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Integrability of Lie algebroids by proper Lie groupoids

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Resumo

Um critério clássico diz-nos que uma álgebra de Lie (real) $\mathfrak{g}$ é a álgebra de Lie de um grupo de Lie G **compacto** se e só se existe um produto interno em $\mathfrak{g}$ que é invariante para a acção adjunta de $\mathfrak{g}$ em si mesma. O objectivo deste trabalho foi o de investigar se este resultado poderia ser extendido para algebróides de Lie, obtendo-se um critério análogo caracterizando quando é que um algebróide de Lie $\mathcal{A}$ é o algebróide de Lie de um grupóide de Lie $\mathcal{G}$ **próprio**.

Teve-se como base a formulação de uma conjectura de trabalho afirmando que a existência de um produto interno em $\mathcal{A}$ satisfazendo uma certa propriedade de invariância seria uma condição necessária e suficiente para que $\mathcal{A}$ fosse integrado por um grupóide próprio. O trabalho consistiu então em decidir se a conjectura era válida, e caso contrário, porquê e como falhava.

Numa direcção, provámos que a existência de um grupóide de Lie próprio integrando $\mathcal{A}$ e satisfazendo algumas condições razoáveis implica a existência de um produto interno em $\mathcal{A}$ nas condições da nossa conjectura.

Dedicámo-nos então de seguida a saber se a nossa condição aplicada a um algebróide de Lie integrável $\mathcal{A}$ implicaria a existência de um grupóide de Lie próprio integrando $\mathcal{A}$. Para um algebróide de Lie transitivo provámos que esse é de facto o caso. Para um algebróide de Lie geral, no entanto, vimos que tal é falso. Apresentamos a esse propósito três contra-exemplos de naturezas distintas, que sugerem que a integrabilidade por grupóides próprios deve requerer condições de outra natureza.

Palavras Chave: *Grupóide, Algebróide, Próprio, Integrabilidade.*

Abstract

A classical criterion states that a (real) Lie algebra $\mathfrak{g}$ is the Lie algebra of a **compact** Lie group G if and only if there exists an inner product on $\mathfrak{g}$ which is invariant under the adjoint action of $\mathfrak{g}$ on itself. Our aim in this work was to investigate if this result could be extended to Lie algebroids, giving an analogous criterion characterizing when is a Lie algebroid $\mathcal{A}$ the Lie algebroid of a **proper** Lie groupoid $\mathcal{G}$.

First, a working conjecture was formulated stating that the existence of an inner product on $\mathcal{A}$ satisfying a certain invariance property should be a necessary and sufficient condition for $\mathcal{A}$ to integrate to a proper Lie groupoid. Our work then consisted in deciding if the conjecture held and, if not, why and how it failed.

In one direction, we proved that the existence of a proper Lie groupoid integrating $\mathcal{A}$ and satisfying some mild conditions implies the existence of an inner product on $\mathcal{A}$ as in our conjecture.

We then studied whether our condition on an integrable Lie algebroid $\mathcal{A}$ implied the existence of a proper Lie groupoid integrating $\mathcal{A}$. For a transitive Lie algebroid we proved that this is indeed the case. For a general Lie algebroid, however, we show that this is false. We present three counter examples of distinct natures that suggest that integrability by proper groupoids must require conditions of a different nature.

Keywords: *Groupoid, Algebroid, Proper, Integrability.*

Contents

Introduction

The aim of this work was to attempt to generalize a well known result characterizing the Lie algebras that can be integrated by **compact** Lie groups, namely the following result:

Theorem 0.1. *Let* $\mathfrak{g}$ *be a (finite-dimensional) Lie algebra (over* $\mathbb{R}$*). Then* $\mathfrak{g}$ *is the Lie algebra of a compact Lie group* G *if and only if there exists an inner product* $\langle\cdot,\cdot\rangle$ *on* $\mathfrak{g}$ *such that, for all* $X, Y, Z \in \mathfrak{g}$*, we have*

$$\langle [X, Y], Z\rangle + \langle Y, [X, Z]\rangle = 0 \tag{0.2}$$

Obviously our attempt at generalization replaces the Lie group G and the Lie algebra $\mathfrak{g}$ by a Lie groupoid $\mathcal{G}$ and a Lie algebroid $\mathcal{A}$, respectively.

The compactness of G, however, should not be replaced by compactness of $\mathcal{G}$ but by **properness**. This is because, while both compact and proper groupoids are a generalization of compact groups [1], properness is the best generalization: indeed, a proper groupoid will be compact if and only if the base space is compact, which makes properness more interesting, since it does not impose conditions on the base space itself. This also ends up resulting in compact groupoids not being significantly more manageable that proper ones. Note this reflects what happens with actions of groups[2], where proper actions are a generalization of actions of compact groups that is more or less just as "workable" as the later. It is therefore fair to say that proper groupoids are the "spiritual" generalization of compact groups.

As for the rest of the theorem, a rethinking of the objects that appear in it is needed for it to even have a chance to be true.

The inner product on $\mathfrak{g}$ in the theorem will naturally be replaced by an internal product on $\mathcal{A}$ (which, $\mathcal{A}$ being a vector bundle, means an inner product on each fiber).

It is slightly less clear what to replace the elements $X, Y, Z \in \mathfrak{g}$ with. Since in

[1] In the sense that the notions coincide when restricted to groups.
[2] Note that, as we will see, each group action has a corresponding groupoid.

the original theorem we are interested in considering both the internal product and the Lie bracket of such elements, we should replace $X, Y, Z \in \mathfrak{g}$ by $X, Y, Z \in \Gamma(\mathcal{A})$, since the Lie bracket in $\mathcal{A}$ is defined for elements of $\Gamma(\mathcal{A})$.

Next, the formula (0.2) must itself be changed. To understand the changes, one should first understand the "spirit" of (0.2) (or at least, one of its possible interpretations) [3]. As we know, the Lie algebra $\mathfrak{g}$ is simply the Lie algebra of right invariant vector fields on G with the bracket of vector fields as Lie bracket. Also, on any Lie algebra, the Lie bracket is a derivation. Therefore, letting $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ be the right invariant vector fields in G corresponding to X, Y and Z, (0.2) tells us that there exists an (right invariant) inner product on G which satisfies the **Leibniz rule (or the rule of the derivative of the product)** relatively to the derivative $[\cdot,\cdot]$, that is

$$0 = \tilde{X} \cdot \langle \tilde{Y}, \tilde{Z} \rangle = \langle [\tilde{X}, \tilde{Y}], \tilde{Z} \rangle + \langle \tilde{Y}, [\tilde{X}, \tilde{Z}] \rangle$$

This will therefore be the motivation for our generalization of (0.2). So if now we have $X, Y, Z \in \Gamma(\mathcal{A})$ and set $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ as the corresponding right invariant vector fields on $\mathcal{G}$, we have:

$$\tilde{X} \cdot \langle \tilde{Y}, \tilde{Z} \rangle = (\rho(X) \cdot \langle Y, Z \rangle) \circ \mathbf{t}$$

(this is an immediate consequence of the fact that $\tilde{X}$ and $\langle \tilde{Y}, \tilde{Z} \rangle$ are related via the target map $\mathbf{t}$ to $\rho(X)$ and $\langle Y, Z \rangle$, respectively)

So we conclude that the 0 in (0.2) should be replaced by $\rho(X) \cdot \langle Y, Z \rangle$.

However, this is not enough, since now $[\cdot,\cdot]$ is not a derivative, that is, it is not a connection. So the $[X, Y]$ and $[X, Z]$ in (0.2) must be replaced by $\nabla_X Y$ and $\nabla_X Z$ for an appropriate $\mathcal{A}$-connection ∇ on $\mathcal{A}$. Obviously, the choice of ∇ can not be random. ∇ should not only be related to $[\cdot,\cdot]$, but actually represent $[\cdot,\cdot]$ in some sense. The key to achieve this is the fact that, although $[\cdot,\cdot]$ is not a connection, it is a connection up to homotopy[4], and that all connections up to homotopy are chain homotopic to a real connection ∇, and we therefore use such a connection. In practice we will demand that

$$\nabla_X Y = [X, Y] + \tilde{\nabla}_{\rho(Y)} X$$

[3]From an algebraic point of view, the need to correct the formula comes from the fact that the Lie bracket $[\cdot,\cdot]$ is not $C^\infty(M)$-linear which implies that (0.2) can not be true, since by multiplying X, Y or Z by functions we obtain equalities we can not guarantee to be true.

[4]In an informal sense, a connection up to homotopy is something that satisfies the same axioms as a connection except the one about linearity in the first coordinate, which fails, but in a controlled way.

for $\tilde{\nabla}$ a certain TM-connection on $\mathcal{A}$.

So our working conjecture was:

Conjecture 0.3 (Main conjecture)**.** Let $\mathcal{A}$ be a Lie algebroid. Then $\mathcal{A}$ is the Lie algebroid of a **proper** Lie groupoid $\mathcal{G}$ if and only if there exists an inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{A}$ and a TM-connection $\tilde{\nabla}$ on $\mathcal{A}$ such that, for all $X, Y, Z \in \Gamma(\mathcal{A})$, we have

$$\langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle = \rho(X) \cdot \langle Y, Z \rangle \tag{0.4}$$

where ∇ is the $\mathcal{A}$-connection on $\mathcal{A}$ given by

$$\nabla_X Y = [X, Y] + \tilde{\nabla}_{\rho(Y)} X$$

In this work we will see that the conjecture, in this generality, is false. One of the directions, that for proper groupoids adequate inner products and connections can be found, is however true, at least for fairly general groupoids, and we prove this in Section 3.2. As for the other direction, it holds at least for transitive algebroids, as we see in Section 3.3.1, but needs the additional hypothesis that the algebroid $\mathcal{A}$ is already integrable[5], as we will see by example in Section 3.3.2. One might then attempt to restate the conjecture with the hypothesis that $\mathcal{A}$ be integrable. However, in Section 3.3.3 we present three integrable counter examples to the conjecture that "show" not only that the conjecture is false, but that it is unlikely that integrability by proper groupoids can be controlled by a condition of the simplicity and nature of the one present in the conjecture.

[5]That is, that it is already the algebroid of some groupoid.

Remark

A remark should be done about the definition of properness used (Definition 1.26). As was pointed to me after the conclusion of this work by one of the elements of the Jury, Prof. Gustavo Granja, it should be added to the definition that a proper groupoid must be Hausdorff. This is because of the (correct) definition of a proper map and the proposition that follow (a reference for this is "Transformation groups" from Tom Dieck):

Definition 0.5. A continuous map $f\colon X \to Y$ is said to be proper if one of the following equivalent conditions is satisfied:

- the map $f \times \mathrm{id}_Z$ is closed for any topological space Z.
- f is closed and $f^{-1}(y)$ is compact for every $y \in Y$.

Proposition 0.6. *If X and Y are Hausdorff, and Y is locally compact, then $f\colon X \to Y$ being proper is equivalent to $f^{-1}(K)$ being compact for every compact set $K \subset Y$.*

It should be noted that changing Definition 1.26 to include the Hausdorfness of $\mathcal{G}$ does not alter any of the results obtained in our work. This is immediately clear for the results in Section 3.2 and the examples presented in Sections 3.3.2 and 3.3.3. To see this does not affect the conclusion of Section 3.3.1 one needs the following result:

Proposition 0.7. *Let $\mathcal{G}$ be a transitive groupoid. Then $\mathcal{G}$ is Hausdorff.*

Proof. We want to see that any two points $g_1, g_2 \in \mathcal{G}$ can be separated. Since M is Hausdorff, if g_1 and g_2 have different sources or targets we are done. We can therefore assume we have $g_i\colon x \to y$ for $i = 1, 2$. Since bisections induce local diffeomorphisms of $\mathcal{G}$ we can actually assume $g_i \in \mathcal{G}_x$ for $i = 1, 2$. Now let U, V be neighborhoods of x and α and β as in the proof of Proposition 1.30, assuming further that $\alpha(x) = \beta(x) = 1_x$. We have that the sets $\alpha(U) \cdot W_1 \cdot \beta(V)$ and $\alpha(U) \cdot W_2 \cdot \beta(V)$ will intersect if and only if W_1 and W_2 already intersect. Since $\mathcal{G}_x$ is Hausdorff (Remark 1.4) we will be done by choosing neighborhoods

$W_i \subset \mathcal{G}_x$ of the g_i that do not intersect (that $\alpha(U) \cdot W_i \cdot \beta(V)$ then contains a neighborhood of g_i in $\mathcal{G}$ is a consequence of the map $(u, g, v) \mapsto \alpha(u)g\beta(v)$ having bijective differential on points of the form (x, g, x)). $\square$

Chapter 1

Lie groupoids

In this section we present the definition of groupoid and the results about them that are for our work. The chapter strongly follows the structure of the Lecture 1 in [1]. Most of the content is in fact found there. The extra content consists essentially of the proofs of some results (often left as exercises in [1]) and some small results we proved and that we use in Chapter 3 but that we felt would be better placed in this chapter.

1.1 Definition of groupoid (and Lie groupoid)

This is the shortest definition of a groupoid:

Definition 1.1. A **groupoid** $\mathcal{G}$ is a (small) category in which every arrow is invertible.

This isn't, however, the most convenient way to think of a groupoid, at least in the context we are interested in. Since we are interested in defining Lie groupoids, that is, groupoids with a differential structure, we will need a definition displaying the sets that should be required to be manifolds and the functions that should be required to be smooth (note this is exactly what one does when defining topological or Lie groups, or actions of such groups). Those sets and functions are:

- The set $\mathcal{G}$ of all arrows (denoted by the same letter as the groupoid itself) and the set $M_{\mathcal{G}}$ of all objects, abbreviated to M when there is no danger of ambiguity.
- The **source** and **target** maps:

$$\mathbf{s}, \mathbf{t} \colon \mathcal{G} \to M$$

associating to each $g \in \mathcal{G}$ its source $\mathbf{s}(g)$ and its target $\mathbf{t}(g)$.

We will write $g\colon x \to y$, $x \xrightarrow{g} y$ or even $y \xleftarrow{g} x$ to indicate g is an arrow from x to y (this last one being preferred since it avoids some confusion when multiplying elements).

- The **multiplication (or composition) map**:

$$m\colon \mathcal{G}_2 \to M$$

defined on the set $\mathcal{G}_2$ of pairs of composable arrows

$$\mathcal{G}_2 = \{(g,h) \in \mathcal{G} \times \mathcal{G}\colon s(g) = t(h)\}$$

For a pair $(g,h) \in \mathcal{G}_2$ of composable arrows, $m(g,h)$ is the composition $g \circ h$, which will also be written gh.

- The **unit map**:

$$u\colon M \to \mathcal{G}$$

which sends $x \in M$ to the identity $M_{\mathcal{G}} \in \mathcal{G}$ over x. u will often be regarded as an inclusion of M in $\mathcal{G}$.

- The **inverse map**:

$$i\colon \mathcal{G} \to \mathcal{G}$$

which sends g to g^{-1}.

Furthermore, these functions must also satisfy the following identities:

- **law of composition**: if $x \xleftarrow{g} y \xleftarrow{h} z$, then $x \xleftarrow{gh} z$.
- **law of associativity**: if $x \xleftarrow{g} y \xleftarrow{h} z \xleftarrow{k} u$, then $g(hk) = (gh)k$.
- **law of units**: $x \xleftarrow{1_x} x$ and, for all $x \xleftarrow{g} y$, $1_x g = g 1_y = g$.
- **law of inverses**: if $x \xleftarrow{g} y$, then $y \xleftarrow{g^{-1}} x$ and $gg^{-1} = 1_y$, $g^{-1}g = 1_x$.

So we have the following (alternative) definition:

Definition 1.2 (Groupoid)**.** A **groupoid** consists of a set $\mathcal{G}$, a set M and functions $\mathbf{s}$,$\mathbf{t}$,m,u and i satisfying all of the properties above.

It is perhaps useful to note that (loosely speaking) a groupoid is something that satisfies exactly the same algebraic rules as a group, except that the product of two elements may not always be defined [1].

We also define Lie groupoid:

[1]Note however that this does not completely characterize groupoids, since the rule that specifies when products are or not defined is somewhat rigid.

Definition 1.3 (Lie groupoid)**.** A **Lie groupoid** is a groupoid $\mathcal{G}$ such that both the set $\mathcal{G}$ of arrows and the set M of objects are manifolds, $\mathcal{G}$ not necessarily Hausdorff, and the maps $\mathbf{s}$, $\mathbf{t}$, m, u and i are smooth, and $\mathbf{s}$ and $\mathbf{t}$ are submersions.

Remark 1.4. The relaxation of the Hausdorffness of $\mathcal{G}$ can be justified as a necessary condition to allow more Lie algebroids to have an associated Lie groupoid. It allows, for instance, for all bundles of Lie algebras (to be defined in the examples of Lie algebroids) to have such a groupoid.

It should also be noted that no non-Hausdorff Lie groups are allowed. This is because a topological group is Hausdorff if and only if the set $\{e\}$ containing only the identity is closed, and when that is not the case there will be points $g \neq e$ contained in every neighborhood of e, so that e can not have a neighborhood homeomorphic to $\mathbb{R}^n$.

The last condition in the definition, that $\mathbf{s}$ and $\mathbf{t}$ should be submersions, should be interpreted as saying that the differential structure on $\mathcal{G}$ reflects the differential structure and topology on M. For instance, given an arrow $g\colon x \to y$ one should have that for any neighborhood of $g \in \mathcal{G}$, then for any choice of $\tilde{x}$ sufficiently close to x then there should exist a $\tilde{g}\colon \tilde{x} \to \tilde{y}$ in that neighborhood of g (where the $\tilde{y}$ itself can be chosen close to y[2], although this follows from $\mathbf{t}$ being continuous and not from being submersion) [3].

On the other hand, the fact that $\mathbf{s}$ and $\mathbf{t}$ are submersions is also relevant because of the following proposition:

Proposition 1.5. *The following are equivalent:*

1. $\mathbf{s}$ *is a submersion*
2. $\mathbf{t}$ *is a submersion*
3. *the map* $\mathbf{s} \times \mathbf{t}\colon \mathcal{G} \times \mathcal{G} \to M \times M$ *is transverse to the diagonal*

Proof.

- (1) $\Leftrightarrow$ (2) is obvious since $\mathbf{s} = \mathbf{t} \circ i$, and i is a diffeomorphism
- (1)&(2) $\Leftrightarrow$ (3) is immediate since then $\mathbf{s} \times \mathbf{t}$ is a submersion.

[2] $\tilde{y}$ can not however be chosen arbitrarily in a neighborhood of y. That would imply that the groupoid was (locally) transitive.

[3] This remark also applies to the case of topological groupoids, that are groupoids such that $\mathcal{G}$ and M are topological spaces, $\mathbf{s}$, $\mathbf{t}$, m, u and i are continuous maps, and $\mathbf{s}$ and $\mathbf{t}$ are open maps.

- (3) ⇔ (1) Note that
$$\mathrm{d}\mathbf{s}(T_g\mathcal{G}) = \mathrm{d}\mathbf{t}(T_{g^{-1}}\mathcal{G}),$$
and that
$$\mathrm{d}(\mathbf{s}\times\mathbf{t})(T_{(g,g^{-1})}\mathcal{G}\times\mathcal{G}) = \mathrm{d}\mathbf{s}(T_g\mathcal{G})\oplus\mathrm{d}\mathbf{t}(T_{g^{-1}}\mathcal{G}) = \mathrm{d}\mathbf{s}(T_g\mathcal{G})\oplus\mathrm{d}\mathbf{s}(T_g\mathcal{G})$$
which has dimension $2\times\dim(\mathrm{d}\mathbf{s}(T_g\mathcal{G}))$. Since on the other hand the intersection with the tangent space to the diagonal has dimension $\dim(\mathrm{d}\mathbf{s}(T_g\mathcal{G}))$ we conclude that transversality occurs precisely when $\mathrm{d}\mathbf{s}(T_g\mathcal{G}) = T_{\mathbf{s}(g)}M$.

□

This proposition tells us that $\mathbf{s}$ or $\mathbf{t}$ being submersions is a sufficient condition for $\mathcal{G}_2$, the set of pairs of composable arrows, to be a submanifold of $\mathcal{G}\times\mathcal{G}$, which is necessary for the condition of m being smooth to make sense[4].

Definition 1.6 (Lie groupoid morphism)**.** Given Lie groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$ over M_1 and M_2, respectively, a **morphism** between them is just a co-variant functor, that is, a pair of functions $F\colon\mathcal{G}_1\to\mathcal{G}_2$ and $f\colon M_1\to M_2$ compatible with all the structure (that is, with sources, targets, units, composition and (as a consequence) inverses). In the Lie case the functions are naturally taken to be smooth.

The map $\mathbf{s}\times\mathbf{t}$ considered in the previous proposition is always a Lie groupoid morphism onto the so called pair groupoid of the base space M, which will be defined in the next section (the function between the sets of objects being the identity map).

We finish this section by fixing some notations:

The sets
$$\mathbf{s}^{-1}(x),\quad \mathbf{t}^{-1}(x)$$
are called **the s-fiber at** x and **the t-fiber at** x, respectively. Note that the inverse map i induces a bijection (in the Lie case, a diffeomorphism) between these.

Given a $g\colon x\to y$, the **right multiplication** by g induces a bijection (in the Lie case, diffeomorphism) between the $\mathbf{s}$-fibers at y and x:
$$R_g\colon\mathbf{s}^{-1}(y)\to\mathbf{s}^{-1}(x)$$

[4] Actually it can be seen that $\mathbf{s}$ and $\mathbf{t}$ being submersions is a sufficient condition for the set of n-tuples $(g_1,\ldots,g_n)$ of composable arrows (that is, such that $\mathbf{s}(g_i)=\mathbf{t}(g_{i+1})$ for $i=1,\ldots,n-1$) to be a manifold $\mathcal{G}_n$ for all n (and we indeed have that the product on $\mathcal{G}_n$ is smooth provided the product on $\mathcal{G}_2$ is).

Likewise, **left multiplication** induces an isomorphism between the **t**-fibers at x and at y.

Next, the intersection

$$\mathcal{G}_x = \mathbf{s}^{-1}(x) \cap \mathbf{t}^{-1}(x)$$

is a group, called the **isotropy group at** x.

Finally, we define on M the following equivalence relation: $x \sim y$ if and only if there exists a $g : x \to y$ in $\mathcal{G}$. The equivalence class of x is called the **orbit through** x:

$$\mathcal{O}_x = \{\mathbf{t}(g) \colon g \in \mathbf{s}^{-1}(x)\}$$

and the quotient set

$$M/\mathcal{G} = M/\sim = \{\mathcal{O}_x \colon x \in M\}$$

is called the **orbit set of** $\mathcal{G}$.

It is worth noticing that this equivalence relation is a generalization of the equivalence relation defined for group actions. Indeed, as we will see in the next section, for each group action there is a corresponding groupoid, and the induced equivalence relations will be the same.

In fact, in analogy with the group actions, we have the following definition:

Definition 1.7. A groupoid $\mathcal{G}$ is called **transitive** if $M/\mathcal{G}$ consists of a single point.

In the Lie case we have the following result (that comes from [1], where it is left as an exercise):

Proposition 1.8. *Given a Lie groupoid $\mathcal{G}$ over M and $x \in M$:*

(a) the isotropy groups $\mathcal{G}_x$ are Lie groups

(b) the orbits $\mathcal{O}_x$ are (regularly immersed[5]) submanifolds of M

(c) the unit map $u \colon M \to \mathcal{G}$ is an embedding

(d) $\mathbf{t} \colon \mathcal{G}(x, -) \to \mathcal{O}_x$ is a principal $\mathcal{G}_x$-bundle.

The following is another important concept:

Definition 1.9. A **local bisection** is a map $b \colon U \to \mathcal{G}$, for $U \subset M$ an open set, such that $\mathbf{s} \circ b$ and $\mathbf{t} \circ b$ are both diffeomorphisms.

If b is defined on all of M it is said to be a global bisection, or just a bisection.

[5] An immersion $i \colon N \to M$ is called *regular* if for any map $f \colon P \to N$ the composition $i \circ f \colon P \to M$ is smooth if and only if f is smooth.

Since in practice the image of the bisection is more important than the function itself, we will always assume that $\mathbf{s} \circ b$ is the identity map, which is always possible by considering the bisection $\tilde{b} = b \circ (\mathbf{s} \circ b)^{-1}$.

We will now see that for any $g \colon x \to y$ there is a local bisection b defined on a neighborhood of x such that $b(x) = g$. This will be a consequence of the following result from linear algebra:

Proposition 1.10. *Let V, W be vector spaces and $\pi_1 \colon V \to W$, $\pi_2 \colon V \to W$ be two surjective linear maps. Then there exists a subspace L of V that is mapped bijectively to W by both the π_i.*

Proof. Choose $v_1, \ldots, v_d$ in V ($d = \dim(W)$) such that $\{\pi_1(v_j)\}$ is a basis of W. If $\{\pi_2(v_j)\}$ is linear independent, we are done. If not, we can assume without loss of generality that its span is generated by the first k vectors, $\pi_2(v_1), \ldots, \pi_2(v_k)$ ($k = 0$ being a possibility). But since π_2 is surjective, there must exist $\xi_{k+1}, \ldots, \xi_d \in V$ such that $\pi_2(v_1), \ldots, \pi_2(v_k), \pi_2(\xi_{k+1}), \ldots, \pi_2(\xi_d)$ form a basis of W. It is possible to choose the $\xi_j \in \mathrm{Ker}(\pi_1)$, since $\mathrm{Ker}(\pi_1)$ is a complement of the span of the v_j. One then takes as L the span of $v_1, \ldots, v_k, v_{k+1} + \xi_{k+1}, \ldots, v_d + \xi_d$.

□

Corollary 1.11. *Let $g \colon x \to y$ be an element of $\mathcal{G}$. Then there is a local bisection b defined on a neighborhood of x such that $b(x) = g$.*

Proof. Notice that in the previous proposition the codomains of the π_i do not need to be chosen the same, just to have the same dimension. One then chooses local coordinates in $\mathcal{G}$ and M for which $\mathbf{s}$ is a linear projection from $\mathbb{R}^{\dim(\mathcal{G})}$ to $\mathbb{R}^{\dim(M)}$. The proposition guarantees the existence of a linear left inverse of $\mathbf{s}$, say b, such that $\mathrm{d}_x(\mathbf{t} \circ b)$ is an isomorphism, so that b will be a local bisection for a small enough neighborhood of x.

□

Corollary 1.12. *The projection $\pi \colon M \to M/\mathcal{G}$ is an open map (for $M/\mathcal{G}$ with the quotient topology).*

Proof. We want to prove that for an open set $U \subset M$ the set V of the orbits of points in U is also open. A point y is in V if there exists a $g \colon x \to y$ for some $x \in U$. But then the existence of a local bisection b such that $b(x) = g$ shows that V contains a neighborhood of y.

□

1.2 Examples

We now present the first examples:

Example 1.13 (Groups)**.** Any group G is obviously a groupoid, taking as the set of objects a set with a single element, that is $M = \{*\}$

Example 1.14 (Pair groupoids)**.** For any set X, one can give $X \times X$ a groupoid structure, with $\mathcal{G} = X \times X$ and $M_{\mathcal{G}} = X$, by setting

$$\mathbf{s}((x,y)) = y$$

and

$$\mathbf{t}((x,y)) = x$$

Since (x,y) is the unique arrow from y to x, this determines the other functions (due to the identities they must satisfy):

$$m((x,y),(y,z)) = (x,y)(y,z) = (x,z)$$

$$u(x) = (x,x)$$

$$i((x,y)) = (y,x)$$

Example 1.15 (bundles of Lie groups)**.** A bundle of Lie groups is a groupoid for which the source and target maps are the same. In other words, it is a family of groups $\mathcal{G}_x$ for each $x \in M$ varying smoothly.

Example 1.16 (General linear groupoids)**.** For E a vector bundle over the manifold M, the general linear groupoid $GL(E)$ is the groupoid with object space M and arrow space the set of all linear isomorphisms between the fibers. More specifically, an element $g\colon x \to y$ is a linear isomorphism $g : E_x \to E_y$. The multiplication in the groupoid is obviously simply the composition of maps.

It is worth noting that general linear groupoids are a generalization of the general linear groups of vector spaces, and also that $GL(E)$ can be seen as a subcategory of the Lie category $End(E)$ of linear maps between the fibers of E.

Example 1.17 (Action groupoids)**.** Given an action of a Lie group G on a manifold M, say, $A\colon G \times M \to M$, there is an associated action groupoid with arrow space $\mathcal{G} = G \times M$ and object space M. The source and target maps are given by

$$\mathbf{s}(g,x) = x, \qquad \mathbf{t}(g,x) = A(g,x) = g \cdot x$$

and the composition of arrows given by

$$(h,y)(g,x) = (hg,x)$$

Notice that the action groupoid encloses all the information relative to the action A.

The example above shows that groupoids generalize not only the notion of group but also the notion of group action. When we define actions of groupoids we will see that they are also completely characterized by an associated action groupoid, so that one could say that groupoids are in some sense "closed" for the notion of action.

Example 1.18 (The fundamental groupoid of a manifold)**.** One fine example of a naturally appearing groupoid is the fundamental groupoid of a manifold M, denoted $\Pi_1(M)$, which consists of homotopy classes of paths with fixed end points (we assume that M is connected), with composition the concatenation of paths whenever defined. This is a more natural object than the fundamental group: if one thinks about how the later is usually defined, it is clear that what is usually done is to define the fundamental groupoid first (although probably without calling it so) and then defining the fundamental group as any of the isotropy groups, which is why a base point is required[6].

Note also that, when M is simply-connected, $\Pi_1(M)$ is isomorphic to the pair groupoid $M \times M$ (since the homotopy class of a path is determined by its end points). In general, the obvious homomorphism of Lie groupoids $\Pi_1(M) \longrightarrow M \times M$, will be a local diffeomorphism.

Example 1.19 (The fundamental groupoid of a foliation)**.** More generally, let $\mathcal{F}$ be a foliation on a manifold M. The fundamental groupoid of $\mathcal{F}$, denoted by $\Pi_1(\mathcal{F})$, consists of the leafwise homotopy classes of paths (with fixed end points):

$$\Pi_1(\mathcal{F}) = \{[\gamma] : \gamma : [0,1] \to M \text{ a path lying in a leaf}\}.$$

The structural maps are given in exactly the same way as the fundamental group (indeed all that changes is the equivalence relation on the set of paths, since now only leafwise homotopies are allowed). Obviously, the orbit O_x is the leaf L through x, while the isotropy group $\Pi_1(\mathcal{F})_x$ is the fundamental group $\pi_1(L, x)$.

1.3 Groupoid actions

Definition 1.20 (Groupoid actions)**.** Given a groupoid $\mathcal{G}$ over M, a **$\mathcal{G}$-space** is defined as a set E, a map $\mu\colon E \to M$, called the **momentum map**, and a

[6]It is however worth noticing that this does not mean that the fundamental groupoid is more useful than the fundamental group. This is because for decent enough cases one can expect the fundamental group to be a simple purely algebraic object, where the groupoid can not be so, since its definition includes M itself as the base space.

map

$$A\colon \mathcal{G} \times_M E = \{(g,e) \in \mathcal{G} \times E \colon \mathbf{s}(g) = \mu(e)\} \to E$$

which is the action itself. $A((g,e))$ is also written as $g \cdot e$. Furthermore, the following identities are to be satisfied:

(i) **law of composition**: $\mu(g \cdot e) = \mathbf{t}(g)$

(ii) **law of associativity**: $g \cdot (h \cdot e) = (gh) \cdot e$ for all $g, h \in \mathcal{G}$ and $e \in E$ for which the expression makes sense

(iii) **law of units**: $1_{\mu(e)} \cdot e = e$ for all $e \in E$

(where the names of the laws come from their obvious counterparts in the definition of groupoid)

In order to understand the definition of actions of groupoids, it is useful to recall that, in some sense, a groupoid is a group where products are not always defined. Namely, the criteria for the product gh to be defined is that g should "start" where h "ends". As such, to define a $\mathcal{G}$-space a similar notion is necessary. A point $e \in E$ can be though as being "over" the point $\mu(e) \in M$, and so the action $g \cdot e$ will defined precisely when e is "over" the source of g. The law (i) simply tells us that when $g \cdot e$ is defined, it will be "over" the target of g. We can therefore regard the action of g as an isomorphism

$$A_g \colon E_{\mathbf{s}(g)} \to E_{\mathbf{t}(g)}$$

(where $E_x = \mu^{-1}(x)$ is the fiber over x)

Laws (ii) and (iii) are obvious generalizations of the corresponding laws for groups.

In the Lie case the functions are naturally required to be smooth (that $\mathcal{G} \times_M E$ is a manifold is guaranteed by $\mathbf{s}$ being a submersion).

Remark 1.21. The concept defined is actually the concept of left $\mathcal{G}$-space. The definition of right $\mathcal{G}$-space is entirely analogous, but it is worth noticing that in that case an element $g : \mathbf{s}(g) \to \mathbf{t}(g)$ will carry $E_{\mathbf{t}(g)}$ to $E_{\mathbf{s}(g)}$, the reverse of what happens in the case of left actions. This is because in the case of right actions the computations should make sense if the action is written on the right, that is, if the action of g on e is written $e \cdot g$. We then conclude that for the **law of composition** $(e \cdot g) \cdot h = e \cdot gh$ to even make sense each g should act on $E_{\mathbf{t}(g)}$ instead of $E_{\mathbf{s}(g)}$.

It should however be noted that, as happens in the group case, there is actually little point in distinguishing left from right actions, since we have an anti-automorphism on $\mathcal{G}$ given by the inverse map i. Specifically, to a right action A corresponds the left action $\tilde{A}$ such that $\tilde{A}_g = A_{i(g)}$ (and vice-versa).

Example 1.22 (Actions by left and right multiplication)**.** As happens for groups, any groupoid $\mathcal{G}$ acts on itself by left and right multiplication. In the case of the left multiplication action we have $\mu = \mathbf{t}$ and the action of g on h is $g \cdot h = gh$. In the case of the right multiplication action we have instead $\mu = \mathbf{s}$ and $g \cdot h = hg^{-1}$.

Remark 1.23. There is however no adjoint action of $\mathcal{G}$ on itself. This is because the expression ghg^{-1} can only make sense for an h such that $\mathbf{s}(h) = \mathbf{t}(h)$. So an adjoint action of $\mathcal{G}$ will only ever be defined on $\bigcup_{x \in M} \mathcal{G}_x$, which in the Lie case will not even necessarily be a manifold.

The non existence of the adjoint action of $\mathcal{G}$ on itself, or on the corresponding Lie algebroid (the analogous of the Lie algebra of a Lie group) is actually one of the main difficulties found when attempting to prove the main conjecture of our work, since that action is one of the main ingredients used in the proof of the original result for Lie groups.

As we saw earlier group actions give rise to, and are completely characterized by, a corresponding groupoid, called the action groupoid. The next example shows that actually the same also happens to groupoid actions.

Example 1.24 (Action groupoids (again))**.** Given a $\mathcal{G}$-space $\mu\colon E \to M$, one defines the corresponding action groupoid $\mathcal{H}$ such that

$$\mathcal{H} = \mathcal{G} \times_M E = \{(g, e) \in \mathcal{G} \times E \colon \mathbf{s}(g) = \mu(e)\}, \qquad M_{\mathcal{H}} = E,$$

the source, target and multiplication being given exactly as in the group case

$$\mathbf{s}(g, x) = x, \qquad \mathbf{t}(g, x) = g \cdot x, \qquad (h, y)(g, x) = (hg, x)$$

We also define the concept of representation of a groupoid $\mathcal{G}$, which is just a linear action:

Definition 1.25 (Representation)**.** Given a groupoid $\mathcal{G}$ over M, a **representation** of $\mathcal{G}$ is an action of $\mathcal{G}$ on a vector bundle $\mu\colon E \to M$ such that the action of each $g\colon x \to y$ is a **linear** isomorphism from E_x to E_y.

Note that, generalizing what happens in the group case, a representation of $\mathcal{G}$ on E is exactly the same as a groupoid morphism $\mathcal{G} \to GL(E)$.

1.4 Proper groupoids

We now define the type of groupoids we will be more interested in:

Definition 1.26 (Proper groupoid)**.** A groupoid $\mathcal{G}$ is said to be **proper** if the map $\mathbf{s} \times \mathbf{t} \colon \mathcal{G} \to M \times M$ is a proper map.

Notice that the concept of proper groupoid generalizes both compact groups and proper actions (of groups). A group is a proper groupoid if and only if it is compact, and indeed in a proper groupoid $\mathcal{G}$ all the isotropy groups $\mathcal{G}_x$ must be compact since $\mathcal{G}_x = (\mathbf{s} \times \mathbf{t})^{-1}(x, x)$. And by definition an action is proper if and only if the corresponding groupoid is, and indeed the definition of proper action looks far more natural if we introduce the action groupoid, suggesting that it is somehow natural to do so.

One of the advantages of working with compact groups is that it is possible to take averages, which is the reason why, for instance, for any representation V of a proper group G there exists an invariant inner product (this fact is used in the proof of the original theorem that motivates our Main Conjecture (0.3)).

This is done by the use of the Haar measure on G, that is, a measure on G that is invariant by the left action of G [7]. Such a measure exists for all Lie groups and we will have something analogous for Lie groupoids. Since the left action of an element $g \colon x \to y$ does not induce a bijection on $\mathcal{G}$ it does not make sense to say that g preserves a measure on $\mathcal{G}$. g gives us instead a bijection between the target fibers $\mathbf{t}^{-1}(x)$ and $\mathbf{t}^{-1}(y)$, so what one should expect is that there exist measures λ^x on each target fiber $\mathbf{t}^{-1}(x)$ and that the left action preserves these measures. The existence of such a system of measures is the content of the following theorem (see, for instance, [7]):

Theorem 1.27 (Existence of a smooth Haar system)**.** *For any groupoid $\mathcal{G}$ there exists a (left-invariant) smooth Haar system λ, that is, a family $\lambda = \{\lambda^x \colon x \in M\}$ of smooth measures λ^x each supported in the manifold $\mathbf{t}^{-1}(x)$, satisfying the following properties:*

1. *for any $\phi \in C_c^\infty(\mathcal{G})$,*

$$I_\lambda(\phi)(x) = \int_{\mathbf{t}^{-1}(x)} \phi(g) \mathrm{d}\lambda^x(g)$$

defines a smooth function $I_\lambda(\phi)$ on M. This condition should be interpreted as saying that the measures λ^x vary smoothly with x.

2. *λ is left invariant, that is, for any $g \colon x \to y$, and any $\phi \in C_c^\infty(\mathbf{t}^{-1}(y))$, we have*

$$\int_{\mathbf{t}^{-1}(x)} \phi(gh) \mathrm{d}\lambda^x(h) = \int_{\mathbf{t}^{-1}(y)} \phi(h) \mathrm{d}\lambda^y(h)$$

[7] One might also take the right invariant Haar measure. However these may not be the same for non compact G.

Furthermore, when $\mathcal{G}$ is proper there exists a "cut-off" function for $\mathcal{G}$, that is, a smooth function $c\colon M \to \mathbb{R}$, $c \geq 0$, such that:

3. *$\mathbf{t}\colon supp(c \circ \mathbf{s}) \to M$ is proper ($supp(c \circ \mathbf{s})$ being the closure of the set where $c \circ \mathbf{s}$ is different from zero).*

4. *$\int_{\mathbf{t}^{-1}(x)} c(\mathbf{s}(g)) \mathrm{d}\lambda^x(g) = 1$ for all $x \in M$.*

(the integral in condition (4) is well defined by condition (3))

It should be noted that condition (4) is a normalization condition necessary to allow c to be used to calculate "averages".

Remark 1.28. To obtain a right invariant Haar system (a system of measures on each source fiber $\mathbf{s}^{-1}(x)$ and invariant by the right action of $\mathcal{G}$ on itself), one just transforms a left invariant one by the inverse map i. In the proper case, the "cut-off" function can be taken to be the same, although one would want to rewrite (3) and (4) in the obvious way.

As an application of this theorem we show that for any representation of a proper groupoid there is an invariant inner product:

Proposition 1.29. *Let $\mathcal{G}$ be a proper groupoid over M and $\mu\colon E \to M$ a representation of $\mathcal{G}$. Then there exists an invariant inner product $\langle \cdot, \cdot \rangle$ on E, that is, an inner product on each fiber such that*

$$\langle g \cdot v, g \cdot w \rangle = \langle v, w \rangle$$

for all $g \in \mathcal{G}$ and $v, w \in E$ such that the expression makes sense (in particular it must be $\mu(v) = \mu(w)$).

Proof. Choose $\langle \cdot, \cdot \rangle_0$ **any inner** product on E and let λ and c be a **right** Haar system and "cut-off" function. Define, for any $v, w \in E$ such that $\mu(v) = \mu(w) = x$

$$\langle v, w \rangle = \int_{\mathbf{s}^{-1}(x)} \langle g \cdot v, g \cdot w \rangle_0 c(\mathbf{t}(g)) \mathrm{d}\lambda^x(g)$$

That the integral is well defined is once again a consequence of (3) (the integrand function having compact support). Bilinearity of $\langle \cdot, \cdot \rangle$ follows from the linearity of integrals and positivity follows from the fact that $c \geq 0$ and (4). That $\langle \cdot, \cdot \rangle$ is also smooth follows from (1). Finally, the invariance $\langle \cdot, \cdot \rangle$ follows from the following computation (where $g\colon x \to y$):

$$
\begin{aligned}
\langle g\cdot v, g\cdot w\rangle &= \int_{\mathbf{s}^{-1}(y)} \langle hg\cdot v, hg\cdot w\rangle_0 c(\mathbf{t}(h))\mathrm{d}\lambda^y(h)\\
&= \int_{\mathbf{s}^{-1}(y)} \langle hg\cdot v, hg\cdot w\rangle_0 c(\mathbf{t}(hg))\mathrm{d}\lambda^y(h)\\
&= \int_{\mathbf{s}^{-1}(x)} \langle h\cdot v, h\cdot w\rangle_0 c(\mathbf{t}(h))\mathrm{d}\lambda^x(h)\\
&= \langle v, w\rangle
\end{aligned}
$$

□

We now state another relevant result about proper groupoids. To explain the motivation for this result we start by setting the notation $\mathcal{G}(U,V)$ for the arrows from V to U[8]. For $y,z\in\mathcal{G}$, $\mathcal{G}(y,z)$ will either be empty or diffeomorphic to $\mathcal{G}_y$ (and $\mathcal{G}_z$). This suggests that having all the isotropy groups compact, although not necessarily sufficient, will "almost" guarantee a groupoid to be proper. The following theorem says that for transitive groupoids this is indeed the case:

Theorem 1.30. *Let $\mathcal{G}$ be a transitive groupoid over M such that for some (and therefore all) $x\in\mathcal{G}$ we have $\mathcal{G}_x$ compact. Then $\mathcal{G}$ is a proper groupoid.*

Proof. Each orbit $\mathcal{O}_x$ of a groupoid $\mathcal{G}$ is the image of $\mathbf{s}^{-1}(x)$ by the target map $\mathbf{t}$. It is easy to see that $\mathbf{t}$ will then be a map of constant rank so that $\mathcal{O}_x$ will be an immersed manifold of M with tangent space the image of $\mathrm{d}\mathbf{t}$ on $\mathbf{s}^{-1}(x)$. It follows that if $\mathcal{G}$ is transitive then $\mathrm{d}\mathbf{t}$ must be surjective on $\mathbf{s}^{-1}(x)$ (and $\mathrm{d}\mathbf{s}$ must be surjective on $\mathbf{t}^{-1}(x)$.

Now fix any $x\in M$. For all $y,z\in M$ one can then choose neighborhoods U,V of y and z and maps $\alpha\colon U\to\mathcal{G}$ and $\beta\colon V\to\mathcal{G}$ such that $u\xleftarrow{\alpha(u)}x$ and $x\xleftarrow{\beta(v)}v$ for $u\in U, v\in V$. It is then clear that:

$$
\mathcal{G}(U,V)=\alpha(U)\cdot\mathcal{G}_x\cdot\beta(V)
$$

It follows that when U and V are both compact $\mathcal{G}(U,V)$ will be compact, from which it follows that $\mathcal{G}$ is proper (large compacts $\tilde{U}$ and $\tilde{V}$ may of course need to be broken into a finite number of such compacts U and V). □

The compactness of all the isotropy groups does not however guarantee properness for general groupoids. To see this check Example 3.23 and Example I both in Section 3.3.3.

[8]This is in accordance with the convention that arrows should be represented as $y\xleftarrow{g}z$, that is, from right to left.

1.5 The Lie algebroid of a Lie groupoid

The definition of the Lie algebroid of a Lie groupoid will be a generalization of the concept of Lie algebra of a Lie group. As a vector space, the Lie algebra $\mathfrak{g}$ of the Lie group G is just the tangent space at the identity. As for the Lie bracket, there are two equivalent ways of defining it:

- The Lie bracket is the derivative of the adjoint action of G on $\mathfrak{g}$, which is induced by the adjoint action of G on itself.
- The Lie bracket is the bracket of vector fields applied to the right invariant vector fields (which are identified with $\mathfrak{g}$).

As was noted before there is no adjoint action of a groupoid $\mathcal{G}$ on itself, so it is unclear how to extend the first definition.

The second definition however is much nicer. One first has to define what a right invariant vector field is. Since the right action of an element is actually a diffeomorphism between two of the source fibers, it follows that $\mathcal{G}$ can only be said to act on the sub-bundle $T^s\mathcal{G}$ of $T\mathcal{G}$ defined as

$$T^{\mathbf{s}}\mathcal{G} = \mathrm{Ker}(d\mathbf{s}) \subset T\mathcal{G}$$

Letting R_g denote the right action of g we now define the space of right invariant vector fields:

$$\mathfrak{X}^s_{\mathrm{inv}} = \{X \in \Gamma(T^s\mathcal{G})\colon X_{gh} = R_h(X_g), \quad \forall (g,h) \in \mathcal{G}_2\}$$

Similarly to how invariant vector fields on groups are determined by their value at the identity, invariant vector fields on a groupoid are determined by their values at the identities, that is, at $M \subset \mathcal{G}$ (where we identify M with its image under the unit map $u\colon M \to \mathcal{G}$), since for X invariant:

$$X_g = R_g(X_{1_{\mathbf{s}(g)}})$$

In other words, there is a bijection between $\mathfrak{X}^s_{\mathrm{inv}}$ and $\Gamma(\mathcal{A})$, the sections of the vector bundle over M

$$\mathcal{A} = T^s\mathcal{G}|_M$$

Notice now that $\mathfrak{X}^s_{\mathrm{inv}}$ is an (infinite dimensional) Lie algebra when given the Lie bracket of vector fields. This is because, given $X, Y \in \mathfrak{X}^s_{\mathrm{inv}}$, since they are tangent to each $\mathbf{s}^{-1}(x)$, the value of the bracket on $\mathbf{s}^{-1}(x)$ is merely the value of the bracket of vector fields on $\mathbf{s}^{-1}(x)$, and because given $g\colon x \to y$ the vector fields on $\mathbf{s}^{-1}(x)$ are the pull-backs by R_g of the vector fields on $s^{-1}(y)$.

By the identification of $\mathfrak{X}^s_{\text{inv}}$ with $\Gamma(\mathcal{A})$ we can assume the bracket to be defined on this second space. This will be our Lie bracket on the Lie algebroid. However, a novelty arises with respect to the group case: $\Gamma(\mathcal{A})$ is not the algebroid itself. Instead the algebroid is $\mathcal{A}$. This makes perfect sense if one thinks that Lie algebras are intended to be infinitesimal versions of Lie groups, the result of some linearization of the structure of the group, so that the algebroid should be a manifold of the same dimension as the groupoid[9].

We gather up this discussion in the following definitions:

Definition 1.31. Given a Lie algebroid $\mathcal{G}$ over M the vector bundle of the corresponding Lie algebroid is the vector bundle over M

$$\mathcal{A} = T^s\mathcal{G}|_M$$

Definition 1.32. The **Lie bracket** $[\cdot,\cdot]$ on $\mathcal{A}$ is the bracket on $\Gamma(\mathcal{A})$ induced by its isomorphism with $\mathfrak{X}^s_{\text{inv}}$.

For a $\alpha \in \Gamma(\mathcal{A})$ we shall denote by $\tilde{\alpha}$ its unique extension to an element of $\mathfrak{X}^s_{\text{inv}}$.

Finally a last (and new) "ingredient" is necessary do define the structure of a Lie algebroid. This "ingredient" will be needed to express the relation of $[\cdot,\cdot]$ with the multiplication of its arguments by functions $f \in C^\infty(M)$ (something not necessary on the group case, because since in that case $M = \{*\}$, all such functions are constant).

Definition 1.33. The **anchor map** of $\mathcal{A}$ is the vector bundle map

$$\rho\colon \mathcal{A} \to TM$$

obtained by the restriction of the map $\mathrm{d}\mathbf{t}\colon T\mathcal{G} \to TM$.

The next proposition is the relation mentioned before. It should be noted that it is a Leibniz-type identity. [10]

Proposition 1.34. *For all $\alpha, \beta \in \Gamma(\mathcal{A})$ and all $f \in C^\infty(M)$,*

$$[\alpha, f\beta] = f[\alpha,\beta] + (\rho(\alpha)\cdot f)\beta$$

Proof. We first note that $\widetilde{f\beta} = (f \circ \mathbf{t})\tilde{\beta}$. The result then follows from the following computation, using the original Leibniz identity for vector fields and the fact that the vector fields $\tilde{\alpha}$ and $\rho(\alpha)$ are $\mathbf{t}$-related:

[9]Note also that, by the tubular neighborhood theorem, $\mathcal{A}$ is diffeomorphic to a neighborhood of M in $\mathcal{G}$

[10]We denote by $X \cdot f$ the Lie derivative of the function $f \in C^\infty(M)$ along the vector field $X \in \mathfrak{X}(M)$

$$
\begin{aligned}
\widetilde{[\alpha, f\beta]} &= [\tilde{\alpha}, \widetilde{f\beta}] \\
&= [\tilde{\alpha}, (f \circ \mathbf{t})\tilde{\beta}] \\
&= (f \circ \mathbf{t})[\tilde{\alpha}, \tilde{\beta}] + (\tilde{\alpha} \cdot (f \circ \mathbf{t}))\tilde{\beta} \\
&= (f \circ \mathbf{t})\widetilde{[\alpha, \beta]} + ((\rho(\alpha) \cdot f) \circ \mathbf{t})\tilde{\beta} \\
&= \widetilde{f[\alpha, \beta]} + (\widetilde{\rho(\alpha) \cdot f})\beta
\end{aligned}
$$

□

We finally define the Lie algebroid as "the whole package":

Definition 1.35. Given a Lie groupoid $\mathcal{G}$ its **Lie algebroid** is the vector bundle $\mathcal{A} = T^s\mathcal{G}|_M$, together with the anchor map ρ and the Lie bracket $[\cdot, \cdot]$ on $\Gamma(\mathcal{A})$.

We will denote the Lie algebroid by the same symbol as the corresponding vector bundle (generally $\mathcal{A}$).

To finish this section we will define the exponential map of a section $\alpha \in \Gamma(\mathcal{A})$. We first define:

Definition 1.36. For $x \in M$, we put

$$\phi^t_\alpha(x) := \phi^t_{\tilde{\alpha}}(1_x) \in \mathcal{G}$$

where $\phi^t_{\tilde{\alpha}}$ is the flow of the right invariant vector field $\tilde{\alpha}$ induced by α. We call ϕ^t_α **the flow** of α[11].

The **exponential map of** α is then defined as the map

$$e^\alpha \colon M \to \mathcal{G}$$

$$x \mapsto \phi^1_\alpha(x)$$

at least for all $x \in M$ for which $\phi^1_\alpha(x)$ is defined. Actually we have the following:

Proposition 1.37. *Let $\alpha \in \Gamma(\mathcal{A})$. Then e^α is defined in all of M if and only if $\rho(\alpha)$ is a complete vector field*[12].

Proof. Due to invariance, when $\phi^t_{\tilde{\alpha}}(1_x)$ is defined $\phi^t_{\tilde{\alpha}}(g)$ will also be defined for all g such that $\mathbf{t}(g) = x$. This tells us that e^α being defined is equivalent to $\phi^1_{\tilde{\alpha}}$ being defined in $\mathcal{G}$, which is itself equivalent to $\tilde{\alpha}$ being complete.

[11]Note that this is coherent with the identification between $\Gamma(\mathcal{A})$ and $\mathfrak{X}^s_{\text{inv}}$.

[12]Recall that a vector field is said to be complete if its flow is defined for all times.

Obviously $\tilde{\alpha}$ being complete implies that $\rho(\alpha)$ is complete, since for a path $g(t)$ integrating $\tilde{\alpha}$ we will have that $\mathbf{t}(g(t))$ is a path integrating $\rho(\alpha)$. The proof of Proposition 2.5 below implies that $\tilde{\alpha}$ is complete when $\rho(\alpha)$ is.

□

Note that, when defined, e^{α} will be a bisection (Definition 1.9).

1.6 s-connected and s-simply connected groupoids

Just as compactness for groups is generalized by properness for groupoids, other properties of groups have non-obvious generalization. This is the case with the concepts of connectedness and simply connectedness.

Definition 1.38. Let $\mathcal{G}$ be a Lie groupoid. $\mathcal{G}$ is said to be **s-connected** if all its source fibers are connected and **s-simply connected** if all its source fibers are connected and simply connected.

To see that $\mathbf{s}$-connectedness is the correct generalization of connectedness, remember that for all groups G, the connected component of the identity, G^0, is not only a subgroup with the same Lie algebra, but the smallest such subgroup, as it is generated by the image of the exponential map. For a groupoid $\mathcal{G}$ a similar role is played by the largest $\mathbf{s}$-connected subset containing the identities, which we denote by $\mathcal{G}^0$. In other words, $\mathcal{G}^0 \subset \mathcal{G}$ consists of those arrows $g : x \longrightarrow y$ of $\mathcal{G}$ which are in the connected component of $\mathbf{s}^{-1}(x)$ containing 1_x. This is indeed a subgroupoid:

Proposition 1.39. *For any Lie groupoid, $\mathcal{G}^0$ is an open subgroupoid of $\mathcal{G}$.*

Proof. Right multiplication by an arrow $g : x \longrightarrow y$ is a homeomorphism from $\mathbf{s}^{-1}(y)$ to $\mathbf{s}^{-1}(x)$ and hence maps connected components to connected components. If g belongs to the connected component of $\mathbf{s}^{-1}(x)$ containing 1_x, then right multiplication by g maps 1_y to g, so it maps the connected component of 1_y to the connected component of 1_x, proving that $\mathcal{G}^0$ is closed under multiplication. Moreover, g^{-1} is mapped to 1_x, so that g^{-1} belongs to the connected component of 1_y, and hence $\mathcal{G}^0$ is closed under inversion. This shows that $\mathcal{G}^0$ is a subgroupoid.

To check that $\mathcal{G}^0$ is open, it is enough to observe that each point $x \in M \subset \mathcal{G}$ has an open neighborhood contained in $\mathcal{G}^0$. This is obvious by the local normal form for submersions: $\mathbf{s}$ is identified with a linear projection from $\mathbb{R}^{\dim(\mathcal{G})}$ to $\mathbb{R}^{\dim(M)}$, and hence there is an $\mathbf{s}$-connected open neighborhood of x. □

It follows that $\mathcal{G}^0$ has the same Lie algebroid $\mathcal{A}$ as $\mathcal{G}$, and it is fairly easy to see that $\mathcal{G}^0$ is generated by the images of the exponentials of sections $\alpha \in$

$\Gamma(\mathcal{A})$. From this it follows that, similarly to what happens for Lie groups, a Lie group homomorphism between Lie groupoids $\mathcal{G}$ and $\mathcal{H}$, with $\mathcal{G}$ **s**-connected is completely determined by the induced map between the algebroids.

To see that **s**-simply connectedness is the correct generalization of simply connectedness, notice that a simply connected Lie group is a "maximal" connected Lie group with a given Lie algebra in the sense that it is a covering of any other connected Lie group with the same Lie algebra. The following theorem shows that **s**-simply connected groupoids generalize this (this result is Proposition 6.5 in [5]):

Theorem 1.40. *Let $\mathcal{G}$ be a s-connected Lie groupoid. There exists a Lie groupoid $\tilde{\mathcal{G}}$ and a surjective homomorphism $\mathcal{F} : \tilde{\mathcal{G}} \to \mathcal{G}$ such that:*

(i) $\tilde{\mathcal{G}}$ is s-simply connected.

(ii) $\tilde{\mathcal{G}}$ and $\mathcal{G}$ have the same Lie algebroid.

(iii) $\mathcal{F}$ is a local diffeomorphism.

Moreover, $\tilde{\mathcal{G}}$ is unique up to isomorphism.

The following theorem generalizes one of the main features of the relation between simply connected Lie groups and their corresponding Lie algebras (see [5]):

Theorem 1.41 (Lie II)**.** *Let $\mathcal{G}$ and $\mathcal{H}$ be Lie groupoids with algebroids $\mathcal{A}$ and $\mathcal{B}$ and $F : \mathcal{A} \to \mathcal{B}$ be a morphism of Lie algebroids[13]. Then, if $\mathcal{G}$ is* **s***-simply connected, there exists a (unique) morphism of Lie groupoids $\mathcal{F} : \mathcal{G} \to \mathcal{H}$ integrating F.*

[13]To be defined in the next section.

Chapter 2

Lie algebroids

In this section we present the definition of algebroid and the results about them that are for our work. The chapter strongly follows the structure of the Lecture 2 in [1]. Most of the content is in fact found there. The extra content consists essentially of the proofs of some results (often left as exercises in [1]) and some small results we proved and that we use in Chapter 3 but that we felt would be better placed in this chapter. The exception to this is Section 2.3.2 which is adapted from Section 3 of [2].

2.1 Definition of Lie algebroid

The definition of a Lie algebroid is merely an abstraction of the structure of the Lie algebroid of a Lie groupoid:

Definition 2.1. A **Lie algebroid** over a manifold M consists of a vector bundle $\mathcal{A}$ with a bundle map $\rho\colon \mathcal{A} \to TM$ called the anchor and a Lie bracket $[\cdot,\cdot]$ on the space of sections $\Gamma(\mathcal{A})$, satisfying the Leibniz identity

$$[\alpha, f\beta] = f[\alpha,\beta] + (\rho(\alpha)\cdot f)\beta$$

for all $\alpha,\beta \in \Gamma(\mathcal{A})$ and $f \in C^\infty(M)$.

Although not obvious, this definition implies that the bracket $[\cdot,\cdot]$ and the anchor ρ must also satisfy the following relation:

Proposition 2.2. *Let $\alpha,\beta \in \Gamma(\mathcal{A})$, then*

$$\rho([\alpha,\beta]) = [\rho(\alpha),\rho(\beta)]$$

where the bracket on the right side is the usual bracket of vector fields.

Proof. Let $\alpha, \beta, \gamma \in \Gamma(\mathcal{A})$ and $f \in C^\infty(M)$. We have

$$[[\alpha, \beta], f\gamma] = f[[\alpha, \beta], \gamma] + (\rho([\alpha, \beta]) \cdot f)\gamma$$

$$\begin{aligned}[[\beta, f\gamma], \alpha] &= [f[\beta, \gamma], \alpha] + [(\rho(\beta) \cdot f)\gamma, \alpha] \\ &= f[[\beta, \gamma], \alpha] - (\rho(\alpha) \cdot f)[\beta, \gamma] + (\rho(\beta) \cdot f)[\gamma, \alpha] - (\rho(\alpha) \cdot (\rho(\beta) \cdot f))\gamma\end{aligned}$$

$$\begin{aligned}[[f\gamma, \alpha], \beta] &= [f[\gamma, \alpha], \beta] - [(\rho(\alpha) \cdot f)\gamma, \beta] \\ &= f[[\gamma, \alpha], \beta] - (\rho(\beta) \cdot f)[\gamma, \alpha] - (\rho(\alpha) \cdot f)[\gamma, \beta] + (\rho(\beta) \cdot (\rho(\alpha) \cdot f))\gamma\end{aligned}$$

Adding up all these equations (ignoring the intermediate computations) and using the Jacobi identity, one obtains

$$0 = (\rho([\alpha, \beta]) \cdot f)\gamma - (\rho(\alpha) \cdot (\rho(\beta) \cdot f))\gamma + (\rho(\beta) \cdot (\rho(\alpha) \cdot f))\gamma = (\rho([\alpha, \beta]) \cdot f)\gamma - ([\rho(\alpha), \rho(\beta)] \cdot f)\gamma$$

which gives us the desired result (since it must be true for all f and γ). $\square$

This proposition tells us that the bracket and the anchor commute or, as will be clear when we define Lie algebroid morphism, that the anchor ρ is a homomorphism from the algebroid $\mathcal{A}$ to the algebroid TM.

In the same way that for a groupoid $\mathcal{G}$ each $x \in M$ has an associated group $\mathcal{G}_x$, when we have a Lie algebroid $\mathcal{A}$ each point has an associated Lie algebra $\mathfrak{g}_x(\mathcal{A})$ which, when $\mathcal{A}$ is the algebroid of $\mathcal{G}$, is the Lie algebra of $\mathcal{G}_x$ (that this is so is an exercise in [1]):

Definition 2.3. At each point $x \in M$ the Lie algebra

$$\mathfrak{g}_x(\mathcal{A}) = \mathrm{Ker}(\rho_x)$$

is called the **isotropy Lie algebra** at x.

Recall now that a groupoid $\mathcal{G}$ divides the base space into orbits M. It turns out that these orbits are essentially determined by the algebroid, and indeed, for all Lie algebroid $\mathcal{A}$ over M it is possible to divide M into orbits using only the information contained in the algebroid. This uses the notion of $\mathcal{A}$-path:

Definition 2.4. Given a Lie algebroid $\mathcal{A}$ over M, an **$\mathcal{A}$-path** consists of a pair (a, γ) where $\gamma : I \to M$ is a path in M, $a : I \to \mathcal{A}$ is a path in $\mathcal{A}$([1]), such that

(i) a is a path above γ, i.e., $a(t) \in \mathcal{A}_{\gamma(t)}$ for all $t \in I$.

[1]Letting I denote the unit interval $[0, 1]$.

(ii) $\rho(a(t)) = \frac{\mathrm{d}\gamma}{\mathrm{d}t}(t)$, for all $t \in I$.

Obviously a determines γ, so that when referring to the $\mathcal{A}$-path we will usually mention only a.

It is worth thinking of the tangent bundle TM, which is one of the simplest examples of Lie algebroid, when trying to understand the definition above. For TM, the definition simply tells us that a is the derivative of γ, and for a general $\mathcal{A}$ it tells us that a should "lift" that derivative via ρ, so that a is a sort of "$\mathcal{A}$-derivative of γ". Also worth noticing is that considering the (singular) distribution $\mathrm{Im}(\rho)$ on M, γ will be a path integrating that distribution (with a being "proof" that this is so).

The equivalence relation on M is now simply defined by saying that $x, y \in M$ are equivalent if there exists an $\mathcal{A}$-path a such that $\gamma(0) = x$ and $\gamma(1) = y$. As usual an equivalence class will be called an **orbit** of $\mathcal{A}$, and when ρ is surjective we say that $\mathcal{A}$ is a **transitive Lie algebroid**. In this case the orbits of $\mathcal{A}$ are the connected components of M.

The following proposition shows that, as previously mentioned, for the Lie algebroid of a groupoid, the two sets of orbits are the same.

Proposition 2.5. *Let $\mathcal{G}$ be a s-connected Lie groupoid over M and let $\mathcal{A}$ be its Lie algebroid. Then the orbits of $\mathcal{G}$ in M coincide with the orbits of $\mathcal{A}$ in M.*

Proof. Suppose $x, y \in M$ are in the same orbit of $\mathcal{G}$, that is, that there exists $g\colon x \to y$. Since $\mathcal{G}$ is $\mathbf{s}$-connected there is a path $g(t) : I \to \mathbf{s}^{-1}(x)$ with $g(0) = 1_x$ and $g(1) = g$. Then $a(t) = \mathrm{d}_{g(t)}R_{g(t)^{-1}}(\frac{\mathrm{d}g}{\mathrm{d}t}(t))$ is clearly an $\mathcal{A}$-path connecting x and y.

To prove the other direction, one should construct the path $g(t)$ from the $\mathcal{A}$-path a. One starts by choosing a 1-parameter family α_t of sections of $\mathcal{A}$ such that $\alpha_t(\gamma(t)) = a(t)$. This gives rise to a corresponding 1-parameter family of invariant vector fields $\tilde{\alpha}_t$ in $\mathcal{G}$. We will be done if the solution of $\frac{\mathrm{d}g}{\mathrm{d}t}(t) = \tilde{\alpha}_t(t)$ with starting condition $g(0) = 1_x$ is defined for all $t \in I$. But considering the 1-parameter family of starting conditions $g_t(t) = 1_{\gamma(t)}$ we have that, I being compact, the joint solution $g_t(s)$ to all these problems will be defined for all s such that $|s - t| < \epsilon$ with $\epsilon > 0$. The paths g_t need not be the same, in the sense that they need not "glue", but are instead right translates of each other, and therefore by gluing translates we obtain the desired result. [2] □

[2] Underlying this proof is also another of the motivations for the concept of $\mathcal{A}$-path. When $\mathcal{A}$ is the Lie algebroid of a groupoid $\mathcal{G}$, then an $\mathcal{A}$-path can be thought of as being the derivative of a source constant path in $\mathcal{G}$, by noting that all the tangent vectors in the (usual) derivative of the path can be uniquely identified with a vector of $\mathcal{A}$ via right translation.

The following proposition, which we will not prove, states that even when $\mathcal{A}$ is not the Lie algebroid of a Lie groupoid, the orbits are just as well behaved (this is Theorem 8.5.1 in [3]) :

Proposition 2.6. *Let $\mathcal{A}$ be a Lie algebroid over M. Then each orbit $\mathcal{O}$ of $\mathcal{A}$ is an immersed submanifold of M, which integrates $Im(\rho)$, i.e., $T_x\mathcal{O} = Im(\rho_x)$ for all $x \in \mathcal{O}$.*

The following is useful terminology:

Definition 2.7. A Lie algebroid A is called **integrable** if it is isomorphic to the Lie algebroid of a Lie groupoid $\mathcal{G}$. For such a $\mathcal{G}$, we say that $\mathcal{G}$ integrates A.

This definition would make little sense for (finite dimensional) Lie algebras as they are all integrable. It turns out, however, that this is not the case for algebroids. We present an example of non integrable algebroid in the following section, although we do not prove that it is so.

Even so, we at least have the following result:

Theorem 2.8 (Lie I)**.** *If A is integrable, then there exists an unique (up to isomorphism) s-simply connected Lie groupoid $\mathcal{G}$ integrating A.*

Proof. This is a consequence of the fact that for each groupoid $\mathcal{G}$ there exists a unique (up to isomorphism) Lie groupoid $\widetilde{\mathcal{G}}$ which is s-simply connected and which has the same Lie algebroid as $\mathcal{G}$. □

With this and Theorem 1.41 we see that, if we restrict ourselves to integrable algebroids, we have an equivalence between the category of integrable algebroids and the category of **s**-simply connected groupoids.

We now define morphisms of Lie algebroids:

Definition 2.9. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be Lie algebroids over M_1 and M_2. A **morphism of Lie algebroids** is a vector bundle map

$$\begin{array}{ccc} A_1 & \xrightarrow{F} & A_2 \\ \downarrow & & \downarrow \\ M_1 & \xrightarrow[f]{} & M_2 \end{array}$$

which is compatible with the anchors and the brackets.

The compatibility however requires some explaining:

Compatibility with the anchor means that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A}_1 & \xrightarrow{F} & \mathcal{A}_2 \\ {\scriptstyle \rho_1}\downarrow & & \downarrow{\scriptstyle \rho_2} \\ TM_1 & \xrightarrow[\mathrm{d}f]{} & TM_2 \end{array}$$

Compatibility with the bracket is a little cumbersome to describe. The difficulty here is that, in general, sections of $\mathcal{A}_1$ cannot be pushed forward to sections of $\mathcal{A}_2$ (for such sections it is just a matter of demanding that the bracket "commutes" with push-forwards). Instead we have to work at the level of the pull-back bundle $f^*\mathcal{A}_2$. First note that from sections α of $\mathcal{A}_1$ or α' of $\mathcal{A}_2$, we can produce new sections $F(\alpha)$ and $f^*(\alpha')$ of $f^*\mathcal{A}_2$ by:

$$F(\alpha) = F \circ \alpha, \;\; f^*(\alpha') = \alpha' \circ f.$$

Now, given any section $\alpha \in \Gamma(A_1)$, we can express (at least locally) its image under F as a (non-unique) finite linear combination

$$F(\alpha) = \sum_i c_i f^*(\alpha_i),$$

where $c_i \in C^\infty(M_1)$ and $\alpha_i \in \Gamma(\mathcal{A}_2)$. By compatibility with the brackets we mean that, if $\alpha, \beta \in \Gamma(\mathcal{A}_1)$ are sections such that their images are expressed as finite combinations as above, then their bracket is a section whose image can be expressed as:

$$F([\alpha,\beta]_{\mathcal{A}_1}) = \sum_{i,j} c_i c'_j f^*[\alpha_i, \beta_j]_{\mathcal{A}_2} + \\ + \sum_j (\rho(\alpha)\cdot c'_j) f^*(\beta_j) - \sum_i (\rho(\beta)\cdot c_i) f^*(\alpha_i).$$

It is a simple computation to see that this does not depend on how one expresses $F(\alpha)$.

Notice that, in the easy case where the sections can be pushed forward to sections $\alpha', \beta' \in \Gamma(\mathcal{A}_2)$, meaning that $F(\alpha) = \alpha' \circ f$ and $F(\beta) = \beta' \circ f$, this just means that:

$$F([\alpha,\beta]_{\mathcal{A}_1}) = [\alpha', \beta']_{\mathcal{A}_2} \circ f.$$

Remark 2.10. There are essentially two cases for which the definition above is much clearer:

The first is the case when the algebroids have the same basis M and the map between the bases is the identity. In this case push-forwards always exist, so that the condition to be satisfied is

$$F([\alpha,\beta]) = [F(\alpha), F(\beta)] \qquad \text{for all } \alpha,\beta \in \Gamma(\mathcal{A}_1).$$

The second is the case when $\mathcal{A}_1 = f^*\mathcal{A}_2$ and F is the natural map from $f^*\mathcal{A}_2$ to $\mathcal{A}_2$, when the condition to be satisfied is

$$[f^*\alpha, f^*\beta] = f^*[\alpha,\beta] \qquad \text{for all } \alpha,\beta \in \Gamma(\mathcal{A}_2).$$

It is worth noticing in this case that the $f^*\alpha$ form a (local) $C^\infty(M)$ basis for $\Gamma(f^*\mathcal{A}_2)$.

Now, to understand the general case, one notes that it is always possible to define Lie algebroid structures on $f^*\mathcal{A}_2$ compatible with the Lie algebroid structure on $\mathcal{A}_2$, that is, satisfying the relation above. In fact, it is easy to see that to give such a structure is exactly the same as choosing a compatible anchor. The general case then results of "composing" the two cases above. A Lie algebroid map from $\mathcal{A}_1$ to $\mathcal{A}_2$ can be thought of as a sort of composition of a morphism from $\mathcal{A}_1$ to $f^*\mathcal{A}_2$ and a morphism from $f^*\mathcal{A}_2$ to $\mathcal{A}_2$. The only reason why is isn't exactly so is because the anchor of $\mathcal{A}_1$ might not be compatible with any of the possible anchors on $f^*\mathcal{A}_2$. In a very informal sense we can think we have such morphisms but that the anchor used on $f^*\mathcal{A}_2$ varies depending on the sections for which we are calculating the bracket.

As expected, morphisms of Lie groupoids induce morphisms of the corresponding Lie algebroids (this result is left as an exercise in [1]):

Proposition 2.11. *Let $\mathcal{F} : \mathcal{G} \to \mathcal{H}$ be a homomorphism of Lie groupoids with $\mathcal{A}$ and $\mathcal{B}$ the corresponding Lie algebroids. Then the restriction of $D\mathcal{F}$ to $\mathcal{A}$ is a Lie algebroid homomorphism from $\mathcal{A}$ to $\mathcal{B}$.*

2.2 Examples

The first two examples represent two of the main types of behavior found in algebroids:

Example 2.12 (Lie algebras)**.** Obviously any Lie algebra $\mathfrak{g}$ is a Lie algebroid over a point, which will be integrated by a Lie group.

Example 2.13 (tangent bundles)**.** For any manifold M the tangent space TM is obviously a Lie algebroid, with the bracket of vector fields and the identity map as anchor. In fact, for any other Lie algebroid the anchor map is a Lie algebroid morphism to TM. Tangent bundles are transitive algebroids and have trivial isotropy Lie algebras. Both pair groupoids $M \times M$ and fundamental groupoids

$\Pi_1(M)$ integrate TM, the second being the unique $\mathbf{s}$-simply connected groupoid integrating TM.

The previous examples have the following generalizations:

Example 2.14 (bundles of Lie algebras)**.** A bundle of Lie algebras over M is a vector bundle $\mathcal{A}$ over M together with a smooth varying family of Lie algebra brackets on each fiber. It is easy to see this is precisely the same as a Lie algebroid with zero anchor map. It turns out that bundles of Lie algebras are always integrable, the integrating groupoid being of course a bundle of Lie groups[3].

Example 2.15 (integrable distributions)**.** Remember that a distribution, that is, a subbundle $\mathcal{A} \subset TM$, is integrable if and only if it is involutive, that is, if the bracket of two sections of $\mathcal{A}$ is always a section of $\mathcal{A}$ [4], or, in other words, if and only if $\mathcal{A}$ is a Lie subalgebroid of TM.

It is easy to see that, for $\mathcal{F}$ the foliation integrating $\mathcal{A}$ as a distribution, $\Pi_1(\mathcal{F})$ integrates $\mathcal{A}$ as a Lie algebroid.

General Lie algebroids can be seen as combining aspects from both of the previous types of algebroids.

The following example is particularly important since it gives examples of non-integrable Lie algebroids:

Example 2.16 (Two forms)**.** Any closed 2-form ω on a manifold M has an associated Lie algebroid, denoted $\mathcal{A}_\omega$ defined as follows:

As a vector bundle,

$$\mathcal{A}_\omega = TM \oplus \mathbb{R},$$

the anchor is the projection on the first component, while the

bracket on sections $\Gamma(\mathcal{A}_\omega) \simeq \mathfrak{X}(M) \times C^\infty(M)$ is defined by:

$$[(X, f), (Y, g)] = ([X, Y], \mathcal{L}_X(g) - \mathcal{L}_Y(f) + \omega(X, Y)).$$

(the closedness of the form is required for the Jacobi identity to hold)

It turns out that if M is simply connected, then integrability of $\mathcal{A}_\omega$ is equivalent to the group

$$\Gamma_\omega = \left\{ \int_\gamma \omega : \gamma \in \pi_2(M) \right\} \subset \mathbb{R}.$$

of spherical periods of ω being discrete (see for instance [1]. This fact is mentioned in Lecture 3, Example 3.1 and again in Example 3.28, when this is finally proved).

[3] The associated groupoid will however not always be Hausdorff, although each of the isotropy groups will.

[4] This is the content of the Frobenius Integrability Theorem.

This shows that considering, for example, $M = \mathbb{S}^2 \times \mathbb{S}^2$ and $\omega = dS \oplus \lambda\, dS$, where dS is the standard area form on $\mathbb{S}^2$, then for $\lambda \in \mathbb{R} - \mathbb{Q}$ the the group of spherical periods $\Gamma_\omega = \mathbb{Z} + \lambda\mathbb{Z}$ will not be discrete, so that the corresponding Lie algebroid is not integrable.

2.3 Connections

2.3.1 Connections

When defining a TM connection on a vector bundle E, the reason why the first vector bundle is taken to be TM is because the definition includes a Leibniz rule that requires the possibility to derive functions in $C^\infty(M)$ by sections of that first bundle. But since for a Lie algebroid $\mathcal{A}$ the anchor map ρ allows us to do just that, we have the following definition[5]:

Definition 2.17. Given a Lie algebroid $\mathcal{A}$ over M and a vector bundle E over M, an **$\mathcal{A}$-connection** on E is a bilinear map

$$\nabla\colon \Gamma(A) \times \Gamma(E) \to \Gamma(E)$$

$$(\alpha, s) \mapsto \nabla_\alpha(s)$$

which is $C^\infty(M)$ linear on α, and which satisfies the Leibniz rule with respect to s:

$$\nabla_\alpha(fs) = f\nabla_\alpha(s) + (\rho(\alpha) \cdot f)s.$$

As in the case of TM-connections, we define the **curvature** of the connection as the map

$$R_\nabla\colon \Gamma(\mathcal{A}) \times \Gamma(\mathcal{A}) \to \mathrm{Hom}(\Gamma(E), \Gamma(E))$$

$$R_\nabla(\alpha, \beta)(X) = \nabla_\alpha \nabla_\beta X - \nabla_\beta \nabla_\alpha X - \nabla_{[\alpha,\beta]} X$$

which is $C^\infty(M)$ linear in all the arguments (α, β and X), by the same formal calculations as in the usual case.

Also in analogy with the usual case, the connection is called **flat** if $R_\nabla = 0$.

2.3.2 Connections up to homotopy

This section is adapted from Section 3 of [2].

The Lie bracket $[\cdot, \cdot]$ is not an $\mathcal{A}$-connection on $\mathcal{A}$. Although the Leibniz identity is satisfied, the bracket is not $C^\infty(M)$ linear in the first variable. This motivates the following:

[5]The definition makes sense for all vector bundles A with linear maps $\rho\colon A \to TM$ (so the Lie bracket on $\Gamma(\mathcal{A})$ is irrelevant for this definition) .

Definition 2.18. A $\mathbb{R}$-bilinear map

$$\nabla \colon \Gamma(A) \times \Gamma(E) \to \Gamma(E)$$

$$(\alpha, s) \mapsto \nabla_\alpha(s)$$

verifying

$$\nabla_\alpha(fs) = f\nabla_\alpha(s) + (\rho(\alpha) \cdot f)s.$$

and which is local in α [6] is called a **non-linear connection**.

Obviously $[\cdot,\cdot]$ is a non-linear $\mathcal{A}$-connection on $\mathcal{A}$. Fortunately, the non-linearity of $[\cdot,\cdot]$ is controlled by the Leibniz identity. To deal with this controlled non-linearity, we start by defining the following:

Definition 2.19. A **supercomplex of vector bundles** (E, ∂) is a pair of vector bundles E^0 and E^1 with maps

$$E^0 \underset{\partial}{\overset{\partial}{\rightleftarrows}} E^1$$

such that $\partial^2 = 0$.

Remark 2.20. Notice that a supercomplex of vector bundles induces a complex of vector spaces

$$\Gamma(E^0) \underset{\partial}{\overset{\partial}{\rightleftarrows}} \Gamma(E^1)$$

Definition 2.21. Let $\mathcal{A}$ be a Lie algebroid and (E, ∂) a supercomplex of vector bundles. A **non-linear connection ∇ of $\mathcal{A}$ on E** is a pair of non-linear connections ∇^0 and ∇^1 on E^0 and E^1, respectively, such that for each $\alpha \in \Gamma(\mathcal{A})$ we have that ∇_α is a chain map (for the complex of sections).

Such a ∇ is said to be a **connection up to homotopy** if for each $\alpha \in \Gamma(\mathcal{A})$ and $f \in C^\infty(M)$ the chain maps $\nabla_{f\alpha}$ and $f\nabla_\alpha$ are homotopic, that is, if we have

$$\nabla_{f\alpha} = f\nabla_\alpha + H_\nabla(f,\alpha)\partial + \partial H_\nabla(f,\alpha)$$

where $H_\nabla(f,\alpha) \in \Gamma(\mathrm{End}(E))$ are odd maps[7] that are $\mathbb{R}$-linear and local in α and f.

These notions allow us to turn the bracket $[\cdot,\cdot]$ into a connection up to homotopy. For this consider the supercomplex of vector bundles

$$\mathrm{Ad}(\mathcal{A})\colon\ \Gamma(\mathcal{A}) \underset{0}{\overset{\rho}{\rightleftarrows}} \Gamma(TM)$$

[6]Meaning that the value at any point only depends on the values of α on arbitrarily small neighborhoods

[7]That is, taking E^0 to E^1 and E^1 to E^0.

and define a connection up to homotopy ∇^{ad} by

$$\nabla^{\text{ad}}_\alpha \beta = [\alpha, \beta] \qquad \nabla^{\text{ad}}_\alpha X = [\rho(\alpha), X] \qquad \text{for } \alpha, \beta \in \Gamma(\mathcal{A}), X \in \Gamma(TM)$$

(the homotopy H_∇ is given by $H_\nabla(f, \alpha)(\beta) = 0$ and $H_\nabla(f, \alpha)(X) = -(X \cdot f)\alpha$).

One final notion is necessary to understand understand why in the conjecture we use a connection to replace the bracket $[\cdot, \cdot]$.

Definition 2.22. Two non-linear connections ∇ and ∇' are said to be **equivalent** if for all $\alpha \in \Gamma(\mathcal{A})$ the chain maps ∇_α and ∇'_α are homotopic, that is, if

$$\nabla_\alpha = \nabla'_\alpha + \theta(\alpha)\partial + \partial\theta(\alpha)$$

where the $\theta(\alpha) \in \Gamma(\text{End}(E))$ are odd maps that are $\mathbb{R}$-linear and local in α.

The following lemma justifies the terminology "connection up to homotopy":

Lemma 2.23. *A non-linear connection ∇ is a connection up to homotopy if and only if it is equivalent to a (linear) connection.*

Proof. If ∇ is equivalent to a linear connection ∇', it is a simple computation to see that ∇ is a connection up to homotopy by setting $H_\nabla(f, \alpha) = \theta(f\alpha) - f\theta(\alpha)$.

On the other hand, if ∇ is a connection up to homotopy, first assume $\mathcal{A}$ is trivial as a vector bundle, so that one has a basis $\{e_1, \ldots, e_r\}$ for the sections. One then sets $\nabla'_{e_i} = \nabla_{e_i}$ and extends by linearity (that is, $\nabla'_{\sum f^i e_i} = \sum f^i \nabla_{e_i}$). Setting $\alpha = \sum f^i e_i$, $\theta(\alpha)$ is given by $\sum H_\nabla(f^i, e_i)$.

For a general $\mathcal{A}$ one just chooses a covering of M by open sets U_i over which $\mathcal{A}$ is trivial, and a subordinate partition of unity τ_i. For each open set in the cover we then obtain associated ∇'_i and θ_i. We set $\nabla' = \sum \tau_i \nabla'_i$ and $\theta = \sum \tau_i \theta_i$. □

This lemma shows that ∇^{ad} is equivalent to an actual (linear) connection ∇. A concrete description of such a connection can be obtained from any TM-connection $\tilde{\nabla}$ on $\mathcal{A}$ by setting

$$\nabla'_\alpha \beta = \tilde{\nabla}_{\rho(\beta)}\alpha + [\alpha, \beta] \qquad \nabla'_\alpha X = \tilde{\nabla}_X \alpha + [\rho(\alpha), X]$$

for $\alpha, \beta \in \Gamma(\mathcal{A}), X \in \Gamma(TM)$.

Simple calculations show this is a linear connection and an equivalence $\theta(\alpha)$ between ∇' and ∇^{ad} is obtained by setting $\theta(\alpha)(\beta) = 0$, $\theta(\alpha)(X) = \tilde{\nabla}_X \alpha$.

2.4 Representations

A representation of a Lie algebroid $\mathcal{A}$ can be defined as follows:

Definition 2.24. A **representation** of a Lie algebroid $\mathcal{A}$ over M consists of a vector bundle E over M together with a flat $\mathcal{A}$-connection ∇, i.e., a connection such that

$$\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha = \nabla_{[\alpha,\beta]},$$

for all $\alpha, \beta \in \Gamma(A)$.

This is not the most natural way to define a representation, at least when thinking of the relationship of representations of Lie groups and Lie algebras, which tells us that a representation of a Lie algebroid should be the infinitesimal version of a representation of a corresponding Lie groupoid (if such a groupoid exists). In other words, a representation of the algebroid $\mathcal{A}$ should be a Lie algebroid homomorphism to the Lie algebroid $\mathfrak{gl}(E)$ [8] covering the identity map on M.

To see that this is indeed the case, we will need the following definition and lemma:

Definition 2.25. Given a vector bundle E over M, a **derivation** of E is a pair (D, X) consisting of a vector field X on M and a linear map

$$D : \Gamma(E) \to \Gamma(E),$$

satisfying the Leibniz identity

$$D(fs) = fDs + (X \cdot f)s,$$

for all $f \in C^\infty(M)$, $s \in \Gamma(E)$. We denote by $\mathrm{Der}(E)$ the Lie algebra of derivations of E, where the Lie bracket is given by

$$[(D, X), (D', X')] = (DD' - D'D, [X, X']).$$

Lemma 2.26. *Given a vector bundle E over M, the Lie algebra of sections of the algebroid $\mathfrak{gl}(E)$ is isomorphic to the Lie algebra Der(E) of derivations on E, where the anchor of $\mathfrak{gl}(E)$ is identified with the projection $(D, X) \mapsto X$.*

Proof. Given a section α of $\mathfrak{gl}(E)$, its flow gives linear maps

$$\phi^t_\alpha(x) : E_x \to E_{\phi^t_{\rho\alpha}(x)}.$$

defined for small enough t.

[8] The Lie algebroid of the Lie groupoid $GL(E)$.

In particular, we obtain a map at the level of sections,

$$(\phi^t_\alpha)^* : \Gamma(E) \to \Gamma(E),$$
$$(\phi^t_\alpha)^*(s) = \phi^t_\alpha(x)^{-1}(s(\phi^t_{\rho\alpha}(x))).$$

The derivative of this map induces a derivation D_α on $\Gamma(E)$:

$$D_\alpha s = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\phi^t_\alpha)^*(s).$$

Conversely, given a derivation D, we find a 1-parameter group ϕ^t_D of automorphisms of E, sitting over the flow ϕ^t_X of the vector field X associated to D, as the solution of the equation

$$Ds = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\phi^t_D)^*(s).$$

Viewing $\phi^t_D(x)$ as an element of $\mathcal{G}$, and differentiating with respect to t at $t = 0$, we obtain a section α_D of $\mathfrak{gl}(E)$. Clearly, the correspondences $\alpha \mapsto D_\alpha$ and $D \mapsto \alpha_D$ are inverse to each other, hence we have an isomorphism between $\Gamma(\mathfrak{gl}(E))$ and $\mathrm{Der}(E)$. To see that this preserves the bracket one remarks that the correspondences we have defined are local and hence we may assume that E is trivial as a vector bundle. This reduces the problem to a simple computation involving matrices which we skip (but what we have is the following: $\mathfrak{gl}(E)$ can be written as $TM \oplus \mathfrak{gl}(E_x)$ [9], and the bracket is given by the bracket of vector fields in the first coordinate and by the pointwise bracket of Lie algebras in the second. Furthermore, for $\alpha = (X, A) \in \Gamma(\mathfrak{gl}(E)) = \Gamma(TM \oplus \mathfrak{gl}(E_x))$ and $s \in \Gamma(E)$ a **constant** section (for the trivialization used), then $D_\alpha s = As$, where the left side represents the pointwise multiplication of matrices and vectors). □

Remark 2.27. For a point $x \in M$ and $\alpha \in \Gamma(\mathfrak{gl}(E))$, $D_\alpha s(x)$ is then given by

$$\begin{aligned} D_\alpha s(x) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} ((\phi^t_\alpha)^*(s))(x) = \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \phi^t_\alpha(x)^{-1}(s(\phi^t_{\rho\alpha}(x))) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} c(t)^{-1}(s(\mathbf{t}(c(t)))) \end{aligned}$$

where $c(t)$ is a path on $\mathrm{GL}(E)$ with $c(0) = 1_x$. The expression $c(t)^{-1}(s(\mathbf{t}(c(t))))$ can be rewritten as $m(c(t)^{-1}, s(\mathbf{t}(c(t))))$, where $m : \mathrm{GL}(E) \times_M E$ is the product. It is then clear that $D_\alpha s(x)$ only depends on $c'(0)$, and therefore any other path c with the same derivative (at zero) will give the same result.

[9]Here $\mathfrak{gl}(E_x)$ is the general linear Lie algebra of the typical fiber of E.

Noticing now that a connection is precisely the same as a map

$$\nabla\colon \Gamma(\mathcal{A}) \to \mathrm{Der}(E)$$

that is $C^\infty(M)$ linear and anchor compatible (because of the Leibniz identities), and that such a map is a representation precisely if it is also a Lie algebra homomorphism, it follows that a representation is exactly the same as a map from $\Gamma(\mathcal{A})$ to $\Gamma(\mathfrak{gl}(E))$ that is $C^\infty(M)$ linear, a Lie algebra homomorphism and anchor compatible, and this is exactly the same as a Lie algebroid homomorphism between $\mathcal{A}$ and $\mathfrak{gl}(E)$ covering the identity.

Chapter 3

Results on the conjecture

In this chapter we present the results obtained concerning our conjecture. We start by restating it:

Conjecture 3.1 (Main conjecture)**.** Let $\mathcal{A}$ be a Lie algebroid. Then $\mathcal{A}$ is the Lie algebroid of a proper Lie groupoid $\mathcal{G}$ if and only if there exists an inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{A}$ and a TM-connection $\tilde{\nabla}$ on $\mathcal{A}$ such that, for all $X, Y, Z \in \Gamma(\mathcal{A})$, we have

$$\langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle = \rho(X) \cdot \langle Y, Z \rangle \tag{3.2}$$

where ∇ is the $\mathcal{A}$-connection on $\mathcal{A}$ given by

$$\nabla_X Y = [X, Y] + \tilde{\nabla}_{\rho(Y)} X \tag{3.3}$$

For the sake of avoiding repetition, we make the following definition:

Definition 3.4. A triple $(\langle \cdot, \cdot \rangle, \nabla, \tilde{\nabla})$ as in the conjecture above is called an **invariant triple** on $\mathcal{A}$.

3.1 Proof of the original theorem

We now recall the original theorem we are trying to generalize as well as its proof (which is adapted from the proof of the more informative Theorem 3.6.2 in [4]), since it inspires some parts of our proofs in the following sections. Remark (3.7) in particular will be explicitly used.

Theorem 3.5. *Let $\mathfrak{g}$ be a (finite-dimensional) Lie algebra (over $\mathbb{R}$). Then $\mathfrak{g}$ is the Lie algebra of a compact Lie group G if and only if there exists an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ such that, for all $X, Y, Z \in \mathfrak{g}$, we have*

$$\langle [X, Y], Z \rangle + \langle Y, [X, Z] \rangle = 0 \tag{3.6}$$

Proof. Suppose first a compact G exists. Then we can choose an inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ invariant by the adjoint action Ad of G on $\mathfrak{g}$, that is

$$\langle \mathrm{Ad}(g)(Y), \mathrm{Ad}(g)(Z)\rangle = \langle Y, Z\rangle, \qquad \text{for all } g \in G \text{ and } Y, Z \in \mathfrak{g}$$

(3.6) is simply the infinitesimal expression of this invariance:

$$\begin{aligned}
0 = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \langle Y, Z\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} &\langle \mathrm{Ad}(e^{tX})(Y), \mathrm{Ad}(e^{tX})(Z)\rangle \\
&= \langle \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathrm{Ad}(e^{tX})(Y), Z\rangle + \langle Y, \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathrm{Ad}(e^{tX})(Z)\rangle \\
&= \langle [X,Y], Z\rangle + \langle Y, [X,Z]\rangle
\end{aligned}$$

proving the first direction.

We now assume we have an inner product $\langle\cdot,\cdot\rangle$ such that (3.6) is satisfied and let G_0 be the simply connected group integrating $\mathfrak{g}$. We have (using the computation above and the properties of the exponential):

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=t_0} &\langle \mathrm{Ad}(e^{tX})(Y), \mathrm{Ad}(e^{tX})(Z)\rangle = \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \langle \mathrm{Ad}(e^{tX})(\mathrm{Ad}(e^{t_0X})(Y)), \mathrm{Ad}(e^{tX})(\mathrm{Ad}(e^{t_0X})(Z))\rangle \\
&= \langle [X, \mathrm{Ad}(e^{t_0X})(Y)], \mathrm{Ad}(e^{t_0X})(Z)\rangle + \langle \mathrm{Ad}(e^{t_0X})(Y), [X, \mathrm{Ad}(e^{t_0X})(Z)]\rangle
\end{aligned}$$

proving that $\langle \mathrm{Ad}(e^{tX})(Y), \mathrm{Ad}(e^{tX})(Z)\rangle$ is constant in t. From this we conclude that for all $g \in G_0$ in a neighborhood of the identity $e \in G_0$, the adjoint action on $\mathfrak{g}$ is by isometries. It follows that all G_0 acts by isometries.

This means we can think of Ad as a homomorphism $\mathrm{Ad}\colon G_0 \to O(\mathfrak{g})$ where $O(\mathfrak{g})$ is the compact group of the isometries of $\mathfrak{g}$ (relative to the inner product $\langle\cdot,\cdot\rangle$). It is possible (although it's quite a bit of work and we won't do it. See Section 3.6 in [4] for details) to see that the image of Ad is a closed subgroup of $O(\mathfrak{g})$, call it H. H would be our solution if only Ad was injective at the infinitesimal level.

Let $\xi \subset \mathfrak{g}$ be the kernel of ad (the derivative of Ad), which is a Lie subalgebra with the trivial bracket, and $\xi^\perp$ its orthogonal space. $\xi^\perp$ is itself a Lie subalgebra since for $X, Y \in \xi^\perp$ and $Z \in \xi$:

$$0 = \langle [X,Y], Z\rangle + \langle Y, [X,Z]\rangle = \langle [X,Y], Z\rangle$$

([X,Z] being zero since $Z \in \xi$). Since $[\xi, \xi^\perp] = 0$, it follows that $\mathfrak{g} = \xi \oplus \xi^\perp$ (in the sense of Lie algebras), and that $\xi^\perp$ is the Lie algebra of H. ξ obviously is the Lie algebra of a torus T, and so we finally set

$$G = H \times T$$

□

Remark 3.7. The second half of the proof gives us G as a direct product of a quotient of G_0 (by its center) and a torus. This will not however be the most useful format of $\mathcal{G}$ for us. Instead we consider the induced surjective map

$$G_0 \to G = H \times T$$

from which we see that G is a quotient of G_0 by a discrete subgroup L **contained** in the center of G. Note in fact that, letting C be the center of G_0, from the proof is follows that any L such that C/L is compact will "do the trick".

3.2 First direction: A proper groupoid has an invariant triple

In this section we present the results we obtained in the one direction of the conjecture that is true (at least for fairly general proper Lie groupoids). The proof is split into three cases of increasing generality, that are not independent: each case will use the previous one.

3.2.1 Transitive proper groupoids

The easiest case is the case of transitive groupoids:

Proposition 3.8. *Let $\mathcal{G}$ be a proper and transitive Lie groupoid and let $\mathcal{A}$ be its Lie algebroid. Then there exists an invariant triple on $\mathcal{A}$.*

Proof.

- Since $\mathcal{G}$ is transitive it must be the case that ρ is surjective at all points of M and therefore $\mathrm{Ker}(\rho)$ is a subbundle of $\mathcal{A}$. Furthermore, although it is not possible to define the adjoint action of $\mathcal{G}$ on itself, it is possible to define the adjoint action of $\mathcal{G}$ on $\bigcup_{x \in M} \mathcal{G}_x$, the subgroupoid formed by the union of the isotropy groups, which has $\mathrm{Ker}(\rho)$ as Lie algebroid. So $\mathcal{G}$ has an adjoint action on $\mathrm{Ker}(\rho)$, and since $\mathcal{G}$ is proper that means a $\mathcal{G}$-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathrm{Ker}(\rho)$ exists, by Proposition (1.29).

- We extend $\langle \cdot, \cdot \rangle$ to $\mathcal{A}$ in any way (and represent the extension by the same symbol)

- For $X \in \Gamma(\mathcal{A})$ and $Y, Z \in \Gamma(\mathrm{Ker}(\rho))$ we therefore have:

$$\langle [X,Y], Z\rangle + \langle Y, [X,Z]\rangle = \rho(X)\cdot\langle Y, Z\rangle$$

  which is the intended formula in this case since whenever $Y \in \Gamma(\mathrm{Ker}(\rho))$ we must have $\nabla_X Y = [X,Y]$.

- In the case $X \in \Gamma(\mathcal{A})$, $Y \in \Gamma(\mathrm{Ker}(\rho))$ and $Z \in \Gamma(\mathrm{Ker}(\rho)^\perp)$ the equation (3.2) will be satisfied as long as we can choose ∇ such that ∇_X always sends $\Gamma(\mathrm{Ker}(\rho)^\perp)$ to itself.

- We therefore define ∇_X on $\Gamma(\mathrm{Ker}(\rho)^\perp)$ as any $\mathcal{A}$-connection on $\mathrm{Ker}(\rho)^\perp$ satisfying (3.2) for $Y, Z \in \Gamma(\mathrm{Ker}(\rho)^\perp)$. Such a connection certainly exists: on any local trivialization of $\mathcal{A}$ just chose an orthogonal basis of $\mathrm{Ker}(\rho)^\perp$ and have the Christoffel symbols be zero in that basis (that this suffices in this local setting is a consequence of the fact that the equation (3.2) is $C^\infty(M)$-linear). Then just glue this local connections using a partition of unity.

- ∇_X is therefore defined by the sum of its expression on $\Gamma(\mathrm{Ker}(\rho))$ and on $\Gamma(\mathrm{Ker}(\rho)^\perp)$

- It then suffices to see there exists a $\tilde\nabla$ such that (3.3) is satisfied. This is a direct consequence of $\rho\colon \mathrm{Ker}(\rho)^\perp \to TM$ being an isomorphism: define $\tilde\nabla$ as

$$\tilde\nabla_\alpha X = \nabla_X(\rho^{-1}(\alpha)) - [X, \rho^{-1}(\alpha)]$$

  That this is a connection follows from the following computations:

$$\begin{aligned}\tilde\nabla_{f\alpha} X = \nabla_X(f\rho^{-1}(\alpha)) - [X, f\rho^{-1}(\alpha)] = f\nabla_X(\rho^{-1}(\alpha))\\ + (X\cdot f)\rho^{-1}(\alpha) - f[X,\rho^{-1}(\alpha)] - (X\cdot f)\rho^{-1}(\alpha) = f\tilde\nabla_\alpha X\end{aligned}$$

$$\begin{aligned}\tilde\nabla_\alpha fX = \nabla_{fX}(\rho^{-1}(\alpha)) - [fX, \rho^{-1}(\alpha)] = f\nabla_X(\rho^{-1}(\alpha))\\ - f[X,\rho^{-1}(\alpha)] + (\rho(\rho^{-1}(\alpha))\cdot f)X = f\tilde\nabla_\alpha X + (\alpha\cdot f)X\end{aligned}$$

  The $\tilde\nabla$ thus defined satisfies (3.3) since for $Y \in \mathrm{Ker}(\rho)$ we already knew (3.3) to be true and for $\mathrm{Ker}(\rho)^\perp$ it follows directly by definition of $\tilde\nabla$.

□

Remark 3.9. In the previous proof, when choosing ∇_X in $\Gamma(\mathrm{Ker}(\rho)^\perp)$, we can make a slightly more careful choice so that when $X \in \Gamma(\mathrm{Ker}(\rho)^\perp)$ then ∇ is the

Levi-Civita connection (using the isomorphism $\rho\colon \mathrm{Ker}(\rho)^{\perp} \to TM$ to view the inner product on $\mathrm{Ker}(\rho)^{\perp}$ as an inner product on TM). This shows that the inner product can be chosen to satisfy the "torsion free property"

$$\rho(\nabla_X Y) - \rho(\nabla_Y X) = \rho([X, Y]).$$

3.2.2 Action groupoids (actions of transitive proper groupoids)

Using the previous case we can now prove the analogous result for any action groupoid corresponding to the action of a transitive and proper groupoid [1]:

Proposition 3.10. *Let $\mathcal{G}$ be a proper and transitive groupoid and $\mu\colon E \to M$ be a $\mathcal{G}$-space. Set $\mathcal{H} = \mathcal{G} \times_M E$ the action groupoid and let $\mathcal{B}$ be the corresponding Lie algebroid. Then there exists and invariant triple on $\mathcal{B}$.*

Proof. Denote as usual the Lie groupoid of $\mathcal{G}$ by $\mathcal{A}$.

We start by noting that for $e \in E$

$$\mathbf{s}_{\mathcal{H}}^{-1}(e) = \{(g, e)\colon \mathbf{s}(g) = \mu(e)\} \simeq \{g\colon \mathbf{s}(g) = \mu(e)\} = \mathbf{s}_{\mathcal{G}}^{-1}(\mu(e))$$

from which it follows that we have

$$\mathcal{B}_e \simeq \mathcal{A}_{\mu(e)}$$

or in other words, that

$$\mathcal{B} = \mu^*(\mathcal{A})$$

the pull-back of $\mathcal{A}$ by $\mu\colon E \to M$. Furthermore, since the projection on the first coordinate from $\mathcal{H} = \mathcal{G} \times_M E$ to $\mathcal{G}$ is a Lie groupoid morphism inducing the natural map from $\mathcal{B} = \mu^*(\mathcal{A})$ to $\mathcal{A}$, it follows that the pull-back of sections is compatible with both the Lie bracket and the anchors, that is, that:

$$\mu^*([\alpha, \beta]) = [\mu^*(\alpha), \mu^*(\beta)] \qquad \mathrm{d}\mu(\rho_{\mathcal{B}}(\mu^*(\alpha))) = \rho_{\mathcal{A}}(\alpha)$$

The rest of the proof is now very simple: we simply pull-back the $\langle \cdot, \cdot \rangle$, ∇ and $\tilde{\nabla}$ that we know to exist on $\mathcal{A}$ (we will denote the pull-back by the same symbols as the original to simplify notation). The pull-back of $\langle \cdot, \cdot \rangle$ is merely the usual pull-back of inner products. As for the pull back of the connections, those are simply the connections characterized by the following compatibility with pull-backs of sections, that is, for $\alpha, \beta, \in \Gamma(\mathcal{A})$, $X \in \Gamma(TE)$:

$$\nabla_{\mu^*(\alpha)} \mu^*(\beta) = \mu^*(\nabla_\alpha \beta), \qquad \tilde{\nabla}_X \mu^*(\beta) = \mu^*(\tilde{\nabla}_{\mathrm{d}\mu(X)} \beta)$$

[1] An action groupoid of a proper groupoid is proper. The proof is analogous to that of the fact that the action of any compact group is proper.

where the second equality uses slightly abusive notation: naturally $\mathrm{d}\mu(X)$ can not be expected to be a vector field. But at a point p the value of $\tilde{\nabla}_{\mathrm{d}\mu(X)}\beta$ is determined by $\mathrm{d}\mu(X)|_p$ and β, so that it is clear how to interpret the second equation.

That these expressions actually define unique connections is the content of a lemma we will present in a moment. But let's first see that the $\langle\cdot,\cdot\rangle$, ∇ and $\tilde{\nabla}$ so defined are as intended.

As we noted before, (3.2) and (3.3) are compatible with the multiplication of its arguments by functions, so that it suffices to verify the equations for a basis of the sections of $\mathcal{B}$, which means that it suffices to consider pull-backs of sections of $\mathcal{A}$.

$$\begin{aligned}\langle\nabla_{\mu^*(\alpha)}\mu^*(\beta),\mu^*(\gamma)\rangle + \langle\mu^*(\beta),\nabla_{\mu^*(\alpha)}\mu^*(\gamma)\rangle &= \\ &= \langle\mu^*(\nabla_\alpha\beta),\mu^*(\gamma)\rangle + \langle\mu^*(\beta),\mu^*(\nabla_\alpha\gamma)\rangle \\ &= (\langle\nabla_\alpha\beta,\gamma\rangle + \langle\beta,\nabla_\alpha\gamma\rangle)\circ\mu \\ &= (\rho(\alpha)\cdot\langle\beta,\gamma\rangle)\circ\mu \\ &= \rho(\mu^*(\alpha))\cdot\langle\mu^*(\beta),\mu^*(\gamma)\rangle\end{aligned}$$

$$\begin{aligned}\nabla_{\mu^*(\alpha)}\mu^*(\beta) = \mu^*(\nabla_\alpha\beta) &= \mu^*([\alpha,\beta] + \tilde{\nabla}_{\rho(\beta)}\alpha) \\ &= [\mu^*(\alpha),\mu^*(\beta)] + \tilde{\nabla}_{\rho(\mu^*(\beta))}\mu^*(\alpha)\end{aligned}$$

□

The following lemma is the existence of "pull-backs" of connections that was used above:

Lemma 3.11 (Pull-back of connections)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be algebroids over M and N, respectively, and $F\colon \mathcal{B}\to\mathcal{A}$ (with $f\colon N\to M$ the corresponding map between the bases). Further, let E be a vector bundle over M on which there is a $\mathcal{A}$-connection ∇. Then there exists a unique $\mathcal{B}$-connection $\tilde{\nabla}$ on $f^*(E)$ satisfying:*

$$\tilde{\nabla}_X f^*(\alpha) = f^*(\nabla_{F(X)}\alpha) \quad \text{for all } X\in\Gamma(\mathcal{B}),\alpha\in\Gamma(E)^2 \tag{3.12}$$

Proof. It is obvious that only one such connection can exist, since (3.12) specifies the value of $\tilde{\nabla}_X$ for all sections that are pull-backs, and these are a (local) basis of the sections. Since, once given a basis of sections, giving a connection is the same as giving the Christoffel symbols, it follows that (3.12) will define a

[2]Where the remarks about how the right hand side should be interpreted made during the previous proof still apply.

connection as long as for any fixed (local) basis $\alpha_1,\dots,\alpha_k$ (of $\mathcal{A}$) and any other section β the value of $\tilde{\nabla}_X f^*(\beta)$ is the same when computed via (3.12) as via the Christoffel symbols. Writing $\beta = \sum g_i\alpha_i$:

$$\begin{aligned}
\tilde{\nabla}_X f^*(\beta) &= \tilde{\nabla}_X f^*(\sum g_i\alpha_i) \\
&= \tilde{\nabla}_X \sum (g_i \circ f) f^*(\alpha_i) \\
&= \sum ((g_i \circ f)\tilde{\nabla}_X f^*(\alpha_i) + (\rho(X)\cdot(g_i\circ f))f^*(\alpha_i)) \\
&= \sum ((g_i \circ f) f^*(\nabla_{F(X)}\alpha_i) + ((\mathrm{d}f(\rho(X))\cdot g_i)\circ f)f^*(\alpha_i)) \\
&= \sum f^*(g_i\nabla_{F(X)}\alpha_i + ((\rho(F(X)))\cdot g_i)\alpha_i) \\
&= f^*(\sum \nabla_{F(X)}(g_i\alpha_i)) \\
&= f^*(\nabla_{F(X)}\beta)
\end{aligned}$$

□

Remark 3.13. In the previous lemma the Lie bracket structure on $\mathcal{A}$ and $\mathcal{B}$ was not necessary. The lemma holds for vector bundles with anchor maps and a vector bundle map F compatible with those.

3.2.3 The (fairly) general case

Finally, we finish proving the direction we have been working on in a fairly general case. The crucial ingredient is the following theorem from Zung ([8]) saying that under some conditions proper groupoids can be locally linearized, that is, are isomorphic to the groupoid of a linear action (of a transitive groupoid).

Theorem 3.14. *Let $\mathcal{G}$ be a Hausdorff source-locally trivial proper Lie groupoid over M, and $\mathcal{O}$ an orbit of finite type [3]. Then there is an invariant neighborhood $\mathcal{U}$ of $\mathcal{O}$ in M such that the restriction $\mathcal{G}_{\mathcal{U}}$ of $\mathcal{G}$ to $\mathcal{U}$ is isomorphic to the action groupoid $\mathcal{G}_{\mathcal{O}} \times \nu(\mathcal{O})$.*

Although we will not present a proof of this result here, we describe the action of $\mathcal{G}_{\mathcal{O}}$ on the normal bundle $\nu(\mathcal{O})$ of $\mathcal{O}$ in M:

For any $g\colon x \to y$ in $\mathcal{G}_{\mathcal{O}}$ choose any local bisection b (Definition 1.9) such that $b(x) = g$. Then the action is defined by:

$$g\cdot[v] = [\mathrm{d}_x(\mathbf{t}\circ b)(v)]$$

(where $[v] \in \nu_x(\mathcal{O})$ is the class of $v \in T_xM$)

[3]A manifold N is said to be of finite type if there exists a proper map $f\colon N \to \mathbb{R}$ having only finite critical points.

Proposition 3.15. *The action described above is well defined.*

Proof. It suffices to prove that the action is well defined (that is, does not depend on the bisection chosen) for all identities 1_x, for which the action must be the identity map. Now notice that we have

$$T_xM = T_x\mathcal{O} \oplus \nu_x(\mathcal{O}) \qquad T_x\mathcal{G} = T_x\mathbf{s}^{-1}(x) \oplus T_xM$$

(where in the first expression we choose the representatives of $\nu_x(\mathcal{O})$ arbitrarily and in the second expression we use x as 1_x and M as its image under the unit map u.)

Now let $b\colon M \to \mathcal{G}$ be any (local) bisection such that $b(x) = 1_x$ and $\mathbf{s} \circ b$ is the identity map. It follows from the second condition that for any $v \in T_xM$ we have

$$\mathrm{d}b(v) = w \oplus v$$

for some $w \in T_x\mathbf{s}^{-1}(x)$.

On the other hand, we have that $\mathrm{d}\mathbf{t}(T_x\mathbf{s}^{-1}(x)) = T_x\mathcal{O}$ and that $\mathrm{d}\mathbf{t}$ is the "identity" on T_xM (since $\mathbf{t} \circ u$ is the identity), so that

$$\mathrm{d}(\mathbf{t} \circ b)(v) = \mathrm{d}\mathbf{t}(\mathrm{d}b(v)) = \mathrm{d}\mathbf{t}(w \oplus v) = v + r$$

for $r \in T_x\mathcal{O}$, which finishes the proof. □

We now use Theorem 3.14 to prove that an invariant triple does exist when the theorem is applicable to all orbits. This will merely be a matter of using the result for action groupoids and "gluing" the inner products and connections obtained using an **invariant** partition of unity. The existence of such a partition of unity is the content of the following proposition:

Proposition 3.16. *Let $\mathcal{G}$ be a Hausdorff source-locally trivial proper Lie groupoid over M, such that all orbits have finite type. Then given a cover $\{U_\alpha\}$ by open invariant sets there exists a subordinate partition of unity $\{\tau_\alpha\}$ by functions constant on orbits.*

Proof. The proof will essentially consist of proving that the orbit space $M/\mathcal{G}$ is paracompact, so that all covers by invariant sets have locally finite refinements. A well known sufficient condition for paracompactness is that the topological space be regular and Lindelof (Theorem 41.5, [6]). The second condition is immediate because M is second countable and the quotient map is open (Corollary 1.12), and this implies that $M/\mathcal{G}$ is also second countable (the countable basis for the topology being given as the image of the countable basis for the topology of M by the quotient map). We now prove regularity:

Let $\mathcal{O}$ be a orbit, ν its normal bundle, which we identify with the neighborhood $\mathcal{U}$ in Theorem 3.14. We now choose an inner product γ on ν invariant by the action of $\mathcal{G}|_{\mathcal{O}}$. We claim that for any (finite) ϵ, the set B_ϵ of the vectors of length less than or equal to ϵ is a closed set of M [4]:

Let v_i be a sequence in B_ϵ convergent to, say, v. Fix any $x \in \mathcal{O}$ and write $v_i = g_i \cdot w_i$ for w_i belonging to $B_{\epsilon,x}$, which we can assume have a limit w, and g_i in $\mathcal{G}|_{\mathcal{O}}$ (which exist because $\mathcal{G}|_{\mathcal{O}}$ is transitive). Then the set $S = \{(v_i, w_i)\}$ is contained in a compact set, since it has a limit, and since $\{g_i\} \subset (\mathbf{s} \times \mathbf{t})^{-1}(S)$ we have by properness that the g_i are contained in a compact set, and therefore have a limit g. But then it must be $v = g \cdot w$, so that v is in B_ϵ, as intended, proving B_ϵ closed.

This shows, firstly, that orbits are closed, or in other words, that points in $M/\mathcal{G}$ are closed, and secondly that $M/\mathcal{G}$ is indeed regular, since any invariant closed set not intersecting $\mathcal{O}$ will not intersect B_ϵ for small ϵ (This obviously must be the case for every fiber ν_x, that is, it must be true that the open set doesn't intersect $B_{\epsilon,x}$ for small enough ϵ. But then invariance implies that the open set does not intersect B_ϵ either).

Now take a cover of M by open sets of the form ν_α, one for each orbit, and give each vector bundle ν_α an invariant inner product γ_α (with respect to the action of $\mathcal{G}|_{\mathcal{O}_\alpha}$). Let also V_α be the cover corresponding to the vectors of modulo less than 1. Since this cover (as well as the first) corresponds to an open cover of $M/\mathcal{G}$ one can take a locally finite saturated refinement W_α. By the "shrinking lemma" (see for instance Lemma 41.6 of [6]) one can take a further saturated refinement Z_α such that closure$(Z_\alpha) \subset W_\alpha$. Now pick any fiber ν_x of each ν_α, and in that fiber define a non negative function being strictly positive on $\nu_x \cap Z_\alpha$ and supported on $\nu_x \cap W_\alpha$. Average the function by the isotropy group of x (which is compact) and extend the function to all ν_α by invariance, thus obtaining an invariant non negative function f_α. Then $\sum f_\alpha$ is well defined since the f_α are supported on a locally finite cover and is a strictly positive function since the W_α cover M. We then have that $\tau_\beta = \frac{f_\beta}{\sum f_\alpha}$ is the desired partition of unity constant on orbits for the special cover ν_α.

For a general cover, a standard trick is used. From the initial invariant cover $\{U_\alpha\}_{\alpha \in A}$ one constructs a refinement $\{V_\beta\}_{\beta \in B}$ of open sets of the form above, and fixes a function $f \colon B \to A$ between the sets of indexes such that $V_\beta \subset U_{f(\beta)}$. Letting $\{\pi_\beta\}_{\beta \in B}$ be the partition of unity subordinate to the V_β the partition τ_α is obtained by $\tau_\alpha = \sum_{\beta \in f^{-1}(\alpha)} \pi_\beta$.

□

[4] Although it is obviously closed in $\nu \subset M$, closedness in M requires the properness of the groupoid.

The result that follows is now a straight-forward computation:

Proposition 3.17. *Let $\mathcal{G}$ be a Hausdorff source-locally trivial proper Lie groupoid over M, such that all orbits have finite type, and $\mathcal{A}$ its Lie algebroid. Then there exists and invariant triple on $\mathcal{A}$.*

Proof. We take a cover of M by open sets of the form ν_α for which we have inner products $\langle\cdot,\cdot\rangle_\alpha$, connections ∇^α and $\tilde{\nabla}^\alpha$ and an invariant partition of unity τ_α. We simply define:

$$\langle\cdot,\cdot\rangle = \sum \tau_\alpha \langle\cdot,\cdot\rangle_\alpha \qquad \nabla = \sum \tau_\alpha \nabla^\alpha \qquad \tilde{\nabla} = \sum \tau_\alpha \tilde{\nabla}^\alpha$$

(where the $\langle\cdot,\cdot\rangle_\alpha$, ∇^α and $\tilde{\nabla}^\alpha$ receive as arguments the restrictions of the arguments of $\langle\cdot,\cdot\rangle$, ∇ and $\tilde{\nabla}$)

All we have to do is check (3.2) and (3.3):

$$\begin{aligned}
\rho(X)\cdot\langle Y,Z\rangle &= \rho(X)\cdot\left(\sum \tau_\alpha\langle Y,Z\rangle_\alpha\right) = \\
&= \sum\left(\tau_\alpha(\rho(X)\cdot\langle Y,Z\rangle_\alpha) + (\rho(X)\cdot\tau_\alpha)\langle Y,Z\rangle_\alpha\right) \\
&= \sum \tau_\alpha(\rho(X)\cdot\langle Y,Z\rangle_\alpha) \qquad\qquad \text{(since } \tau_\alpha \text{ is invariant)} \\
&= \sum \tau_\alpha(\langle\nabla^\alpha_X Y,Z\rangle + \langle Y,\nabla^\alpha_X Z\rangle) \\
&= \left\langle\sum(\tau_\alpha\nabla^\alpha_X Y),Z\right\rangle + \left\langle Y,\sum(\tau_\alpha\nabla^\alpha_X Z)\right\rangle \\
&= \langle\nabla_X Y,Z\rangle + \langle Y,\nabla_X Z\rangle
\end{aligned}$$

$$\begin{aligned}
\nabla_X Y &= \sum \tau_\alpha \nabla^\alpha_X Y \\
&= \sum \tau_\alpha([X,Y] + \tilde{\nabla}^\alpha_{\rho(Y)} X) \\
&= \sum \tau_\alpha[X,Y] + \sum \tau_\alpha \tilde{\nabla}^\alpha_{\rho(Y)} X \\
&= [X,Y] + \tilde{\nabla}_{\rho(Y)} X
\end{aligned}$$

□

Remark 3.18. Note that in our proofs we only used the conclusion of Theorem 3.14, so that the results we proved will hold for any proper groupoid such that that conclusion is true.

3.3 Second direction: Invariant triple implies proper groupoid

In this section we present the results we obtained concerning the second direction of our conjecture. Unfortunately, the conjecture no longer holds in general. It does hold for transitive groupoids, as we see in the first subsection, but one needs to assume that the transitive groupoid is already integrable, as we see in the second subsection. Finally, in the third and last subsection we present three integrable counter examples of a distinct nature, that suggest that conditions of a different nature must be necessary to guarantee the existence of a proper integrating groupoid.

3.3.1 Transitive algebroids

As happened with the first direction, transitive algebroids are easier to work with, and indeed the conjecture is true for these (for integrable algebroids):

Proposition 3.19. *Let $\mathcal{A}$ be a* **transitive** *and* **integrable** *Lie algebroid for which there exists an invariant triple. Then $\mathcal{A}$ is the Lie algebroid of a* **proper** *groupoid $\mathcal{G}$.*

This will follow from the following stronger proposition, noticing that the existence in $\mathcal{A}$ of an invariant triple guarantees that all the isotropy algebras $\mathfrak{g}_x(\mathcal{A})$ satisfy the condition of Theorem 3.5.

Proposition 3.20. *Let $\mathcal{A}$ be a* **transitive** *and* **integrable** *Lie algebroid. Then $\mathcal{A}$ is the Lie algebroid of a transitive* **proper** *groupoid $\mathcal{G}$ if and only if for some (and therefore any) of the Lie algebras $\mathfrak{g}_x(\mathcal{A})$ there exists an inner-product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}_x(\mathcal{A})$ such that, for all $X, Y, Z \in \mathfrak{g}_x(\mathcal{A})$, we have*

$$\langle [X,Y], Z\rangle + \langle Y, [X,Z]\rangle = 0 \qquad (3.21)$$

Proof. One of the directions was of course proved in the previous section.

For the other one, let $\mathcal{G}^0$ be any groupoid integrating $\mathcal{A}$. Let $\mathfrak{g}_x(\mathcal{A})$ be the Lie algebra over some point $x \in M$, that is the Lie algebra of $\mathcal{G}^0_x$ (or the kernel of ρ restricted to the fiber of $\mathcal{A}$ over x). (3.21) tells us that $\mathfrak{g}_x(\mathcal{A})$ can be integrated by a compact group, from which it follows, from Remark 3.7, that there exists a discrete subgroup L_x of the center of $\mathcal{G}^0_x$ such that $\mathcal{G}^0_x/L_x$ is compact [5]. Let L the set of the orbits of L_x by the adjoint action of $\mathcal{G}$ on $\bigcup_{x\in M}\mathcal{G}_x$ (this is

[5]Remark 3.7 tells us this is true for the simply connected version G of $\mathcal{G}^0_x$. But $\mathcal{G}^0_x$ is a quotient of G by a subgroup of its center, and by Remark 3.7 a larger subgroup L such that G/L is compact exists.

supposed to be some sort of "normal" subgroupoid generated by L_x). It is clear that L is a subgroupoid of $\bigcup_{x\in M}\mathcal{G}_x$, and that its fiber over x is just L_x, as a consequence of L_x being contained in the center of $\mathcal{G}^0_x$. In fact, any fiber L_y of L over any y will be contained in the center of $\mathcal{G}^0_y$ and $\mathcal{G}^0_y/L_y$ will be compact. That L is indeed smooth can be proved copying the proof of Theorem 1.30, which also shows that the subset of the identities forms one of the connected components of L (this condition is the groupoid equivalent of a subgroup being discrete in the case of groups). From this it follows that

$$\mathcal{G} = \mathcal{G}^0/L$$

is a groupoid with the same Lie algebroid as $\mathcal{G}^0$, and which is proper by Theorem 1.30.

□

Remark 3.22. It should be noted that in this case considering connections in order to compensate the fact that the Lie bracket does not provide an action of the Lie algebroid on itself is unnecessary, since the result we proved shows that all that matters is the bracket at any of the isotropy Lie algebras.

3.3.2 Integrability

In this section we see that the requirement used in the last section that the algebroid be integrable is necessary.

To see this we will show that for all the algebroids $\mathcal{A}_\omega$ associated with two-forms ω (Example 2.16) invariant triples do exist.

To see this, start by choosing any inner product $\langle\cdot,\cdot\rangle$ on TM, and extend it to an inner product on $\mathcal{A}_\omega$ by:

$$\langle (X, f_X), (Y, f_Y)\rangle = \langle X, Y\rangle + f_X f_Y$$

(we denote the inner product on $\mathcal{A}_\omega$ by the same symbol to simplify notation)

The connection $\tilde{\nabla}$ of TM on $\mathcal{A}_\omega$ is then defined as:

$$\tilde{\nabla}_Y(X, f_X) = (\tilde{\nabla}_Y X, Y\cdot f_X + \omega(Y, X))$$

(where the $\tilde{\nabla}$ on the right-hand side is the Levi-Civita connection of the inner product on TM and we once again commit a slight abuse to simplify notation) This is indeed a connection:

$$\tilde{\nabla}_{gY}(X, f_X) = (\tilde{\nabla}_{gY}X, g(Y\cdot f_X) + g(\omega(Y, X))) = g\tilde{\nabla}_Y(X, f_X)$$

$$\tilde{\nabla}_Y g(X, f_X) = (\tilde{\nabla}_Y(gX), Y\cdot(gf_X)+g(\omega(Y,X))) = g\tilde{\nabla}_Y(X, f_X)+(Y\cdot g)(X, f_X)$$

With these choices we have:

$$\begin{aligned}\nabla_{(X,f_X)}(Y,f_Y) &= [(X,f_X),(Y,f_Y)] + \tilde{\nabla}_{\rho(Y,f_Y)}(X,f_X)\\ &= ([X,Y], X\cdot f_Y - Y\cdot f_X + \omega(X,Y)) + \tilde{\nabla}_Y(X,f_X)\\ &= ([X,Y], X\cdot f_Y - Y\cdot f_X + \omega(X,Y)) + (\tilde{\nabla}_Y X, Y\cdot f_X + \omega(Y,X))\\ &= (\tilde{\nabla}_X Y, X\cdot f_Y)\end{aligned}$$

(where in the last step we have used the fact that $\tilde{\nabla}$ is a Levi-Civita connection)

And hence (3.2) follows:

$$\begin{aligned}\langle\nabla_{(X,f_X)}(Y,f_Y),(Z,f_Z)\rangle + \langle(Y,f_Y),\nabla_{(X,f_X)}(Z,f_Z)\rangle\\ &= \langle(\tilde{\nabla}_X Y, X\cdot f_Y),(Z,f_Z)\rangle + \langle(Y,f_Y),(\tilde{\nabla}_X Z, X\cdot f_Z)\rangle\\ &= \langle\tilde{\nabla}_X Y, Z\rangle + f_Z(X\cdot f_Y) + \langle Y,\tilde{\nabla}_X Z\rangle + f_Y(X\cdot f_Z)\\ &= X\cdot\langle Y,Z\rangle + X\cdot f_Y f_Z\\ &= X\cdot\langle(Y,f_Y),(Z,f_Z)\rangle\\ &= \rho(X,f_X)\cdot\langle(Y,f_Y),(Z,f_Z)\rangle\end{aligned}$$

Since there are algebroids of the form $\mathcal{A}_\omega$ that are not integrable (as we mentioned before), this shows that the existence of an invariant triple does not guarantee integrability even in the transitive case for which we know that the condition plus integrability is equivalent to the existence of a proper integrating groupoid.

3.3.3 Counter-examples

In this section we present three (essentially distinct) integrable counter examples that our conjecture, implying that it must be corrected (if possible), for it to hold.

Before presenting the counter examples, however, we note that to determine whether an integrable algebroid has an integrating proper groupoid it may not suffice to try to find a proper quotient of the integrating **s**-simply connected groupoid. To this effect we present a proper groupoid $\mathcal{G}$ for which the **s**-connected component $\mathcal{G}^0$ is not proper[6]:

[6] Note that the reason why in the case of groups and algebras we can restrict the search of an integrating compact group to quotients of the simply connected integrating one is because if G is compact then G^0 also is.

Example 3.23. Start by considering the manifold

$$N = \mathbb{R} \times \mathbb{P}^2$$

on which we consider the foliation $\mathcal{F}$ by vertical spaces (that is, the leaves are copies of $\mathbb{P}^2$). Let

$$\mathcal{H} = \Pi_1(\mathcal{F})$$

be the fundamental groupoid of this foliation. This is clearly a proper groupoid (and Hausdorff too). Now set

$$M = \mathbb{R} \times \mathbb{P}^2 - \{0\} \times \mathbb{P}^1$$

$$\mathcal{G} = \mathcal{H}|_M$$

$\mathcal{G}$ is of course still proper (and Hausdorff) , being a restriction of a proper groupoid. It is however no longer $\mathbf{s}$-connected: letting $\pi_1 \colon M \to \mathbb{R}$ be the projection on the first coordinate, we have that $\mathcal{G}|_{\pi_1^{-1}(0)}$ is isomorphic to $\Pi_1(\mathbb{R}^2) \times \mathbb{Z}_2$, so that for $x \in \pi_1^{-1}(0)$ we have that $\mathbf{s}^{-1}(x)$ now has two connected components. To obtain $\mathcal{G}^0$ one simply removes $\Pi_1(\mathbb{R}^2) \times \{1\}$. This results in $\mathcal{G}^0$ not being proper, since the union of the isotropies over $[0,1] \times \{p\}$, for p any element of $\mathbb{P}^2 - \mathbb{P}^1$, is diffeomorphic $[0,1] \sqcup]0,1]$, and properness would require it to be compact.

Do note however that the $\mathcal{G}^0$ above has a quotient that is proper, namely the quotient by the subgroupoid $L = \bigcup_{x \in M} \mathcal{G}^0_x$ formed by all the isotropy groups. This means that this example does not invalidate the possibility that to study the existence of an integrating proper groupoid it may suffice to look at quotients of the $\mathbf{s}$-simply connected integrating one. Whether this is the case or not is certainly an important question that must be answered before the problem of integrability by proper groupoids can be given a definite answer.

We now present the counter examples:

Example I

The first counter example results from the fact that the properness of a groupoid implies certain properties about the orbits $\mathcal{O}_x$. Since those orbits are however essentially determined by the the algebroid (Proposition 2.5), only algebroids with adequate orbits can be algebroids of proper groupoids.

One such property is the fact that the orbits of a proper groupoid must be closed.

For such an example consider the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with one of its irrational foliations $\mathcal{F}$, that is, any of the foliations induced by a foliation of $\mathbb{R}$ by parallel lines of equations $x + \lambda y = c$ (where the λ a fixed irrational number for all the lines). There exists an invariant triple for the distribution associated to this foliation: simply consider the standard metric on the torus and the corresponding Levi-Civita connection (restricted to $\mathcal{F}$).

However, since the orbits of this distribution are not closed, this implies it can not be integrated by a proper $\mathbf{s}$-connected groupoid by Proposition 2.5. To see no integrating proper groupoid can exist at all, suppose we had one, say, $\mathcal{G}$. For any $x \in M$, the orbit $\mathcal{O}_x$ is $\mathcal{O}_x = \mathbf{t}(\mathbf{s}^{-1}(x))$. Each of the connected components of $\mathbf{s}^{-1}(x)$ can, by right translation (by the inverse of any of its elements), be transformed into a $\mathbf{s}$-fiber of $\mathcal{G}^0$, and therefore its image by $\mathbf{t}$ is one of the leafs of $\mathcal{F}$. But second countability implies $\mathbf{s}^{-1}(x)$ has countably many components, so that $\mathcal{O}_x$ will be the union of countably many leafs, and can not therefore be closed.

Example II

The next counter example explores the following fact: If $\mathfrak{g}$ is a Lie algebra and G is a compact connected group which integrates $\mathfrak{g}$, then G is a quotient of $\tilde{G}$, its simply connected form, by a discrete subgroup containing a sublattice of maximum rank of the maximal torus of $\tilde{G}$.

Now let $\mathcal{A}$ be an algebroid integrable by a proper groupoid $\mathcal{G}$ and denote by $\tilde{\mathcal{G}}$ the $\mathbf{s}$-simply connected integrating groupoid. One has that $\mathcal{G}^0$, which is a quotient of $\tilde{\mathcal{G}}$ by Theorem 1.40, although not necessarily proper, must have compact isotropy groups, since any isotropy group $\mathcal{G}^0_x$ is given as the union of some of the components of $\mathcal{G}_x$, and since the second is compact, so is the first.

That means that for such an $\mathcal{A}$ it must be possible to find a subgroupoid L of $\tilde{\mathcal{G}}$ contained in $\bigcup_{x\in M} \tilde{\mathcal{G}}_x$ and such that $L \cap \tilde{\mathcal{G}}_x$ is a discrete subgroup containing a sublattice of maximum rank of the maximal torus of $\tilde{\mathcal{G}}_x$. This counter example explores the difficulty resulting from the possibility that the dimension of the centers may vary, which may make it impossible for the union of such subgroups to be a smooth submanifold.

The counter example is the Lie algebra bundle $\mathcal{A} = \mathbb{R} \times \mathbb{R}^3$ over $\mathbb{R}$ (its first factor) with the bracket

$$[v, w]_\epsilon = \epsilon \cdot (v \times w),$$

where $\times$ denotes the cross product of vectors in $\mathbb{R}^3$.

The $\mathbf{s}$-simply connected groupoid integrating this algebroid is a Lie group bundle with the fibers over $\epsilon \neq 0$ isomorphic to the group of unitary quaternions, which topologically is $\mathbb{S}^3$ and has only two points in its center, while the fiber over zero is just $\mathbb{R}^3$ with addition. It is then clear that it is impossible to choose a smooth subgroupoid in the conditions above, while clearly an invariant triple does exists by taking the inner product which is the standard inner product on $\mathbb{R}^3$ on each fiber (any TM-connection on $\mathcal{A}$ will do) [7].

Example III

This final counter example explores the impossibility to guarantee the existence of a subgroupoid of the **s**-simply connected groupoid in the conditions mentioned in the previous example even when the centers all have the same dimension, simply because the vector bundle of the centers of the Lie algebras might not admit the necessary lattices.

Consider the Lie algebroid $T\mathbb{S}^2$ with zero anchor and Lie bracket (i.e., we view $T\mathbb{S}^2$ as a bundle of abelian Lie algebras). This algebroid is the **s**-simply connected groupoid integrating itself. Suppose now the desired subgroupoid did exist, which in this case would just be a smooth family of lattices of rank 2. This family would then be a covering of $\mathbb{S}^2$. Since $\mathbb{S}^2$ is simply connected, the covering would just be a disjoint union of manifolds diffeomorphic to $\mathbb{S}^2$ itself. But then any of those copies other than the one corresponding to the zeros of the fibers would give us a vector field on $\mathbb{S}^2$ without zeros, which is a contradiction. Notice that of course invariant triples do exist: in fact, since the bracket and the anchor are both identically zero any inner product and TM-connection will work.

Remark 3.24. Notice that since this last example has zero Lie bracket and anchor it seems that no alternative condition involving inner products that one might have chosen in the conjecture would fail to be true for this algebroid, so that it seems that conditions of a different nature are necessary to account for this particular counter example.

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[7]Another argument for why this algebroid can not be integrated by a proper groupoid $\mathcal{G}$ is that, if that were the case, the source (or target, since they are the same) map would be a proper submersion, hence it would be a locally trivial fibration, and therefore $\mathcal{G}_0$ and $\mathcal{G}_\epsilon$ would be diffeomorphic for small enough ϵ. But this is impossible since the universal cover of $\mathcal{G}_0$ is $\mathbb{R}^3$ and the universal cover of $\mathcal{G}_\epsilon$ is $\mathbb{S}^3$.

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