supset C$$

which are stationary (i.e., independent of $t \in I$) outside as small a neighborhood of C as we like. This is made precise in the definition and lemma below.

DEFINITION 50. *Let P be a compact polyhedron, $C \subset V$ compact and*

$$h : P \times I \to \Phi(U), \quad U \supset C$$

a homotopy, and

$$C \subset U_0 \subset \mathrm{Cl}(U_0) \subset U$$

*A C-**compression** of h into U_0 is a new homotopy*

$$\widetilde{h} : P \times [0, 1] \to \Phi(U)$$

of the same $h|P \times \{0\}$, which is independent of $t \in [0, 1]$ outside U_0, and which coincides with h near C.

A homotopy h as above will be called **compressible** *if, for arbitrarily small $U_0 \supset C$, there exist C-compressions $\widetilde{h}$ of h into U_0.*

LEMMA 51. *Let Φ be a sheaf of quasi-spaces. Then Φ is flexible iff for every $C \subset V$ compact, all homotopies*

$$h : P \times I \to \Phi(U), \quad U \supset C$$

are C-compressible.

PROOF. Suppose Φ is flexible. Choose some $U_0 \subset \mathrm{Cl}(U_0) \subset U$ and consider the diagram

$$\begin{tikzcd} P \times \{0\} \arrow[r, "f"] \arrow[d] & \Phi(\mathrm{Cl}\, U_0) \arrow[d] \\ P \times [0,1] \arrow[r, "h"'] \arrow[ur, dashed, "h'"] & \Phi(C \bullet U_0) \end{tikzcd}$$

where $C \bullet U_0 := C \cup (\mathrm{Cl}(U_0) - U_0)$ and the bottom arrow is defined by requiring that it be stationary on $(\mathrm{Cl}(U_0) - U_0)$ and restrict to $h|C$ over C.

Out of h', define $\widetilde{h}$ as

$$\widetilde{h}(p, t)|v = \begin{cases} h'(p, t)|v & \text{if } v \in U_0 \\ h(p, t)|v & \text{otherwise} \end{cases}$$

Conversely, suppose we are given a diagram

$$\begin{tikzcd} P \times \{0\} \arrow[r, "f"] \arrow[d] & \Phi(C') \arrow[d] \\ P \times [0,1] \arrow[r, "h"'] \arrow[ur, dashed, "h'"] & \Phi(C) \end{tikzcd}$$

and we know that h is compressible. Then choose $U_0 \supset C$ with $\mathrm{Cl}(U_0) \subset C'$, and a C-compression $\widetilde{h}$ of h into U_0. Then define

$$h'(p, t)|v = \begin{cases} \widetilde{h}(p, t)|v & \text{if } v \in U_0 \\ f(p)|v & \text{otherwise.} \end{cases}$$

This correctly defines the required lift. $\square$

The importance of flexibility is described in the following

THEOREM 52. *Suppose Φ, Ψ are sheaves of quasi-spaces on V, $\dim V = n$, that $\Psi(V) \neq \varnothing$ and that, for some $k \geqslant 1$,*

$$\varphi : \Phi \to \Psi$$

is a local $(n+k)$-equivalence of sheaves, i.e., that it induces $(n+k)$-equivalences of stalks

$$\varphi_v : \Phi_v \to \Psi_v$$

If Φ, Ψ are flexible, then $\varphi_V : \Phi(V) \to \Psi(V)$ is a k-equivalence.

The claim will be proved in a sequence of lemmas.

Suppose to begin with, let Φ be a sheaf of quasi-spaces on V, and for a finite polyhedron P, let Φ^P be the P-parametric sheaf on $P \times V$.

PROPOSITION 53. *Φ^P is flexible if Φ is flexible.*

PROOF. In view of Lemma 49, it suffices to prove that if $P = \Delta^k$ and $C \subset C'$ are compact sets in V, then $(C' \times P, C \times P \cup C' \times \partial P)$ is Φ^P-flexible.

Let then

$$\begin{tikzcd} Q \times \{0\} \arrow[r] \arrow[d] & \Phi^P(C' \times P) \arrow[d] \\ Q \times I \arrow[r] \arrow[ur, dashed] & \Phi^P(C \times P \cup C' \times \partial P) \end{tikzcd}$$

be given.

These correspond to maps

$$f : P \times Q \to \Phi(C')$$

and

$$\begin{tikzcd} \mathrm{Op}(\partial P) \times Q \times I \arrow[r, "\partial g'"] \arrow[d] & \Phi(C') \arrow[d] \\ P \times Q \times I \arrow[r, "g"'] & \Phi(C) \end{tikzcd}$$

with

$$\partial g' = f \quad \text{on } \operatorname{dom}(f) \cap \operatorname{dom}(\partial g') = \mathrm{Op}(\partial P) \times Q \times \{0\}$$

Therefore, $\partial g'$ and f together define a

$$\begin{tikzcd} (P \times Q \times I)' \arrow[r, "g''"] \arrow[d] & \Phi(C') \arrow[d] \\ P \times Q \times I \arrow[r, "g"'] \arrow[ur, dashed, "g'"] & \Phi(C) \end{tikzcd}$$

where

$$(P \times Q \times I)' := (P \times Q \times \{0\}) \cup (\mathrm{Op}(\partial P) \times Q \times I) \subset P \times Q \times I$$

Since $P \times Q \times I$ deformation retracts onto $(P \times Q \times I)'$, when Φ is flexible we can always solve the latter diagram, and the solution g' corresponds to a $Q \times I \to \Phi^P(C')$ solving our original problem. □

Let now Ψ be another sheaf of quasi-spaces on V, and

$$\varphi : \Phi \to \Psi$$

be a morphism.

Then just as in the case of topological spaces, there is a path-space construction to φ :

$$\widetilde{\Phi} : \mathcal{O}(V)^{\mathrm{op}} \to \mathrm{QTop}$$
$$\widetilde{\Phi}(U) \subset \Phi(U) \times \Psi^I(U)$$
$$\widetilde{\Phi}(U) = \{(\phi, \gamma) : \gamma(0) = \varphi(\phi)\}$$

which comes equipped with a fibration

$$\widetilde{\varphi} : \widetilde{\Phi} \to \Psi$$
$$\widetilde{\varphi}_U(\phi, \gamma) = \gamma(1)$$

and a canonical homotopy equivalence

$$p : \widetilde{\Phi} \leftrightarrows \Phi : j$$
$$p_U(\phi, \gamma) = \phi$$
$$j_U(\phi) = (\phi, \phi)$$

so that

$$\begin{tikzcd}
\Phi \arrow[rr, "\varphi"] \arrow[dr, "j"'] & & \Psi \\
& \widetilde{\Phi} \arrow[ur, "\widetilde{\varphi}"'] &
\end{tikzcd}$$

commutes.

Observe that if P is a finite polyhedron, the assignment

$$\varphi^P : \Phi^P \to \Psi^P$$

commutes with that of path-spaces in the sense that the path-space construction applied to φ^P coincides with applying the parametric functor $-^P$ to the path-space construct of φ.

Lemma 54. *If Φ is flexible, then so is $\widetilde{\Phi}$.*

Proof. The proof of the lemma consists of the observation that $\mathrm{ev}_0 : \Psi^I \to \Psi$ is a fibration, hence $\mathrm{pr} : \widetilde{\Phi} \to \Phi$ is also a fibration, whence h'' below

$$\begin{tikzcd}
P \times \{0\} \arrow[r, "f"] \arrow[d] & \widetilde{\Phi}(C') \arrow[r] \arrow[d] & \Phi(C') \arrow[d] \\
P \times I \arrow[r] \arrow[ur, dashed, "h'"] \arrow[urr, dashed, "h''"] & \widetilde{\Phi}(C) \arrow[r] & \Phi(C)
\end{tikzcd}$$

can be found due to the flexibility of Φ, and this produces h' as a solution to

$$\begin{tikzcd}
P \times \{0\} \arrow[r, "f"] \arrow[d] & \widetilde{\Phi}(C') \arrow[d] \\
P \times I \arrow[r, "h''"'] \arrow[ur, dashed, "h'"] & \Phi(C')
\end{tikzcd}$$

□

Now suppose that $\Phi(V) \ni \psi$ and consider the fibre sheaf

$$\Omega : \mathcal{O}(V)^{\mathrm{op}} \to \mathrm{QTop}$$
$$\Omega(U) := \widetilde{\varphi}^{-1}(\psi|U)$$

Lemma 55. *Ω is also flexible.*

PROOF. The proof consists of the following observation : a lifting problem for $\Omega(C') \to \Omega(C)$ can be written as the problem of finding h' below

$$\begin{tikzcd}
P \times \{0\} \arrow[r] \arrow[d] & \widetilde{\Phi}(C') \arrow[r, "\widetilde{\varphi}"] \arrow[d] & \Psi(C') \arrow[d] \\
P \times I \arrow[r, "\widetilde{\varphi}"'] \arrow[ur, dashed, "h'"] \arrow[urr, dashed, "h''"'] & \widetilde{\Phi}(C) \arrow[r] & \Psi(C)
\end{tikzcd}$$

where the upper composite is the constant map at $\psi|C'$ and the bottom one, that at $\psi|C$. Now $\widetilde{\Phi} \to \Psi$ a fibration means that we can solve

$$\begin{tikzcd}
P \times \{0\} \arrow[r] \arrow[d] & \widetilde{\Phi}(C') \arrow[d] \\
P \times I \arrow[r, "\psi|C'"'] \arrow[ur, dashed, "h'"] & \Psi(C')
\end{tikzcd}$$

and thus solve the original problem. ☐

LEMMA 56. *Let Φ be a flexible sheaf on V, and $f : V \to W$ be a proper map. Then $f_*\Phi$ is a flexible sheaf on W. In particular, push-forwards of flexible sheaves over compact manifolds are always flexible.*

PROOF. Obvious. ☐

LEMMA 57. *If Ω is a flexible, locally 1-connected sheaf on a compact subset $C \subset \mathbb{R}$, then $\Omega(C)$ is connected.*

PROOF. There is no loss of generality in assuming C connected, i.e., $C = [a, b]$. Let ω_0, ω_1 be two global sections. Observe that, for each $v \in V$, there is $\varepsilon = \varepsilon(v, \omega_0, \omega_1) > 0$ such that

$$\omega_0|B_{\frac{\varepsilon(v)}{2}}(v) \text{ can be joined by a path } \lambda_v(t) \text{ to } \omega_1|B_{\frac{\varepsilon(v)}{2}}(v)$$

Let then $0 < 1/N \leqslant \varepsilon(v)$ for all $v \in C$, and decompose C into $C_1 \cup C_2$, where

$$C_1 = \coprod [2i/N, (2i+1)/N]$$
$$C_2 = \coprod [(2i-1)/N, (2i)/N]$$

Thus we have homotopies $\lambda_1(t)$ between $\omega_0|C_1$ and $\omega_1|C_1$, and $\lambda_2(t)$ between $\omega_0|C_2$ and $\omega_1|C_2$, but they do not necessarily agree on the *finite* set $C_1 \cap C_2 = \{i/N\}$.

But since we assume that Ω is locally 1-connected, there exists a homotopy ν between $\lambda_1|C_1 \cap C_2$ and $\lambda_2|C_1 \cap C_2$:

$$\begin{tikzcd}
I \times \{0\} \arrow[r, "\lambda_1"] \arrow[d] & \Omega(C_1) \arrow[d] \\
I \times I \arrow[r, "\nu"'] \arrow[ur, dashed, "\widetilde{\nu}"] & \Omega(C_1 \cap C_2) \simeq \prod \Omega(i/N)
\end{tikzcd}$$

A solution $\widetilde{\nu}$ is a homotopy of $\lambda_1 \in \Omega(C_1)$ to some λ_1' which agrees with λ_2 around $C_1 \cap C_2$, so we can glue them into a single path

$$\lambda : (I, 0, 1) \to (\Omega(C), \omega_0, \omega_1)$$

☐

COROLLARY 58. *If Ω is a flexible, locally $(k+1)$-connected sheaf on a compact subset $C \subset \mathbb{R}$, then $\Omega(C)$ is k-connected.*

PROOF. Follows from the fact that Ω^P is (locally) 1-connected for all polyhedra of dimension at most k iff it is (locally) $(k+1)$-connected, and that Ω^P is flexible by lemma ??. $\square$

Let us make use of the following auxiliary notion : let $\mathfrak{K}(V)$ denote the poset of compact subsets of V, and let

$$\mu : \mathfrak{K}(V) \to \mathbb{N}$$

assign to each $C \in \mathfrak{K}(V)$ the least $m \in \mathbb{N}$ such that C embeds into $\mathbb{R}^m$. This is well-defined as V itself embeds into some finite-dimensional Euclidian space.

LEMMA 59. *Let Ω be a flexible, locally $(k+n)$-connected flexible sheaf on V, and let C be a compact subset. Then $\Omega(C)$ is k-connected, and $\Omega(C')$ is $(k+1)$-connected for all non-empty C' obtained by intersecting C with a hypersurface in V.*

PROOF. If $\mu(C) = 1$, this boils down to the Corollary 58. Suppose then that the claim has been proved for all C' with $\mu(C') < m$, and let $\mu(C) = m$. Choose a hyperplane $H \subset \mathbb{R}^m$ and consider the orthogonal projection

$$f : \mathbb{R}^m = H \oplus \mathbb{R} \supset C \to \mathbb{R}$$

By Lemma 56, the sheaf $f_*\Omega$ is a flexible sheaf on $f(C) \subset \mathbb{R}$.

Note that the stalk $(f_*\Omega)(c)$ over a $c \in f(C) \subset \mathbb{R}$ is $\Omega(f^{-1}(c) \cap C)$. But by construction, $\mu(f^{-1}(c) \cap C) < m$, so that by the inductive hypothesis we know that $\Omega(f^{-1}(c) \cap C)$ is $(k+1)$-connected. Thus $f_*\Omega$ is a locally $(k+1)$-connected flexible sheaf on $f(C)$, so that Lemma 58 once again applies, and proves $\Omega(C) = (f_*\Omega)(f(C))$ to be k-connected. $\square$

PROOF OF 52. Choose an exhaustion of V by compact subsets :

$$V = \bigcup_n C_i$$
$$C_i \subset C_{i+1}$$

Observe that we are assuming that the fibre sheaf Ω (which is flexible itself) is $(n+k-1)$-connected, and everything boils down to showing that $\Omega(V)$ is $(k-1)$-connected.

Since all the $\Omega(C_i)$'s are $(k-1)$-connected by Lemma 59 and all $\Omega(C_{i+1}) \to \Omega(C_i)$'s are fibrations, then

$$\Omega(\lim_{\to} C_i) = \lim_{\leftarrow} \Omega(C_i)$$

is also $(k-1)$-connected. Hence $\Phi(V) \to \Psi(V)$ is a k-equivalence, as claimed.

$\square$

3. Microflexibility

We now turn to the "micro" version of the concepts discussed above. This is especially useful since this is a form of "local" version of flexibility which is easier to detect.

DEFINITION 60. *A homotopy*

$$h : P \times I \to \Phi(U), \quad U \supset C \text{ compact}$$

*will be called C-**microcompressible** if there exists $\varepsilon = \varepsilon(h) > 0$ such that C-compressions of $h|[0, \varepsilon]$ exist into arbitrarily small neighborhoods of C.*

That is, it is required that there be $\varepsilon > 0$ such that, for small enough $U_0 \supset C$, there is a new homotopy

$$\widetilde{h} : P \times [0, \varepsilon] \to \Phi(U)$$

of $h|P \times \{0\}$ which coincides with $h|P \times [0, \varepsilon]$ around C and is independent of t outside U_0.

The set of all points $v \in V$ for which the value of $\widetilde{h}(p,t)|v$ depends on t for some choice of $p \in P$ will be called the **support** *of $\widetilde{h}$,* $\operatorname{supp}(\widetilde{h}) \subset U_0$.

REMARK 61. *Let us just highlight an important subtlety of this definition : while the homotopy $\widetilde{h}$ itself is allowed to depend (and actually always does) on the particular choice of neighborhood U_0, the ε should be* independent *of this choice.*

Clearly, every C-compressible homotopy is also C-microcompressible. Therefore, in a flexible sheaf Φ, all homotopies of sections over compact sets are C-microcompressible. In fact, the converse also holds :

PROPOSITION 62. *If all homotopies*

$$P \times I \to \Phi(C)$$

are C-microcompressible, then all such homotopies are C-compressible as well.

PROOF. Let

$$h : P \times I \to \Phi(U), \quad U \supset C$$

represent a such homotopy.

Define

$$h' : P \times I \times I \to \Phi(U)$$
$$h'(p,t,s)|v := h(p, \min(1, t+s))$$

Fix $U_0 \supset C$; and let

$$\widetilde{h'} : P \times I \times [0,\varepsilon] \to \Phi(U)$$

be a compression of $h'|P \times I \times [0,\varepsilon]$ into U_0. This means that upon restricting the original h to $P \times [a,b]$, for any $b - a \leqslant \varepsilon$, is C-compressible into U_0.

Fix then $N > 1/\varepsilon$ and suppose for $i < i_0$, we can find a C-compression $\widetilde{h}$ of $h|P \times [0, i/N]$ into U_0. By definition of C-compression, h and $\widetilde{h}$ coincide around C, on some $U_1 \supset C$, say. Then a C-compression $\widetilde{h_{i+1}}$ of $\widetilde{h'}|P \times I \times [i/N, (i+1)/N]$ into U_1 can be glued to $\widetilde{h}$, and this produces a C-compression of $h|[0,(i+1)/N]$ into U_0.

Hence, by induction, the whole h C-compresses into U_0. ☐

The lemma shows that we do not obtain a broader class of sheaves by relaxing the compressibility property to that of mcrocompressibility.

There is, however, a way to in fact enlarge this class of sheaves by the following observation : the microcompressibility property for homotopies of sections around a compact C provides a *universal* $\varepsilon > 0$ such that the restricted homotopy can be compressed into an *arbitrarily small* neighborhood U_0 of C.

We could thus consider the less stringent condition on a sheaf Φ that :

> Given a homotopy h of sections around a compact C *and* an open $U_0 \supset C$, there is a $\varepsilon = \varepsilon(h, U_0) > 0$ such that $h|[0,\varepsilon]$ is compressible into U_0.

That is, we can allow ε to depend on U_0 as well.

DEFINITION 63. *Call the sheaves of quasi-spaces Φ that satisfy the property above* **microflexible**.

PROPOSITION 64. *A sheaf of quasi-spaces Φ is microflexible if and only if*

$$\Phi(C') \to \Phi(C)$$

is a Serre microfibration *for each pair of nested compact sets $C' \supset C$, i.e., given P a finite polyhedron and commutative diagram*

$$\begin{tikzcd} P \times \{0\} \arrow[r] \arrow[d] & \Phi(C') \arrow[d] \\ P \times I \arrow[r] & \Phi(C) \end{tikzcd}$$

there is $\varepsilon > 0$ with

$$\begin{tikzcd} & P \times \{0\} \arrow[r] \arrow[dl] \arrow[d] & \Phi(C') \arrow[d] \\ P \times [0,\varepsilon] \arrow[r] \arrow[urr, dashed] & P \times I \arrow[r] & \Phi(C) \end{tikzcd}$$

solvable.

Proof. Suppose first that Φ is microflexible, and that

$$h : P \times I \to \Phi(U), \quad U \supset C \text{ compact}$$

and U_0 are given.

We can assume that $\mathrm{Cl}(U_0) \subset U$ is compact and then consider the lift

$$f : P \times \{0\} \to \Phi(\mathrm{Cl}(U_0))$$

of the restriction of $h|P \times \{0\}$ to $C \cup (\mathrm{Cl}\, U_0 - U_0)$. A lift $\widetilde{h}$ of $h|[0,\varepsilon]$ provides the required C-compression into U_0.

Conversely, suppose given any homotopy h and open set U_0, there is a C-compression of $h|[0,\varepsilon]$ into U_0. Given the problem

$$\begin{tikzcd} P \times \{0\} \arrow[r] \arrow[d] & \Phi(C') \arrow[d] \\ P \times I \arrow[r, "h"'] \arrow[ur, dashed, "h'"] & \Phi(C) \end{tikzcd}$$

choose $\mathrm{Cl}\, U_0 \subset \mathrm{dom}(h)$, C-compress it to $\widetilde{h}$ and define h' by

$$h'(p,t)|v = \begin{cases} \widetilde{h}(p,t)|v & \text{if } v \in U_0 \\ f(p)|v & \text{otherwise.} \end{cases}$$

$\square$

Example 7. *Let $\mathcal{R} \subset J^r E$ be an* open *differential relation. Then its sheaf of solutions Φ is microflexible.*

As an useful technical remark, let us point out in the following lemma the following “duality” between domains of definition and support, in the presence of microflexibility :

Proposition 65. *Suppose*

$$h : P \times I \to \Phi(U)$$

for $C \subset U$, is given.

If Φ is microflexible, given $U_0 \supset C$, we can find $\varepsilon > 0$ and a C-deformation

$$\widetilde{h} : P \times I \to \Phi(U)$$

of h which is independent of t outside U_0 if $t \leqslant \varepsilon$.

Proof. Consider the homotopy

$$\overline{h} : P \times I \times I \to \Phi(U)$$
$$\overline{h}(p,s,t)|v := h(p, \min(1, s+t))|v$$

Microflexibility provides $\varepsilon(h, U_0) > 0$ and a

$$\overline{h}' : P \times I \times [0,\varepsilon] \to \Phi(U)$$

which coincides with $\overline{h}$ around C, starts at $h|P \times I \times \{0\}$, and is independent of t outside U_0. Let then

$$\widetilde{h} : P \times I \to \Phi(U)$$
$$\widetilde{h} := \overline{h}'|P \times I \times \{\varepsilon\}$$

This $\widetilde{h}$ has the required properties. □

4. Sharp Actions by Diffeotopies

In this section we describe a condition which ensures flexibility of germs of microflexible *equivariant* sheaves Φ along submanifolds.

Let $\mathfrak{D} \subset \mathrm{diff}(V)$ be a sub-pseudogroup of local diffeomorphisms of V.

The notion of **$\mathfrak{D}$-diffeotopy** of an inclusion $U \to U'$ of open sets is then defined in the obvious way, i.e., as a smooth

$$d : U \times I \to U'$$
$$d_t \in \mathfrak{D}, \quad d_0 = \text{ the inclusion } U \hookrightarrow U'$$

Suppose then that $\mathfrak{I}$ is a collection of diffeotopies

$$d_t : U \to U', \quad U \supset V_0$$

where V_0 is a closed subset in U, and fix a Riemannian metric on V.

Definition 66. *Call $\mathfrak{I}$* **strictly moving** *a subset $S \subset V_0$ if there is*

$$\mathrm{disp}(\mathfrak{I}, S, V_0) > 0$$

such that for all $d_t \in \mathfrak{I}$,

$$t \geqslant 1/2 \Rightarrow \mathrm{dist}(d_t(S), V_0) \geqslant \mathrm{disp}(\mathfrak{I}, S, V_0)$$

Definition 67. *$\mathfrak{I}$ is called* **sharp** *at S if, for each $\nu > 0$, $\mathfrak{I}$ contains a $\mathfrak{D}$-diffeotopy strictly moving S, which is stationary if $t \geqslant 1/2$ or $\mathrm{dist}(v, S) \geqslant \nu$.*

Thus $\mathfrak{I}$ is sharp at S (roughly) when it contains isotopies which can strictly move S and yet can be S-"compressed". Finally,

Definition 68. *A submanifold $V_0 \subset V$ is called* **sharply movable** *by $\mathfrak{D}$ if each point v in V_0 admits a neighborhood $U \subset V$ such that, for every closed hypersurface S in $U \cap V_0$, there is a set $\mathfrak{I} = \mathfrak{I}(v, U, S)$ of $\mathfrak{D}$-diffeotopies $V_0 \cap U \to U$ strictly moving $S \subset V_0$, and which are sharp at S.*

Suppose now that, for a fixed smooth manifold E,

$$\Phi_E : \mathcal{O}(V)^{\mathrm{op}} \to \mathrm{QTop}$$
$$\Phi_E(U) = C^\infty(U, E)$$

DEFINITION 69. *Say that a $\mathfrak{D}$-diffeotopy $d_t : U \to U'$* **acts** *on a subsheaf $\Phi \subset \Phi_E$ if the assignment*

$$\Phi_E(U') \to \Phi_E^I(U)$$
$$\varphi' \mapsto d_t^*\varphi'$$

preserves Φ.

THEOREM 70. *Suppose Φ is a microflexible sheaf of quasi-spaces on V, with $V_0 \subset V$ sharply movable. Then $\Phi|V_0$ is flexible.*

PROOF. According to corollary 49, it suffices to prove that $\Phi|V_0$ is locally flexible. That is, that given any point $v \in V_0$, there is $v \in U \subset V$ such that, given any compact $C \subset U$ and homotopy

$$h : P \times I \to \Phi(U)$$

there is a C-compression of $h|(V_0 \cap U)$ into arbitrarily small neighborhoods U_0 of C. Moreover, according to Proposition 62, it is enough to find a C-*micro*compression.

Choose then U as the neighborhod of v in V ensured by Definition 68.

Let U_0 denote a small neighborhood of C in $V_0 \cap U_0$. Choose M (depending on U_0) to be a compact codimension-zero submanifold of U_0 which contain C in its interior. By invoking the sharp movability hypothesis, we can find a set $\mathfrak{I}$ of $\mathfrak{D}$-diffeotopies $V_0 \cap U \to U$, strictly moving $\partial M \subset V_0 \cap U$, and sharp at ∂M.

Localizing h in U: Let first $\operatorname{disp}(\mathfrak{I}, S, V_0) > \mu > 0$ be such that the closure of the μ-neighborhood $B_\mu(C)$ of C is contained in U, and contains U_0. (Bear in mind that the construction need only hold for *sufficiently small* U_0). Observe that, by definition of microflexibility, we can C-microcompress h into $B_\mu(C)$, i.e., there is $\varepsilon = \varepsilon(h, \mu) > 0$ and homotopy

$$\widetilde{h}' : P \times [0, \varepsilon] \to \Phi(U)$$

of $h|P \times \{0\}$, agreeing with h around C whenever they are both defined.

Further compression: By means of Proposition 65, we can assume that, for some $\delta = \delta(h, U_0) > 0$,

$$\operatorname{supp}(\widetilde{h}'|[0, \delta]) \cap V_0 \subset U_0$$

Observe that, were the choice of δ independent of U_0 (as ε is), we would be done. Our task then is to find a U_0-compression of the *whole* of $\widetilde{h}'$, not just up to time δ.

Using $\mathfrak{I}$: Let now

$$0 < \nu < \min(\operatorname{dist}(\partial M, C), \operatorname{dist}(\partial M, \partial U_0))$$

and use sharpness at ∂M of $\mathfrak{I}$ to find a $d_t \in \mathfrak{I}$ such that $d_t(v)$ is independent of t if either $t \geqslant 1/2$ or $v \notin B_\nu(\partial M)$.

Now note that d_t/δ is is stationary for $t \geqslant \delta/2$, so we can regard it as being defined on $[0, \varepsilon]$ instead of $[0, \delta]$.

Let us now define

$$\widetilde{h} : P \times [0, \varepsilon] \to \Phi(V_0 \cap U)$$
$$\widetilde{h}(p, t)|v := \begin{cases} \widetilde{h}'(p, t)|d_{t/\delta}(v) & \text{if } v \in M \\ \widetilde{h}'(p, \min(t, \delta))|d_{t/\delta}(v) & \text{otherwise.} \end{cases}$$

Observe that, by the choice of μ,

$$d_1(\partial M) \cap \operatorname{supp}(\widetilde{h}') = \varnothing$$

so that if $v \in \partial M$,

$$\widetilde{h}'(p,t)|d_{t/\delta}(v) = \widetilde{h}'(p, \min(t,\delta))|d_{t/\delta}(v)$$

for $t \geqslant \delta$. So the formula above is well-defined.

Now, since $d_0(v) = v$ for $v \in V_0 \cap U$, this defines a homotopy of $h|P \times \{0\}$; moreover, by the choice of ν, $d_t(v) = v$ for $v \in \mathrm{Op}(C)$, so $\widetilde{h}$ and $\widetilde{h}'$ agree on a neighborhood of C. Finally, $\mathrm{supp}(\widetilde{h}) \subset U_0$; indeed, this is ensured, for $t \leqslant \delta$, by the "Further compression" step, and, for $t \geqslant \delta$, by the choice of ν. □

5. Open manifold pairs

DEFINITION 71. *A manifold V is called* **open** *if it admits a proper, positive Morse function*

$$f : V \to \mathbb{R}$$

without local maxima.

Let V be an open manifold, and V_0 a subpolyhedron of V – i.e., a subcomplex of a smooth triangulation of V.

DEFINITION 72. *(V, V_0) is an* **open manifold pair** *if the following holds : there exists a subpolyhedron of positive codimension K of V, called a (V, V_0)*-**core**, *such that, for every open neighbourhood $V_0 \cup K \subset U'$, there is a smaller open neighbourhood $U \subset U'$ of $V_0 \cup K$ and a diffeotopy*

$$d_t : V \to V$$
$$d_t|(V_0 \cup K) = \mathrm{id}_{(V_0 \cup K)}$$

of the identity map $d_0 = \mathrm{id}_V$ to a map d_1 that sends V into U.

The crucial example for our applications is this :

PROPOSITION 73. *Let*

$$f : V \to \mathbb{R}$$

be a proper Morse function without local maxima. Then $(V, \mathrm{Morse}(f))$ is an open manifold pair, where $\mathrm{Morse}(f)$ *denotes the Morse complex of f.*

PROOF. Just note that the hypothesis on the absence of local maxima says that $\mathrm{Morse}(f)$ has positive codimension in V, so (for some choice of Riemannian metric on V) the flow of the gradient $-\nabla f$ contracts V into as small a neighborhood of $\mathrm{Morse}(f)$ as we want. □

6. Classical Corollaries

Let us first observe a few practical consequences of what has been developed so far.

First, suppose $\Phi : \mathcal{O}(V)^{\mathrm{op}} \to \mathrm{QTop}$ is *any* sheaf of quasi-spaces.

PROPOSITION 74. *$\Phi^\flat$ is a* flexible *sheaf.*

PROOF. Let $C \subset V$ be any compact, and let

$$h : P \times I \to \Phi^\flat(U), \quad U \supset C$$

be given. Choose

$$C \subset U_0 \subset \mathrm{Cl}\, U_0 \subset U$$

and a smooth

$$\varrho : U \to \mathbb{R}$$
$$\varrho|\,\mathrm{Op}\, C = 1$$

supported in U_0. Now define the U_0-compression

$$\widetilde{h} : P \times I \to \Phi^\flat(U)$$
$$\widetilde{h}(p,t)|v := h(p, \varrho(v))|v$$

□

COROLLARY 75. *If Φ is a flexible sheaf of quasi-spaces, $\Phi(V) \to \Phi^\flat(V)$ is a weak equivalence, hence the sheaf-theoretic h-principle holds true.*

PROOF. Immediate from Theorem 52 and Proposition 74. □

Next observe the :

PROPOSITION 76. *Suppose*

$$\pi : N \to V_0$$

is a line bundle over a manifold V_0. Let D_π denote the sub-pseudogroup of $\operatorname{diff}(N)$ consisting of π-fibred local diffeomorphisms, i.e., giving commutative

$$\begin{tikzcd} \operatorname{dom} d \arrow[rr, "d", "\simeq"'] \arrow[dr, "\pi"'] & & \operatorname{codom} d \arrow[dl, "\pi"] \\ & V_0 & \end{tikzcd}$$

Then $\mathcal{D}_\pi$ sharply moves V_0.

PROOF. Obvious. □

COROLLARY 77. *If $V_0 \subset V$ is a subpolyhedron of positive codimension, and Φ is a microflexible sheaf of quasi-spaces on V, then all forms of the sheaf-theoretic h-principle holds for $\Phi|V_0$.*

PROOF. Direct consequence of Theorem 70 refined through Lemma 49, and Proposition 76. □

COROLLARY 78 (Holonomic Approximation). *Let (V, V_0) be an open manifold pair, K a (V, V_0)-core and $A = K \cup V_0$. Given*

$$F : (D^k, \operatorname{Op} \partial D^k) \to (\Gamma(\operatorname{Op} A, J^r E), j^r\Gamma(\operatorname{Op} A, E))$$

with

$$F|\operatorname{Op}(V_0) : D^k \to j^r\Gamma(\operatorname{Op} V_0, E)$$

and arbitrary continuous positive functions $\varepsilon, \delta : V \to \mathbb{R}_+$, there exist

(1) *A continuous family of diffeotopies*

$$d : (D^k, \operatorname{Op} \partial D^k) \to (\operatorname{Diff}(V), \mathrm{id}_V)$$

which are stationary on $D^k \times \operatorname{Op} V_0 \cup \operatorname{Op} \partial D^k \times V$ and are δ-small in the C^0-sense – i.e., are such that

$$\operatorname{dist}(p, d_{(t,q)}(p)) < \delta(p) \text{ for all } (t,p,q) \in I \times V \times D^k;$$

(2) *A continuous*

$$\widetilde{F} : D^k \times V \supset \operatorname{Op}(d_1 A) \to J^r E$$

with

$$\widetilde{F}|\,(\mathrm{Op}(d_1 A) \cap \{q\} \times V) =: \widetilde{F}_q \in j^r\Gamma(\mathrm{Op}\, d_{(1,q)}A, E)$$
$$\widetilde{F}_q|\,\mathrm{Op}\, V_0 = F_q|\,\mathrm{Op}\, V_0$$
$$\widetilde{F}_q = F_q \textit{ for } q \in \mathrm{Op}(\partial D^k)$$
$$\mathrm{dist}(F_q(p), \widetilde{F}_q(p)) < \varepsilon(p) \textit{ for all } p \in \mathrm{Op}(d_{(1,q)}A), q \in D^k$$

Corollary 79 (Approximation by exact forms). *Let (V, V_0) be an open manifold pair, K a (V, V_0)-core and $A = K \cup V_0$. Given*

$$\omega \in \Omega^p(\mathrm{Op}\, A)$$

with

$$\omega|\,\mathrm{Op}(V_0) \in \Omega^p_{exact}(\mathrm{Op}\, V_0)$$

and arbitrary continuous positive functions $\varepsilon, \delta : V \to \mathbb{R}_+$, there exist

(1) *A diffeotopy*

$$d : (I, 0) \to (\mathrm{Diff}(V), \mathrm{id}_V)$$

which is stationary around V_0 and is δ-small in the C^0-sense, and

(2) *A*

$$\widetilde{\omega} \in \Omega^p_{exact}(\mathrm{Op}(d_1 A))$$

with

$$\widetilde{\omega}|\,\mathrm{Op}\, V_0 = \omega|\,\mathrm{Op}\, V_0$$
$$\omega|\,\mathrm{Op}(d_1 A) \quad C^0\textit{-close to} \quad \widetilde{\omega}$$

(We state the non-parametric version of the corollary; for full generality insert the proof below into the full statement of Corollary 78).

Proof. Observe first that

$$d : \Omega^q(V) \to \Omega^{q+1}(V)$$

factors as

$$d = \widetilde{\mathrm{symb}(d)} \circ j^1$$

where

$$j^1 : \Omega^q(V) = \Omega^q(V) \to \Gamma(V, J^1\Lambda^q T^*V)$$
$$\widetilde{\mathrm{symb}(d)} : \Gamma(V, J^1\Lambda^q T^*V) \to \Omega^{q+1}(V)$$

where we denoted by $\widetilde{\mathrm{symb}(d)}$ the map, at the level of sections, induced by the symbol of d :

$$\mathrm{symb}(d) : J^1\Lambda^q T^*V \to \Lambda^{q+1}T^*V$$

Note that $\mathrm{symb}(d)$ is a *fibration with contractible fibres*; this means that, for any $(q-1)$-form

$$\eta : V \to \Lambda^{q-1}T^*V$$

we can find a lift

$$F_{\omega,\eta} : V \to J^1\Lambda^{q-1}T^*V$$
$$\widetilde{\mathrm{symb}(d)} F_{\omega,\eta} = \omega, \quad p^1_0 F_{\omega,\eta} = \eta$$

Now employ Corollary 78 to produce the C^0-diffeotopy d_t and the holonomic approximation $\widetilde{F_{\omega,\eta}}$ along A :

$$\widetilde{F_{\omega,\eta}}|\operatorname{Op}(d_1 A) = j^1\widetilde{\eta}$$

Then $\widetilde{\omega} := d\widetilde{\eta}$ C^0-approximates $\omega|\operatorname{Op}(d_1 A)$, coincides with it around V_0, and $\widetilde{\eta}$ itself is C^0-close to $\eta|\operatorname{Op}(d_1 A)$. ☐

REMARK 80. *If $V_0 = \varnothing$ and $\xi \in H^p_{dR}(V)$, we can approximate any $\omega \in \Omega^p(\operatorname{Op} A)$ by a closed $\widetilde{\omega} \in \xi$ by applying the previous corollary to the form $\omega - \overline{\omega}$, for some $\overline{\omega} \in \xi$, and then considering $\widetilde{\omega} + \overline{\omega}$.*

Let now $\mathcal{F} \in \operatorname{Fol}^q(W)$ and let

$$\operatorname{Trans}(\cdot, \mathcal{F}) : \mathcal{O}(V)^{\mathrm{op}} \to \mathrm{Top}$$
$$\operatorname{Trans}(\cdot, \mathcal{F}) \subset C^\infty(\cdot, W)$$
$$f \in \operatorname{Trans}(U, \mathcal{F}) \Leftrightarrow f \pitchfork \mathcal{F}$$

Similarly, let

$$\operatorname{Trans}(T\cdot, T\mathcal{F}) \subset \operatorname{Hom}(T\cdot, TW)$$
$$F \in \operatorname{Trans}(TU, T\mathcal{F}) \Leftrightarrow F \pitchfork T\mathcal{F}$$

Observe the natural continuous map

$$d : \operatorname{Trans}(V, \mathcal{F}) \to \operatorname{Trans}(TV, T\mathcal{F})$$

As an exercise in the taxonomy put forth in Chapter 1, Section 5, we state the next corollary in abbreviated form :

COROLLARY 81 (Gromov-Phillips). *The relative, C^0-dense form of the h-principle holds locally around $V_0 \subset V$ of positive codimension.*

Note that if in the statement above (V, V_0) is assumed to be an open manifold pair, then the solution claimed by the corollary can be globalized, if we allow the C^0-approximation to be dropped; cf. Section 5.

7. h-principle as obstruction theory

The formalism of Gromov structures describes a certain "contravariant closure" operation for sheaves of quasi-spaces

$$\Phi \mapsto H^1(\cdot, \Phi)$$

Unlike Φ itself, $H^1(\cdot, \Phi)$ makes sense for all spaces T, and behaves contravariantly with respect to mappings $T' \to T$, and covariantly with respect to $\Phi \to \Phi'$. This allows $H^1(\cdot, \Phi)$ to be *classified* in the sense of Theorem 25, which also identifies $h^1(\cdot, \Phi)$ with $[\cdot, B\Phi]$.

But, be as it may, our ultimate goal is to understand $\Phi(V)$; to this end, recall that we have described a canonical embedding

$$\Phi \to H^1(\cdot, \Phi)$$

which allows us to regard a germ of section of Φ over a $V_0 \subset V$ as a Gromov Φ-structure on V_0; moreover, we described how to recognize, among those $F \in H^1(V, \Phi)$ which correspond to $\varphi \in \Phi(V)$: namely, those whose underlying Hæfliger structure (in the parlance of Section 1) is the differential structure of V.

The advantage of this approach is that we can describe e-principles for *"weak solutions"* in a fully obstruction-theoretic fashion :

A pair $(F, \varphi') \in \Phi^\flat(V) \times_{\Phi^\flat(V_0)} \Phi(V_0)$ gives rise to the (solid) commutative

$$\begin{tikzcd} V_0 \arrow[r, "\varphi'"] \arrow[d] & B\Phi \arrow[d] \\ V \arrow[r, "F"'] \arrow[ur, dashed, "\varphi"] & B\Phi^\flat \end{tikzcd}$$

with the induced $V \to B\Gamma_n$ classifying the differential structure on V.

We replace $\flat : B\Phi \to B\Phi^\flat$ by the path-space fibration

$$\begin{tikzcd} B\Phi \arrow[rr, "\flat"] \arrow[dr, "j"] & & B\Phi^\flat \\ B\Phi \arrow[u, "="] & \widetilde{B\Phi} \arrow[l, "p"] \arrow[ur, "\tilde{\flat}"'] & \end{tikzcd}$$

and we consider the induced commutative

$$\begin{tikzcd} V_0 \arrow[r, "j\varphi'"] \arrow[d] & \widetilde{B\Phi} \arrow[d, "\text{fib}"] \\ V \arrow[r, "F"'] \arrow[ur, dashed, "\varphi''"] & B\Phi^\flat \end{tikzcd}$$

Suppose φ'' is a solution to this latter diagram; then

$$\begin{tikzcd} V_0 \arrow[r, "\varphi'"] \arrow[d] & B\Phi \arrow[d] \\ V \arrow[r, "F"'] \arrow[ur, dashed, "p\varphi''"] & B\Phi^\flat \end{tikzcd}$$

has strictly commutative upper triangle, and homotopy-commutative lower triangle.

This suggests considering $\Phi^\sharp(V) := \mathfrak{H}^{-1}[\tau_V]$:

$$\begin{tikzcd} \Phi(V) \arrow[r] \arrow[d, "\mathfrak{H}"'] & \Phi^\sharp(V) \arrow[d, "\mathfrak{H}"] \\ \{\tau_v\} \arrow[r] & [\tau_V] \end{tikzcd}$$

as a space of "weak solutions" to Φ on V. Observe that for $F \in \Phi^\sharp(V)$, $E(F) \simeq TV$ as a fibre bundle, and that

$$\begin{tikzcd} \Phi^\sharp(V) \arrow[r, "\mathfrak{G}"] \arrow[d, "\mathfrak{H}"'] & H^1(TV, \Phi) \arrow[d] \\ {[\tau_V]} \arrow[r, "\mathfrak{G}"'] & \mathrm{Fol}^n_{\pitchfork}(TV \to V) \end{tikzcd}$$

is a pullback diagram.

The moral so far is that we can ensure that any $F \in \Phi^\flat(V)$ with $F|V_0 \in \Phi(V_0)$ is homotopic rel $\mathrm{Op}\, V_0$ through families of parametric of Φ to a "weak solution" $\varphi'' \in \Phi^\sharp(V)$ extending φ' by imposing only conditions on the dimension of the pair (V, V_0) and the connectivity of the map $B\Phi \to B\Phi^\flat$:

$$\dim(V, V_0) \leqslant \mathrm{conn}(B\Phi \to B\Phi^\flat)$$

Now we need a procedure to "regularize" a weak solution as φ'' above to a true solution $\varphi \in \Phi(V)$
:

THEOREM 82. *Let (V, V_0) be an open manifold pair. Then given*

$$(\varphi', \varphi'') \in \Phi(V_0) \times_{\Phi^\sharp(V_0)} \Phi^\sharp(V)$$

there is a homotopy rel $\operatorname{Op} V_0$ *of* φ'' *in* $\Phi^\sharp(V)$ *to a* $\varphi \in \Phi(V)$.

PROOF. Let

$$V \xrightarrow{s} E(\varphi'') \xrightarrow{p} V$$

be a graph for φ''.

We know that

$$s|\operatorname{Op} V_0 : \operatorname{Op} V_0 \to E(\varphi'')$$

is transverse to the underlying $\mathfrak{H}\mathfrak{G}(\varphi'') = \mathfrak{G}(\mathfrak{H}\varphi'') \in \mathrm{Fol}^n_{\pitchfork}(E(\varphi'') \to V)$, and that $s^*\mathfrak{G}(\mathfrak{H}\varphi'')$ is concordant to τ_V.

Observe that the problem of finding a

$$S \in \operatorname{Trans}(TV, T\mathfrak{G}(\mathfrak{H}\varphi''))$$
$$S|\operatorname{Op} V_0 = ds|\operatorname{Op} V_0$$

is inobstructed by usual transversality theory.

We can then apply Corollary 81 to find a (V, V_0)-core K, a diffeotopy d_t, stationary around V_0, and a

$$\sigma \in \operatorname{Trans}(\operatorname{Op}(d_1(K \cup V_0)), \mathfrak{G}(\mathfrak{H}\varphi''))$$
$$\sigma|\operatorname{Op} V_0 = s|\operatorname{Op} V_0$$

together with a homotopy H of $d\sigma$ to $S|\operatorname{Op}(d_1(K \cup V_0))$ in $\operatorname{Trans}(TV, T\mathfrak{G}(\mathfrak{H}\varphi''))$.

Let h denote the corresponding homotopy of $C^\infty(\operatorname{Op}(d_1(K \cup V_0)), E(\varphi''))$. Then

$$I \to H^1(\operatorname{Op}(d_1(K \cup V_0)), \Phi)$$
$$t \mapsto h_t^*\mathfrak{G}(\varphi'')$$

defines a homotopy rel $\operatorname{Op} V_0$ of φ'' to a $\varphi''' \in \Phi(\operatorname{Op}(d_1(K \cup V_0))$.

Now globalize the solution by means of a compression $V \to \operatorname{Op}(d_1(K \cup V_0))$. □

COROLLARY 83. *If Φ is an invariant, microflexible sheaf of quasi-spaces on V, then*

$$B\Phi \to B\Phi^\flat$$

is an $(n-1)$-equivalence.

REMARK 84. *Actually, as follows from the folding e-principle of Eliashberg (see* [**21**]*), for microflexible Φ, the above is in fact a weak equivalence. But since we are mainly concerned with relative e-principles, we won't stress connectivity above n.*

8. Toy Example : Foliations

The sheaf of foliations Fol^q of codimension q is a typical example of a sheaf of differential-topologic nature whose topology we would like to understand.

The flexible tools developed so far beg the question whether Fol^q is microflexible.

This will be decided in the negative using the Godbillon-Vey invariant to provide obstructed cycles for which the h-principle cannot apply.

Next, we will adress another obstruction to h-principles in the realm of foliations, namely Bott's obstructions. Finally, we sketch a general procedure of Hæfliger's to convert the (very overdetermined) problem of integrability-up-to-homotopy of foliations into a homotopy-theoretic problem that is positively solvable granted certain conditions – a particular case of a more general construction of Gromov's described in the previous section.

We would like in particular to address the following classical question, dating back to Reeb :

Question. *¿When is a codimension-q subbundle*

$$\mathcal{D}_0 \subset TV$$

homotopic trough such subbundles to an involutive one $\mathcal{D}_1 = T\mathcal{F}$ *?*

The Godbillon-Vey class. Let us first make a digression into codimension-one foliations.

Suppose $\mathcal{F} \in \mathrm{Fol}^1(V)$ is a transversally orientable, codimension-one foliation of V. Then there is a one-form

$$\theta \in \Omega^1(V)$$

which annihilates precisely the vectors tangent to $\mathcal{F}$:

$$\ker\theta = T\mathcal{F}$$

Now, the fact that $T\mathcal{F}$ is involutive is equivalent to

$$d\theta = \alpha \wedge \theta$$

for some other one-form α; upon differentiating this last equation we see that

$$d\alpha \wedge \theta = 0 \Rightarrow d\alpha = \theta \wedge \beta, \beta \text{ a one-form.}$$

Note that

$$d(\alpha \wedge d\alpha) = (d\alpha)^2 = (\theta \wedge \beta)^2 = 0$$

so that $\alpha \wedge d\alpha$ defines a class

$$\mathrm{GV}(\theta, \alpha) \in H^3_{\mathrm{dR}}(V)$$

Observe that, given θ, α is defined modulo θ, and if we let

$$\alpha' := \alpha + f\theta$$

then

$$\alpha' \wedge d\alpha' = \alpha \wedge d\alpha - d(f\alpha \wedge \theta)$$

so $\mathrm{GV}(\theta, \alpha) = \mathrm{GV}(\theta, \alpha + f\theta)$.

Moreover, θ itself is only conformally well-defined; if

$$\theta' := f\theta, \quad f \neq 0$$

then

$$d\theta' = \theta' \wedge (\alpha - d(\log f))$$

so taking $\alpha' := \alpha - d(\log f)$, we get

$$\alpha' \wedge d\alpha' = \alpha \wedge d\alpha - d(\log f d\alpha)$$

This shows that $\mathrm{GV}(\theta, \alpha)$ only depends on the foliation $\mathcal{F}$, $\mathrm{GV} = \mathrm{GV}(\mathcal{F})$[1]. Moreover, if $\mathcal{F}$ is *not* transversally orientable, we pass to its double covering and observe that the expression

$$\alpha \wedge d\alpha$$

descends, so that $\mathrm{GV}(\mathcal{F})$ is defined for all codimension-one foliations.

Definition 85. *The cohomology class* $\mathrm{GV}(\mathcal{F}) \in H^3_{dR}(V)$ *is called the* **Godbillon-Vey class** *of* $\mathcal{F}$*.*

[1]In fact, only depends on the *cobordism* type of $\mathcal{F}$, but this is irrelevant for us at this point.

Observe that GV is natural in the sense that[2]

$$\mathrm{GV}(f^*\mathcal{F}) = f^*\,\mathrm{GV}(\mathcal{F})$$
$$f : V' \to V \text{ transverse to } \mathcal{F}$$

Therefore, a foliation with non-trivial Godbillon-Vey class cannot be induced from one with trivial Godbillon-class; in particular, no such foliation can be induced from a space with no non-trivial 3 cocycles.

Now we can address the question of microflexibility :

THEOREM 86. Fol^1 *is* not *microflexible.*

PROOF. On the Lie group $SL_2(\mathbb{R})$ there exists a basis of left-invariant 1-forms

$$\{\theta, \alpha, \beta\}$$

such that :

$$d\theta = \theta \wedge \alpha, \quad d\alpha = \theta \wedge \beta, \quad d\beta = \alpha \wedge \beta$$

Let now $\Gamma \subset SL_2(\mathbb{R})$ be a cocompact subgroup. Then θ descends to $V = SL_2(\mathbb{R})/\Gamma$, and defines a foliation $\mathcal{F} \in \mathrm{Fol}^1(V)$ whose Godbillon-Vey class is represented by the left-invariant 3-form

$$\alpha \wedge d\alpha = \theta \wedge \alpha \wedge \beta$$

and hence is non-zero as a de Rham cohomology class.

Embed V in some large enough Euclidian space $\mathbb{R}^N$, $N > 4$, and extend $\mathcal{F}$ by means of a tubular neighborhood to a codimension-one foliation $\mathcal{F}'$ of a neighborhood of V in $\mathbb{R}^N$, $\mathcal{F}' \cap V = \mathcal{F}$. Now, $\mathcal{F}'$ can be extended to a codimension-one distribution $\mathcal{D}$ on the ambient space $\mathbb{R}^N$. But by naturality of GV, $\mathcal{D}$ cannot be deformed through such distributions, relative to $\mathrm{Op}\,V$, to a codimension-one foliation. Hence the h-principle must fail over some $(D^4, \partial D^4)$, and so Fol^1 cannot be microflexible. $\square$

Bott's obstruction to integrability. Roughly, it imposes certain vanishing conditions on the characteristic classes of a distribution on V which is abstractly isomorphic to a foliation. More precisely, it consists of the following observation : let $\mathcal{D}$ is an integrable distribution on V, and

$$p : TV \longrightarrow \nu(\mathcal{D}) := TV/\mathcal{D}$$

be the projection onto the normal bundle to $\mathcal{D}$. We can define

$$\begin{aligned} \hat{\nabla} : \Gamma(\mathcal{D}) \times \Gamma(\nu(\mathcal{D})) &\longrightarrow \Gamma(\nu(\mathcal{D})) \\ (W, Z) &\mapsto p[W, \tilde{Z}] \\ \text{where} \quad p(\tilde{Z}) &= Z \end{aligned}$$

That this is well-defined (i.e., independent of the lift $\tilde{Z}$) follows immediately from the fact that $p[\Gamma(\mathcal{D}), \Gamma(\mathcal{D})] = 0$ since $\mathcal{D}$ is involutive. Note that this has all formal properties a connection on $\nu(\mathcal{D})$ would enjoy were it not for the fact that W is restricted to lie in $\mathcal{D}$.

Now, if $\tilde{\nabla}$ is any connection on $\nu(\mathcal{D})$, and upon a choice of Riemannian metric we identify $\nu(\mathcal{D})$ with the orthogonal complement to $\mathcal{D}$ in TV, we can define a **basic** connection on $\nu(\mathcal{D})$ by

$$\begin{aligned} \nabla : \Gamma(TV) \times \Gamma(\nu(\mathcal{D})) &\longrightarrow \Gamma(\nu(\mathcal{D})) \\ (W, Z) &\mapsto \hat{\nabla}_{p(W)} Z + \tilde{\nabla}_{W - p(W)} Z \end{aligned}$$

(A connection is called *basic* if it amounts to $p[W, Z]$ when $W \in \Gamma(\mathcal{D})$). *Hence basic connections exist on the normal bundle to involutive subbundles.*

[2]The transversality condition can be dropped and GV can be extended to all Γ_1-structures... but we don't need this here.

Let now $\triangledown$ be a basic connection on $\nu(\mathcal{D})$ and denote by $F_\triangledown$ its curvature. The latter can be regarded as a two-form with values in endomorphisms of $\nu(\mathcal{D})$ which, since the connection is *basic*, vanish when restricted to $\Gamma(\Lambda^2\mathcal{D})$ since the Jacobi identity holds for sections of $\mathcal{D}$:

$$F_\triangledown(W, W')(Z) = p[W, [W', \widetilde{Z}]] - p[W', [W, \widetilde{Z}]] - p[[W, W'], \widetilde{Z}]$$

Thus, if $I(\mathcal{D})$ denotes the differential ideal in $\Omega^\bullet(V)$ spanned by the 1-forms which annihilate $\mathcal{D}$, we see that $F_\triangledown \in \Gamma(I(\mathcal{D}) \otimes \mathrm{End}(\nu(\mathcal{D})))$.

Suppose now that $\mathcal{D}$ has codimension q, i.e., that $I(\mathcal{D})$ is locally free in q generators. Then any product of $q+1$ elements in $I(\mathcal{D})$ must vanish identically. If we now recall that via Chern-Weil theory the real Pontryagin ring $\mathrm{Pont}(\nu(\mathcal{D})) < H^\bullet_{\mathrm{dR}}(V;\mathbb{R})$ is generated by all cohomology classes represented by elements of the form $P(F_\triangledown)$, where P is an invariant polynomial, we see that the real Pontryagin ring of $\nu(\mathcal{D})$ must vanish above dimension $2q$. Since the real Pontryagin ring only depends on the isomorphism type of $\nu(\mathcal{D})$, we have :

Theorem 87. *If $\mathcal{D}$ is a codimension-q distribution of V which is isomorphic to an involutive distribution, then*

$$\mathrm{Pont}^k(\nu(\mathcal{D})) = 0 \quad \textit{for } k > 2q$$

Hæfliger theory. So far, we have only described *negative* results concerning the problem described at the beginning : Godbillon-Vey shows that in general the h-principle is obstructed, and Bott's obstructions impose conditions on the Pontryagin ring of $\nu(\mathcal{D}_0)$ for $\mathcal{D}_0$ to be *isomorphic* (and, in particular, homotopic) to an involutive subbundle $\mathcal{D}_1$.

But we can provide a *positive* answer in the following form : denote as usual the groupoid of germs of local diffemorphisms of $\mathbb{R}^q$ by $\Gamma \rightrightarrows \mathbb{R}^q$, and by

$$d : \Gamma_q \to \mathrm{GL}_q$$

the constinuous map which assigns to such a germ (the translation of) its differential.

Theorem 88 (Hæfliger [**40**]). *Let $\mathcal{D}_0$ be a a codimension-q distribution in an open n-manifold V and let $\tau : V \to B\,\mathrm{GL}_n$, $\xi : V \to B\,\mathrm{GL}_{n-q}$ and $\alpha : V \to B\,\mathrm{GL}_q$ denote the classifying maps for the bundles TV, $\mathcal{D}_0$ and $TV/\mathcal{D}_0$, respectively. Then there exists a homotopy $\mathcal{D}_t$ of distributions starting at $\mathcal{D}_0$ and ending at a foliation $\mathcal{D}_1 = T\mathcal{F}$ if and only if the following diagram can be solved :*

$$\begin{tikzcd}
 & B\Gamma_q \times B\,\mathrm{GL}_{n-q} \arrow[d, "Bd\times 1"] \\
 & B\,\mathrm{GL}_q \times B\,\mathrm{GL}_{n-q} \arrow[d, "\oplus"] \\
V \arrow[uur, dashed] \arrow[ur, "\xi\times\alpha"] \arrow[r, "\tau"'] & B\,\mathrm{GL}_n
\end{tikzcd}$$

where $\oplus$ denotes the arrow (in the homotopy category $hTop$) inducing direct sum of vector bundles.

Moreover, Bd is $(q+1)$-connected.

Proof. Note that Fol^q is germifiable, and denote by Dist^q the sheaf of codimension-q distributions, and by Vect_q the sheaf of rank-q vector bundles[3]on $\mathbb{R}^n$.

[3]within a specified set-theoretic universe... but let's not get bogged down by unnecessarily precise language.

Observe the natural maps

$$\begin{gathered} T : \mathrm{Fol}^q \to \mathrm{Dist}^q \\ \mathcal{F} \mapsto T\mathcal{F} \\ V : \mathrm{Dist}^q \to \mathrm{Vect}_q \times \mathrm{Vect}_{n-q} \\ E \mapsto (E^\perp, E) \\ \oplus : \mathrm{Vect}_q \times \mathrm{Vect}_{n-q} \to \mathrm{Vect}_n \\ (N, E) \mapsto N \oplus E \end{gathered}$$

where $E \mapsto E^\perp$ is induced by some auxiliary Riemannian metric.

Note that

$$\oplus^{-1}(TV \to V) = \mathrm{image}\,(\mathrm{Dist}^q \to \mathrm{Vect}_q \times \mathrm{Vect}_{n-q})$$

and consider

$$B\,\mathrm{Dist}^q \to B\,\mathrm{Vect}_q \times B\,\mathrm{Vect}_{n-q}$$

Now observe that $\mathrm{Top}(\cdot, GL_q)$ is the local equivariant model of Vect_q, so

$$B\,\mathrm{Vect}_q = B\,\mathrm{Top}(\cdot, GL_q) \simeq BGL_q$$

so we replace the map above by the corresponding

$$B\,\mathrm{Dist}^q \to BGL_q \times BGL_{n-q}$$

Now, there is a natural continuous map

$$d : \Gamma_q \to \mathrm{GL}_q$$

assigning to each germ of diffeomorphism (the translation to the origin of) its differential.

Now, the universal Fol^q-structure

$$\vartheta_{\mathrm{Fol}^q} \in H^1(B\,\mathrm{Fol}^q, \mathrm{Fol}^q)$$

defines a

$$B\,\mathrm{Fol}^q \to B\Gamma_q \times BGL_{n-q}$$

making

$$\begin{tikzcd} B\,\mathrm{Fol}^q \arrow[r] \arrow[d] & B\Gamma_q \times BGL_{n-q} \arrow[d] \\ B\,\mathrm{Dist}^q \arrow[r] & BGL_q \times BGL_{n-q} \end{tikzcd}$$

homotopy-commutative, and it is easy to see that the above diagram yields a fibre weak equivalence.

Hence homotopy types of lifts of the diagram in the statement of the theorem correspond bijectively to homotopy classes of deformations of $\mathcal{D}_0$ to a Gromov Fol^q-structure on V lying above τ_V. By the discussion in the previous section, such a structure is always concordant to a foliation. Since the q-manifold $S^i \times \mathbb{R}^{q-i}$ has one single smooth, codimension-q foliation, this implies that the homotopy fibre of $B\,\mathrm{Fol}^q \to B\,\mathrm{Dist}^q$ is q-connected, and thus so is that of $B\Gamma_q \to BGL_q$.

Let now an element $\xi \in h^1(S^q; \mathrm{Fol}^q)$ be given and represent it by some Fol^q-structure on $S^q = \partial D^{q+1} \subset U$, where U is an open subset of the $(q+1)$-dimensional disk, which extends as a $\mathrm{Fol}^{q\flat}$-structure on D^{q+1}.

Now, applying the reasoning of Section 7 we can assume that ξ indeed comes from a *foliation* $\mathcal{F}_1$ on U. Now pick a

$$F \in \mathrm{Epi}(TU, T\mathbb{R}^q) \sim \mathrm{Trans}(TU, T\mathcal{F}_{\mathrm{pt}})$$

and use Gromov-Phillips to find a submersion $g : U \to \mathbb{R}^q$ whose differential dg is homotopic in $\mathrm{Epi}(TU, \mathbb{R}^q)$ to F.

Now, the submersion

$$f : \mathrm{Op}(\frac{1}{2}S^q) \to \mathbb{R}^q$$
$$f(x) = g(2x)$$

defines a Fol^q-structure ξ' on $\mathrm{Op}(\frac{1}{2}D^{q+1})$ which is a true foliation on $\mathrm{Op}(\frac{1}{2}S^q)$ codimension-q foliation $\mathcal{F}_0$. Observe moreover that by the construction of f, there is a non-singula vector field X defined on a neighborhood of $1/2 \leqslant \|x\| \leqslant 1$ with $X| \mathrm{Op}(S^q) = T\mathcal{F}_0$ and $X| \mathrm{Op}(\frac{1}{2}S^q) = T\mathcal{F}_1$. Since all 1-dimensional smooth disributions are involutive, the trajectories of X define a smooth, codimension-q foliation $T\mathcal{F}$ on that neighborhood.

Now let $\widetilde{\xi} \in H^1(D^{q+1}, \mathrm{Fol}^q)$ be defined by glueing $\mathcal{F}$ and ξ'. This is the nullhomotopy $\xi \sim 0$ we sought. □

Corollary 89. *On an open n-manifold V with $H^i(V;\mathbb{Z}) = 0$ for all $i > q+1$, any codimension-q distribution is homotopic to a foliation.*

CHAPTER 3

Poisson Geometry

In this chapter we apply the machinery described in the preceding chapters to the differential relation controlling *Poisson* structures.

1. Poisson Manifolds

Consider the commutative algebra $C^\infty(V) := \Gamma^\infty(V, V \times \mathbb{R})$ of smooth functions on a manifold V. A **Poisson bracket** on a smooth manifold V is a Lie algebra structure $\{\cdot,\cdot\}$ on $C^\infty(V)$ which is a derivation of the commutative structure. Explicitly, it is an antisymmetric $\mathbb{R}$-bilinear pairing

$$\{\cdot,\cdot\} : C^\infty(V) \times C^\infty(V) \to C^\infty(V)$$

satisfying

(1) $\underset{f,g,h}{\circlearrowright} \{\{f,g\},h\} = 0$[1] (Jacobi identity)

(2) $\{f, gh\} = \{f,g\}h + g\{f,h\}$ (Leibniz rule)

for all $f, g, h \in C^\infty(V)$.

A Poisson bracket on V defines a **Poisson bivector** $\pi \in \Gamma(V, \Lambda^2 TV)$ by

$$\{f,g\} := \pi(df \wedge dg)$$

The condition on a bivector[2] π to define, via the above rule, a Poisson bracket (i.e., that it satisfy Jacobi's identity) is that its Schouten-Nijenhuis bracket[3] with itself vanishes identically :

$$[\pi,\pi] = 0$$

see Section 3 for further details.

We will use the term **Poisson structure** to refer either to a Poisson bracket or its associated Poisson bivector.

In terms of a system of local coordinates $x_1, ..., x_n$ on V, a bivector

$$\pi = \frac{1}{2}\sum_{i,j}\pi_{ij}\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$$

induces

$$\{f,g\} = \sum_{i,j}\pi_{ij}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} = \sum_{i<j}\pi_{ij}\left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_i}\right)$$

---

[1]The symbol $\circlearrowright$ always stands for *cyclic sum*; e.g.

$$\underset{f,g,h}{\circlearrowright} \{\{f,g\},h\} = \{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\}$$

[2]We will employ the same notation for a bivector and its associated skew-symmetric bundle map $T^*V \to TV$.

[3]We recall that the Schouten-Nijenhuis bracket is the unique extension of the Lie bracket of vector fields to a graded bracket on the space of alternating multivector fields that makes the alternating multivector fields into a Gerstenhaber algebra. See Section 3

which is a Poisson bracket iff

$$\circlearrowleft_{i,j,k} \sum_l \frac{\partial \pi_{ij}}{\partial x_l} \pi_{lk} = 0 \tag{1}$$

We remark in passing that this is a (heavily) overdetermined system of non-linear (quasi-linear, in fact) partial differential equations. The corresponding first-order differential relation will be denoted

$$\Pi \subset J^1 \Lambda^2 TV$$

We will denote by Poiss^{C^r} the sheaf of C^r solutions to Π, $r \geqslant 1$, and by Poiss that of smooth solutions. Notice that this is never void, as the zero bivector is always Poisson.

A differentiable map

$$\varrho : V \to W$$

between Poisson manifolds is called a **Poisson map** if the corresponding

$$\varrho^* : C^\infty(W) \to C^\infty(V)$$

preserves the Poisson bracket : $\varrho^*\{f, g\} = \{\varrho^* f, \varrho^* g\}$; this is tantamount to saying that the corresponding bivectors are ϱ-related.

Clearly, Poisson manifolds and Poisson maps between them form a category, denoted $\mathfrak{Poiss}$.

Observe that a Poisson structure $\{\cdot, \cdot\}$ defines a map

$$\begin{aligned} \{\cdot, \cdot\} : C^\infty(V) &\to \mathfrak{X}(V) \\ f &\mapsto \{f, \cdot\} =: X_f \end{aligned}$$

as follows from the Leibniz rule; this is called the **Hamiltonian vector field of** f. The Jacobi identity implies that this is a homomorphism of Lie algebras :

$$[X_f, X_g] = X_{\{f,g\}}$$

The **rank** of a bivector $\pi \in \mathfrak{X}^2(V)$, not necessarily Poisson, at a point $x \in V$, $\mathrm{rank}_x \pi$, is defined to be the dimension of the span of Hamiltonian vector fields at x. A point x is called **π-regular** if the rank of π is locally constant at x; otherwise x is said to be singular. Clearly, π-regular points form a dense open subset $\mathrm{Reg}(P)$ in V.

A bivector is called **regular** if $\mathrm{Reg}(\pi) = V$; in this case, $\mathcal{D} := \pi^\sharp(T^*V)$ is a subbundle of TV of rank $k = \mathrm{rank}\, \pi$. Notice that π endows its image $\mathcal{D}$ with an almost symplectic structure $\omega \in \Gamma(V, \wedge^2 \mathcal{D}^*)$, defined by

$$\omega(\pi(\xi), \pi(\eta)) := \pi(\xi, \eta).$$

Conversely, any subbundle $\mathcal{D} \subset TV$ together with a nondegenerate two-form $\omega \in \Gamma(V, \wedge^2 \mathcal{D}^*)$ defines a regular bivector π: it is the unique bivector such that the following diagram commutes:

$$\begin{tikzcd} T^*V \arrow[r, "\pi^\sharp"] \arrow[d, "i^*"'] & TV \\ \mathcal{D}^* \arrow[r, "\omega^{-1}"'] & \mathcal{D} \arrow[u, "i"'] \end{tikzcd}$$

where $i : \mathcal{D} \hookrightarrow TV$ denotes the inclusion and i^* its transpose.

Observe furthermore that the Lie homomorphism $f \mapsto X_f$ above says, in the regular case, that $\mathcal{D}$ is involutive, i.e., the tangent space of a foliation $\mathcal{F}$, and the $\mathcal{D}$-non-degenerate two-form ω above is easily seen to be closed on each leaf of $\mathcal{F}$.

This is still true even in the non-regular case : the Hamiltonian vector fields of π define a *singular foliation* on V, in the sense that every point x of V lies in a unique maximal integral (immersed) submanifold $\mathcal{L}$ – the *leaf* through x; each such leaf $\mathcal{L}$ is equipped by π with a symplectic form $\omega_{\mathcal{L}}$ defined as in the regular case, with the immersion $\mathcal{L} \hookrightarrow V$ a Poisson map. This provides a nice geometric description of a Poisson structure, but we will not pursue this point of view to avoid dealing with the monstrous technical difficulties of *singular* foliations; we content ourselves with quoting a local structure theorem which will be employed elsewhere :

THEOREM 90 (Weinstein's Splitting Theorem, [**70**]). *If* $\pi \in \mathrm{Poiss}(V)$ *has rank* $2r$ *at some point* p*, then there exists* $p \in U \subset V$ *and a Poisson isomorphism*

$$\varphi = \varphi_S \times \varphi_N : U \xrightarrow{\simeq} S \times N$$

where S *is a symplectic manifold of dimension* $2r$*, and* N *is a Poisson manifold which has rank* 0 *at* $\varphi_N(p)$*.*

PROOF. If $r = 0$ the statement is vacuous. For $r \neq 0$, take $f_1, g_1' \in C^\infty(\mathrm{Op}\, p)$ with $\pi(df_1, dg_1') = X_{f_1} g_1' \neq 0$; by the fundamental theorem of ordinary equations, we can find $g_1 \in C^\infty(\mathrm{Op}\, p)$ with $X_{f_1} g_1 = 1$. Therefore $\{X_{f_1}, X_{g_1}\}$ Lie-commute, so we can complete $\{f_1, g_1\}$ to a local system of coordinates $\{f_1, g_1, y_3, ..., y_{\dim V}\}$ with $\pi(dy_i, df_1) = 0 = \pi(dy_i, dg_1)$.
The obvious induction establishes then the claim. □

Observe finally that, when $p \in \mathrm{Reg}(\pi)$, N has the trivial Poisson structure.

2. Symplectic Realizations or Poisson as Folded Symplectic

We now turn to a discussion of symplectic realization of Poisson bivectors. From a conceptual point of view, this allows to regard Poisson structures as a non-degenerate structure abiding by some flatness conditions, namely, projectable symplectic structures. This will find some concrete application in Sections 7-8. We follow closely the expositions of [**16**], [**24**] and also [**70**].

DEFINITION 91. *A* **symplectic realization** *of a Poisson manifold* $(V, \{\cdot,\cdot\})$ *is a surmersion*[4]

$$\varrho : S \to V$$

of a symplectic *Poisson manifold* (S, Π) *which is a Poisson map.*

DEFINITION 92. *A surmersion* $\varrho : S \to V$ *of a symplectic manifold* (S, ω) *is called* **symplectically complete** *if the symplectic orthogonal to* $\ker d\varrho$ *is a foliation.*

THEOREM 93. *A symplectic realization of a Poisson manifold* (V, π) *is symplectically complete. Conversely, if a surmersion*

$$\varrho : S \to V$$

of a symplectic manifold (S, ω) *is symplectically complete and has connected fibres, there is a Poisson strucure* π *on* V *which is realized by* (S, ϱ, ω)*.*

PROOF. This is just this very simple remark : if

$$\varrho : S \to V$$

is a surmersion of a symplectic manifold (S, ω), then

$$\Pi(\ker d\varrho^0) = \ker d\varrho^\omega$$

where $\omega =: \Pi^{-1}$. Otherwise said, the distribution $(\ker d\varrho)^\omega$ ω-orthogonal to the foliation $\ker d\varrho$ consists precisely of the Π-Hamiltonians $X_{\varrho * f}$ of pullbacks of functions on V.

Therefore, if ϱ is a symplectic realization of π, then involutivity of $\ker d\varrho^\omega$ follows from the Jacobi identity for π.

Conversely, if (S, ω, ϱ) is symplectically complete, then

$$[X_{\varrho * f}, X_{\varrho * g}] \in \Gamma(\ker d\varrho^\omega), \quad \forall f, g \in C^\infty(V)$$

implies that $[X_{\varrho * f}, X_{\varrho * g}]$ is *constant* on the leaves of $\ker d\varrho^\omega$, and thus of the form $\varrho^* h$ for some $h \in C^\infty(V)$ if ϱ has connected fibres. □

[4]This language was coined by Thom; surmersion = surjective submersion

REMARK 94. *Observe that the corank of a Poisson tensor* π *realized by* (S, ω, ϱ) *at a point* $p \in V$ *is the difference*

$$\operatorname{rank}_p \pi = \dim V - (\ker d_p \varrho) \cap (\ker d_p \varrho)^\omega$$

A key fact is that *every* Poisson manifold admits a symplectic realization by some (exact !) symplectic structure in a neighbourhood of the zero section in its cotangent bundle. To explain the construction in a slightly invariant fashion, we must introduce some concepts first. Cf. [**24, 16**] for further references.

DEFINITION 95. *Let* (V, π) *be Poisson. A* **contravariant connection** *on* T^*V *is a bilinear map*

$$\nabla : \Omega^1(V) \times \Omega^1(V) \to \Omega^1(V)$$

satisfying

(1) $\nabla_{f\xi}\eta = f\nabla_\xi \eta$;
(2) $\nabla_\xi f\eta = f\nabla_\xi\eta + \left(L_{\pi(\xi)}f\right)\eta$

for all $(f, \xi, \eta) \in C^\infty(V) \times \Omega^1(V) \times \Omega^1(V)$. *A contravariant connection is* **torsion-free** *if*

$$\nabla_\xi\eta - \nabla_\eta\xi = \pi(\xi \wedge \eta) \tag{2}$$

If in addition T^*V *is equipped with a Riemannian metric* $\langle\cdot,\cdot\rangle$ *(as we will henceforth assume) we can introduce the notion of* **metric** *contravariant connection* ∇ *if*

$$L_{\pi(\zeta)}\langle\xi, \eta\rangle = \langle\nabla_\zeta\xi, \eta\rangle + \langle\xi, \nabla_\zeta\eta\rangle \tag{3}$$

The fundamental theorem of Riemannian geometry has its counterpart in the contravariant world :

THEOREM 96. *There is a unique metric, torsion-free contravariant connection* ∇ *on* T^*V*, called the* **Levi-Civita connection** $(V, \pi, \langle\cdot,\cdot\rangle)$.

From now on, we will always assume that ∇ is *the* Levi-Civita connection of $(\pi, \langle\cdot,\cdot\rangle)$.

DEFINITION 97. *A curve* $a : I \to T^*V$ *will be called a* **Poisson path** *if*

$$\frac{d\gamma_a}{dt}(t) = \pi(a(t)), \quad \gamma_a := \operatorname{pr} \circ a. \tag{4}$$

A Poisson path a *is called a* **geodesic** *if*

$$\nabla_a a = 0 \tag{5}$$

along $\gamma_a : I \to V$.

As in the Riemannian case, the condition on a to define a geodesic turns out to be an ordinary differential equation, so that there is always a unique maximal geodesic a_ξ starting at a chosen initial condition $\xi \in T^*V$.

DEFINITION 98. *Let* $\mathcal{G} \subset T^*V \times \mathbb{R}$ *be the subspace of all* (ξ, t) *for which* $a_\xi(t)$ *is defined. The* **geodesic flow** *of* ∇ *is defined by*

$$G : \mathcal{G} \to T^*V \tag{6}$$
$$G(\xi, t) = a_\xi(t) \tag{7}$$

THEOREM 99 ([**16**]). *Let* (V, π) *be a Poisson manifold,* $\operatorname{pr} : T^*V \to V$ *the canonical projection, and* ω_0 *the tautological symplectic structure of* T^*V*. Then if*

$$\omega := \int_0^1 G_t^*\omega_0 dt \tag{8}$$

there is an open disk subbundle S *of* T^*V *such that* $(S, \operatorname{pr}, \omega)$ *is a symplectic realization of* (V, π).

A slightly more down-to-earth description of the above ω is to be found in [**70**], and goes like this : let $x_1, ..., x_n$ be coordinates on an open subset $U \subset V$ and $y_1, ..., y_n$ fibre coordinates on $T^*U \simeq U \times \mathbb{R}^n$.

Consider the map

$$f : T^*U \to \mathbb{R}$$
$$f(x,y) := x \cdot y = \sum_1^n x_i y_i$$

We regard f as an n-parametric family of maps

$$y \mapsto [U \ni x \mapsto f(x,y)]$$

and denote by ξ_y the corresponding Hamiltonian vector field, $\xi_y := \pi(df_y)$.

Let ψ_y^s denote the time-s flow of ξ_y. Then set

$$\varphi_i : T^*U \to \mathbb{R}$$
$$\varphi_i(x,y) := \int_0^1 x_i \circ \psi_y^s(x) ds$$

Hence $\varphi_i(x,y)$ is the average value of the x_i-coordinate of the flow of the Hamiltonian field ξ_y which starts at $x \in U$. Note that $\varphi_i(x,0) = x_i$.

So the claim of Theorem 99 boils down to saying that

$$\varphi \in \Omega^1(T^*U)$$
$$\varphi := \sum_1^n \varphi_i dy_i$$

is a primitive for a symplectic realization of π :

$$d\varphi = \begin{pmatrix} 0 & -a^\tau \\ a & b \end{pmatrix}$$
$$a = \left(\frac{\partial \varphi_j}{\partial x_i}\right)_{i,j}, \quad b = \left(\frac{\partial \varphi_j}{\partial x_i} - \frac{\partial \varphi_i}{\partial x_j}\right)_{i,j} \qquad 1 \leqslant i,j \leqslant n$$
$$\pi = -a^{-1} b a^{-\tau}$$

We wish now to point out that the construction of a non-degenerate structure (S, ω) lying above π still makes sense, *whether π is Poisson or not*, provided we give up on either insisting that ω be closed or that ω^{-1} project down to π, as is necessary in light of Theorem 93.

The first possibility is to define the primitive φ as we did above, and then modify $d\varphi$ to some ω' by setting

$$\omega' = \begin{pmatrix} 0 & -a^\tau \\ a & b' \end{pmatrix}$$

where

$$b' := -a \pi a^\tau$$

Then ω' might no longer be closed, but by construction it is non-degenerate and ω^{-1} projects down via pr to π. As π and a are globally defined, this local formula for ω' globalizes appropriately by uniqueness.

This tentative replacement for "symplectic realization" seems tempting in that it is much easier to deal with the integrability condition $d\omega = 0$ than that of $[\pi, \pi] = 0$, as seen in Chapter 2, Section

6. However, imposing additionally that the resulting non-degenerate bivector be *projectable*, which means concretely that, in the notation above

$$\frac{\partial}{\partial y_k} a^{-1} b a^{-\tau} = 0 \quad \text{for all } 1 \leqslant k \leqslant n$$

renders this pseudo-realization useless from the perspective of h-principles, as it converts the problem of integrability into an even more overdetermined one.

Indeed, the projectability of $(d\varphi)^{-1}$ is controlled by a differential relation

$$\mathcal{R}_{\mathrm{proj}} \subset J^2(d^{-1}\Omega^2_{nd}) \subset J^2 T^* S$$
$$\mathcal{R}_{\mathrm{proj}} = \ker(\mathrm{symb}\,\mathrm{Proj})$$
$$\mathrm{Proj} : d^{-1}\Omega^2_{nd}(S) \to \oplus_1^n \mathfrak{X}^2(T^*S)$$
$$\mathrm{Proj} : \varphi \mapsto \left(\frac{\partial (d\varphi)^{-1}}{\partial y_1}, ..., \frac{\partial (d\varphi)^{-1}}{\partial y_n} \right)$$

as we can always assume that the fibres of $S \to V$ are connected; observe that

$$\mathrm{codim}_{J^2T^*S}\, \mathcal{R}_{\mathrm{proj}} = n \binom{2n}{2}$$

so $\mathcal{R}_{\mathrm{proj}}$ is even more overdetermined than the corresponding Poisson relation $\Pi \subset J^1\Lambda^2 TV$, since

$$\mathrm{codim}_{J^1\Lambda^2TV}\, \Pi = \binom{n}{3}$$

On the other hand, we may give up on projectability, and just consider the exact symplectic form ω defined in the statement of Theorem 99. The only thing preserved in this approach is that

$$\mathrm{pr}_*(\omega^{-1}|V) = \pi$$

Definition 100. *Call a*

$$p : S \to V$$

equipped with a section $V \to S$, a symplectic structure ω on S where

$$p_*(\omega^{-1}|V) = \pi$$

a **symplectic pseudo realization** *of the bivector $\pi \in \mathfrak{X}^2(V)$.*

We will see applications of this later on.

3. Poisson as a Dirac Geometry

We follow the excellent exposition of [**62**] of the results obtained in [**48**], simplified through the later work of [**69**].

Courant algebroids.

Definition 101. *A* **real Courant algebroid** *on a manifold V consists of :*

(1) *A vector bundle $\mathbb{E} \to V$;*
(2) *A bundle map $\rho : \mathbb{E} \to TV$;*
(3) *A non-degenerate symmetric $C^\infty(V)$-bilinear pairing $\langle \cdot, \cdot \rangle$ on the fibres of $\mathbb{E}$;*
(4) *A bracket*

$$[\![\cdot, \cdot]\!] : \Gamma(\mathbb{E}) \times \Gamma(\mathbb{E}) \to \Gamma(\mathbb{E})$$

such that, for every triple $e_1, e_2, e_3 \in \Gamma(\mathbb{E})$, we have

C1. Derivation of $[\![\cdot, \cdot]\!]$:

$$[\![e_1, [\![e_2, e_3]\!]]\!] = [\![[\![e_1, e_2]\!], e_3]\!] + [\![e_2, [\![e_1, e_3]\!]]\!]$$

C2. Derivation of $\langle\cdot,\cdot\rangle$:

$$\rho(e_1)\langle e_2,e_3\rangle = \langle [\![e_1,e_2]\!],e_3\rangle + \langle e_2,[\![e_1,e_3]\!]\rangle$$

C3. $\langle\cdot,\cdot\rangle$ controls skew-symmetric anomaly of $[\![\cdot,\cdot]\!]$:

$$[\![e_1,e_2]\!] + [\![e_2,e_1]\!] = \natural^{-1}\rho^*\left(d\langle e_1,e_2\rangle\right)$$

where

$$\rho^* : T^*V \to \mathbb{E}^*$$

is the map dual to ρ and

$$\natural : \mathbb{E} \to \mathbb{E}^*$$

is the isomorphism induced by $\langle\cdot,\cdot\rangle$.

REMARK 102. *Observe that $[\![\cdot,\cdot]\!]$ is* not *a Lie bracket on $\Gamma(\mathbb{E})$.*

From these axioms, it follows formally that :

(1) $[\![e_1,fe_2]\!] = f[\![e_1,e_2]\!] + (\rho(e_1)f)\,e_2$;
(2) $\rho([\![e_1,e_2]\!]) = [\rho(e_1),\rho(e_2)]$;
(3) $\rho\circ\natural^{-1}\circ\rho^* = 0$;
(4) $[\![e,\natural^{-1}\circ\rho^*(\mu)]\!] = \natural^{-1}\circ\rho^*\left(\mathcal{L}_{\rho(e)}\mu\right)$;
(5) $[\![\natural^{-1}\circ\rho^*(\mu),e]\!] = -\natural^{-1}\circ\rho^*\left(\iota_{\rho(e)}d\mu\right)$

for all $e,e_1,e_2\in\Gamma(\mathbb{E})$, $f\in C^\infty(V)$ and $\mu\in\Omega^1(V)$.

EXAMPLE 8. *Let $\mathbb{T}V := TV\oplus T^*V$ and define $\langle\cdot,\cdot\rangle$ to be the natural symmetric (n,n)-pairing*

$$\langle X+\xi,Y+\eta\rangle := (\iota_X\eta+\iota_Y\xi)$$

anchor $\rho:\mathbb{T}V\to TV$ projection to TV and bracket

$$[\![X+\xi,Y+\eta]\!] := [X,Y] + \mathcal{L}_X\eta - \iota_Y d\xi$$

This is the **standard Courant structure** *of the* **generalized tangent bundle** *of V.*

Dirac structures.

DEFINITION 103. *We call a subbundle $L<\mathbb{E}$* **(Courant) involutive** *if $\Gamma(L)$ is closed under the bracket $[\![\cdot,\cdot]\!]$.*

It is called **isotropic** *if*

$$\langle e_1,e_2\rangle = 0$$

for any two sections $e_1,e_2\in\Gamma(L)$, and **Lagrangian** *if it is maximal among isotropic subbundles.*

An involutive Lagrangian subbundle $L<\mathbb{E}$ will be called a **(real) Dirac structure on** $\mathbb{E}$.

DEFINITION 104. *Let $L<\mathbb{T}V$ be a Lagrangian subbundle, and denote by*

$$\mathrm{pr}_T:\mathbb{T}V\to TV,\quad \mathrm{pr}_{T^*}:\mathbb{T}V\to T^*V$$

the induced projections. The **type** *of L at a $p\in V$ is the codimension of $\mathrm{pr}_T(L_p)$ in TV; its* **cotype** *at p is the codimension of $\mathrm{pr}_{T^*}(L_p)$ in T^*V.*

DEFINITION 105. *A* **Lie algebroid** *over a manifold V consists of the following data :*

(1) *A vector bundle $A\to V$, equipped with a bundle morphism $\sharp : L\to TV$ over V, the* **anchor***;*
(2) *A Lie bracket*

$$[\cdot,\cdot]_A : \Gamma(A)^2\to\Gamma(A)$$

satisfying the Leibniz-type condition

$$[fe_1,e_2]_A = f[e_1,e_2]_A - (\sharp e_2 f)e_1,\quad f\in C^\infty(V), e_i\in\Gamma(A)$$

and for which the induced

$$\sharp : \Gamma(A) \to \mathfrak{X}(V)$$

is a Lie algebra homomorphism.

REMARK 106. *Clearly, Lagrangian subbundles must have rank half that of* $\mathbb{E}$. *Moreover, axioms* **C1-C3** *imply that* $[\![\cdot,\cdot]\!]$ *restricts to a* Lie *bracket on sections of a Dirac structure on* $\mathbb{E}$. *In other words, Dirac structures inherit a natural structure of* **Lie algebroid**.

REMARK 107. *More generally, if* $L < \mathbb{E}$ *is a Lagrangian subbundle, then*

$$\mathrm{Jac}_L : \Gamma(L)^3 \to \mathbb{R}$$
$$\mathrm{Jac}_L(e_1, e_2, e_3) := \langle e_1, [\![e_2, e_3]\!]\rangle$$

is tensorial and anti-symmetric, i.e.

$$\mathrm{Jac}_L \in \Gamma\left(\Lambda^3 L^*\right)$$

This tensor controls the involutivity of L *in the sense that*

$$\mathrm{Jac}_L = 0 \iff L \text{ is involutive.}$$

Now, any Lie algebroid $(A, \sharp, [\cdot,\cdot]_A)$ comes equipped with a de Rham-type operator on forms :

$$d_A : \Gamma\left(\Lambda^p A^*\right) \to \Gamma\left(\Lambda^{p+1} A^*\right) \qquad p \geqslant 0$$
$$(d_A\omega)(e_0, e_1, ..., e_p) := \sum_i (-1)^i \rho(e_i)\omega(e_0, ..., \hat{e}_i, ..., e_p) + \\ + \sum_{i<j} (-1)^{i+j} \omega\left([e_i, e_j], e_0, ..., \hat{e}_i, ..., \hat{e}_j, ..., e_p\right)$$

squaring to zero, $d_A^2 = 0$, and thus defining cohomology groups

$$H^p(A) := \frac{\ker\left(d_A : \Gamma\left(\Lambda^p A^*\right) \to \Gamma\left(\Lambda^{p+1} A^*\right)\right)}{\mathrm{image}\left(d_A : \Gamma\left(\Lambda^{p-1} A^*\right) \to \Gamma\left(\Lambda^p A^*\right)\right)}$$

REMARK 108. *Observe that a Lie bracket on* $\Gamma(A)$ *extends uniquely to a bracket*

$$[\cdot,\cdot]_A : \Gamma\left(\Lambda^p A\right) \times \Gamma\left(\Lambda^q A\right) \to \Gamma\left(\Lambda^{p+q-1} A\right) \qquad p, q \geqslant 0$$

by requiring that it be graded anti-symmetric, a derivation of $\wedge$, *and satisfies the graded Jacobi identity :*

Graded anti-symmetric:

$$[\zeta, \chi]_A = -(-1)^{(|\zeta|-1)(|\chi|-1)}[\chi, \zeta]_A$$

Derivation of $\wedge$**:**

$$[\zeta, \chi \wedge \tau]_A = [\zeta, \chi]_A \wedge \tau + (-1)^{(|\zeta|-1)|\chi|} \chi \wedge [\zeta, \tau]_A$$

Graded Jacobi:

$$\underset{\zeta,\chi,\tau}{\circlearrowleft} (-1)^{(|\zeta|-1)(|\tau|-1)}[\zeta, [\chi, \tau]_A]_A = 0$$

for every triple of homogeneous elements $\zeta, \chi, \tau \in \Gamma\left(\Lambda^\bullet A\right)$.

We shall refer to this extension as the **Schouten** *bracket of* A.

DEFINITION 109. *Let* A *be a Lie algebroid and* $e \in \Gamma(A)$. *The* **Lie derivative** *with respect to* e *is the degree* 0 *operator*

$$L_e^A : \Gamma\left(\Lambda^\bullet A^*\right) \to \Gamma\left(\Lambda^\bullet A^*\right)$$

defined by the supercommutator

$$L_e^A := [d_A, \iota_e]$$

where

$$i_e : \Gamma(\Lambda^\bullet A^*) \to \Gamma(\Lambda^\bullet A^*)$$

denotes the (degree -1*) operator of contraction with* e*.*

REMARK 110. *Observe the supercommutation relations*

$$\begin{aligned} [d_A, d_A] &= 0 & [d_A, L_e^A] &= 0 \\ [d_A, \iota_e] &= L_e^A & [L_e^A, L_{e'}^A] &= L^A_{[e,e']_A} \\ [\iota_e, \iota_{e'}] &= 0 & [L_e^A, \iota_{e'}] &= \iota_{[e,e']_A} \end{aligned}$$

EXAMPLE 9. *The tangent bundle* TV *is a Dirac structure in the generalized tangent bundle* $\mathbb{T}V$*. The anchor is* id_{TV} *and has for bracket the Lie bracket of vector fields.*

EXAMPLE 10. *The cotangent bundle* TV, T^*V *is a Dirac structures in* $\mathbb{T}V$*. It has trivial anchor and bracket.*

EXAMPLE 11. *If* $\mathcal{D}$ *is a subbundle of* TV*, then*

$$\mathcal{D} \oplus \mathrm{Ann}(\mathcal{D})$$

is Lagrangian in $\mathbb{T}V$*. It is Dirac iff* $\mathcal{D}$ *is the tangent bundle of a foliation* $\mathcal{F}$*.*

EXAMPLE 12. *If* ω *is a two-form on* V*, then*

$$e^\omega TV := \{X + \iota_X\omega : X \in TV\}$$

is Lagrangian in $\mathbb{T}V$*. It is Dirac iff* ω *is closed.*

EXAMPLE 13. *If* π *is a bivector on* V*, then*

$$e^\pi T^* := \{\iota_\xi\pi + \xi : \xi \in T^*V\}$$

is Lagrangian in $\mathbb{T}V$*. It is Dirac iff* π *is Poisson.*

The examples justify our interest in Dirac geometries, which simultaneously encode foliations, presymplectic and Poisson structures as presymplectic singular foliations. (See [**9**])

A especially important feature of the Dirac formalism is allowing to regard certain geometric structures of very singular type (e.g., a Poisson or a presymplectic structure of non-constant rank) as the "singular" shadow $\mathrm{pr}_T(L)$ a "regular" object L.

We have yet to say what we allow as "morphisms" of Dirac geometries. In view of the examples that this formalism encodes, it is natural to demand that its morphisms should generalize maps of "covariant" objects (such as presymplectic structures) and that of "contravariant" objects (like Poisson structures). For that purpose we introduce two notions.

DEFINITION 111. *If* $L_i \subset \mathbb{T}V_i$ *are Dirac structures, a smooth map*

$$f : V_0 \to V_1$$

is called :

Backward Dirac $(V_0, L_0) \to (V_1, L_1)$**:** *if*

$$L_0 = f^*L_1 := \{(X_0, f^*\xi_1) : (f_*X_0, \xi_1) \in L_1\}$$

Forward Dirac $(V_0, L_0) \to (V_1, L_1)$**:** *if*

$$L_1 = f_*L_0 := \{(f_*X_0, \xi_1) : (X_0, f^*\xi_1) \in L_0\}$$

REMARK 112. *Observe that the above definitions describe* relations*, and not* operations*, of Dirac structures. In fact,* f_*L_0 *need not be even well-defined, and* f^*L_1 *need not be a* smooth *subbundle*[5]

[5]But a sufficient condition to ensure smoothness is that f and $\mathrm{Ann}\, f_*TV_0 \cap L$ have both constant rank, see [**7**]

In the specific case of Dirac structures arising from *Poisson* structures, the remark above specializes to the following :

- $f_* L_{\pi_0}$ is well-defined iff π_0 if f-related to some $\pi_1 \in \mathfrak{X}^2(V_1)$, i.e., iff
$$f_{*x}\pi_0(x) = \pi_1(fx), \quad \text{for all } x \in V_0$$
- $f^* L_{\pi_1}$ is *pointwise* well-defined, but $f^* L_{\pi_1}$ has cotype zero at x iff
$$\pi_1(\operatorname{Ann} f_{*x}T_xV_0) \cap (f_{*x}T_xV_0) = 0$$

Call f a **pointwise Poisson-Dirac immersion** if f is an immersion and the condition above is satisfied. Note that, in that case, $f^* L_{\pi_1}$ corresponds to a bivector $\pi_0(x) \in \Lambda^2 T_xV_0$ at all $x \in V_0$; concretely,

$$\pi_0(x)(\alpha, \beta) := \pi_1(f(x))(\alpha', \beta')$$
$$f^*\alpha' = \alpha, \quad f^*\beta' = \beta$$

Note however that the assignment

$$x \mapsto \pi_0(x)$$

need not be smooth; when it is, we call f a **Poisson-Dirac immersion**.

EXAMPLE 14. *A simple example is given in* [**15**] *and reads as follows : in* $\mathbb{C}^3$*, the complex* 1*-forms*

$$dz_2, \quad dz_3 - z_2 dz_1$$

define a foliation $\mathcal{F}$ *by complex lines; in particular,* $\mathcal{F} \in \mathrm{SympFol}^4(\mathbb{C}^3)$ *corresponds to a corank-four Poisson tensor. Now*

$$\mathbb{C}^2 \to\subset \mathbb{C}^3$$
$$(z_1, z_2) \mapsto (z_1, z_2, 0)$$

is an embedding which is pointwise Poisson-Dirac and as such defines, for each $p \in \mathbb{C}^2$*, a* $\pi(p) \in \Lambda^2 T_p\mathbb{C}^2$.

But observe that $(z_2 = 0) \subset \mathbb{C}^2$ *is a leaf of* $\mathcal{F}$*, so that* $\operatorname{rank}\pi(p) = 2$ *for* $z_2(p) = 0$*, whereas all other leaves of* $\mathcal{F}$ *meet* $\mathbb{C}^2$ *at a single point – and thus* $\operatorname{rank}\pi(p) = 0$ *for* $z_2(p) \neq 0$.

This example shows that $p \mapsto \pi(p)$ *is not even lower semicontinuous !*

Lie bialgebroids.

DEFINITION 113. *Let* $(A, a, [\cdot,\cdot]_A)$ *be a Lie algebroid over* V*, and suppose its dual bundle* A^* *is also endowed with the structure of a Lie algebroid* $(A^*, a_*, [\cdot,\cdot]_{A^*})$*. We cay that the pair* (A, A^*) *is a* **Lie bialgebroid** *if* d_A *is a derivation of* $[\cdot,\cdot]_{A^*}$*, i.e., if*

$$d_A[\mu, \nu]_{A^*} = [d_A\mu, \nu]_{A^*} + [\mu, d_A\nu]_{A^*} \quad \textit{for all } \mu, \nu \in \Gamma(A^*)$$

By a result of [**50**], this notion is self-dual in the sense that

$$(A, A^*) \text{ Lie bialgebroid } \iff (A^*, A) \text{ Lie bialgebroid}$$

Suppose now $(A, a, [\cdot,\cdot]_A)$ is a Lie algebroid, and that its dual bundle also carries a Lie algebroid structure $(A^*, a_*, [\cdot,\cdot]_{A^*})$. Then define the bundle

$$\mathbb{E} := A \oplus A^*$$

This bundle comes equipped with two non-degenerate symmetric pairings :

$$\langle X + \xi, Y + \eta\rangle_\pm := (\iota_Y\xi \pm \iota_X\eta)$$

On $\Gamma(\mathbb{E})$ define a bracket $[\![\cdot,\cdot]\!]$ by

$$[\![X + \xi, Y + \eta]\!] := \{[X, Y]_A + \mathcal{L}^{A^*}_\xi Y - \mathcal{L}^{A^*}_\eta X - \frac{1}{2} d_{A^*}\langle X + \xi, Y + \eta\rangle_-\} +$$
$$+ \{[\xi, \eta]_{A^*} + \mathcal{L}^A_X\eta - \mathcal{L}^A_X\eta + \frac{1}{2} d_{A^*}\langle X + \xi, Y + \eta\rangle_-\}$$

and an anchor

$$\rho : \mathbb{E} \to TV$$
$$\rho(X + \xi) := a(X) + a_*(\xi)$$

THEOREM 114 (Thm 2.5, [**48**]). *The quadruple* $(\mathbb{E}, \rho, [\![\cdot,\cdot]\!], \langle\cdot,\cdot\rangle_+)$ *defines a Courant algebroid if* (A, A^*) *is a Lie bialgebroid.*

THEOREM 115 (Thm 2.6, [**48**]). *In a Courant algebroid* $(\mathbb{E}, \rho, [\![\cdot,\cdot]\!], \langle\cdot,\cdot\rangle)$*, suppose* $L_1, L_2 < \mathbb{E}$ *are transverse Dirac structures. Then* (L_1, L_2) *is a Lie bialgebroid, where* L_2 *is considered as the bundle dual to* L_1 *under the pairing* $\langle\cdot,\cdot\rangle$*.*

REMARK 116. *Observe that* $e^\omega T$ *is transverse to* T^*V *is* ω *is a symplectic form on* V*; hence, by Theorem 115,* $(e^\omega T, T^*V)$ *is a Lie bialgebroid.*

Hamiltonian operators. Suppose we are given a Lie bialgebroid (A, A^*) and we construct the Courant algebroid of Theorem 114.

Given a A-two-form $B \in \Gamma\left(\Lambda^2 A^*\right)$, let $e^B A < \mathbb{E}$ be the subbundle

$$e^B A := \{X + \iota_X B : X \in A\}$$

Observe that $e^B A$ is again Lagrangian :

$$\begin{aligned} \langle X + \iota_X B, Y + \iota_Y B\rangle_+ &= (\iota_X \iota_Y B + \iota_Y \iota_X B) \\ &= 0 \end{aligned}$$

Now :

THEOREM 117 (Thm. 6.1 [**48**]). *The Lagrangian subbundle* $e^B A$ *is Dirac if and only if the following Maurer-Cartan equation holds :*

$$d_A B + \frac{1}{2}[B, B]_{A^*} = 0$$

There are a few points we wish to highlight concerning the Dirac formalism as a tool to handle *integrability/involutivity* of structures up-to-homotopy.

First, for a two-form $B \in \Omega^2(V)$, the condition that $e^B(e^\pi T^*V)$ be the graph of a bivector π' is that

$$\mathrm{pr}_{T^*}\, e^B L_\pi = T^*V \Leftrightarrow \mathrm{id}_{T^*V} + B\pi \in \mathrm{End}(T^*V) \text{ is invertible}$$

In that case, according to the theorem above, π Poisson implies

$$e^B \pi = \pi(\mathrm{id}_{T^*V} + B\pi)^{-1} \text{ Poisson iff } dB = 0 \text{ on each leaf of } \pi$$

The effect of a B-transform on a Dirac structure L is that of twisting the presymplectic form along its leaves, and the algebraic condition above ensures that the symplectic structure on the leaves of the Poisson bivector π is twisted in such a way as to remain non-degenerate. Note in particular that the rank of π is *pointwise* preserved under a B-transform.

4. Soft remarks on the hard nature of Poiss

We are primarily concerned with the following:

PROBLEM 118. *Specify subsheaves* $\Phi \subset$ Poiss *for which* some *form of the h-principle holds.*

Of course, recall that there is a natural $\mathbb{R}$-action

$$\mathbb{R} \times \mathrm{Poiss}(V) \to \mathrm{Poiss}(V)$$
$$(t, \pi) \mapsto t\pi$$

so the sheaf of Poisson structures is *star-shaped* and hence *universally contractible.*

In particular every bivector is homotopic to a *Poisson* one. Therefore we must impose extra-requirements to obtain a meaningful h-principle. We list a few of the possibilities :

Extension Problem: One requires a *relative* version of the h-principle, proving that, say, on all submanifolds $V_0 \subset V$ of a certain type, germs of Poisson bivectors along V_0 can be *extended* to global Poisson structures;

Approximation Problem: Here one starts with a given germ of bivector along submanifolds of a given type, and shows that one can C^0-approximate such germ by a germ of Poisson structure;

Sheaves with Topology: Specify sheaves which have non-trivial topology, such as sheaves of Poisson structures with bounded or fixed corank, admitting only specified singularity types, unimodular structures etc.

Model example. A model example of a such subsheaf is $\Phi := \mathrm{Poiss}_{\leqslant 2}$, the sheaf of Poisson structures of rank no greater than 2. This sheaf, regardless of being microflexible or not, is easily seen to abide by the h-principle as follows : if $\pi \in \mathrm{Poiss}_{\leqslant 2}(V)$, then given *any* function

$$f : V \to \mathbb{R}$$

the bivector $f\pi$ is again Poisson :

$$[f\pi, f\pi] = f^2[\pi,\pi] + f\pi \wedge \pi(df)$$

where the latter term vanishes since $\pi \wedge X = 0$ for all X in the image of π.

Hence if we are given an extension problem

$$\begin{cases} \pi' \in \mathrm{Poiss}_{\leqslant 2}(\partial D) \\ ¿\exists \pi \in \mathrm{Poiss}(D), \pi|\,\mathrm{Op}(\partial D) = \pi'? \end{cases}$$

we choose a $f : \mathrm{Op}\, D \to \mathbb{R}$ with germ 1 along ∂D, and zero germ along a concentric smaller sphere still in the domain of definition of π'; then $\pi := f\pi' \in \mathrm{Poiss}(D)$ extends π'. This produces relative h-principles for the sheaf $\mathrm{Poiss}_{\leqslant 2}$ by the obvious inductive argument.

Sheaf-theoretic h-principle. Next, observe that the fact that if $P \in \mathrm{Poiss}(\mathbb{R}^n)$, then for all $t \in \mathbb{R}$, the **streched bivector**

$$\mathrm{stretch}\, P : x \mapsto P(tx)$$

where $P(tx)$ is regarded as an element in $\Lambda^2 T_x V$ by translation, is again Poisson.

So if we are given a

$$P \in \mathfrak{X}^2(D^n), \quad D^n \subset M$$
$$P|\partial D \in \mathrm{Poiss}(\partial D)$$

we can define a

$$\widetilde{P} \in \mathrm{Poiss}^\flat(D)$$
$$\widetilde{P}|\partial D = P^\flat|\partial D$$

by means of a choice of smooth function

$$\varrho : M \to I$$
$$\varrho|\,\mathrm{Op}(\partial D) = 1$$

with $\mathrm{supp}(\varrho)$ contained in the interior of the subspace where P is Poisson.

Explicitly, $\widetilde{P}$ is represented by

$$P' \in \mathrm{Poiss}^M(\Delta_D)$$
$$P'(v_1, v_2) := P(\varrho(v_1)v_2 + (1 - \varrho(v_1))v_1)$$

Note that

$$P'(v_1, v_2) = \begin{cases} P(v_2) & \text{if } \varrho(v_1) = 1 \\ P(v_1) & \text{if } \varrho(v_1) = 0 \end{cases}$$

and that

$$P(v_1, \cdot)|\operatorname{Op}(v_1) \in \operatorname{Poiss}(v_1)$$

for all $v_1 \in V$.

By means of sufficiently fine triangulations (and with the obvious induction), this method allows to homotope any section $F \in \Gamma(M, \Pi)$ to the image of some $\widetilde{P}$ by the natural map

$$\operatorname{Poiss}^\flat \to \Gamma(\cdot, \Pi)$$

in such a way that $\operatorname{germ}_{V_0} F$ holonomic implies $\operatorname{germ}_{V_0} \widetilde{P} = P^\flat$ for some $P \in \operatorname{Poiss}(V_0)$, $j^1 P = F$.

One easily sees the the same argument applies at the level of Gromov structures, so that

PROPOSITION 119. *The Poisson relation is germifiable. Hence $B\operatorname{Poiss}^\flat$ is weakly contractible.*

1-equivalence $B\operatorname{Poiss} \to B\operatorname{Poiss}^\flat$. Assume given a Poisson structure $P \in \operatorname{Poiss}(V)$.

PROPOSITION 120. *$B\operatorname{Poiss} \to B\operatorname{Poiss}^\flat$ is a 1-equivalence.*

PROOF. It suffices to show that $B\operatorname{Poiss}$ is connected since, according to Proposition 119, and thus $B\operatorname{Poiss} \to B\operatorname{Poiss}^\flat$ induces (trivial) surjections on homotopy groups.

So suppose we are given a germ $\varphi \in \operatorname{Poiss}_x$ for some $x \in \mathbb{R}^n$. Represent it by a $\pi \in \operatorname{Poiss}(U)$ for some $x \in U$ and let $x' \in U$ be π-regular. If $\operatorname{rank}_{x'} \pi = 0$, choose a smooth curve C connecting x to x' inside U, and take $\operatorname{germ}_C \pi$. This shows that, for all π and all x, $\operatorname{germ}_x \pi$ is connected to $\operatorname{germ}_{x'} 0$ for some x' as close to x as we wish.

In the case where $\operatorname{rank}_{x'} \pi \geqslant 2$, we use Theorem 90 to find a small $x' \in U''$ not containing x, and a system of coordinates $(p_1, q_1, ..., q_r, y_1, ..., y_{n-2r})$ centred at x', where π is expressed as

$$\pi|U'' = \sum_1^r \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i}$$

Let now $\varrho : \mathbb{R} \to \mathbb{R}$ be a smooth monotone function with $\varrho(t) = 0$ for $t \leqslant \varepsilon/8r$ and $\varrho(t) = 1$ for $t \geqslant \varepsilon/4r$, where $\varepsilon > 0$ is some number with

$$\sum_i p_i^2 + q_i^2 + \sum y_j^2 \leqslant \varepsilon \quad \Longrightarrow \quad (p_1, ..., y_{n-2r}) \in U''$$

Define

$$P_i : U'' \to \mathbb{R}$$
$$P_i(p_1, q_1, ..., y_{n-2r}) := \varrho\left(p_i^2 + q_i^2 + \sum_1^{n-2r} y_j^2\right)$$

and set

$$P := \sum_1^r P_i \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i}$$

Observe that $P \in \operatorname{Poiss}(U'')$ and that

$$\operatorname{germ}_{x''} P = \operatorname{germ}_{x''} \pi$$

at those points x'' of U'' where all P_i's have germ 1. It is easy to see that

$$W := (U - U') \cup (U' - \cup_1^r \operatorname{Cl} P_i^{-1}[0, 1))$$

is connected, so we can choose a smooth curve C connecting x to x' such that :

$$C|[0,1/2] \text{ lies in } W$$
$$C|[1/2,1] \text{ lies in } U'$$

Then connect $\mathrm{germ}_x\,\pi$ to $\mathrm{germ}_{x'}\,0$ by taking the germ along $C([0,1/2])$ of π, and along $C([1/2,1])$ of P. □

5. h-principles, I : regular Poisson structures

In this section we will prove the h-principle for symplectic foliations on open manifolds.

Given a foliation $\mathcal{F}$ on a manifold V, by an $\mathcal{F}$-foliated form ω of degree p we mean a section $\omega \in \Gamma(V, \wedge^p T^*\mathcal{F})$. We can always extend ω to a p-form $\widetilde{\omega} \in \Gamma(V, \wedge^p T^*V)$ (not uniquely) and often we will not distinguish between ω and an extension $\widetilde{\omega}$. The de Rham differential induces a differential $d_{\mathcal{F}}$ on foliated forms, so a form ω is $d_{\mathcal{F}}$-closed if $d\widetilde{\omega}$ pullbacks to zero on each leaf of $\mathcal{F}$.

Definition 121. *A pair $(\mathcal{F},\omega)$ where ω is an $\mathcal{F}$-foliated, non-degenerate, 2-form will be called an* **almost symplectic foliation**. *If, additionally, ω is $d_{\mathcal{F}}$-closed then we call $(\mathcal{F},\omega)$ a* **symplectic foliation**.

Theorem 122. *Let V be an open manifold, $(\mathcal{F}_0,\omega_0)$ a codimension-q almost symplectic foliation and ξ a class in $H^2_{dR}(V;\mathbb{R})$. Then there is a homotopy $(\mathcal{F}_t,\omega_t)$ of almost-symplectic, codimension-q foliations, such that:*

(1) $(\mathcal{F}_1,\omega_1)$ *is* symplectic *and*
(2) ω_1 *can be represented by a* global *closed two-form lying in ξ.*

Proof. A codimension-q distribution on V is simply a section of the Grassmannian bundle $\mathrm{Gr}_{n-q}(TV)$, which we topologize as usual with the compact-open topology. We also topologize $\Gamma(V, \wedge^2 T^*V)$ with the compact-open topology and we consider the subspace topology on the space of almost symplectic, codimension-q distributions :

$$\mathrm{QSympDist}^q(V) \subset \mathrm{Gr}_{n-q}(V) \times \Omega^2(V),$$
$$\mathrm{QSympDist}^q(V) = \left\{(\mathcal{D},\omega) : (\omega|\mathcal{D})^{\frac{n-q}{2}} \neq 0\right\}.$$

On the space $\mathrm{QSympFol}^q$ of almost symplectic, codimension-q foliations and on the space $\mathrm{SympFol}^q$ of symplectic, codimension-q foliations we take also the induced topologies :

$$\mathrm{QSympFol}^q(V) := \mathrm{Fol}^q(V) \times \Gamma(V, \wedge^2 T^*V) \cap \mathrm{QSympDist}^q(V),$$
$$\mathrm{SympFol}^q(V) := \{(\mathcal{F},\omega) \in \mathrm{QSympFol}^q(V) : \omega \text{ is } \mathcal{F}-\text{symplectic}\}$$

Thus our initial data $(\mathcal{F}_0,\omega_0)$ lives in $\mathrm{QSympFol}^q(V)$.

Let us observe that, since non-degeneracy of a form is an open condition, there exists a positive function

$$\varepsilon : V \to \mathbb{R}$$

such that

$$\mathrm{dist}(\omega(y),\omega_0(y)) < \varepsilon(y)$$

implies ω $\mathcal{F}_0$-leafwise non-degenerate.

Hence, applying approximation by closed forms (Corollary 79), we can choose some smooth positive function ρ and a closed form ϕ such that $[\phi] = \xi$ and which is ε-close to ω_0 on the ρ-neighbourhood U_ρ of a core K of V.

Next, we choose a smooth function $\chi : V \longrightarrow [0,1]$, which is identically zero outside U_ρ and identically 1 on $U_{\rho/2}$, and we define a homotopy of 2-forms by setting:

$$\omega : [0,1/2] \longrightarrow \Gamma(V; \wedge^2 T^*V)$$

$$t \mapsto \omega_0 + 2t(\phi - \omega_0)\chi$$

Then ω is a continuous map such that :

(1) $\omega(0) = \omega_0$
(2) $\omega(t)$ is $\mathcal{F}_0$-leafwise non-degenerate for all $t \in [0, 1/2]$;
(3) $\omega(1/2)$ is closed on $U_{\rho/2}$.

We let $\mathcal{F}(t)$ be the stationary homotopy at $\mathcal{F}_0$, for $t \leqslant 1/2$; note then that $t \mapsto (\mathcal{F}(t), \omega(t))$ takes values in $\mathrm{QSympFol}^q(V)$.

In order to define the second half of the homotopy, we choose a compression $g_t : V \to V$ between $g_0 = 1_V$ and $g_1 : V \to U_{\rho/2}$ and we define a continuous path $(\mathcal{F}(t), \omega(t)) \in \mathrm{QSympFol}^q(V)$ for $t \geqslant 1/2$ by setting:

$$t \mapsto ((g_{2t-1})^*)\mathcal{F}(1/2), (g_{2t-1})^*\omega(1/2))$$

The concatenated homotopy

$$t \mapsto (\mathcal{F}(t), \omega(t)), \qquad t \in [0, 1]$$

is the one we sought, since $(\mathcal{F}(1), \omega(1)) \in \mathrm{SympFol}^q(V)$ and the 2-form $\omega(1)$ lies in $g_1^*\xi = \xi$. $\square$

Remark 123. *Note that the concatenated homotopy is only* continuous *in t, but a standard argument involving reparameterization with vanishing derivatives at the end points makes it smooth.*

Obstructions to integrability. Recall that Δ_q denotes the space of almost symplectic, codimension-q distributions and that Ω_q denotes the space of symplectic, codimension-q foliations on the manifold V. In this section, we wish to address the following question :

> Are there strictly topological conditions that one can imposed upon V so as to ensure that $\pi_0(\mathrm{SympFol}^q(V)) \to \pi_0(\mathrm{QSympDist}^q(V))$ is an isomorphism ?

Notice that the previous theorem implies that $\mathrm{SympFol}^q(V) \hookrightarrow \mathrm{QSympFol}^q(V)$ induces an isomorphism at the level of π_0. To handle the map

$$\pi_0(\mathrm{QSympFol}^q(V)) \to \pi_0(\mathrm{QSympDist}^q(V))$$

we will invoke Hæfliger's obstruction theory, Theorem 88.

That result shows that the integrability problem for distributions on open manifolds can be completely reduced to one in obstruction theory, namely, solving

$$\begin{tikzcd}
& B\Gamma_q \times B\,\mathrm{GL}_{n-q} \arrow[d, "Bd\times 1"] \\
& B\,\mathrm{GL}_q \times B\,\mathrm{GL}_{n-q} \arrow[d, "\oplus"] \\
V \arrow[r, "\tau"'] \arrow[ur, "\xi\times\alpha"'] \arrow[uur, dashed] & B\,\mathrm{GL}_n
\end{tikzcd}$$

In order to apply this to our setting, let us spell out what Haefliger's theorem says: given a distribution $\mathcal{D}_0$ such that the diagram above can be solved one can find a path of bundle isomorphisms

$$\varphi_t : TV \longrightarrow \mathcal{D}_0 \oplus (\mathcal{D}_0)^\perp$$

where φ_1 maps $T\mathcal{F}$ onto $\mathcal{D}_0$ for some codimension-q foliation $\mathcal{F}$. That is : the path

$$t \mapsto \varphi_t^*\mathcal{D}_0 =: \mathcal{D}_t$$

ends at $T\mathcal{F}$. If an almost-symplectic $\mathcal{D}_0$-form ω_0 has been provided, one can transport this two-form along the path φ_t by setting

$$\omega_t := \varphi_t^*\omega_0$$

so that, at all times we have $(\mathcal{D}_t, \omega_t) \in \mathrm{QSympDist}^q(V)$ and at the end-point $(\mathcal{D}_1, \omega_1) \in \mathrm{QSympFol}^q(V)$. Now we can use Theorem 122 to construct a continuous path starting at $(\mathcal{D}_1, \omega_1)$ and ending at some element in $\mathrm{SympFol}^q(V)$. This yields:

Proposition 124. *Given a distribution $\mathcal{D}_0$ on an open manifold V with a non-degenerate 2-form ω_0, there is a homotopy $(\mathcal{D}_t, \omega_t) \in \mathrm{QSympDist}^q(V)$ starting at $(\mathcal{D}_0, \omega_0)$ and ending at a $\mathcal{F}_1$-foliated* symplectic *form ω_1 if and only if the distribution $\mathcal{D}_0$ is homotopic to a foliation.*

Let us recast the previous result in terms of Poisson geometry. Let π be regular of rank k.

It is well known that a regular bivector π is a **Poisson structure** (i.e., $[\pi, \pi] = 0$) if and only if $\mathcal{D} := \mathrm{image}\, \pi^\sharp$ is the tangent bundle of a foliation $\mathcal{F}$ and the foliated 2-form ω is $d_{\mathcal{F}}$-closed (in other words, it is a foliated symplectic structure); this is easily worked out from Theorem 90.

Hence, Proposition 124 is equivalent to

Theorem 125. *Let π_0 be a regular bivector on an open manifold V. Then there is a path $t \mapsto \pi_t$ of regular bivectors on V starting at π_0 and ending at a* Poisson *bivector π_1 if and only if the distribution $\mathcal{D}_0 = \mathrm{image}\, \pi_0^\sharp$ is homotopic to an integrable distribution.*

Remark 126. *According to* [**15**, Corollary 14]*, if the leafwise symplectic form of a regular Poisson stucture extends to a* global *closed 2-form, then its leaves are submanifolds of a very special type, called Lie-Dirac submanifolds, and the Poisson manifold is integrable to a smooth symplectic groupoid. See* [**15**] *for more details.*

Also, we have, in line with Corollary 89:

Corollary 127. *On an open n-manifold V satisfying $H^i(V; \mathbb{Z}) = 0$ for all $i > q + 1$, any regular codimension-q bivector is homotopic, through regular bivectors, to a Poisson structure.*

Remark 128. *David Martínez Torres has pointed out that this approach can be used to prove foliated versions of the h-principle for conformal symplectic structures, contact structures and, more generally, Jacobi structures (one needs the analog of Theorem 78 concerning approximation by d_θ-closed forms, where d_θ denotes the twisted differential).*

An Example. Our main result states that the integrability of a regular bivector is controlled by the integrability of the underlying distribution. However, it could still be the case that the existence of a non-degenerate 2-form on this distribution would force its integrability. In other words, one might wonder whether we are any closer to solving the obstruction problem pertaining to integrability of subbundles if we have already solved that of providing an almost symplectic structure (which amounts to lifting the classifying map to the bundle into $B\mathrm{Sp}$). In this last section we modify a classical example of Bott to construct an example of a regular bivector on an open manifold which is not homotopic to a regular Poisson structure, showing that not all the integrability obstructions are encoded in the sympletic ones.

Let E be a vector bundle over the manifold V and denote by $\mathrm{Pont}(E) \subset H^\bullet_{\mathrm{dR}}(V; \mathbb{R})$ the Pontryagin ring of E. Recall Bott's theorem :

Theorem 129 (Bott [**4**]). *If a codimension-q distribution $\mathcal{D}$ on V is homotopic to an involutive distribution $T\mathcal{F}$, then its normal bundle $\nu(\mathcal{D})$ satisfies:*

$$\mathrm{Pont}^k(\nu(\mathcal{D})) = 0, \quad \textit{for } k > 2q.$$

Recall also that the Pontryagin classes $p_i(E)$ of a (real) vector bundle E are related to the Chern classes $c_i(E \otimes \mathbb{C})$ of its complexification $E \otimes \mathbb{C}$ by:

$$p_i(E) = (-1)^i c_{2i}(E \otimes \mathbb{C}).$$

On the other hand, the Euler class and the top Chern class of a complex vector bundle coincide, so the square of the Euler class $e(E)^2 = e(E \otimes \mathbb{C})$, lies in the Pontryagin ring of E.

In order to construct our example, we start with the trivial complex vector bundle $\underline{\mathbb{C}^{2n}} = \mathbb{C}P^{2n-1} \times \mathbb{C}^{2n}$ over $\mathbb{C}P^{2n-1}$ and split it holomorphically as the sum of the tautological line bundle J and its orthogonal :

$$\underline{\mathbb{C}^{2n}} = J \oplus J^\perp,$$

where:

$$J = \{(x, v) \in \mathbb{C}P^{2n-1} \times \mathbb{C}^{2n} : v \in x\},$$
$$J^{\perp} = \{(x, v) \in \mathbb{C}P^{2n-1} \times \mathbb{C}^{2n} : v \perp x\}.$$

We denote by H the line bundle dual to J, and by H^2 its symmetric square, whose fibre at a point $x \in \mathbb{C}P^{2n-1}$ consists of all degree 2 homogeneous functions $J_x \longrightarrow \mathbb{C}$. Finally, let $\theta_1, ..., \theta_{2n}$ be a basis of $(\mathbb{C}^{2n})^*$ and define on $\mathbb{C}^{2n}$ the non-degenerate two-form

$$\theta = \theta_1 \wedge \theta_2 + \cdots + \theta_{2n-1} \wedge \theta_{2n}$$

The 2-form θ defines a holomorphic surjection

$$\theta_* : \mathrm{Hom}(J, J^{\perp}) \longrightarrow H^2, \quad \theta_*(\varphi)(\xi) = \theta(\xi, \varphi(\xi))$$

where $\varphi \in \mathrm{Hom}(J, J^{\perp})$ and $\xi \in J$. Under the well-known isomorphism $T\mathbb{C}P^{2n-1} \simeq \mathrm{Hom}(J, J^{\perp})$ we see that $\mathcal{D} := \ker(\theta_*)$ is a holomorphic subbundle of (complex) codimension one. If $\nu(\mathcal{D})$ denotes its normal bundle, then by multiplicativity of the Euler class, one has

$$0 \neq 2n = e(T\mathbb{C}P^{2n-1}) = e(\mathcal{D})e(\nu(\mathcal{D}))$$

so that $0 \neq e(\nu(\mathcal{D})) \in H^2(\mathbb{C}P^{2n-1}; \mathbb{R})$. Thus $e^4(\nu(\mathcal{D})) = e(\nu(\mathcal{D}))^2 e(\nu(\mathcal{D}))^2 \in \mathrm{Pont}^4(\nu(\mathcal{D}))$ as we pointed out above. Since the real cohomology ring of $\mathbb{C}P^{2n-1}$ is the truncated polynomial ring $\mathbb{R}[t]/t^{2n}\mathbb{R}[t]$, where t has degree 2, choosing $n > 2$ guarantees that $e(\nu(\mathcal{D}))^4 \neq 0$. By Bott's result, we conclude that $\mathcal{D}$ is a (real) codimension-2 distribution which is not homotopic to a foliation.

Finally, we consider the open manifold $V := \mathbb{C}P^{2n-1} \times \mathbb{C}$. The projection in the first factor $p : V \to \mathbb{C}P^{2n-1}$ gives an injection:

$$p^* : H^{\bullet}_{\mathrm{dR}}(\mathbb{C}P^{2n-1}; \mathbb{R}) \to H^{\bullet}_{\mathrm{dR}}(V; \mathbb{R})$$

which maps the Euler class of $\nu(\mathcal{D})$ to that of $p^*\nu(\mathcal{D})$, which is the normal bundle to the holomorphic codimension-1 subbundle $p^*\mathcal{D} \subset TV$. This means that $p^*\mathcal{D}$ also cannot be homotopic to a foliation in V. A metric on $p^*\mathcal{D}$, together with its complex structure, yields a non-degenerate 2-form on $\mathcal{D}$. Hence, this gives an example of a regular bivector on an open manifold which is not homotopic, through regular bivectors, to a Poisson structure.

6. h-principles, II : b-Poisson structures

Another prototype of the h-principles we wish to prove in the realm of Poisson geometry arises in the context of b-geometry, as proposed by Melrose and further developed by Guillemin, Miranda and Pires. By imposing a (rather stringent) regularity condition on bivectors, we are able to sucessfully reduce the Poisson relation to a microflexible one.

The category ${}^b\,\mathrm{Mfd}$.

Definition 130. *The category of* b**-manifolds** *and* b**-maps** ${}^b\,\mathrm{Mfd}$ *is the one having as objects pairs* (V, Z)*, where* Z *is a codimension-one submanifold of* V*, and whose morphisms*

$$(V, Z) \to (V', Z')$$

are those smooth maps

$$f : V \to V'$$

which are transverse to Z'*, and such that*

$$f^{-1}Z' = Z$$

Definition 131. *A smooth function*

$$x : V \to \mathbb{R}$$

which has zero as a regular value is called a **defining function** *for the hypersurface* $x^{-1}(0)$. *If* $Z \subset V$ *is a codimension-one submanifold and* $U \subset V$ *is an open subset, we call a defining function*

$$x : U \to \mathbb{R}$$

for $U \cap Z$ *to* **define** Z **locally on** U.

From now on, (V, Z) will denote an m-dimensional b-manifold. Observe that every b-manifold (V, Z) is locally b-isomorphic, around points $p \in Z$, to the local model

$$(V, Z) \underset{\text{loc}}{\simeq} \left(\mathbb{R}^m, \{0\} \times \mathbb{R}^{m-1}\right)$$

around the origin, in the sense that there are open sets

$$p \in U \subset V$$

and

$$0 \in U' \subset \mathbb{R}^m$$

and a b-isomorphism

$$\begin{tikzcd} U \arrow[r, "\simeq"] & U' \\ U \cap Z \arrow[u] \arrow[r, "\simeq"] & U' \cap \{0\} \times \mathbb{R}^{m-1} \arrow[u] \end{tikzcd}$$

Let then $(x, y_1, ..., y_n)$ be coordinates in U arising out of this b-isomorphism; we shall refer to such local coordinates as **adapted** to (V, Z).

We begin our study of b-manifolds with a very simple lemma which is going to be useful later on :

Lemma 132. *If* $h \in C^\infty(V)$ *vanishes identically on* Z, *then (locally) defining functions for* Z *divide* h.

Proof. This is obviously a local matter, so we can assume that

$$V = \mathbb{R} \times \mathbb{R}^n = \{(x, y_1, ..., y_n) : x, y_i \in \mathbb{R}\}$$

Now if

$$h : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$$
$$h(0, y) = 0$$

we write the Taylor expansion of h with integral remainder :

$$h(x, y) = h(0, y) + \frac{\partial h}{\partial x}(0, y)x + x^2 \int_0^1 (1-t)\frac{\partial^2 h}{\partial x^2}(t, y)dt = \tag{9}$$

$$= x \cdot \left(\frac{\partial h}{\partial x}(0, y) + x \int_0^1 (1-t)\frac{\partial^2 h}{\partial x^2}(t, y)dt \right) \tag{10}$$

thus proving our claim. $\square$

The (co-)tangent bundle of a b-manifold. Let (V, Z) be a b-manifold, and denote by $\mathfrak{X}(V)$ the Lie algebra of vector fields on V; as is well-known, this corresponds to the space of smooth sections of the tangent bundle of V. Recall furthermore that, for any point $p \in V$, T_pV is described by

$$T_pV = \mathfrak{X}(V)/I_p(V) \cdot \mathfrak{X}(V)$$

where $I_p(V) \subset C^\infty(V)$ denotes the ideal of smooth functions vanishing at p.

Now let $\mathfrak{X}(V, Z) \subset \mathfrak{X}(V)$ be the Lie subalgebra consisting of those vector fields $X \in \mathfrak{X}(V)$ which are tangent to Z. A miraculous fact, first pointed out by Melrose, is that this subalgebra can also be identified with the space of smooth sections of a certain vector bundle over V, which we denote by $T(V, Z) \to V$:

$$\mathfrak{X}(V, Z) = \Gamma(V, T(V, Z))$$

Let us give a more concrete description of this vector bundle :

Proposition 133. *On*

$$T(V, Z) := \coprod_{p \in V} \mathfrak{X}(V, Z)/I_p(V) \cdot \mathfrak{X}(V, Z)$$

there is a unique vector bundle structure such that under the natural (vector bundle) map

$$T(V, Z) \to TV$$

the Lie subalgebra $\mathfrak{X}(V, Z)$ pulls back to $\Gamma\,(V, T(V, Z))$.

Proof. Note that, for each p, the ideal $I_p(V) \cdot \mathfrak{X}(V, Z)$ consists of finite sums $\sum f_i X_i$, where f_i are smooth functions vanishing at p, and X_i are vector fields tangent to Z. Thus, for $p \in V - Z$, $I_p(V) \cdot \mathfrak{X}(V, Z)$ coincides with $I_p(V) \cdot \mathfrak{X}(V)$, whence $T_p(V, Z) = T_pV$ for all such points.

Write a $X \in \mathfrak{X}(U, U \cap Z)$ in these coordinates :

$$X = a\frac{\partial}{\partial x} + \sum_1^n b_i \frac{\partial}{\partial y_i}$$

where $a, b_1, ..., b_n \in C^\infty(U)$.

Now

$$X|U \cap Z \in \Gamma(U \cap Z, T(U \cap Z))$$

immediately implies that $a|U \cap Z = 0$, so by Lemma (132) we have that x divides a, say,

$$X = \alpha x\frac{\partial}{\partial x} + \sum_1^n b_i \frac{\partial}{\partial y_i}$$

so that $x\frac{\partial}{\partial x}, \frac{\partial}{\partial y_1}, ..., \frac{\partial}{\partial y_n}$ forms a local basis for $T(V, Z)$ in adapted coordinate charts.

It remains to check that the change of adapted coordinate charts is actually smooth; for this purpose, let $x', y'_1, ..., y'_n$ be another such system of local coordinates. Then

$$(x, y) \mapsto (e^{f(x,y)}x, g_1(x, y), ..., g_n(x, y)) = (x', y')$$

for some smooth function f, so that

$$x\frac{\partial}{\partial x} = \left(1 + x\frac{\partial f}{\partial x}\right) x' \frac{\partial}{\partial x'} + x\sum_{i=1}^n \frac{\partial g_i}{\partial x}\frac{\partial}{\partial y'_i}$$

$$\frac{\partial}{\partial y_j} = \left(\frac{\partial f}{\partial y_j}\right) x' \frac{\partial}{\partial x'} + \sum_{i=1}^n \frac{\partial g_i}{\partial y_j}\frac{\partial}{\partial y'_i}$$

□

In passing, we note that, along Z, the direction spanned by $x\frac{\partial}{\partial x}$ is *independent* of the adapted chart, and thus defines a canonical non-vanishing section $\nu \in \Gamma\left(Z, T(V,Z)|Z\right)$ which gives rise to a line bundle ${}^{b}N(V,Z) \subset T(V,Z)|Z$, said to be the ***b*-normal line field** of (V,Z), and which appears as the kernel of

$$T(V,Z)|Z \to TZ$$

induced by the natural surjective map

$$\mathfrak{X}(V,Z) \to \mathfrak{X}(Z)$$

Definition 134. *The* **tangent bundle** *of the b-manifold* (V,Z) *is the vector bundle* $T(V,Z) \to V$*, whereas its* **cotangent bundle** $T^*(V,Z) \to V$ *is the dual to* $T(V,Z)$.

Similarly, we define ***p*-multivector fields** *and* ***p*-differential forms** *on the b-manifold* (V,Z) *as sections of* $\Lambda^p T(V,Z)$ *and* $\Lambda^p T^*(V,Z)$*, respectively. For notational convenience, we set*

$$\mathfrak{X}^p(V,Z) := \Gamma\left(V, \Lambda^p T(V,Z)\right)$$
$$\Omega^p(V,Z) := \Gamma\left(V, \Lambda^p T^*(V,Z)\right)$$

Note that, in an adapted coordinate chart as above, the cotangent bundle of (V,Z) is spanned by $\frac{dx}{x}, dy_1, ..., dy_n$.

Remark 135. *Observe that if*

$$\begin{tikzcd} A \arrow[rr, "\sharp"] \arrow[dr] & & TV \arrow[dl] \\ & V & \end{tikzcd}$$

is any transitive *Lie algebroid and* $Z \subset V$ *a codimension-one submanifold, we can transport the "miracle"*

$$\mathfrak{X}(V,Z) = \Gamma\left(V, T(V,Z)\right)$$

to write the Lie subalgebra of sections α *of* A *whose anchor* $\sharp\alpha$ *is tangent to* Z *as the space of all smooth sections of the Lie algebroid*

$$A_Z := A \times_{TV} T(V,Z)$$

which carries natural Lie algebroid morphisms $T(V,Z) \leftarrow A_Z \to A$ *fitting into the commutative diagram*

$$\begin{tikzcd} A \arrow[r, "\sharp"] & TV \\ A_Z \arrow[u] \arrow[r, "\sharp_Z"'] & T(V,Z) \arrow[u] \end{tikzcd}$$

This could be used to treat "non-singularly" in A_Z *sections* $\alpha \in \Gamma(A)$ *with* $\sharp\alpha$ *vanishing linearly along* Z*. This section is devoted to the crucial example* $A = TV$*,* $\sharp = \mathrm{id}_{TV}$.

Definition 136. *Let* V *be a* $2n$*-dimensional manifold.*

A **transversely non-degenerate** *bivector* $\pi \in \mathfrak{X}^2(V) := \Gamma\left(V, \Lambda^2 TV\right)$ *is a bivector with the property that its n-th (=top) exterior power* π^n *is transverse to the zero section*

$$z : V \to \Lambda^{2n} TV$$

The space of all transversely non-degenerate bivectors will be denoted by

$$\mathfrak{X}^2_{\pitchfork}(V) \subset \mathfrak{X}^2(V)$$

Observe that by the transversality requirement, a such transversely non-degenerate bivector π defines a codimension-one submanifold,

$$Z(\pi) := (\pi^n)^{-1} z(V) \subset V$$

which we call the **singular locus** of π; hence $\pi \in \mathfrak{X}^2_{\pitchfork}(V)$ gives rise to the b-manifold $(V, Z(\pi))$.

The space of all transversely non-degenerate bivectors with singular locus Z will be written

$$\mathfrak{X}^2_{\pitchfork}(V, Z) \subset \mathfrak{X}^2_{\pitchfork}(V)$$

Definition 137. *A transversely non-degenerate bivector π will be called a b-**bivector** if it is also a bivector on the b-manifold $(V, Z(\pi))$, i.e.,*

$$\pi \in \mathfrak{X}^2_{\pitchfork}(V) \cap \mathfrak{X}^2(V, Z)$$

The space of all b-bivectors will be denoted by $\mathfrak{X}^2_b(V)$.

Suppose $\pi \in \mathfrak{X}^2(V)$, $Z \subset V$ of codimension one as usual, and fix a point $p_0 \in Z$. In an open neighbourhood of p_0 in V, we choose a defining function x for Z and write π in the form

$$\pi = \frac{\partial}{\partial x} \wedge \pi(dx) + \nu \tag{11}$$

where $\nu(dx)$ vanishes identically. Call this decomposition of π **adapted** to the defining function x. Then clearly

$$\pi\left(T^*_{p_0} V\right) \subset T_{p_0} Z$$

if and only if

$$\pi\left(d_{p_0} x\right) = 0$$

Lemma 138. *Let $Z \subset V$ be a connected codimension-one submanifold and $\pi \in \mathfrak{X}^2(V)$. Then the following are equivalent :*

1. *$\pi(T^*_p V) \subset T_p Z$ for all $p \in Z$;*
2. *$\pi(d_p x) = 0$ for all (locally) defining function x and all $p \in Z$;*
3. *$\pi \in \mathfrak{X}^2(V, Z)$.*

Proof. (1) $\Leftrightarrow$ (2) is our remark above, while (3) $\Rightarrow$ (1) is obvious.

As for (2) $\Rightarrow$ (3) : Applying Lemma 132 in the present case yields

$$\pi(dx)|Z = 0 \Rightarrow \pi(dx) = xX$$

for some $X \in \mathfrak{X}(V)$; hence $\pi(\frac{dx}{x})$ is defined and thus $\pi_p \in \Lambda^2 T_p(V, Z)$. $\square$

Let us now point out an important **exclusion principle** for transversely non-degenerate bivectors :

Proposition 139. *Let π be transversely non-degenerate with singular locus Z, and set*

$$\aleph(\pi) := \{p \in Z : \pi(T^*_p V) \subset T_p Z\}$$

Then $\aleph(\pi)$ is open and closed in Z.

Proof. $\aleph(\pi)$ is clearly closed by its very definition.

Now use an adapted decomposition of π as in (11) :

$$\pi = \frac{\partial}{\partial x} \wedge \pi(dx) + \nu$$

and note that π^n is a non-zero multiple of

$$\frac{\partial}{\partial x} \wedge \pi(dx) \wedge \nu^{n-1}$$

Now, if $p_0 \in \aleph(\pi)$, then we know from the previous lemma that $\pi(d_{p_0} x) = 0$. Since π^n is transverse to the zero section, we must have $\nu(p_0)^{n-1} \neq 0$, so that ν^{n-1} is non-vanishing on $\mathrm{Op}(p_0)$.

But

$$\pi^n | (\mathrm{Op}(p_0) \cap Z) = 0$$

then implies that

$$\pi(dx) \text{ divides } \nu^{n-1}$$

on $(\mathrm{Op}(p_0) \cap Z)$, and the only way this would not contradict transversality of π^n is if

$$\pi(dx) | (\mathrm{Op}(p_0) \cap Z)$$

is identically zero. This shows that $\aleph(\pi)$ is also open in Z, whence our claim. □

COROLLARY 140. *To check whether a given transversely non-degenerate bivector $\pi \in \mathfrak{X}^2_{\pitchfork}(V)$ is a b-bivector on $(V, Z(\pi))$ it suffices to check whether*

$$\pi(T_p^* V) \subset T_p Z$$

at some (hence every) point p in each connected component of $Z(\pi)$.

PROOF. Immediate from Lemma 138 and Proposition 139. □

By means of this proposition, we can derive a **rigidity principle** for transversely non-degenerate bivectors :

PROPOSITION 141. *Suppose π_t is a homotopy of transversely non-degenerate bivectors on V. If π_0 is a b-bivector, then every π_t is a b-bivector.*

PROOF. Suppose not; then there is t with $\pi_t(T^*V)$ not contained in TZ_{π_t} at some point $p \in Z_{\pi_t}$. But then

$$\Pi := \pi_t + \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial s} \in \mathfrak{X}^2(V \times \mathbb{R}^2)$$

is also transversely non-degenerate.

Let $\widetilde{Z}$ be the singular locus of Π. The exclusion principle for Π says that either it sends $T^*(V \times \mathbb{R}^2)$ into $T\widetilde{Z}$ at all points or at none. But

$$\Pi(T^*_{(p,t,s)}(V \times \mathbb{R}^2)) = \pi_t(T^*V) + T_{(t,s)}\mathbb{R}^2$$

and by hypothesis we have

$$\Pi(T^*_{(p,0,s)}(V \times \mathbb{R}^2)) = T_p Z_0 + T_{(0,s)}\mathbb{R}^2$$

Hence $\Pi \in \mathfrak{X}^2(V \times \mathbb{R}^2, \widetilde{Z})$ and thus $\pi_t \in \mathfrak{X}^2(V, Z_t)$. □

***b*-de Rham complex.** Recall that a coordinate-free way to define the exterior differential for a (usual) manifold V,

$$d : \Omega^p(V) \to \Omega^{p+1}(V)$$

is by means of the formula

$$(d\omega)(X_0, ..., X_p) := \sum_i (-1)^i X_i \omega(X_0, ..., \widehat{X_i}, ..., X_p) +$$
$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, ..., \widehat{X_i}, ..., \widehat{X_j}, ..., X_p)$$

where $X_0, ..., X_p \in \mathfrak{X}(V)$. Observe that the fact that $d^2 = 0$ is a straightforward consequence of the Jacobi identity on $\mathfrak{X}(V)$.

Now the fact that $\mathfrak{X}(V, Z)$ is a *Lie subalgebra* of $\mathfrak{X}(V)$ implies that the *exact same formula* defines a b-exterior differential

$$^b d : \Omega^p(V, Z) \to \Omega^{p+1}(V, Z)$$

PROPOSITION 142. *The b-de Rham operator factors as the composition of*

$$j^1 : \Gamma(V, \Lambda^p T^*(V,Z)) \to \Gamma(V, J^1(\Lambda^p T^*(V,Z)))$$
$$\widetilde{\mathrm{symb}} : \Gamma(V, J^1(\Lambda^p T^*(V,Z))) \to \Gamma(V, \Lambda^{p+1} T^*(V,Z))$$

where $\widetilde{\mathrm{symb}}$ *is induced by a bundle* epimorphism *with contractible fibres :*

$$\mathrm{symb} : J^1(\Lambda^p T^*(V,Z)) \to \Lambda^{p+1} T^*(V,Z)$$

PROOF. Let us see how to express ${}^b d$ in local (adapted) coordinates : away from Z, it is clear that ${}^b d$ coincides with d, and close to the boundary, we write a general p-form using adapted local charts and multi-index notation

$$\omega = \sum_{|\alpha|=p} \omega_\alpha dy_\alpha + \sum_{|\beta|=p-1} \omega'_\beta \frac{dx}{x} \wedge dy_\beta$$

and one computes

$${}^b d\omega = \sum_{|\alpha|=p} \left(x \frac{\partial \omega_\alpha}{\partial x} \frac{dx}{x} + \sum_j \frac{\partial \omega_\alpha}{\partial y_j} dy_j \right) \wedge dy_\alpha - \sum_{|\beta|=p-1} \frac{\partial \omega'_\beta}{\partial y_j} \frac{dx}{x} \wedge dy_j \wedge dy_\beta$$

The claim follows immediately by inspection of the formula above; note in particular that symb is fibrewise surjective even along $x = 0$. □

So we end up with a b-de Rham complex :

$$\Omega^0(V,Z) \xrightarrow{{}^b d} \Omega^1(V,Z) \xrightarrow{{}^b d} \cdots \xrightarrow{{}^b d} \Omega^p(V,Z) \xrightarrow{{}^b d} \cdots$$

whose homology we define to be the b-de Rham cohomology of (V,Z) :

$${}^b H^p(V,Z) := \frac{\ker\left({}^b d : \Omega^p(V,Z) \to \Omega^{p+1}(V,Z)\right)}{\mathrm{image}\left({}^b d : \Omega^{p-1}(V,Z) \to \Omega^p(V,Z)\right)}$$

Moreover, by continuity of the usual differential, a b-map

$$f : (V,Z) \to (V',Z')$$

defines both a push-forward

$${}^b f_* : T_p(V,Z) \to T_{f(p)}(V',Z')$$

and a pull-back

$${}^b f^* : T^*_{f(p)}(V',Z') \to T_p(V,Z)$$

dual to one another, in a functorial fashion, and it follows automatically that this induces

$${}^b f_* : N_p(V,Z) \to N_{f(p)}(V',Z')$$

Needless to say, the induced

$${}^b f^* : \Omega^p(V',Z') \to \Omega^p(V,Z)$$

commutes with the differential,

$${}^b d({}^b f^* \omega') = {}^b f^*({}^b d\omega')$$

and thus establishes a morphism at the level of b-cohomology :

$${}^b f^* : {}^b H^\bullet(V',Z') \to {}^b H^\bullet(V,Z)$$

b-symplectic structures.

DEFINITION 143. *By a* **non-degenerate** *two-form on a b-manifold (V, Z) we mean a section $\omega \in \Omega^2(V, Z)$ with the property that the induced vector bundle morphism*

$$\omega : T(V, Z) \to T^*(V, Z)$$

is an isomorphism.

Such a two-form is called **symplectic** *if in addition to being non-degenerate it is ${}^b d$-closed : ${}^b d\omega = 0$.*

As in the (usual) manifold case, this is tantamount to saying that $\dim V$ is even, say, $\dim V = 2n$ and that ω^n is non-vanishing as a section of $\Omega^{2n}(V, Z)$.

DEFINITION 144. *A b-**Poisson** structure π on V is a Poisson structure given by a transversely non-degenerate bivector. Hence we set :*

$$\mathrm{Poiss}^b(V) := \mathrm{Poiss}(V) \cap \mathfrak{X}^2_{\pitchfork}(V)$$

PROPOSITION 145. *There is a one-to-one correspondence*

$$\{\textit{symplectic structures } \omega \textit{ on } (V, Z)\} \leftrightarrow \{b\textit{-Poisson structures } \pi \textit{ with singular locus } Z\}$$

PROOF. One direction is trivial : a non-degenerate two-form ω on (V, Z) gives rise to the bivector $\pi = \omega^{-1}$, and by definition of the b-de Rham operator, π automatically satisfies $[\pi, \pi] = 0$. It just remains to check that π is transversely non-degenerate, but this is also easy : ω^n non-vanishing as a section of $\Lambda^{2n}T^*(V, Z)$ implies that π^n is also non-vanishing as a section of $\Lambda^{2n}T(V, Z)$, but, regarded as a section of $\Lambda^{2n}TV$ under the natural map $T(V, Z) \to TV$, vanishes transversely along Z.

The converse requires Weinstein's Splitting theorem. Indeed, suppose $\pi \in \mathfrak{X}^2(V)$ is b-Poisson, with singular locus Z. We claim that in fact $\pi \in \mathfrak{X}^2_b(V)$ (thus justifying the names b-bivector and b-Poisson) and that $\omega = \pi^{-1}$ is a symplectic form on (V, Z).

Indeed, π is obviously symplectic on $V - Z$; at a point $p \in Z$, the rank of π decreases to, say, $2r < 2n$. We apply Weinstein's Theorem to find local coordinates $p_1, q_1, ..., q_r, \xi_{2r+1}, ..., \xi_{2n}$ where π is represented by

$$\pi = \varsigma + \mu$$
$$\varsigma := \sum_i \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i}$$
$$\mu := \sum_{i,j} \mu_{ij} \frac{\partial}{\partial \xi_i} \wedge \frac{\partial}{\partial \xi_j}$$

where $\mu_{ij}(p) = 0$ for all i, j; this means that

$$\pi^n = \varsigma^n + \binom{n}{1}\varsigma^{n-1} \wedge \mu + ... + \binom{n}{n-1}\varsigma \wedge \mu^{n-1} + \mu^n$$

Now, $\varsigma^k = 0$ for $k > r$, and the coefficients of μ^k are sums of products of k functions which vanish at p; hence the coefficients of $\varsigma^{n-k} \wedge \mu^k$ are all divisible by x^k; thus, being b-Poisson implies that $r = n - 1$.

Consequently, π has corank 2 at points p of Z, and thus Z inherits a codimension-one foliation by symplectic codimension-two leaves of π. Hence we can find a local representation

$$\pi = x\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y_1} + \sum_{i=0}^{n-1} \frac{\partial}{\partial y_{2i}} \wedge \frac{\partial}{\partial y_{2i+1}}$$

whence $\omega := (\pi)^{-1}$ is well-defined as a section of $\Omega^2(V, Z)$, and thus automatically symplectic on (V, Z). $\square$

Corollary 146. *A transversely non-degenerate bivector π is homotopic through such bivectors to a Poisson structure only if π is a b-bivector.*

Proof. Follows from the rigidity principle (Proposition 141) and Proposition 145. □

So it is clear from the start that, among transversely non-degenerate bivectors, the h-principle can hold true only for b-bivectors. That this indeed is the case is what we show in the sequel.

h-principle for b-Poisson structures.

Theorem 147. *The Poisson relation abides by the h-principle on open manifolds over b-bivectors.*

Proof. Let π_0 be a b-bivector, and let Z_0 be its singular locus. Then as shown above, $\omega_0 := (\pi_0)^{-1}$ is a non-degenerate two-form on (V, Z).

Let $F_0 : V \to J^1(T^*(V, Z))$ be a lift of ω_0, i.e.,

$$\mathrm{symb} \circ F_0 = \omega_0$$

Choose a core $K \subset V$ and apply Holonomic Approximation to find an isotopy h^t of id_V, and a holonomic

$$F_1 : \mathrm{Op}(h^1(K)) \to J^1(T^*(V, Z))$$

which is so close to $F_0| \mathrm{Op}(h^1(K))$ that there is a homotopy

$$H : \mathrm{Op}(h^1(K)) \times I \to J^1(T^*(V, Z))$$

connecting $F_0| \mathrm{Op}(h^1K)$ to F_1 through lifts of non-degenerate two-forms :

$$\mathrm{symb} \circ H_t \text{ non-degenerate on } (V, Z) \text{ around } \mathrm{Op}(h^1K) \text{ for all } t$$

Now fix a compression

$$g : V \times I \to V$$
$$g_0 = \mathrm{id}_V$$
$$g_1(V) \subset U$$

where U is an open neighbourhood of $\mathrm{Op}(h^1K)$ where all our objects are defined.

Next, observe that the g_t's, being open embeddings, define b-maps

$$g_t : (V, Z_t) \to (V, Z)$$
$$Z_t := g_t^{-1} Z$$

which induce a homotopy

$$H' : t \mapsto g_t^* F_0 \in \Gamma(J^1(T^*(V, Z_t)))$$

between F_0 and $g_1^* F_0| \mathrm{Op}(h^1K)$; note that $\mathrm{symb} \circ H'_t$ is non-degenerate as a two-form on (V, Z_t). Concatenating this by the homotopy $g_1^* H$, we get a homotopy through b-bivectors between the bivector π_0 and

$$\pi_1 := (\widetilde{\mathrm{symb}} g_1^* F_1)^{-1}$$

□

7. h-principles, III : coercibility

In this section we wish to formulate the notion of *coercibility* of a bivector, and show how this sheds some light into the integrability up-to-homotopy of Poisson tensors.

7.1. Coercibility data.

Definition 148. *Let $p : E \to V$ be a $\mathbb{R}^q$-bundle and let also $P \in \mathfrak{X}^2(E)$.*
We say that the bivector

$$Q_0 := p_*(P|V) \in \mathfrak{X}^2(V)$$

is **forced down** *upon V by P.*

Consider the ring C_0^∞ of germs of smooth functions $\mathbb{R} \supset \mathrm{Op}\, 0 \to \mathbb{R}$, and let $C_{\text{flat}}^\infty \subset C_0^\infty$ be the ideal of flat germs, i.e., those in the kernel of

$$\mathrm{Taylor} : C_0^\infty \to \mathbb{R}[t]$$
$$\mathrm{Taylor}(\mathrm{germ}_x f) = \sum_{k \geqslant 0} \frac{f^{(k)}}{k!}(0) t^k$$

Suppose now P is a Poisson bivector on a trivial bundle

$$\mathrm{pr} : V \times \mathbb{R} \to V$$

which forces down $Q_0 \in \mathfrak{X}^2(V)$ on $V = V \times \{0\}$. Assume further that a choice of Riemannian metric $\langle \cdot, \cdot \rangle$ on V has been made.

Observe that $t : V \times \mathbb{R} \to \mathbb{R}$ being well-defined around V allows us to write P in the form

$$P = \frac{\partial}{\partial t} \wedge Y + Q$$

so $Y = P(dt), Q(dt) = 0$ and $Q|V = Q_0$.

Remark 149. *Observe that the condition $[P, P] = 0$ is equivalent to*

$$\begin{cases} L_Y Q = Y \wedge \frac{\partial Y}{\partial t} \\ \frac{1}{2}[Q, Q] = Y \wedge \frac{\partial Q}{\partial t} \end{cases}$$

Observe that, from the second equation, one sees that $[Q, Q] \in \mathfrak{X}^3(V \times \mathbb{R})$ actually lies in $Y \wedge \mathfrak{X}^2(V \times \mathbb{R})$; thus we are implicitly assuming that Q (hence Q_0) already satisfies "most" of the Poisson conditions.

Definition 150. *A smooth*

$$f : \mathrm{Op}\, V \to \mathbb{R}$$

is called a ***t*-coercibility datum of order** *$\delta \in C_{\mathit{flat}}^\infty$ if*

- **C1:** $\frac{\|P(dt)\| \|df\|}{P(dt, df)}$ *is bound above by some positive, smooth $h : \mathrm{Op}\, V \to \mathbb{R}$;*
- **C2:** $j^\infty \left(\frac{\|P(dt)\| \|df\|}{P(dt, df)^2} e^{-1/\delta(t)} \right) \to 0$ *as $t \to 0$;*
- **C3:** $j^\infty \left(\frac{\|P(dt)\|}{P(dt, df)} e^{-1/\delta(t)} \right) \to 0$ *as $t \to 0$.*

Observe that the actual choice of Riemannian metric is immaterial in that if $\langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_2$ and the conditions **C1-C3** above are satisfied by one of them, then it is satisfied by the other (with the *same* order δ. Also, let $\delta_{\text{std}} := \mathrm{germ}_0\, e^{-1/t}$ in what follows.

Example 15. *Suppose $P(dt)$ is a gradient with respect to the Riemannian metric, $\langle P(dt), \cdot \rangle = df$. Then f is a t-coercibility datum of order δ_{std}.*

Example 16. *Suppose that there exist $f \in C^\infty(\mathrm{Op}\, V), C > 0$ such that*

$$P(dt, df) > C \|P(dt)\|, \quad C > 0$$

and that $P(dt)$ is compactly supported. Then f can also be taken to have compact support, and be a t-coercibility datum of order δ_{std}.

EXAMPLE 17. *Let* $V = \mathbb{R}^{2n-1}$ *and*

$$P \in \mathrm{Poiss}^b(V \times \mathbb{R}), \quad (P^n)^{-1}0 = V$$

Then

$$P = t\frac{\partial}{\partial t} \wedge \frac{\partial}{\partial x_1} + \sum \frac{\partial}{\partial x_{2i}} \wedge \frac{\partial}{\partial x_{2i+1}}$$

Then

$$P(dt) = t\frac{\partial}{\partial x_1}, \quad P(dt, dx_1) = t$$

Now, clearly $f = x_1$ *satisfies requirements* **C1-C3** *for being a* t*-coercibility datum of order* δ_{std}.

7.2. Coertion of Hamiltonians. We will describe here the process of **coertion** of a Hamiltonian $P(dt)$ by means of a coercibility datum (f, δ). Coertion always takes place with one extra dimension, but it will turn out that it is much easier to deal with – producing nicer results and imposing less dynamical restrictions – when more extra variables added.

THEOREM 151 (Coertion). *Suppose a Poisson structure* $P \in \mathrm{Poiss}(V \times \mathbb{R})$ *forces down a bivector* $Q_0 \in \mathfrak{X}^2(V)$ *on* $t = 0$*, and that* f *is a* t*-coercibility datum of order* $\delta \in C^\infty_{\mathit{flat}}$.

Then there is a 1*-form*

$$b \in \Omega^1(\mathrm{Op}\, V - V)$$

such that :

(1) *The* db*-transformed Poisson structure*

$$e^{db}P \in \mathrm{Poiss}(\mathrm{Op}\, V - V)$$

extends smoothly to a Poisson structure $\widetilde{P} \in \mathrm{Poiss}(\mathrm{Op}\, V)$*;*

(2) *The leaves of* $\widetilde{P}(dt)$ *are contained in the leaves of* $P(dt)$*, and*

$$j^\infty \widetilde{P}(dt) \to 0 \quad \textit{as} \quad t \to 0$$

(3)

$$\widetilde{Q} := \widetilde{P} - \frac{\partial}{\partial t} \wedge \widetilde{P}(dt)$$

is of the form TQT^τ*, where*

$$T = \mathrm{id} + \frac{e^{1/\delta(t)} P(dt) \otimes \widehat{d}f}{1 + e^{1/\delta(t)P(dt,df)}}$$

$$\widehat{d}f = df - \frac{\partial f}{\partial t}dt$$

(4) V *is a Poisson submanifold of* $(\mathrm{Op}\, V, \widetilde{P})$*, and*

$$\widetilde{Q}_0 := \widetilde{Q}|V = SQS^\tau$$

where

$$S \in \mathrm{End}(T\,\mathrm{Op}(V)), \quad S|t^{-1}(c) \in \mathrm{End}(Tt^{-1}(c))$$

$$S = \mathrm{id} + \frac{P(dt) \otimes \widehat{d}f}{P(dt, df)}$$

PROOF. Let us consider, on $\widetilde{V}' := \widetilde{V} - V \times \{0\}$, the 1-form

$$b := -e^{1/\delta(t)} f dt$$

and let

$$B := db = -e^{1/\delta(t)} df \wedge dt = -e^{1/\delta(t)} \widehat{d}f \wedge dt$$

Observe that the assertion of the theorem is local, in the sense that it suffices to verify that, on each $U_\alpha \subset V$ of a differential atlas $\mathfrak{A} = \{(U_\alpha, \varphi_\alpha)\}$ with precompact domains.

$$\widetilde{U}_\alpha := U_\alpha \times (i - 1/2, i + 1/2), \quad \mathrm{Cl}(U_\alpha) \text{ compact}, \quad i \in \mathbb{Z}$$

the bivector $e^{B_\alpha} P_\alpha$ satisfies the conclusion of the theorem, taking $V := U_\alpha$. (The decoration with subscript α denoting the obvious restriction to $\widetilde{U}_\alpha$, and since B is perfectly well defined outside $V \times \{0\}$, and is exact, and the B-transform of P is thus automatically Poisson outside that subspace, the theorem claims nothing on the $U_\alpha \times (i - 1/2, i + 1/2)$ for $i \neq 0$).

Therefore, we may freely assume that $V = \mathbb{R}^n$, and endow it with global coordinates $x_1, ..., x_n$, by means of which we express

$$P = \begin{pmatrix} 0 & -P(dt)^\tau \\ P(dt) & Q \end{pmatrix}$$
$$B = \begin{pmatrix} 0 & e^{1/\delta(t)} \hat{d} f \\ -e^{1/\delta(t)} \hat{d} f^\tau & 0 \end{pmatrix}$$

and

$$P(dt) = \begin{pmatrix} P(dt)_1 \\ P(dt)_2 \\ \\ P(dt)_n \end{pmatrix} \qquad Q = \begin{pmatrix} 0 & Q_{12} & \cdots & Q_{1n} \\ Q_{21} & 0 & \cdots & Q_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ Q_{n1} & Q_{n2} & \cdots & 0 \end{pmatrix}$$

where $P(dt)_k$, Q_{ij} are given by

$$P(dt) = \sum_k P(dt)_k \frac{\partial}{\partial x_k}$$
$$Q = \sum_{i,j} Q_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$$
$$Q_{ij} + Q_{ji} = 0$$

We compute

$$BP = \begin{pmatrix} e^{1/\delta(t)} P(dt, df) & e^{1/\delta(t)} df(Q) \\ 0 & e^{1/\delta(t)} \hat{d} f^\tau P(dt)^\tau \end{pmatrix}$$

and thus

$$\mathrm{id} + BP = \begin{pmatrix} 1 + e^{1/\delta(t)} P(dt, df) & e^{1/\delta(t)} df(Q) \\ 0 & \mathrm{id} + e^{1/\delta(t)} \hat{d} f^\tau P(dt)^\tau \end{pmatrix}$$

where we the abuse notation to mean the identity $n \times n$ and $(n-1) \times (n-1)$ matrices; so $\mathrm{id} + BP$ is everywhere invertible, with inverse

$$(\mathrm{id} + BP)^{-1} = \begin{pmatrix} \frac{1}{1+e^{1/\delta(t)} P(dt,df)} & -\frac{1}{1+e^{1/\delta(t)} P(dt,df)} e^{1/\delta(t)} df(Q)(\mathrm{id} + e^{1/\delta(t)} \hat{d} f^\tau P(dt)^\tau)^{-1} \\ 0 & (\mathrm{id} + e^{1/\delta(t)} \hat{d} f^\tau P(dt)^\tau)^{-1} \end{pmatrix}$$
$$P(\mathrm{id} + BP)^{-1} = \begin{pmatrix} 0 & \frac{1}{1+e^{1/\delta(t)} P(dt,df)} P(dt) \\ \frac{1}{1+e^{1/\delta(t)} P(dt,df)} P(dt) & \left(\mathrm{id} - \frac{e^{1/\delta(t)} P(dt) \hat{d} f}{1+e^{1/\delta(t)} P(dt,df)}\right) Q \left(\mathrm{id} + e^{1/\delta(t)} \hat{d} f^\tau P(dt)^\tau\right)^{-1} \end{pmatrix}$$

We now remark that there is a simple formula for the inverse of this rank-one update of id :

$$(\mathrm{id} + e^{1/\delta(t)} \hat{d} f^\tau P(dt)^\tau)^{-1} = \mathrm{id} - \frac{e^{1/\delta(t)}}{1 + e^{1/\delta(t)} P(dt, df)} \hat{d} f^\tau P(dt)^\tau$$

so that, letting

$$T := \mathrm{id} - \frac{1}{1+e^{1/\delta(t)}P(dt,df)} e^{1/\delta(t)} P(dt)\widehat{d}f$$

we can write $e^B P$ in this more pleasant way :

$$e^B P = P(\mathrm{id} + BP)^{-1} = \begin{pmatrix} 0 & \frac{1}{1+e^{1/\delta(t)}P(dt,df)} P(dt) \\ \frac{1}{1+e^{1/\delta(t)}P(dt,df)} P(dt) & TQT^\tau \end{pmatrix}$$

Bear in mind that $e^B P$ is only defined outside V ! Also, since by hypothesis P is smooth and B was constructed as an exact form, $e^B P \in \mathrm{Poiss}(\mathrm{Op}(V) - V)$.

Next observe the following estimates which follow from our basic hypothesis :

Estimate A:

$$\|e^B P(dt)\|^2 = \|\frac{1}{1+e^{1/\delta(t)}P(dt,df)} P(dt)\|^2 \leqslant \frac{\|P(dt)\|^2}{(1+e^{1/\delta(t)}P(dt,df))^2} \leqslant$$

$$\leqslant \frac{\|P(dt)\|^2}{P(dt,df)^2} e^{-2/\delta(t)} = \left(\frac{\|P(dt)\|}{P(dt,df)} e^{-1/\delta(t)}\right)^2$$

so $j^\infty e^B P(dt)$ goes to zero as $t \to 0$ by **C3**.

Estimate B: Consider first the endormorphism

$$S := \mathrm{id} - \frac{1}{P(dt,df)} P(dt) \otimes \widehat{d}f$$

which is well defined due to **C1**.

Then :

$$T - S = \left(\mathrm{id} - \frac{1}{1+e^{1/\delta(t)}P(dt,df)} e^{1/\delta(t)} P(dt) \otimes \widehat{d}f\right) - \left(\mathrm{id} - \frac{1}{P(dt,df)} P(dt) \otimes \widehat{d}f\right) =$$

$$= \frac{1}{P(dt,df)(1+e^{1/\delta(t)}P(dt,df))} P(dt) \otimes \widehat{d}f$$

so that

$$\|T-S\|^2 \leqslant \frac{n^2 \|P(dt)\|^2 \|\widehat{d}f\|^2}{P(dt,df)^2(1+e^{1/\delta(t)}P(dt,df))^2} \leqslant \frac{n^2 \|P(dt)\|^2 \|\widehat{d}f\|^2}{P(dt,df)^4 e^{2/\delta(t)}} =$$

$$= n^2 \left(\frac{\|P(dt)\| \|\widehat{d}f\|}{P(dt,df)^2} e^{-1/\delta(t)}\right)^2$$

so **C2** ensures that

$$j^\infty \|T-S\|^2 \to 0 \quad \text{as} \quad t \to 0$$

From **Estimate B** we conclude that the endomorphism

$$T \in \mathrm{End}(T(\mathrm{Op}(V) - V))$$

extends smoothly to an automorphism $\widetilde{T}$ of the full $T\,\mathrm{Op}(V)$, by setting :

$$\widetilde{T}(\xi) = \begin{cases} T(\xi) & \text{if } \xi \in T(\mathrm{Op}(V) - V) \\ S(\xi) & \text{otherwise.} \end{cases}$$

Now, **Estimate A** guarantees that the extension of $e^P B(dt)$ by zero on $V \times \{0\}$ defines a smooth vector field $\widetilde{X} \in \mathfrak{X}(\mathrm{Op}\, V)$. Hence $e^B P$ extends to a smooth bivector which on V reads :

$$\widetilde{P} = \begin{pmatrix} 0 & 0 \\ 0 & SQS^\tau \end{pmatrix}$$

Next, observe that the condition of being Poisson is *closed*, in the sense that if $P' \in \mathfrak{X}^2(M)$ (M a manifold), then for any open subset $U \subset M$

$$j^\infty P'(U) \subset \Pi^\infty \quad \Rightarrow \quad j^\infty P'(\operatorname{Cl} U) \subset \Pi^\infty$$

where, as usual, $\Pi^\infty \subset J^\infty \Lambda^2 TM$ denotes the infinite prolongation of the Poisson relation $\Pi \subset J^1 \Lambda^2 TM$. Thus $\widetilde{P}$ is necessarily Poisson.

This completes the proof. □

Observe that the theorem is interesting even with $V = \mathbb{R}^n$.

REMARK 152 (t-relative version). *Observe the following : if instead of $b = -e^{1/t} f dt$ we had chosen $b' := \varrho f dt$, for some*

$$\varrho : \operatorname{Op} V \to \mathbb{R}$$
$$\operatorname{germ}_V \varrho = -e^{1/t}$$

then again the same conclusions of Theorem 151 remain valid, since all we need to control is the behaviour of $e^B P$ in a (deleted) neighborhood of V.

In particular, we can choose ϱ to be zero *outside a fixed neighborhood U of V, thus ensuring that $\widetilde{P} = P$ outside U.*

REMARK 153 (Finite smoothness). *We also point out that an analgous statement holds as that of Theorem 151 holds (upon replacing j^∞) in case P is merely merely C^r-smooth.*

8. h-principles, IV : symplectic germs along spheres

We now turn to an *unstable* application of the methods discussed in the previous section; namely, we discuss the problem of representing a germ of *symplectic* structure P along a submanifold $V_0 \subset V$ by a *global* Poisson structure $\overline{P}$.

We provide a *positive* answer to this problem if $V_0 = \partial D$ is of codimension at least two, and $\overline{P}$ is allowed to have merely finite differentiability along a hypersurface.

THEOREM 154. *Given $P \in \operatorname{Symp}(\partial D)$, for $D \subset V$ an embedded disk of positive codimension, there exists a Poisson bivector $\overline{P} \in \operatorname{Poiss}(D)$ representing P.*

PROOF. Choose a tubular neighborhood $N \supset \partial D$, and choose also an identification

$$N \simeq \partial D \times (1-a, 1+a) \times \mathbb{R}^{n-k} \subset \mathbb{R}^n, \quad a > 0$$

Let $r : N \to \mathbb{R}$ stand for the radial coordinate (hence $\partial D = D \cap r^{-1}(1)$).

Now write

$$P \in \operatorname{Symp}(N)$$
$$P = Q + \frac{\partial}{\partial r} \wedge Y$$

and observe that $Y = P(dr)$ is non-singular.

Consider the differential relation of order one on functions $f : N \to \mathbb{R}$ governing the inequality

$$Yf > 0$$

This relation is open, hence microflexible. Now, hypothesis $n - k > 0$ comes in to ensure that

$$\partial D \times (1-a, 1+a) \times \{0\} \subset \partial D \times (1-a, 1+a) \times \mathbb{R}^{n-k}$$

be sharply movable.

Hence we can find a C^0-small diffeotopy

$$d_t : \partial D \times (1-a, 1+a) \times \mathbb{R}^{n-k} \to \partial D \times (1-a, 1+a) \times \mathbb{R}^{n-k}$$
$$d_t(p, r, z) = (p, r, z'(p, r, z, t))$$
$$d_t(\partial S) \cap \partial D = \varnothing, \quad S := (1-a/2)D$$

stationary around ∂D, and a function

$$f : \mathrm{Op}(d_1 \partial S) \to \mathbb{R}$$

solving $Yf > 0$ on $\mathrm{Op}(d_1 \partial S)$.

Note moreover that by compactness of ∂S, we can find a positive constant C with

$$Yf \geqslant C\|Y\|$$

So we are in position to coerce the Hamiltonian Y by f. Since we have $d_1 \partial S \cap \partial D = \varnothing$, we can invoke Remark 152 to produce a $\widetilde{P} \in \mathrm{Poiss}(U)$, $U \supset d_1(D - \int S)$, such that

$$\widetilde{P} = P \quad \text{on} \quad U - \mathrm{Op}\, d_1 S$$
$$j^\infty \widetilde{P}(dr) = 0 \quad \text{on } r^{-1}(1-a/2) \cap U$$

For notational convenience, let us set $R := r - (1-a/2)$, so that $R^{-1}(a/2) = r^{-1}(1)$ and $R^{-1}(0) = r^{-1}(1-a/2)$.

Observe that, if in the usual decomposition

$$\widetilde{P} = \frac{\partial}{\partial R} \wedge \widetilde{Y} + \widetilde{Q}$$

we had

$$(\star) \qquad \frac{\partial^q \widetilde{Q}}{\partial R^q} = 0 \quad \text{on } R = 0$$

for all $q \geqslant 1$, then a Poisson extension to $\mathrm{Op}\, D \times \mathbb{R}^{n-k}$ would be a collar of the form

$$\overline{P}(\theta, R, z) = \begin{cases} \widetilde{P}(\theta, R, z) & \text{if } R \geqslant 0 \\ \lambda(R)\widetilde{Q}(\theta, 0, z) & \text{if } R \leqslant 0 \end{cases}$$

where

$$\lambda : [a/2 - 1, 0] \to [0, 1]$$
$$\lambda |\, \mathrm{Op}[a/2 - 1, a - 1] = 0, \quad \lambda |\, \mathrm{Op}\, 0 = 1$$

(Note that for this collaring procedure, we *ignore* whatever happens on $R < 0$).

Let us show that this situation can be achieved; choose a smooth function $\varpi : (0, a/2) \to (0, a/2)$ satisfying

$$\frac{d\varpi}{dt}(t) > 0$$
$$\varpi(t) = \begin{cases} \frac{-1}{\log t} & \text{if } t \leqslant b \\ t & \text{if } t \geqslant b' \end{cases}$$

where $0 < b < b' < a/2$.

$$\psi : \partial D \times (0, a) \times \mathbb{R}^{n-k} \to \partial D \times (0, a) \times \mathbb{R}^{n-k}$$
$$\psi(\theta, R, z) = (\theta, \varpi(R), z)$$

Observe that ψ is a smooth diffeomorphism, that $\psi = \mathrm{id}$ for $R \geqslant b'$, that for $R \leqslant b$ its inverse is of the form

$$\psi^{-1}(\theta, R, z) = (\theta, e^{-1/R}, z)$$

and so

$$(\psi_* \frac{\partial}{\partial R})(\theta, R, z) = R^2 e^{1/R} \frac{\partial}{\partial R}$$

for $R \leqslant b$. Thus the push-forward $\psi_* \frac{\partial}{\partial R}$ *cannot* be extended to $R \leqslant 0$.

Nevertheless, if we look at $(\psi_* \widetilde{P})(\theta, R, z)$, then it decomposes as

$$(\psi_* \widetilde{P})(\theta, R, z) = R^2 e^{1/R} \frac{\partial}{\partial R} \wedge \widetilde{Y}'(\theta, R, z) + \widetilde{Q}'(\theta, R, z)$$

where

$$\widetilde{Y}'(\theta, R, z) = \widetilde{Y}(\theta, e^{-1/R}, z)$$
$$\widetilde{Q}'(\theta, R, z) = \widetilde{Q}(\theta, e^{-1/R}, z)$$

so that

$$\frac{\partial^q \widetilde{Q}'}{\partial R^q}(\theta, 0, z) = 0, \quad \text{for all } q \geqslant 1$$

But our hypotesis on $\widetilde{Y}$ implies that, for each q, there is a compact neighborhood $K_q \supset R^{-1}(0)$ and a $C_q > 0$ such that

$$\|\widetilde{Y}(\theta, R, z)\|^2 \leqslant C_q |R|^{2q} \quad \text{on } K_q$$

In particular,

$$\|\widetilde{Y}(\theta, R, z)\|^2 \leqslant C_2 |R|^4$$

on $\mathrm{Op}(R^{-1}0)$, whence there we have

$$\|R^2 e^{1/R} \widetilde{Y}'(\theta, R, z)\|^2 \leqslant C_2 R^2 e^{2/R} e^{-4/R} = C_2 R^2 e^{-2/R}$$

so

$$\frac{\partial^q}{\partial R^q}(R^2 e^{1/R} \widetilde{Y}')(\theta, 0, z) = 0, \quad \text{for all } q \geqslant 1$$

This shows that ($\bigstar$) can always be attained, and thus the smooth collaring discussed above yields a smooth extension to $\mathrm{Op}(d_1 D)$ of the original $P|\,\mathrm{Op}(\partial D)$. Now use diff-invariance of Poiss and $d_1|\,\mathrm{Op}\,\partial D = \mathrm{id}_{\mathrm{Op}\,\partial D}$ to produce a Poisson extension of $P|\,\mathrm{Op}(\partial D)$ to an opening of the original D. $\square$

In the parlance of Gromov structures, this can be stated as the inclusion of the sheaf of symplectic structures on $\mathbb{R}^n$ into the sheaf of Poisson structures induces trivial maps

$$\pi_q(B\,\mathrm{Symp}) \to \pi_q(B\,\mathrm{Poiss}), \quad q \leqslant n-2$$

REMARK 155. *Observe that the only reason for assuming that P is symplectic is to ensure that the Hamiltonian Y is non-singular, but clearly the construction applies* ipsis litteris *if we relax the symplectic hypothesis to $Y \neq 0$.*

REMARK 156. *Observe furthermore that the codimension $\geqslant 2$ in the statement of the theorem is the best possible result in full generality, through the technique of coertion; indeed, observe that no coercibility data can be found if the corresponding Hamiltonian vector field has a* periodic *orbit; this forces us to look for solutions to*

$$Yf \geqslant C\|Y\|^2 \geqslant C' \geqslant 0$$

along positive-codimensional subpolyhedra.

9. Final comments

Microflexibility. The reader has surely noticed that we have been conspicuosly silent about the microflexibility of Poiss. And for very good reason : we don't know whether Poiss is microflexible or not. I am personally inclined to believe that it is *not*, but that nevertheless procedures as those described in Sections 7-8 could still salvage some meaaningful form the h-principle for this sheaf.

One idea of how one could try to *disprove* Poisson microflexibility could work along these lines : recall Thurston's example [**67**] of a 1-parametric, non-trivial family of codimension-one foliations

$$\begin{aligned}(0,+\infty) &\to \mathrm{Fol}^1(S^3)\\ t &\mapsto \mathcal{F}_t\end{aligned}$$

whose Godbillon-Vey *number* $\mathrm{gv}(\mathcal{F}_t) := \langle \mathrm{GV}(\mathcal{F}_t), [S^3]\rangle \in \mathbb{R}$ varies *continuously* :

$$\mathrm{gv}(\mathcal{F}_t) = t$$

This suggests that we might try to detect lack of microflexibility of Poisson structures by enriching an example as the above by finding a family of germs of symplectic foliations at $S^3 \subset \mathbb{R}^5$:

$$\mathcal{F}'_\bullet : [0,+\infty) \to \mathrm{SympFol}^1(S^3)$$

such that :

$$\mathcal{F}'_0 = \mathcal{F}_0 |\, \mathrm{Op}(S^3), \quad \mathcal{F}_0 \in \mathrm{SympFol}^1(D^4)$$

If such a family can be found, this would prove failure of the microflexibility test for a certain (small) homotopy of the corank-one Poisson structure corresponding to $\mathcal{F}_0$.

(But this might be very hard to accomplish, as it seems as yet unknown whether the Thurston epimorphism

$$\pi_3(B\Gamma_1) \twoheadrightarrow \mathbb{R}$$

is injective as well. But we might try to apply the techniques of 5 to this problem).

In the other direction, one might try to develop a well-behaved theory of *Poisson-Dirac immersion* to try to descend microflexibility of immersions to that of Poisson structures, or substructures of a given type.

By "well-behaved" we essentially mean a criterion for determining when an embedding

$$f : V \to (W, P)$$

of V into a Poisson manifold (W, P), which induces *pointwise* bivectors f^*P on V, is automatically smooth.

The example I know of a pointwise Poisson-Dirac embedding which does not produce a smooth induced f^*P does *not* stratify as a continuous bivector, i.e., the rank of f^*P is not lower semicontinuous. But it might be worthwhile for the mentioned problem to check whether something like the following criterion holds true :

CRITERION 157. *Suppose P is a Poisson bivector on W, $f : V \to W$ a smooth embedding, and suppose f is pointwise Poisson-Dirac. If f^*P* **stratifies like a bivector***, i.e., if*

$$\mathrm{rank}_x(f^*P) \leqslant \mathrm{rank}_{x'}(f^*P), \quad x' \in \mathrm{Op}(x)$$

then $Q := f^*P$ is automatically smooth and hence Poisson.

Germ realizability and sufficiency. The notion of "softness" of a differential relation proposed by Gromov and discussed above deals primarily with the *holonomic* obstructions to solving a given relation; we agreed to interpret as *soft* those relations for which pertinent versions of the h-principle, as formulated in Sections 5 and 6, held.

But there are other competing notions which arguably shed some light into what we would intuitively understand as "softness" of a given differential problem – and thus, obviously, on its "rigidity" properties as well.

Here we discuss one such alternative, that of *germ realizability and sufficiency.*

Let $\mathcal{R} \subset J^r E$ be a differential relation.

Definition 158. *Define the* **first prolongation** $\mathcal{R}^1 \subset J^{r+1}E$ *as*

$$\mathcal{R}^1 := j^1\left(\Gamma(V, \mathcal{R})\right) \cap \mathrm{Hol}(V, J^1(J^r E))$$

where

$$j^1 : \Gamma(V, \mathcal{R}) \subset \Gamma(V, J^r E) \to \Gamma(V, J^1(J^r E))$$

which of course makes sense since

$$J^{r+1}E = j^1\left(\Gamma(V, J^r E)\right) \cap \mathrm{Hol}(V, J^1(J^r E))$$

Observe the natural map $\mathcal{R}^1 \to \mathcal{R}$.

Inductively, the k**-th prlongation** $\mathcal{R}^k \subset J^{r+k}E$ *of* $\mathcal{R}$ *is defined as*

$$\mathcal{R}^k := \left(\mathcal{R}^{k-1}\right)^1$$

Finally, we define the **infinite prolongation** *by*

$$\mathcal{R}^\infty := \lim_{\leftarrow} \mathcal{R}^k$$

Definition 159. *A* **formal solution** $j \in \mathcal{R}^\infty$ *to a differential relation* $\mathcal{R}$ *is a point of its infinite prolongation,* $j \in \mathcal{R}^\infty$.

Observe the obvious fact that

$$j \in \mathcal{R}^\infty \Longrightarrow p_r^\infty j \in \mathcal{R}$$

so that any local holonomic section of formal solutions is also a local solution to the differential relation.

Definition 160. *A formal solution* $j \in \mathcal{R}_x^\infty$ *is called* **realizable to a germ** *if there is a germ of solution to* $\mathcal{R}$,

$$\varphi \in \Phi_x, \quad j_x^\infty \varphi = j$$

The background for our analogy with the previously discussed instance of "softness" occurrs under the auspices of a classic theorem of Borel for the case when $\mathcal{R} = E$:

Theorem 161 (Borel's lemma). *Let* $V = V_0 \times \mathbb{R}$. *The assignment*

$$\mathrm{Borel} : C^\infty(V) \to \prod_0^\infty C^\infty(V_0)$$

$$f \mapsto \left(\frac{\partial^k f}{\partial t^k}|_{t=0}\right)_k$$

has a continuous right inverse.

Observe that when $\mathcal{R}$ is *open*, then all formal structures are realizable to a germ. This should be compared, in light of the above discussion, to the fact that open relations are microflexible, and observe the 1-codimensional hypothesis as in sharp mobility.

Problem 162. *Which formal structures* $j \in \mathcal{R}^\infty$ *are realizable to germs ?*

In line with the ongoing discussion, a relation should be deemed "softer" (or "infinitesimally softer", as seems more fitting) the more formal structures are realizable to germs. We again point out the non-triviality of this notion by means of Lewy's example [**47**], where no formal structure at all is realizable to a germ.

This problem is of particular interest to us in its Poisson incarnation. It seems to be highly non-trivial, and is completely open even in the case of finite (positive) jets over a point. As a matter of fact, even the real algebraic varieties of the finite prolongations of the Poisson relation are too poorly understood.

Turning now in the direction of "rigidity", we discuss the following

DEFINITION 163. *Let Φ be a Γ-equivariant sheaf on V.*

A germ $\varphi \in \Phi(x)$ is called k-**sufficient**, $0 \leqslant k \leqslant \infty$, *if for all $\psi \in \Phi(x)$*

$$j_x^k\varphi = j_x^k\psi$$

implies the existence of a $d : x \to x$ in Γ with $d_*\psi = \varphi$.

It is clear how to interpret this notion as "rigidity" of Φ, for it implies that the behaviour of φ in a (however small) whole neighborhood of x is entirely determined by its k-th contact order data at x alone, so a sheaf should be considered "rigid" if "many" of its germs are sufficient, and all the more so if they are sufficient of "low order".

Observe moreover that sufficiency is a quite strong statement about a germ φ even in the $k = \infty$ case.

In the case of Φ = Poiss, inspection of non-degenerate structures (all of whose germs are equivalent by Darboux's theorem) and rank-two structures (where it is easy to construct examples germs which are not determined even by its infinite jet) shows a wild array of distinct behaviours that might be useful to analyse through this perspective, especially in view of the known results on 1-sufficiency in the presence of rigid isotropy ([**12, 17, 19**]).

Other interesting open problems. We wish to close these remarks by presenting some open problems whose solution would arguably be a great advance in th investigation of "soft" properties of Poiss :

- **Self-interpolation:** Instead of studying general flexibility properties of deformations, one might try to interpolate the canonical linear deformation $t \mapsto t\pi$ connecting a given Poisson structure π to the trivial one. One way to approach this problem is through *symplectic realizations* of Poisson manifolds, which behave rather nicely with respect to this canonical deformation. Through this lens, this self-interpolation problem is converted into one for which certain averaging methods might prove useful. An important consequence of a positive result in this direction would be to settle the problem of locality described in the sequel;
- **Locality:** Given a Poisson structure on the unit ball in Euclidian space, when can it be extended to a global Poisson structure ? What if we require that the structure vanish at infinity ? Solving this would shed some light on which semi-local phenomena are expected to occur on Poisson structures on general manifolds;
- **Extending germs:** In more generality, one may ask the following question : what is the condition on a germ of Poisson structure along a submanifold to ensure existence of a global representative ? (One must point out that this question remains open even in the case where the germ is symplectic, in *all* codimensions of the submanifold, if we demand that the Poisson structure be smooth; otherwise see Section 8.)
- **Topological obstructions:** In similar spirit, one might wonder what the topological obstructions on a manifold are to admit a sufficiently non-degenerate Poisson structure. Some very interesting results in this direction have been obtained by A. Ibort and D. Martínez-Torres, see [**45**]. The very classical, and still open question of which closed manifolds admit symplectic structures is naturally part of this program; however, even the existence of Poisson structures, all of whose pointwise germ are non-trivial, has not yet been settled.

Notation

Categories

$\mathrm{Ob}\,\mathcal{C}$	the objects of a category $\mathcal{C}$
$\mathrm{Ar}\,\mathcal{C}$	the arrows of a category $\mathcal{C}$
$\mathrm{Iso}\,\mathcal{C}$	the isomorphisms of a category $\mathcal{C}$
Sets	the category of sets
$\wp(S)$	poset of all subsets of S, under inclusion
Cat	the category of small category
Top	the category of topological spaces
TopCat	the category of topological categories
Grpd	the category of (discrete) groupoids
TopGrpd	the category of topological groupoids
QTop	the category of quasi-spaces
$^{\mathrm{op}}$	opposite
$E : \mathrm{Top}, \mathrm{QTop} \to \mathrm{Sets}$	forgetful functor

Quasi-spaces

T	a topological space
$\mathrm{Cl}(S)$	closure of a subspace $S \subset T$
$\int S$	interior of a subspace $S \subset T$
$\mathrm{Bd}\,S$	boundary $\mathrm{Cl}\,S - \int \mathrm{Cl}\,S$
$S \bullet S'$	$\mathrm{Cl}\,S \cup \mathrm{Bd}\,S'$
$\mathrm{Op}\,S$	an *opening* of $S \subset T$ in T
$\mathfrak{U}$	an open covering of a topological space
$\mathrm{Cov}(T)$	the category of open coverings $\mathfrak{U}$ of T under "refines"
$\mathcal{O}(T)$	the category of open subsets of T with inclusions
$S \downarrow \mathcal{O}(T)$	the poset of open subsets of T containing S, under inclusions
$\mathfrak{y}$	Yoneda embedding $\mathrm{Top} \to \mathrm{QTop}$
$X(T)$	quasi-continuous maps of a space T into the quasi-space X
$X[T]$	homotopy types of quasi-continuous maps of a space T into the quasi-space X

Sheaf theory

$\mathrm{PSh}(T, \mathcal{C})$	the category of $\mathcal{C}$-valued presheaves on T
$\mathrm{Sh}(T, \mathcal{C})$	the category of $\mathcal{C}$-valued sheaves on T
$^{+}$	plus construction
$\mathrm{Etale} \downarrow T$	the category of étale maps over T
$\mathrm{Etale}\,F$	étalé construction of a (pre-)sheaf F

Simplicial objects

Δ	the simplicial category
d_i, s_j	i-th face and j-th degeneracy maps
SSets	the category of simplicial sets
SSpaces	the category of simplicial spaces
$\mathfrak{N}$	nerve
$\mathfrak{S}\mathcal{C}$	Segal construction of the category $\mathcal{C}$
$\|S\|$	geometric realization of the simplicial space S
$\mathrm{car}(p)$	carrier of a point $p \in \|S\|$
$\mathrm{Lin}(T, \|S\|)$	linear homotopy types of maps $T \to \|S\|$
$B\mathcal{C}$	classifying (simplicial) space of a topological category $\mathcal{C}$

Bundle theory

$H^1(T, G)$	set of Hæfliger G-structures on T
$h^1(T, G)$	set of concordance classes of Hæfliger G-structures on T
$\mathrm{Num}(T, G)$	set of numerated G-structures on T
$\mathrm{num}(T, G)$	set of concordance classes of numerated G-structures on T
$\mathrm{PBun}_G(T)$	category of isomorphism types of (left) principal G-bundles

Differential topology

V	a smooth manifold
$E \to V$	a smooth fibre bundle
$J^r E$	the r-jet bundle of a smooth fibre bundle E
j^r	the r-jet prolongation of a section of E
p^r_s	natural affine projection $J^r E \to J^s E$
$\mathfrak{X}^k(V)$	k-multivector fields on V, $\mathfrak{X}(V) := \mathfrak{X}^1(V)$
$\Omega^k(V)$	differential k-forms fields on V
$[\cdot, \cdot]$	Schouten bracket
Z^p_{dR}	sheaf of closed, differential p-forms
B^p_{dR}	sheaf of exact, differential p-forms
dist	distance function with respect to some choice of Riemannian metric
$\mathrm{Morse}(f)$	Morse complex of a Morse function $f : V \to \mathbb{R}$
$\mathfrak{D}$	a pseudo-group of diffeomorphisms
d_t	a $\mathfrak{D}$-diffeotopy
$\mathfrak{I}$	a set of $\mathfrak{D}$-diffeotopies
$\mathfrak{K}(T)$	set of all compact subsets of a space T
$C^0_{\mathrm{Whit}}(V, E)$	the space of continuous maps $V \to E$ under the Whitney/strong topology
∇	a (covariant or contravariant) connection
$G : \mathcal{G} \to TV, T^*V$	geodesic flow of a covariant/contravariant connection ∇
${}^\tau$	transpose
C^∞_0	ring of germs of smooth real functions at zero
C^∞_{flat}	ideal C^∞_0 of of flat germs
$\mathbb{R}[t]$	ring of formal power series in one variable
Taylor	Taylor expansion
Borel	Borel map

Foliations, bundles, characteristic classes

$\mathrm{Dist}^q(V)$	codimension-q distributions on V, i.e., codimension-q subbundles of TV
$\mathcal{D}$	a distribution on V
$\mathrm{QSympDist}^q(V)$	space of codimension-q distributions equipped with a non-degenerate form
$(\mathcal{D}, \omega)$	an element of $\mathrm{QSympDist}^q(V)$
$\nu(\mathcal{D})$	normal bundle to a distribution $\mathcal{D}$
$^{\perp}$	orthogonal bundle (to a subbundle of a bundle with a fibred metric)
$\mathrm{Ann}(\mathcal{D})$	the annihilator of $\mathcal{D}$
$\mathrm{Pont}(E)$	Pontryagin ring of a real vector bundle E
$p_i(E)$	i-th Pontryagin class of E
$c_i(E)$	i-th Chern class of a complex vector bundle E
$e(E)$	Euler class of a real vector bundle E
$\mathrm{Fol}^q(V)$	space of smooth codimension-q foliations on V
$\mathcal{F}$	a smooth foliation
$\mathcal{L}$	leaf of a (possibly singular) foliation
$\mathrm{Trans}(V, \mathcal{F})$	space of maps $V \to W$ which are transverse to $\mathcal{F} \in \mathrm{Fol}^q(W)$
$\mathrm{Trans}(TV, T\mathcal{F})$	space of bundle maps $TV \to W$ which are transverse to $T\mathcal{F} \subset TW$
$\mathrm{Fol}^n_{\pitchfork}(E \to W)$	space of codimension-n smooth foliations on the total space E
$\mathrm{QSympFol}^q(V)$	space of codimension-q foliations equipped with a leafwise non-degenerate form
$(\mathcal{F}, \omega)$	an element of $\mathrm{QSympFol}^q(V)$
$\mathrm{SympFol}^q(V)$	space of codimension-q symplectic foliations of a smooth fibre bundle $E \to W$ consisting of those foliations which are transverse to the fibres of the bundle
$\mathrm{GV}(\mathcal{F})$	Godbillon-Vey class of $\mathcal{F}$

Differential relations

$\mathcal{R}$	a differential relation
$\mathcal{R}^k$	k-th prolongation of $\mathcal{R}$, $1 \leqslant k \leqslant \infty$
$\mathrm{Hol}(\cdot, \mathcal{R})$	sheaf of holonomic sections of $\mathcal{R}$
$\mathrm{symb}(D)$	For a differential operator $D : \Gamma(V, E_0) \to \Gamma(V, E_1)$ of order r, the unique $\Gamma(V, J^r E_0) \to \Gamma(V, E_1)$ with $D = \mathrm{symb}(D) \circ j^r$
$\widetilde{\mathrm{symb}}(D)$	The bundle map $J^r_0 \to E_1$ corresponding to $\mathrm{symb}(D)$
Φ	sheaf of (germs of) solutions of $\mathcal{R}$; a sheaf of quasi-spaces
Φ^P	P-parametric sheaf of Φ
$\Phi^{\davidsstar}$	sheaf of germs of sections of Φ (usually denoted Φ)
$\Phi^{\flat}$	the sheaf of parametric germs of a sheaf of qusi-spaces Φ
$\Phi^{\sharp}$	the sheaf of weak solutions to Φ
$H^1(T, \Phi)$	Gromov Φ-structures on T
$\mathfrak{H} : H^1(\cdot, \Phi) \to H^1(\cdot, \Gamma_n)$	forgetful functor assigning to a Gromov Φ-structure its underlying Γ_n-structure

$\mathfrak{G}(F)$	tautological Gromov structure on the graph $E(F)$ of a Gromov structure F

Poisson geometry

$\{\cdot,\cdot\}$	a Poisson bracket
Poiss^{C^r}	sheaf of C^r Poisson structures
$\circlearrowleft$	cyclic sum
Π	the Poisson differential relation in $J^1\Lambda^2TV$
X_f	Hamiltonian vector field $\{f,\cdot\} = \pi(df,\cdot)$
$\mathfrak{Poiss}$	category of Poisson manifolds and Poisson maps
$\mathrm{rank}_x\,\pi$	rank of π at a point x
$\mathrm{Reg}(\pi)$	subspace of regular points of π
$\mathrm{Poiss}_{\leqslant k}(V)$	Poisson structures of rank at most k
$\mathrm{stretch}\,\pi$	stretch of π

Lie-Dirac geometry

$(\mathbb{E},\rho,\langle\cdot,\cdot\rangle,[\![\cdot,\cdot]\!])$	a Courant algebroid
$\natural$	isomorphism $\mathbb{E}\to\mathbb{E}^*$ by $\langle\cdot,\cdot\rangle$
$\sharp$	anchor of a Lie algebroid
$\mathbb{T}V$	generalized tangent bundle of V
L	a Lagrangian subbundle of $\mathbb{T}V$
Jac_L	the Jacobiator of L
$(A,\sharp,[\cdot,\cdot]_A)$	a Lie algebroid
d_A	de Rham operator of A
$H^\bullet(A)$	de Rham ring of A
$[\cdot,\cdot]_A$	Schouten bracket of A
L^A_e	Lie derivative $\Gamma(\Lambda^\bullet A^*)\to\Gamma(\Lambda^\bullet A^*)$ in the direction of $e\in\Gamma(A)$
ι_e	contraction $\Gamma(\Lambda^\bullet A^*)\to\Gamma(\Lambda^{\bullet-1}A^*)$ by $e\in\Gamma(A)$
$e^\omega T$	graph of a two-form ω
$e^\pi T^*$	graph of a bivector π
(A,A^*)	Lie bialgebroid
e^BA	the graph of A through a two-A-form B in the double Courant algebroid $A\oplus A^*$

Differential b-geometry

${}^b\,\mathrm{Mfd}$	category of b-manifolds and b-maps
$\mathfrak{X}(V,Z)$	vector fields on V which are tangent to the codimension-one submanifold Z
$T(V,Z),T^*(V,Z)$	the b-tangent/cotangent bundle of (V,Z)
${}^bN(V,Z)$	normal line field of (V,Z)

$\mathfrak{X}^k(V, Z)$	k-multivectors on (V, Z)
$\Omega^k(V, Z)$	differential k-forms on (V, Z)
A_Z	b-construction along $Z \subset V$ for a transitive Lie algebroid $A \to V$
$\mathfrak{X}^2_{\pitchfork}(V)$	transversely non-degenerate bivectors on V
$Z(\pi)$	singular locus of $\pi \in \mathfrak{X}^2_{\pitchfork}(V)$
$\mathfrak{X}^2_{\pitchfork}(V, Z)$	transversely non-degenerate π with $Z(\pi) = Z$
$\mathfrak{X}^2_b(V)$	b-bivector on V
${}^b d$	b-de Rham operator
${}^b H^\bullet(V, Z)$	b-de Rham cohomology ring
${}^b f_*$	${}^b df$
$\mathrm{Poiss}^b(V)$	b-Poisson structures

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