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SYMPLECTIC FOLIATIONS, CURRENTS, AND LOCAL LIE GROUPOIDS

BY

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DISSERTATION

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Abstract

This thesis treats two main topics: calibrated symplectic foliations, and local Lie groupoids. Calibrated symplectic foliations are one possible generalization of taut foliations of 3-manifolds to higher dimensions. Their study has been popular in recent years, and we collect several interesting results. We then show how de Rham's theory of currents, and Sullivan's theory of structure currents, can be applied in trying to understand the calibratability of symplectic foliations.

Our study of local Lie groupoids begins with their definition and an exploration of some of their basic properties. Next, three main results are obtained. The first is the generalization of a theorem by Mal'cev. The original theorem characterizes the local Lie groups that are part of a (global) Lie group. We give the corresponding result for local Lie groupoids. The second result is the generalization of a theorem by Olver which classifies local Lie groups in terms of Lie groups. Our generalization classifies, in terms of Lie groupoids, those local Lie groupoids that have integrable algebroids. The third and final result demonstrates a relationship between the associativity of a local Lie groupoid, and the integrability of its algebroid. In a certain sense, the monodromy groups of a Lie algebroid manifest themselves combinatorially in a local integration, as a lack of associativity.

to my parents

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Introduction

This thesis addresses two main topics: calibrations of symplectic foliations, and the associativity of local Lie groupoids. These topics are related to each other through the study of (symplectic) Lie groupoids. On the one hand, symplectic foliations are precisely regular Poisson structures, and Poisson manifolds are the infinitesimal counterpart to symplectic Lie groupoids. On the other hand, local Lie groupoids generalize Lie groupoids.

Let us sketch the broad context in which these topics are studied, and give a brief summary of the main results in this thesis. At the end of this introduction, we will outline the organization of this thesis.

Calibrated symplectic foliations

A symplectic foliation is, roughly speaking, a foliation whose leaves are equipped with a symplectic structure that varies smoothly from leaf to leaf. Alternatively, one can consider a symplectic foliation as a regular Poisson structure. One interesting problem about symplectic foliations is:

Which manifolds admit a symplectic foliation of a given dimension?

For open manifolds, the answer is understood, in the sense that there is an h -principle governing the existence of symplectic foliations [11]. For closed manifolds, however, this problem is very hard, and understanding is limited. Indeed, in codimension 0 the problem reduces to the existence of symplectic structures on closed manifolds, known to be a tough problem. In [2], an example is given of a foliation that does not admit a symplectic structure despite all basic obstructions vanishing. The thesis [28] is dedicated to the study of symplectic foliations in codimension 1.

Because the class of symplectic foliations is large, it is natural to focus attention on a more restricted type of symplectic foliation. One suitable class of structures is that

of *calibrated* symplectic foliations: one requires the existence of a closed 2-form on the manifold that restricts to the given leafwise symplectic form on the foliation. (This condition known by different names in the literature. In [32], these symplectic foliations are called *2-calibrated* to stress that the extra condition on the symplectic foliation is the existence of a certain 2-form. Another common name is that of a symplectic foliation *with a closed extension*, to stress that the extra condition on the symplectic foliation is the existence of a certain closed form.) The question above then leads to the following problem:

Which manifolds admit a calibrated symplectic foliation of a given dimension?

It is the class of calibrated symplectic foliations, and the existence problem just formulated, that chapters 2 and 3 will focus on.

An important motivation for studying calibrated symplectic foliations is the hope that they can provide a suitable generalization of *tautness* of a foliation to higher dimensions. Taut foliations are a special class of codimension-1 foliations that have been studied extensively, particularly in dimension 3. In dimension m , tautness of a foliation amounts to the existence of a closed $(m - 1)$ -form whose restriction to each leaf is non-vanishing. The existence of a taut foliation on a 3-manifold has consequences for the topology of this manifold (see section 1.1). While the study of taut foliations has been very successful on 3-manifolds, they are not as useful in higher dimensions. In particular, the h -principle proved in [22] shows that a compact manifold of dimension at least 4 always admits a taut foliation, given that it admits a codimension-1 foliation. Thus taut foliation lose much of their usefulness because they become too flexible starting in dimension 4. A codimension-1 foliation in dimension 3 is taut precisely if it admits a calibration, and starting in dimension 5, admitting a calibration is a strictly stronger condition than being taut (there are no symplectic codimension-1 foliations in dimension 4). The more limited flexibility of calibrated symplectic foliations compared to taut foliations is exactly analogous to the more limited flexibility of symplectomorphism compared to volume-preserving maps (as exemplified, for example, by Gromov's non-squeezing theorem [13]). The hope, therefore, is that calibrated symplectic foliations are the correct way to generalize tautness in dimension 3 to higher dimensions, and retain sufficient rigidity to be able to deduce interesting consequences about the topology of the ambient manifold.

Associativity of local Lie groupoids

When the theory of Lie groups and Lie algebras was first developed, Lie groups were studied locally, for no suitable notion of manifold was available ([4, pages 286–302]). Once the modern notion of a Lie group has been established, then, it is natural to ask whether every local Lie group is contained in a Lie group. Perhaps surprisingly, the answer is “no”. While it is true that every local Lie group has a neighborhood of the identity that is contained in a Lie group, this is not necessarily true for the entire local Lie group. We say that a local Lie group is *globalizable* if it is contained in a Lie group. The obstruction to globalizability of a local Lie group has long been understood: a theorem by Mal’cev proves that a local Lie group is globalizable precisely if it is *globally associative* [25].

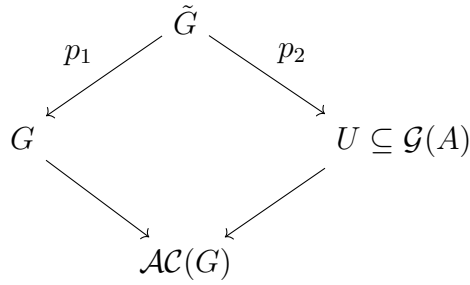
In [27], Olver re-proves Mal’cev’s result, and describes a construction that, starting from Lie groups, can be used to obtain any local Lie group. This effectively gives a sort of classification of local Lie groups. Roughly speaking, every local Lie group is covered by a cover of a globalizable local Lie group. One can wonder whether these results by Mal’cev and Olver have corresponding generalizations to local Lie groupoids.

As shown in chapters 5 and 6 of this thesis, corresponding results do indeed exist. Mal’cev’s theorem follows as a corollary of the following result. In this statement, the associative completion $\mathcal{AC}(G)$ is a groupoid that can be naturally built for any local Lie groupoid G .

Theorem A. *If G is a bi-regular local Lie groupoid with products connected to the axes, then $\mathcal{AC}(G)$ is smooth if and only if $\text{Assoc}(G)$ is uniformly discrete in G (and in that case, $G \rightarrow \mathcal{AC}(G)$ is a local diffeomorphism).*

Olver’s classification also goes through (and we strengthen it slightly). Note that Olver’s construction can only classify those local Lie groupoids whose Lie algebroid is integrable, because the construction to build local Lie groupoids starts from a (global) Lie groupoid, and therefore necessarily from an integrable algebroid. The statement of the classification is as follows.

Theorem B. *Suppose G is a bi-regular s -connected local Lie groupoid over M with integrable Lie algebroid A . Write \tilde{G} for its source-simply connected cover, and write $\mathcal{G}(A)$ for the source-simply connected integration of A . Then we have the following commutative diagram.*



Here, p_1 is the covering map, and the map $G \rightarrow \mathcal{AC}(G)$ is the natural map from a local Lie group to its associative completion. The map p_2 is a generalized covering of local Lie groupoids, which sends a point $g \in \tilde{G}$ to the class of the A -path associated to a \tilde{G} -path from $s(g)$ to g .

The third theorem about local Lie groupoids concerns the theory of integrability. Every Lie algebroid can be integrated to a local Lie groupoid [7]. If one starts with a non-integrable algebroid, and integrates it to a local Lie groupoid, one necessarily obtains a non-globalizable local Lie groupoid. By Mal'cev's theorem, this non-globalizability must manifest itself as a lack of global associativity. Because the integrability of a Lie algebroid is controlled by its monodromy groups, one expects there to be a relationship between the monodromy groups of an algebroid and the associativity of any of its local integrations. As we will prove in the last chapter, this relationship can be made precise: under some technical assumptions, the "associators" of a local Lie groupoid *coincide* with the monodromy groups of its Lie algebroid, at least sufficiently near the identities. Here is the statement.

Theorem C. *Suppose that G is a shrunk local Lie groupoid over M . Let $x \in M$. Consider G_x as a subset of $\mathcal{G}(\mathfrak{g}_x)$ using the natural map $G_x \rightarrow \mathcal{G}(\mathfrak{g}_x)$. Then*

$$\text{Assoc}_x(G) = \tilde{\mathcal{N}}_x \cap G_x.$$

Outline

The main content of this thesis is organized into seven chapters.

In chapter 1, we discuss some of the background material needed for the rest of the thesis. It is intended to be a brief review, and the reader who is unfamiliar with the topics in this first chapter is encouraged to read more comprehensive treatments before moving on. None of the material in this chapter is original work.

In chapter 2, we give an overview of results about calibrated symplectic foliations. We discuss what it means for a symplectic foliation to be calibrated (or calibratable), and give some interpretations. We illustrate by briefly mentioning the special case of cosymplectic structures, and conclude by discussing a result about the transverse geometry of calibrated symplectic foliations due to Martínez-Torres et al. [32]. None of the material in this chapter is new, except perhaps the example on page 27 and proposition 33.

In chapter 3, we discuss how the theory of currents (in the sense of de Rham) can be used to try to understand calibratability of symplectic foliations. This is the last chapter about symplectic foliations. All of the core results in this chapter are due to Sullivan [29]. In this thesis, we explain how Sullivan's theory of structure cycles becomes relevant in understanding calibratability of symplectic foliations. To the best of the author's knowledge, no attempts have been made before at understanding calibratability using Sullivan's theory. It seems likely that the understanding of the local structure of structure cycles should give some type of answer to the existence question for calibrated symplectic foliations, but no deep results have been obtained yet. At the end of the chapter, we unify conjectures by Sullivan and Ghys, and illustrate with a concrete example.

In chapter 4, we start our discussion of local Lie groupoids. We define local Lie groupoids, and explain some of their basic properties. These properties are all straightforward generalizations of corresponding properties for local Lie groups [27]. Then, we explain how one can associate a Lie algebroid to every local Lie groupoid. While some of the results of this chapter do not seem to have been written down earlier, no major theorems are proved.

In chapter 5, we discuss the globalizability of local Lie groupoids. One of the main results of this thesis is the generalization of Mal'cev's theorem to local Lie groupoids. It is obtained in this chapter as a corollary of a theorem describing smoothness of the associative completion of a local Lie groupoid. The associators of a local Lie groupoid are introduced in this chapter, and their relationship with the associative completion is explained. The proof of Mal'cev's theorem for local Lie groupoids can be found in this chapter. The recognition of associative completion as an important operation, and the introduction of associators, are important in this chapter and in the later ones. The most important paper guiding this chapter was [27], where the corresponding results for local Lie groups are proved (though associators are not introduced there).

In chapter 6, associative completions are used as a tool to classify those local Lie groupoids that have an integrable Lie algebroid. We discuss coverings of local Lie groupoids, and we prove the classification result.

In chapter 7, we prove the last of the three theorems, showing how the associators of a local Lie groupoid are related to the monodromy groups of its Lie algebroid. The theorem describing this relationship is the most important result of this thesis. It demonstrates that the monodromy groups of a Lie algebroid manifest themselves in a local integration, in a combinatorial way.

Chapter 1

Preliminaries

In this first chapter, we review some of the basic material that will be needed later on. We will also establish some notations that will be used throughout this thesis.

1.1 Foliations

Suppose that M is a smooth manifold of dimension m . A foliation of M is, roughly speaking, a partition of M into immersed submanifolds of dimension p , in such a way that the partition locally looks like \mathbb{R}^m partitioned into parallel copies of \mathbb{R}^p .

Definitions

There are many ways to define what a foliation is. We will give two.

Definition 1. A *foliation atlas* of dimension p (and codimension $q = m - p$) of M ($0 \leq p \leq m$) is an atlas $(\varphi_i : U_i \rightarrow \mathbb{R}^m)_i$ such that the transition maps are of the form

$$\varphi_{ij}(x, y) = (f_{ij}(x, y), g_{ij}(y))$$

for the decomposition $\mathbb{R}^m = \mathbb{R}^p \times \mathbb{R}^q$.

The important part is that the second component of $\varphi_{ij}(x, y)$ does not depend on x . This definition should be interpreted as follows: \mathbb{R}^m consists of parallel copies of \mathbb{R}^p (a q -parameter family of them). The transition maps of the atlas must respect this decomposition of \mathbb{R}^m into parallel submanifolds (so that points on the same submanifold are mapped to points on the same submanifold). Thus, if a manifold is obtained by gluing together flexible pieces of fabric, then a foliated manifold is obtained by gluing together flexible pieces of striped fabric in such a way that the stripes line up whenever you glue.

Definition 2. A *foliation* of M of dimension p is a maximal foliation atlas of dimension p of M .

We will use the letter \mathcal{F} to denote a foliation. The stripes in a foliation chart are called *plaques*. They are copies of open subsets of \mathbb{R}^p . The plaques fit together globally to partition the manifold into injectively immersed p -dimensional submanifolds of M , called the *leaves* of the foliation. The *leaf space* of \mathcal{F} , written M/\mathcal{F} , is the quotient space of M obtained by identifying points on the same leaf, equipped with the quotient topology. It need not be a well-behaved space.

In some situations, it is useful to choose a foliation atlas that is slightly better behaved than a general foliation atlas. A *regular* foliation atlas is a foliation atlas that satisfies some mild extra conditions:

- the closure of each chart domain is a compact subset of some foliated chart (not necessarily in our atlas),
- the covering by chart domains is locally finite,
- the closure of a plaque of one chart intersects at most one plaque in each other chart.

Every foliation has a regular foliation atlas [6, lemma 1.2.17]. We will make use of regular foliation atlases exactly once, on section 3.2.

A foliation is completely determined by the tangent space to the leaf at each point $x \in M$. These tangent spaces form a subbundle of TM , and this subbundle is integrable because the leaves of \mathcal{F} are integral to this distribution. This gives another way of defining a foliation.

Definition 3 (Alternative definition). A *foliation* of M of dimension p is an integrable subbundle of TM of rank p .

Using this definition, the leaves of the foliation are the maximal integral submanifolds of the integrable distribution. If we want to think of a foliation \mathcal{F} as a subbundle of TM , we will write it as $T\mathcal{F}$ (the tangent bundle of the foliation). Will also use $T^*\mathcal{F}$ for the dual of $T\mathcal{F}$ (the cotangent bundle of the foliation).

We will say that a foliation is *orientable* (*oriented*) if the bundle $T\mathcal{F}$ is orientable (oriented). We will say that a foliation is *co-orientable* (*co-oriented*) if the normal bundle to the foliation, $TM/T\mathcal{F}$, is orientable (oriented). Co-orientability is also called

transverse orientability. If \mathcal{F} is not (co-)orientable, we can obtain a (co-)orientable foliation by going to an orientation cover.

Every co-orientable foliation of codimension 1 can be specified as the kernel of a 1-form. If $\alpha \in \Omega^1(M)$, the kernel of α is an integrable distribution (i.e. a foliation) precisely if $\alpha \wedge d\alpha = 0$.

Examples

Let us go over some basic examples of foliations.

Example (Trivial foliation). If L is a p -dimensional manifold, then $L \times \mathbb{R}^q$ has an obvious foliation. Its leaves are $L \times \{y\}$ for $y \in \mathbb{R}^q$. The leaf space is \mathbb{R}^q . More generally, for any q -dimensional manifold Q , we can foliate $L \times Q$ by leaves $L \times \{q\}$ for $q \in Q$. The leaf space is then Q . Even more generally, if $f : M \rightarrow Q$ is a submersion, the fibers of f form the leaves of a foliation. Then $T\mathcal{F}$ is the kernel of $df : TM \rightarrow TQ$. The leaf space is Q . Foliation of this form are called *simple foliations*. A foliation with a smooth leaf space is automatically simple. ◀

Example (Linear foliation of the 2-torus). Consider

$$M = \mathbb{T}^2 = \frac{\mathbb{R}^2}{\mathbb{Z}^2},$$

the 2-torus, with coordinates $x, y \in \mathbb{R}/\mathbb{Z}$. Let $\theta \in \mathbb{R}$ and consider the foliation given by the 1-form

$$\alpha = -\sin(\theta) dx + \cos(\theta) dy.$$

This form satisfies $\alpha \wedge d\alpha = 0$, and so its kernel is a foliation. If $\tan(\theta) \in \mathbb{Q} \cup \{\infty\}$, this is a foliation by circles. Otherwise, this is a foliation by lines. ◀

Example (Reeb foliation). This example is one of the most important foliations in geometry. Let $f : (-1, 1) \rightarrow \mathbb{R}$ be an even function such that $f(y) \rightarrow +\infty$ as $y \rightarrow \pm 1$. Consider the following foliation of $\mathbb{R} \times [-1, 1]$:

- for each $x \in \mathbb{R}$ the image of $(-1, 1) \rightarrow \mathbb{R} \times [-1, 1] : y \mapsto (x + f(y), y)$ is a leaf,
- the boundary components $\mathbb{R} \times \{\pm 1\}$ are leaves.

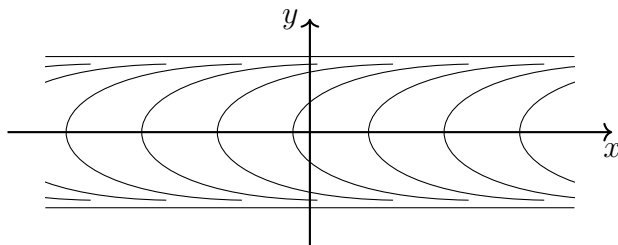


Figure 1.1: A foliation of $\mathbb{R} \times [-1, 1]$. The curved leaves asymptotically approach the boundary components. The Reeb foliation of the solid torus is obtained by spinning this foliation around the x -axis, and then modding out by a translation in the x -direction.

This is a smooth foliation of $\mathbb{R} \times [-1, 1]$ if f is chosen appropriately. (We have not talked about foliations of manifolds with boundary. There are two common types of these: either boundary components are leaves, as in this example, or the foliation is transverse to the boundary. Adapting the definition to the case with boundary is not hard, but we won't need it for anything but this example.) It is shown in figure 1.1. We can spin this foliation around the axis $\mathbb{R} \times \{0\}$ to obtain a foliation of $\mathbb{R} \times \mathbb{D}^2$, and then mod out by translation by 1 unit in the \mathbb{R} -direction to obtain a foliation of $\mathbb{S}^1 \times \mathbb{D}^2$, the so-called *Reeb foliation* of the solid torus, named after the French mathematician Georges Reeb. Specifically, the Reeb foliation of the solid torus has the following leaves:

- for each $x \in \mathbb{S}^1$, the graph of $\mathbb{D}^2 \rightarrow \mathbb{S}^1 \times \mathbb{D}^2 : (r, \theta) \mapsto (x + f(r), (r, \theta))$ is a leaf (where we have used polar coordinates on \mathbb{D}^2),
- the boundary component $\mathbb{S}^1 \times \mathbb{S}^1$ is the only compact leaf.

The Reeb foliation of the solid torus is shown in figure 1.2. One can glue two such foliated tori together along their boundary to obtain a codimension-1 foliation of \mathbb{S}^3 . If one does this in such a way that meridians in one solid torus correspond to longitudes in the other, and vice versa, one obtains the so-called *Reeb foliation* of the 3-sphere. (To ensure smoothness of this foliation, $f(y)$ needs to grow fast enough as $y \rightarrow 1$.) ◀

Example (Lawson's foliations of spheres). In [19], Lawson gave smooth codimension-1 foliations for all spheres of dimension $2^k + 3$ ($k \geq 1$). In the case $k = 1$ this leads to a foliation of \mathbb{S}^5 with exactly one compact leaf, similar to the Reeb foliation of \mathbb{S}^3 . This compact leaf is diffeomorphic to $\mathbb{S}^1 \times L$, where L is a circle bundle over a torus. The compact leaf divides \mathbb{S}^5 into two components, and each of these components is foliated by leaves all of which are diffeomorphic within their component. The foliation is transversely oriented. ◀

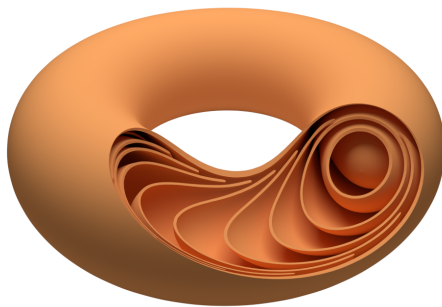


Figure 1.2: A cutaway of the Reeb foliation of the solid torus.

Example (Turbulization, see e.g. [6, example 3.3.11]). Let (M, \mathcal{F}) be a foliated manifold of codimension 1, oriented and co-oriented. Suppose that S is a properly embedded submanifold of M of dimension 1, tranverse to \mathcal{F} (the typical case is when S is a circle tranverse to \mathcal{F}). Fix a foliated tubular neighborhood U of S (this is a tubular neighborhood of S that is isomorphic to $S \times \mathbb{D}^{m-1}$ as foliated manifolds). Use coordinates (z, u, r) on $S \times \mathbb{D}^{m-1}$, where z is the coordinate along S , and the coordinates $u \in \mathbb{S}^{m-2}$ and $r \in [0, 1]$ are the direction and radius in \mathbb{D}^{m-1} . (These coordinates are not smooth where $r = 0$, but this will not matter for us.) The foliation of $S \times \mathbb{D}^{m-1}$ is defined by dz . Now consider the 1-form

$$\alpha = \sin(\lambda(r)) dr + \cos(\lambda(r)) dz,$$

where $\lambda : [0, 1] \rightarrow [0, \pi]$ is smooth, non-decreasing, and satisfies

- $\lambda(r) = 0$ if $0 \leq r \leq 1/4$,
- $\lambda(1/2) = \pi/2$,
- $\lambda(r) = \pi$ if $3/4 \leq r \leq 1$,
- λ is strictly increasing on $[1/4, 3/4]$.

Figure 1.3 shows the graph of such a function. This 1-form satisfies $\alpha \wedge d\alpha = 0$, and therefore defines a foliation. Moreover, this foliation agrees with the original one near $r = 1$ (and also near $r = 0$). We may therefore replace the foliation \mathcal{F} in the tubular neighborhood of S by the one defined by α . This results in a new foliation \mathcal{F}' on M . Note that \mathcal{F}' has a leaf diffeomorphic to $S \times \mathbb{S}^{m-2}$ at $r = 1/2$. This procedure of changing the foliation in a neighborhood of a 1-dimensional submanifold is called *turbulization*.

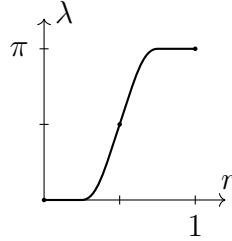


Figure 1.3: The function λ used in the turbulization procedure.

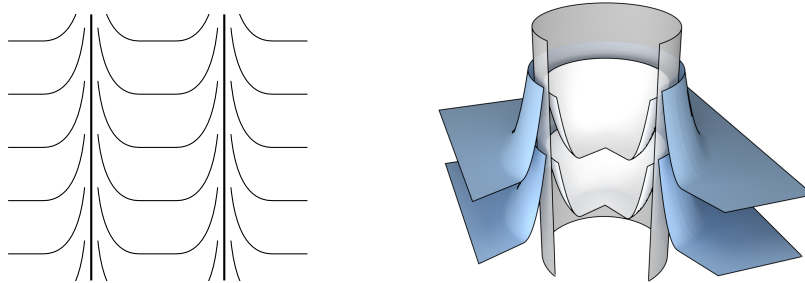


Figure 1.4: *Left:* Turbulization in the two-dimensional case. The original foliation consisted of horizontal lines. The submanifold S was a vertical line centered in the picture. The two vertical leaves correspond to $r = 1/2$. *Right:* Turbulization in the three-dimensional case (cutaway). The original foliation consisted of horizontal planes. The submanifold S is a vertical line, the axis of the cylinder. There is a cylindrical leaf that corresponds to $r = 1/2$. The blue leaves on the outside approach the cylindrical leaf asymptotically. The white cups on the inside are diffeomorphic to the plane, and they also approach the cylindrical leaf. The white cups together with the cylinder form the Reeb foliation of $S^1 \times \mathbb{D}^2$.



Holonomy

Suppose $\gamma : [0, 1] \rightarrow M$ is a curve from $x \in M$ to $y \in M$ that lies in a leaf of \mathcal{F} . Pick a small submanifold T_x of M through x , transverse to \mathcal{F} , and a small submanifold T_y of M through y , transverse to \mathcal{F} . Then the foliation induces a diffeomorphism f from a small neighborhood U_x of x in T_x to a small neighborhood U_y of y in T_y , as follows: suppose $x' \in T_x$ is sufficiently close to x . By moving the curve γ slightly, we obtain a curve γ' that starts at x' , lies in a leaf of the foliation, and ends at a point $y' \in T_y$. Then $f(x') = y'$.

Definition 4. The *holonomy of \mathcal{F} along γ* for the transversals T_x and T_y is the germ of this diffeomorphism $f : U_x \rightarrow U_y$. The *linear holonomy of \mathcal{F} along γ* is the first-order approximation to the holonomy of \mathcal{F} along γ , considered as a map $\nu_x \mathcal{F} \rightarrow \nu_y \mathcal{F}$.

The linear holonomy does not depend on the choice of transversals. Moreover, the holonomy of \mathcal{F} along a curve only depends on the homotopy class of this curve, if we take homotopies in the leaf and relative to the endpoints. The linear holonomy, therefore, also depends only on this homotopy class.

Foliated cohomology

A *foliated differential k -form* on the foliated manifold (M, \mathcal{F}) is a section of $\Lambda^k T^* \mathcal{F}$. A foliated k -form induces an ordinary k -form on each leaf. There is a foliated version of the de Rham differential, called the *leafwise de Rham differential* and written $d_{\mathcal{F}}$, which simply takes the exterior derivative of a foliated form in a leafwise way. The *foliated cohomology* of a foliation \mathcal{F} is the cohomology of the complex

$$\dots \rightarrow \Omega^{k-1}(\mathcal{F}) \xrightarrow{d_{\mathcal{F}}} \Omega^k(\mathcal{F}) \xrightarrow{d_{\mathcal{F}}} \Omega^{k+1}(\mathcal{F}) \rightarrow \dots$$

This cohomology will be denoted by $H^\bullet(\mathcal{F})$.

If E is a vector bundle over M , an \mathcal{F} -*connection* on E (sometimes called a *partial connection*) has the same definition as an ordinary connection, except that one can only

differentiate *along* \mathcal{F} (not in transverse directions). In other words, it is a bilinear map

$$\Gamma(T\mathcal{F}) \times \Gamma(E) \rightarrow \Gamma(E) : (\alpha, s) \mapsto \nabla_\alpha(s)$$

satisfying the usual conditions: it is $C^\infty(M)$ -linear in α , and it satisfies the Leibniz rule

$$\nabla_\alpha(fs) = f\nabla_\alpha(s) + \mathcal{L}_\alpha(f)s$$

with respect to s . If E is a vector bundle over M with a flat \mathcal{F} -connection ∇ , then we can define the cohomology of \mathcal{F} with values in E . It is the cohomology of the complex

$$\dots \rightarrow \Omega^{k-1}(\mathcal{F}; E) \xrightarrow{d_E} \Omega^k(\mathcal{F}; E) \xrightarrow{d_E} \Omega^{k+1}(\mathcal{F}; E) \rightarrow \dots$$

where the differential d_E uses the \mathcal{F} -connection. For $\alpha \in \Omega^k(\mathcal{F}; E)$ it is given by

$$\begin{aligned} (d_E\alpha)(V_0, \dots, V_k) &= \sum_i (-1)^i \nabla_{V_i}(\alpha(V_0, \dots, \hat{V}_i, \dots, V_k)) \\ &+ \sum_{i < j} (-1)^{i+j} \alpha([V_i, V_j], V_0, \dots, \hat{V}_i, \dots, \hat{V}_j, \dots, V_k). \end{aligned}$$

For this differential we have $d_E^2 = 0$ because the connection is flat. The cohomology is denoted by $H^\bullet(\mathcal{F}; E)$.

The normal and conormal bundles of a foliation come equipped with a canonical flat connection, the *Bott connection*, which is such that parallel transport for this connection corresponds to linear holonomy. We will denote the normal bundle to \mathcal{F} by $\nu(\mathcal{F})$ and the conormal one by $\nu^*(\mathcal{F})$. The cohomology with values in these bundles is then written $H^\bullet(\mathcal{F}, \nu(\mathcal{F}))$ and $H^\bullet(\mathcal{F}, \nu^*(\mathcal{F}))$ (where the Bott connection is implicitly used).

Existence problem

A driving problem in the early theory of foliations was the existence problem: which manifolds admit foliations of a certain codimension? For example, the 2-sphere does not have a foliation of dimension 1, because this would imply the existence of a non-vanishing vector field on \mathbb{S}^2 .

The case of open manifolds was settled in 1970 by Haefliger [15], who gave a necessary and sufficient condition under which a distribution can be homotoped to a foliation. Let BO_n be the classifying space of the orthogonal group O_n , and let $B\Gamma_q$ be the classifying

space of the groupoid of germs of diffeomorphisms of \mathbb{R}^q . Then there is a commutative diagram

$$\begin{array}{ccc}
 & & B\Gamma_q \times BO_p \\
 & \swarrow & \downarrow \\
 & BO_q \times BO_p & \\
 & \searrow & \downarrow \\
 & & BO_m
 \end{array}$$

Theorem 5 (Haefliger). *Let M be a smooth manifold. Let $\tau : M \rightarrow BO_m$ be a map classifying the tangent bundle of M . Then there is a natural map from the space of codimension- q foliations on M , up to integrable homotopy (i.e. homotopy through foliations), to the space of homotopy classes of lifts of τ to $B\Gamma_q \times BO_{m-q}$.*

In a diagram, the situation is as follows: the map τ classifies the tangent bundle of M . Splitting this bundle as a sum of a rank- p and rank- q subbundles amounts to finding a lift τ' . The rank- p subbundle is homotopic to an integrable distribution (i.e. a foliation) precisely if τ' can further be lifted to τ'' .

$$\begin{array}{ccc}
 & & B\Gamma_q \times BO_p \\
 & \overset{\tau''}{\dashrightarrow} & \downarrow \\
 M & \overset{\tau'}{\dashrightarrow} & BO_q \times BO_p \\
 & \searrow & \downarrow \\
 & \tau & BO_m
 \end{array}$$

The existence of foliations on closed manifolds was solved in the seventies by Thurston [30, 31]. The result from 1974 settles the case $q > 1$. It uses the language of Haefliger structures (we do not go into detail).

Theorem 6 (Thurston). *Let M be a smooth manifold, \mathcal{H} a Haefliger Γ_q -structure on M with $q > 1$, and $i : \nu(\mathcal{H}) \hookrightarrow TM$ a bundle monomorphism. Then there is a foliation \mathcal{F} of codimension q on M that is homotopic to \mathcal{H} .*

The result from 1976 settles the case $q = 1$.

Theorem 7 (Thurston). *A closed manifold M has a foliation of codimension 1 if and only if its Euler characteristic is zero.*

Taut foliations

We now restrict our attention to the case of co-oriented codimension-1 foliations. Among these, there is a special type of foliation called a *taut* foliation. These special foliations have been extensively studied, particularly in the case $m = 3$, because the existence of taut foliations on 3-manifolds has implications for the topology of the manifold. Tautness of a foliation should be thought of as the counterpart of *tightness* in contact topology.

Definition 8. A co-oriented foliation \mathcal{F} of codimension 1 on M is called *taut* if for each leaf L , there is a loop in M that is transverse to \mathcal{F} and intersects L .

Some authors require that there exist a transverse loop that intersects all leaves of \mathcal{F} . On compact manifolds, the case that is most relevant when studying taut foliations, this definition is equivalent.

There are several criteria that are equivalent to tautness (lemma 6.3.4, proposition 10.4.1 and corollary 10.5.10 in [6]).

Theorem 9. *If \mathcal{F} is a smooth co-oriented codimension-1 foliation on a compact manifold, the following are equivalent:*

- (a) \mathcal{F} is taut,
- (b) no finite family of compact leaves $\{L_i\}_i$ bounds a submanifold $Q \subset M$ such that Q lies on the positive side of every L_i ,
- (c) no non-trivial foliation cycle for \mathcal{F} bounds,
- (d) there is a closed $(m - 1)$ -form which restricts to a volume form on each leaf,
- (e) there is a vector field transverse to \mathcal{F} whose flow preserves some volume form on M ,
- (f) there exists a Riemannian metric on M such that every leaf of \mathcal{F} is a minimal submanifold of M .

These criteria will serve as a useful guide in trying to generalize the notion of tautness in dimension 3 using symplectic foliations.

Let us give an example of how the existence of a taut foliation gives information about the topology of the ambient manifold.

Theorem 10 (Novikov, [26]). *Let \mathcal{F} be a codimension-1 foliation on a compact connected oriented manifold M of dimension 3. If M has finite fundamental group, then \mathcal{F} has a Reeb component. In particular, \mathcal{F} is not taut.*

A *Reeb component* is a solid torus $\mathbb{S}^1 \times \mathbb{D}^2$ embedded in M that is saturated (i.e. is a union of leaves), whose foliation is topologically conjugate to the Reeb foliation of the solid torus (example on page 9). If a foliation contains a Reeb component, it cannot be taut: any transverse curve that crosses into the torus cannot come out again. The Reeb component is effectively a dead end.

Here is another simple observation about taut foliations on simply connected manifolds (not necessarily of dimension 3).

Lemma 11. *Let M be a closed simply connected manifold. If \mathcal{F} is a taut foliation on M , then it has no compact leaves.*

Proof. Suppose L is a compact leaf. We claim that $M \setminus L$ is disconnected. Indeed, a tubular neighborhood of L is diffeomorphic to $L \times (-1, 1)$ (\mathcal{F} must be transversely orientable because M is simply connected). We will use t as a coordinate on $(-1, 1)$. Because \mathcal{F} is taut, there is a path in $M \setminus L$ from a point with $t < 0$ to a point with $t > 0$. Then we can close up the path inside the tubular neighborhood to obtain a loop in M that intersects L exactly once. This shows that L cannot be trivial in $H_{n-1}(M)$. But by $\pi_1(M) = \{*\}$ and Poincaré duality, $H_{n-1}(M)$ is trivial. This contradiction proves the result. \square

1.2 Symplectic foliations and Poisson geometry

In this section, we will combine symplectic structures with foliations and see how this brings us to the world of regular Poisson geometry. We will need the leaves of our foliation to be even-dimensional, so we will assume that M is of dimension $m = 2n + q$.

Definition 12. A *symplectic foliation of codimension q* on M is a codimension- q foliation \mathcal{F} on M together with a leafwise form $\omega \in \Gamma(\Lambda^2 T^* \mathcal{F})$ such that the restriction of ω to every leaf is symplectic.

Local normal form

Suppose (M, \mathcal{F}, ω) is a symplectically foliated manifold. In a sufficiently small foliation chart for \mathcal{F} , we can apply Moser's trick to bring the symplectic forms on all leaves into Darboux form simultaneously. More precisely, we have the following lemma.

Lemma 13. *Let (M, \mathcal{F}, ω) be a manifold with a symplectic foliation of codimension q . Then near any point $p \in M$ we may choose coordinates $(x_1, \dots, x_n, y_1, \dots, y_n, z)$ ($z \in \mathbb{R}^q$) such that in a neighborhood of p :*

- *the foliation \mathcal{F} is given by $z = \text{constant}$,*
- *the leafwise symplectic form is given by $\omega = \sum_i dx_i \wedge dy_i$.*

Regular Poisson structures

Suppose that we have a manifold M equipped with a regular corank- q Poisson structure π . Then M is automatically endowed with a symplectic foliation by taking

- $\mathcal{F} = \text{Im}(\pi^\sharp)$,
- $\omega_x(v, w) = \alpha(w)$ where $x \in M$, $v, w \in T_x\mathcal{F}$, $\alpha \in T_x^*M$ and $\pi^\sharp(\alpha) = v$.

The leaves of this symplectic foliation are known as the *symplectic leaves* of π .

Conversely, given a symplectic foliation (\mathcal{F}, ω) on M , we can endow M with a Poisson structure by taking the Poisson tensor at a point $x \in M$ to be the composition

$$T_x^*M \rightarrow T_x^*\mathcal{F} \xrightarrow{(\omega_x^\flat)^{-1}} T_x\mathcal{F} \rightarrow T_xM.$$

These two constructions (from symplectic foliation to regular Poisson structure and vice versa) are inverse to each other. Thus we have a bijection

$$\left\{ \begin{array}{l} \text{symplectic foliations} \\ \text{of codimension } q \end{array} \right\} \leftrightarrow \{ \text{regular corank-}q \text{ Poisson structures} \}$$

and a symplectic foliation is essentially the same thing as a regular Poisson structure. The local normal form for symplectic foliations is then just the Weinstein splitting theorem in the case of a regular Poisson structure, which says that the Poisson bivector locally looks like

$$\pi = \sum_i \partial x_i \wedge \partial y_i.$$

A deep result by Mitsumatsu states that Lawson’s foliation of \mathbb{S}^5 admits a leafwise symplectic structure.

Theorem 14 (Mitsumatsu [23]). *Lawson’s foliation on the 5-sphere \mathbb{S}^5 admits a smooth leafwise symplectic structure.*

The Reeb foliation shows that \mathbb{S}^3 also admits a symplectic foliation. It is not known whether this is the case for odd-dimensional spheres of dimension at least 7.

1.3 Lie groupoids

Lie groupoids are multi-object generalizations of Lie groups. We briefly review some of the important points and establish notation. We refer the reader to the excellent [24] for a more thorough introduction. The infinitesimal counterpart to a Lie groupoid is a Lie algebroid, but not all algebroids can be integrated to groupoids. A good reference for the theory of integrability of Lie algebroids is [8] or the original article [7].

Definition and examples

There are several ways to define what a groupoid is. The shortest is the following.

Definition 15. A *groupoid* is a small category whose morphisms are all isomorphisms.

A groupoid therefore consists of two sets, M (the set of objects) and \mathcal{G} (the set of arrows), together with structure maps

$$\begin{aligned} s : \mathcal{G} &\rightarrow M && \text{source map} \\ t : \mathcal{G} &\rightarrow M && \text{target map} \\ m : \mathcal{G}^{(2)} &\rightarrow \mathcal{G} && \text{multiplication} \\ i : \mathcal{G} &\rightarrow \mathcal{G} && \text{inversion} \\ u : M &\rightarrow \mathcal{G} && \text{unit map} \end{aligned}$$

that satisfy the axioms of a category. The set $\mathcal{G}^{(2)}$ is $\mathcal{G} \times_s \times_t \mathcal{G}$, the space of composable arrows. (We will more generally use $\mathcal{G}^{(k)} = \mathcal{G} \times_s \times_t \dots \times_s \times_t \mathcal{G}$ for the k -fold fibered product.) Our convention throughout will be that arrows go “from right to left”, meaning that

we can only calculate $gh = g \cdot h = m(g, h)$ if $s(g) = t(h)$. The *orbits* of a groupoid are its isomorphism classes (so they partition M). The *isotropy group* at an object $x \in M$, written \mathcal{G}_x , is the automorphism group of x .

Definition 16. A *Lie groupoid* is a groupoid where \mathcal{G} and M are both manifolds, s and t are smooth submersions, and m, i and u are smooth maps.

Remark. We require that M is Hausdorff, but we do not require that \mathcal{G} is Hausdorff. Indeed, many natural examples of Lie groupoids have non-Hausdorff spaces of arrows. One example is the holonomy groupoid of a foliation that has a vanishing cycle. We do, however, require that the source and target fibers are Hausdorff. This same remark will hold true in the case of local Lie groupoids later on.

It follows from the axioms that $u : M \rightarrow \mathcal{G}$ is an embedding, and we will regard M as an embedded submanifold of \mathcal{G} . This means that, in notation, we will not distinguish a point $x \in M$ and its unit $u(x) \in \mathcal{G}$, and just write x for either. The usual notation for a Lie groupoid with space of arrows \mathcal{G} and space of objects M is $\mathcal{G} \rightrightarrows M$ (the two arrows represent the maps s and t).

Example. A Lie groupoid where M is a singleton is precisely a Lie group. ◀

Example. If \mathcal{G} is a Lie group acting on a manifold M , then the *action Lie groupoid* is the groupoid $\mathcal{G} \times M \rightrightarrows M$. Its space of arrows is $\mathcal{G} \times M$, with $s(g, x) = x$ and $t(g, x) = g \cdot x$. The product is $(h, y) \cdot (g, x) = (hg, x)$, the inversion is $(g, x)^{-1} = (g^{-1}, g \cdot x)$ and the unit map is $u(x) = (e_{\mathcal{G}}, x)$. ◀

Example (Fundamental groupoid of a foliation). Suppose that \mathcal{F} is a foliation of a manifold M . Then there is an important Lie groupoid $\Pi_1(\mathcal{F}) \rightrightarrows M$, called the *fundamental groupoid* of \mathcal{F} . Its space of arrows is

$$\Pi_1(\mathcal{F}) = \{\text{paths in leaves of } \mathcal{F}\} / \sim,$$

where \sim is homotopy in the leaf, relative to the endpoints. The source map assigns to any (class of a) path its starting point. The target map assigns to any (class of a) path its endpoint. The multiplication is concatenation of paths, and the inverse of a path is the same path traversed in the opposite direction.

The isotropy group at a point $x \in M$ is $\pi_1(L, x)$, where L is the leaf containing x . If \mathcal{F} has just one leaf (the entire manifold), then $\Pi_1(\mathcal{F})$ is known as the fundamental groupoid of M . ◀

Example (Holonomy groupoid of a foliation). Another important groupoid associated to a foliation is its *holonomy groupoid*. Its space of arrows is

$$\text{Hol}(\mathcal{F}) = \{\text{paths in leaves of } \mathcal{F}\} / \sim,$$

where two paths are equivalent for \sim if their holonomy is the same. The structure maps are analogous to those of the fundamental groupoid. There is a natural map $\Pi_1(\mathcal{F}) \rightarrow \text{Hol}(\mathcal{F})$ because homotopic paths have the same holonomy. This map is a covering of Lie groupoids (it is a covering map at the level of source fibers). ◀

Example (Symplectic groupoids). Suppose that \mathcal{G} is a Lie groupoid over M . A differential form $\omega \in \Omega^k(M)$ is called *multiplicative* if

$$m^*\omega = p_1^*\omega + p_2^*\omega,$$

where $m : \mathcal{G} \times_s \mathcal{G} \rightarrow \mathcal{G}$ is the multiplication and by $p_i : \mathcal{G} \times_s \mathcal{G} \rightarrow \mathcal{G}$ the projection to the i 'th factor. A *symplectic groupoid* is a pair (\mathcal{G}, ω) , where $\omega \in \Omega^2(\mathcal{G})$ is a symplectic form that is multiplicative. ◀

If $\mathcal{G} \rightrightarrows M$ is a Lie groupoid, then the orbits of this groupoid form a (possibly singular) foliation of M .

While much of the theory of Lie groupoids is similar to that of Lie groups, there are some important differences. The one that is most relevant for us concerns the infinitesimal counterparts to Lie groupoids, which we review next.

Lie algebroids

To every Lie group we associate a Lie algebra by looking at the Lie group structure at the infinitesimal level near the identity. Similarly, for Lie groupoids, there is a structure that allows us to capture a lot about the groupoid at the infinitesimal level. These structures are *Lie algebroids*.

Definition 17. A Lie algebroid over M is a tuple $(A, \rho, [-, -])$ where

- A is a vector bundle over M ,
- $\rho : A \rightarrow TM$ is a bundle map called the *anchor map*,

- $[-, -] : \Gamma(A) \times \Gamma(A) \rightarrow \Gamma(A)$ is a Lie bracket on the space of sections of A ,

such that the anchor map induces a Lie algebra homomorphism $\Gamma(A) \rightarrow \Gamma(TM)$ and the Leibniz identity $[X, fY] = f[X, Y] + \rho(X)(f)Y$ holds for all $X, Y \in \Gamma(A)$ and $f \in C^\infty(M)$.

Remark. This definition gives little intuition about Lie algebroid upon first reading. Here is a short attempt at a more intuitive description. The tangent bundle of a manifold summarizes all the possible directions that one can walk in on this manifold. The fiber $T_x M$ is the vector space of all possible velocities an ant could have when walking on the manifold and passing through x . The Lie bracket of vector fields summarizes the extent to which “ways of walking” (vector fields) fail to commute.

A Lie algebroid is a generalization of the tangent bundle. It gives, for each point, the vector space of all possible ways that you can start walking at that point. The bracket $[-, -]$ of the Lie algebroid summarizes the commutators of these “ways of walking”. The anchor ρ summarizes which effect a certain “way of walking” has on your position, by indicating how your position on the manifold changes if you walk that particular way.

If one considers the arrows of a groupoid as taking you from one point (the source) to another (the target), the Lie algebroid of this Lie groupoid summarizes how these arrows correspond to “ways of walking” at the infinitesimal level.

The Lie algebroid of a Lie groupoid is constructed as follows. Let $T^s \mathcal{G}$ be the kernel of $ds : T\mathcal{G} \rightarrow TM$. Then as a vector bundle, $A = T_M^s \mathcal{G} = T^s \mathcal{G}|_M$. The anchor map is induced from $dt : T\mathcal{G} \rightarrow TM$. Right-invariant vector fields on \mathcal{G} correspond to sections of A , and the bracket $[-, -]$ on $\Gamma(A)$ is the one induced from the usual Lie bracket of these right-invariant vector fields. (We will repeat this construction in greater detail for local Lie groupoids later.)

The theory of Lie groupoids and Lie algebroids is largely analogous to that of Lie groups and Lie algebras. Here are some key facts:

- A morphism of Lie groupoids $\mathcal{G}_1 \rightarrow \mathcal{G}_2$ induces a unique morphism of Lie algebroids $A_1 \rightarrow A_2$. A morphism of Lie algebroids $A_1 \rightarrow A_2$ induces a unique morphism of Lie groupoids $\mathcal{G}_1 \rightarrow \mathcal{G}_2$ if \mathcal{G}_1 is source-simply connected (i.e. its source fibers are simply connected).
- Every Lie groupoid \mathcal{G} is covered by a uniquely determined source-simply connected Lie groupoid. (More precisely, there is a source-simply connected Lie groupoid $\tilde{\mathcal{G}}$

and a morphism $f : \tilde{\mathcal{G}} \rightarrow \mathcal{G}$ that is a local diffeomorphism, such that f induces an isomorphism at the level of Lie algebroids.) This groupoid $\tilde{\mathcal{G}}$ is unique up to isomorphism.

It is a standard result in Lie theory that every Lie algebra is the Lie algebra of some Lie group. This no longer holds for Lie algebroids. We say that a Lie algebroid is *integrable* if it is the Lie algebroid of some Lie groupoid. The obstructions to the integrability of a Lie algebroid were first understood by Crainic and Fernandes [7]. They showed that the integrability of a Lie algebroid is controlled by its *monodromy groups*. The monodromy group at a point $x \in M$ is obtained as an image of $\pi_2(L)$ under a certain map, where L is the orbit of x . Important in the construction of the monodromy groups is the idea of an A -path.

A -paths

Suppose that \mathcal{G} is a Lie group. If we have a path $\gamma : I \rightarrow \mathcal{G}$ starting at the identity, we may differentiate it and right-translate to the origin to get a path $a : I \rightarrow \mathfrak{g}$, where \mathfrak{g} is the Lie algebra of \mathcal{G} . This path effectively gives us the velocity of γ at any time. Conversely, if we have a path $a : I \rightarrow \mathfrak{g}$, we can reconstruct the path in \mathcal{G} that it was obtained from by integrating (assuming it started at the identity). This way, there is a bijective correspondence between \mathcal{G} -paths and \mathfrak{g} -paths.

If we have a family of paths $\gamma_t : I \rightarrow \mathcal{G}$ (all starting at the identity), we can use this differentiation procedure to obtain a family of paths $a_t : I \rightarrow \mathfrak{g}$. If $(\gamma_t)_t$ is a homotopy of paths rel endpoints, this will have some consequences for the family of \mathfrak{g} -paths $(a_t)_t$. It turns out that one can detect, solely from the family $(a_t)_t$ and the Lie algebra structure, whether the original homotopy of \mathcal{G} -paths kept the endpoint of the paths fixed or not. In other words, there is a condition on a family of \mathfrak{g} -paths *that can be expressed using only the structure of the Lie algebra*, that tells us whether the endpoint of the \mathcal{G} -paths was kept fixed. One says that two \mathfrak{g} -paths are *\mathfrak{g} -homotopic* if they can be related to each other by a family of \mathfrak{g} -paths that satisfies this condition. Note that one can speak about \mathfrak{g} -homotopy without ever referring to a Lie group integrating the Lie algebra.

Now if \mathcal{G} is a simply connected Lie group, then \mathcal{G} itself is in bijection with the homotopy classes of \mathcal{G} -paths starting at the identity, rel endpoints. This observation allows one to reconstruct a simply connected Lie group from its Lie algebra. This technique can be used to prove that every Lie algebra is the Lie algebra of some Lie

group (i.e. that every Lie algebra is integrable). One reference where this is done is [24].

While many proofs of the integrability of Lie algebras rely on some sort of classification result about Lie algebras, the one outlined above does not. By generalizing the procedure above to Lie algebroids, Crainic and Fernandes were able to understand the obstructions to integrability of Lie algebroids [7]. Suppose that A is a Lie algebroid over a manifold M .

Definition 18. An A -path is a pair (a, γ) , where $\gamma : I \rightarrow M$ is a path in M and $a : I \rightarrow A$ is a path in A , such that

- a lies over γ , meaning that $a(t) \in A_{\gamma(t)}$ for all $t \in I$,
- $\rho(a(t)) = \frac{d\gamma}{dt}(t)$ for all $t \in I$.

Crainic and Fernandes construct an equivalence relation \sim on the set of all A -paths, called A -homotopy, analogous to the A -homotopy for Lie algebras outlined above.

Definition 19. If A is a Lie algebroid, the quotient

$$\mathcal{G}(A) = \{A\text{-paths}\} / \sim$$

is called the *Weinstein groupoid* of A .

It is a source-simply connected topological groupoid, but it is not necessarily smooth. Next, a morphism of groups $\partial : \pi_2(\mathcal{O}_x) \rightarrow \mathcal{G}(\mathfrak{g}_x)$ is constructed. Here, \mathcal{O}_x is the orbit of a point $x \in M$ and $\mathcal{G}(\mathfrak{g}_x)$ is the simply connected integration of the isotropy Lie algebra \mathfrak{g}_x at x . This morphism is called the *monodromy homomorphism*.

Definition 20. The image of the monodromy homomorphism $\partial : \pi_2(\mathcal{O}_x) \rightarrow \mathcal{G}(\mathfrak{g}_x)$ is called the *monodromy group* of A at x , written $\tilde{\mathcal{N}}_x(A)$.

The result characterizing the integrable Lie algebroids is the following.

Theorem 21 (Crainic-Fernandes, [7]). *For a Lie algebroid A , the following statements are equivalent:*

- A is integrable,
- the Weinstein groupoid $\mathcal{G}(A)$ is smooth,
- the monodromy groups $\tilde{\mathcal{N}}_x(A)$ are uniformly discrete.

Moreover, in this case, $\mathcal{G}(a)$ is the unique s -simply connected Lie groupoid integrating A .

Here, “uniformly discrete” means that there is an open set $U \subset A$ containing the zero section, such that the intersection $U \cap \left(\bigcup_{x \in M} \tilde{\mathcal{N}}_x(A)\right)$ is the zero section. In other words, the non-trivial monodromy is bounded away from the zero section.

Example (Symplectic groupoids integrate Poisson manifolds). If (M, π) is a Poisson manifold, we can consider its *cotangent algebroid*. As a vector bundle, this is T^*M . To turn this into a Lie algebroid, we prescribe the anchor map

$$\pi^\sharp : T^*M \rightarrow TM$$

by $\beta(\pi^\sharp(\alpha)) = \pi(\alpha, \beta)$, and the bracket by

$$[\alpha, \beta] = \mathcal{L}_{\pi^\sharp(\alpha)}(\beta) - \mathcal{L}_{\pi^\sharp(\beta)}(\alpha) - d(\pi(\alpha, \beta)).$$

This makes A into a Lie algebroid over M . If A happens to be integrable (we say that M is an *integrable Poisson manifold*), then its integration \mathcal{G} will be a symplectic Lie groupoid. In other words, because A comes from a Poisson bracket, the integration \mathcal{G} will naturally come with a multiplicative symplectic structure. Conversely, if $\mathcal{G} \rightrightarrows M$ is a symplectic Lie groupoid, then M has the structure of a Poisson manifold, whose cotangent algebroid is the algebroid of \mathcal{G} . ◀

Chapter 2

Calibrated symplectic foliations of codimension 1

In this chapter we consider a special class of symplectic foliations, namely the ones whose leafwise symplectic form can be extended to a closed 2-form on the manifold. *Throughout this chapter, all foliations will be of codimension 1.*

2.1 Extensions of the leafwise symplectic form

Given a symplectic foliation (\mathcal{F}, ω) on M , we can ask about forms $\tilde{\omega} \in \Lambda^2 T^*M$ extending ω . In other words, we want to understand the 2-forms $\tilde{\omega}$ on M such that

$$\tilde{\omega}(v, w) = \omega(v, w) \quad \text{for all } v, w \in T\mathcal{F}.$$

It is easy to see that such an extension always exists. In fact, these extensions are in 1-to-1 correspondence with splittings

$$TM = T\mathcal{F} \oplus \nu,$$

by taking $\nu = \ker(\tilde{\omega})$. (Indeed, any extension has rank at least $2n$ everywhere, and since the rank of a 2-form is always even, the rank must be $2n$. The extension is then determined by specifying its kernel.)

2.2 Existence of a closed extension

We are interested in symplectic foliations for which the leafwise symplectic form has a *closed* extension (closed as elements of $\Omega^2(M)$, not merely leafwise closed, which is assumed of a symplectic foliation). In [32], these are called *2-calibrated*. We will just refer to them as *calibrated*.

Definition 22. A *calibration* for a symplectic foliation (M, \mathcal{F}, ω) is an extension $\tilde{\omega} \in \Omega^2(M)$ of ω such that $d\tilde{\omega} = 0$.

We refer to the tuple $(\mathcal{F}, \tilde{\omega})$ as a *calibrated foliation* on M (it determines ω , of course). A symplectic foliation that admits a calibration will be called *calibratable*.

Remark. One could work with the same definition in higher codimensions. We will restrict ourselves to codimension 1, however. The geometric interpretation of a calibration that we will see in the next section no longer holds in higher codimensions.

The following lemma is immediate.

Lemma 23. *If $(M, \mathcal{F}, \tilde{\omega})$ is a calibrated foliation, then \mathcal{F} is taut.*

Proof. The $2n$ -form $\tilde{\omega}^n$ is a volume form when restricted to any leaf. By theorem 9, this shows that \mathcal{F} is taut. \square

Clearly, the converse does not hold: the submersion $\mathbb{S}^4 \times \mathbb{S}^1 \rightarrow \mathbb{S}^1$ determines a taut foliation whose leaves do not admit symplectic structures.

Even among taut foliations that admit a leafwise symplectic structure, there are some that do not admit calibrations. The following is an example.

Example. We will consider the torus \mathbb{T}^4 as $\mathbb{R}^4/\mathbb{Z}^4$ with coordinates $(\theta_1, \theta_2, \theta_3, \theta_4)$. Let $\phi: \mathbb{T}^4 \rightarrow \mathbb{T}^4$ be the diffeomorphism induced by the linear map $\mathbb{R}^4 \rightarrow \mathbb{R}^4$ with matrix

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Let (M, \mathcal{F}) be the suspension of ϕ with the obvious codimension-1 foliation (so $M = M \times [0, 1]/((p, 0) \sim (\phi(p), 1))$ with the foliation coming from the trivial one on $M \times [0, 1]$).

We claim that (M, \mathcal{F}) has no calibrated symplectic foliation. As a basis for $H^2(\mathbb{T}^4; \mathbb{R})$ we take

$$\{d\theta_1 \wedge d\theta_2, d\theta_1 \wedge d\theta_3, d\theta_1 \wedge d\theta_4, d\theta_2 \wedge d\theta_3, d\theta_2 \wedge d\theta_4, d\theta_3 \wedge d\theta_4\}.$$

(For ease of notation, we left out brackets indicating equivalence classes.) Relative to

this basis, $\phi^* : H^2(\mathbb{T}^4; \mathbb{R}) \rightarrow H^2(\mathbb{T}^4; \mathbb{R})$ is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 \end{pmatrix},$$

which has no non-zero fixed points. This shows that (M, \mathcal{F}) has no calibrated symplectic foliation, because in that case the class of the symplectic form on a leaf has to be invariant under ϕ^* (by Stokes' theorem; see also the geometric interpretation in the next section).

There is, however, a leafwise symplectic structure on (M, \mathcal{F}) . Let

$$\omega_0 = d\theta_1 \wedge d\theta_2 + d\theta_3 \wedge d\theta_4.$$

Let

$$\omega_1 = -d\theta_1 \wedge d\theta_2 - d\theta_1 \wedge d\theta_3 + d\theta_2 \wedge d\theta_4.$$

Note that $\phi^*\omega_1 = \omega_0$. Let

$$\omega_t = t\omega_1 + (1-t)\omega_0.$$

An explicit calculation shows that

$$\omega_t \wedge \omega_t = (3t^2 - 3t + 1)d\theta_1 \wedge d\theta_2 \wedge d\theta_3 \wedge d\theta_4,$$

which does not vanish for $t \in [0, 1]$.

Let $f : [0, 1] \rightarrow [0, 1]$ be smooth such that f equals 0 near 0, and f equals 1 near 1. The family of symplectic forms $(\omega_{f(t)})_{t \in [0, 1]}$ on $\mathbb{T}^4 \times [0, 1]$ descends to a smooth leafwise symplectic structure on

$$M = \frac{\mathbb{T}^4 \times [0, 1]}{(p, 0) \sim (\phi(p), 1)}.$$

(The reparameterization using f is necessary for smoothness of the gluing, but is not essential to the spirit of the example.) ◀

2.3 Geometric interpretation of closedness

One may wonder why it is interesting to look at *closed* extensions of a leafwise symplectic form. It turns out that the closedness has a simple geometric interpretation.

Suppose (M, \mathcal{F}, ω) is a symplectically foliated manifold and $\tilde{\omega}$ a (not necessarily closed) extension of ω . Let

$$TM = T\mathcal{F} \oplus \nu$$

be the splitting determined by $\tilde{\omega}$, so that $\nu = \ker(\tilde{\omega})$. This ν is a line field on M .

Closedness of $\tilde{\omega}$ is a local issue, and we will work in a chart for our interpretation. In the chart, we have local coordinates $x_1, \dots, x_n, y_1, \dots, y_n, z$ with the foliation given by $\ker(dz)$. Every leaf has a symplectic structure. Let X be the vector field tangent to ν such that $dz(X) = 1$. Then the flow of X maps leaves to leaves.

Proposition 24. *The extension $\tilde{\omega}$ is closed if and only if the flow of X preserves the leafwise symplectic structure.*

In other words, $\tilde{\omega}$ is closed if and only if the maps from leaves to leaves induced by the flow of X are symplectomorphisms whenever defined. This proposition is essentially a special case of lemma 6.18 in [21], which says that a connection on a symplectic fibration is symplectic if and only if the connection 2-form is vertically closed.

Proof. If X preserves $\tilde{\omega}$, it clearly preserves ω (since ω is determined by $\tilde{\omega}$ and \mathcal{F} , and \mathcal{F} is preserved by X). Conversely, if X preserves ω , then it preserves $\tilde{\omega}$, since $\tilde{\omega}$ is determined by ω and X itself (and the flow of X preserves X). We therefore want to show that $\mathcal{L}_X \tilde{\omega} = 0$ if and only if $d\tilde{\omega} = 0$.

Now

$$\mathcal{L}_X \tilde{\omega} = di_X \tilde{\omega} + i_X d\tilde{\omega},$$

and $i_X \tilde{\omega} = 0$ because X lies along $\nu = \ker(\tilde{\omega})$. This shows that X preserves the leafwise symplectic form if and only if $i_X d\tilde{\omega} = 0$. But $i_X d\tilde{\omega} = 0$ if and only if $d\tilde{\omega} = 0$ because $\tilde{\omega}$ is already leafwise closed. This proves the result. \square

Remark. Suppose (M, \mathcal{F}, ω) is a symplectically foliated manifold. We can regard it as a regular Poisson manifold. If the symplectic foliation is calibratable, this Poisson manifold is automatically integrable. In fact, its monodromy groups are trivial, as we now briefly explain. We give a shortened version of the explanation in [8, section 5.2]. For more detail, we refer the reader to that section.

For regular Poisson manifolds, the monodromy has a geometric interpretation. Suppose that $x \in M$ lies on a leaf L . Consider a sphere $\gamma : \mathbb{S}^2 \rightarrow L$ inside the symplectic leaf, such that γ maps the north pole $N \in \mathbb{S}^2$ to x . We can calculate the symplectic area $A(\gamma)$ of this sphere by integrating the leafwise symplectic structure over it.

Now take a short transversal to \mathcal{F} at x , parameterized by $t \in (-\varepsilon, \varepsilon)$ (where $t = 0$ corresponds to x). A *deformation of γ_0* is a family $\gamma_t : \mathbb{S}^2 \rightarrow M$ of spheres, parameterized by $t \in (-\varepsilon, \varepsilon)$, such that $\gamma_0 = \gamma$, and such that for each t the sphere γ_t lies entirely in a leaf of \mathcal{F} . The *transversal variation* of γ_t is the element

$$\text{var}(\gamma_t) = \left. \frac{d}{dt} \right|_{t=0} \gamma_t(N) \in \nu_x(\mathcal{F}).$$

It turns out that $\left. \frac{d}{dt} \right|_{t=0} A(\gamma_t)$ only depends on the homotopy class of γ and on $\text{var}(\gamma_t)$, and that it is linear in $\text{var}(\gamma_t)$ (we are using A to denote the symplectic area). This way, we can associate to every $[\gamma] \in \pi_2(L, x)$ an element $A'([\gamma])$ given by

$$\left. \frac{d}{dt} \right|_{t=0} A(\gamma_t) = \langle A'([\gamma]), \text{var}(\gamma_t) \rangle.$$

This quantity $A'([\gamma])$ measures how much the symplectic area of a sphere $[\gamma]$ changes as we move it transversely. As shown in [8, proposition 5.4], we have

$$\tilde{\mathcal{N}}_x = \{A'([\gamma]) \mid [\gamma] \in \pi_2(L, x)\}.$$

By the geometric interpretation of closedness, however, we see that for a calibrated symplectic foliation the symplectic area does not change when moving transversely. This implies that the monodromy of (M, \mathcal{F}, ω) vanishes. In particular, every symplectically foliated manifold that is calibratable is an integrable Poisson manifold (and its isotropy groups are simply $(\mathbb{R}, +)$).

2.4 Local normal form

Using the geometric interpretation of closedness we obtain the following local normal form for calibrated foliations.

Lemma 25. *Let $(M, \mathcal{F}, \tilde{\omega})$ be a manifold with a calibrated foliation. Then near any point*

$p \in M$ we may choose coordinates $x_1, \dots, x_n, y_1, \dots, y_n, z$ such that in a neighborhood of p :

- the foliation \mathcal{F} is given by $dz = 0$,
- the calibration $\tilde{\omega}$ is given by $\tilde{\omega} = \sum_i dx_i \wedge dy_i$.

Note the difference with the local normal form for symplectic foliations: the calibration $\tilde{\omega}$ is also brought to a normal form in this lemma, not just its restriction to leaves.

Proof. Let U be a small open neighborhood of p in its leaf, with Darboux coordinates $x_1, \dots, x_n, y_1, \dots, y_n$. Let X be a section of ν , non-vanishing and parallel for the Bott connection (so that its flow maps leaves to leaves). Write ϕ_t for the time- t flow of X . After shrinking U if necessary and picking $\varepsilon > 0$ small enough, the map

$$U \times (-\varepsilon, \varepsilon) \rightarrow M : q \mapsto \phi_t(q)$$

is a diffeomorphism onto an open neighborhood of p in M . We use (the inverse of) this map as a chart near p , with coordinates x_i, y_i on U and z for the coordinate along $(-\varepsilon, \varepsilon)$. Then the foliation is given by $\ker(dz)$ in this chart. Moreover, translation in the z -direction is a symplectomorphism between leaves. Because the x_i, y_i are Darboux coordinates for U , this means the leafwise symplectic structure is given by $\sum_i dx_i \wedge dy_i$. Also, in this chart, the kernel of $\tilde{\omega}$ is, by construction, given by ∂z . This shows that in fact

$$\tilde{\omega} = \sum_i dx_i \wedge dy_i,$$

proving the result. □

2.5 Calibratability and transverse differentiation

Lemma 26. *A form $\alpha \in \Omega^k(\mathcal{F})$ has a closed extension if and only if its class $[\alpha] \in H^k(\mathcal{F})$ is in the image of the restriction map $H^k(M) \rightarrow H^k(\mathcal{F})$.*

Proof. If α has a closed extension, then clearly it lies in the image of this map. Conversely, suppose $[\alpha]$ is the image of $[\beta] \in H^k(M)$. This means that

$$\beta|_{\mathcal{F}} - \alpha = d_{\mathcal{F}}\gamma$$

for some $\gamma \in \Omega^{k-1}(\mathcal{F})$. Pick an extension $\tilde{\gamma} \in \Omega^{k-1}(M)$ of γ . Set

$$\tilde{\alpha} = \beta + d\tilde{\gamma}.$$

Then $\tilde{\alpha}$ is closed, and its restriction to \mathcal{F} equals α by construction. This proves the result. \square

By definition, $\Omega^\bullet(M, \mathcal{F})$ will be the kernel of the restriction map $\Omega^\bullet(M) \rightarrow \Omega^\bullet(\mathcal{F})$. The following lemma follows from computation.

Lemma 27. *The maps $\phi^k : \Omega^k(M, \mathcal{F}) \rightarrow \Omega^{k-1}(\mathcal{F}, \nu^*)$ given by*

$$\phi^k(\alpha)(v_1, \dots, v_{k-1})(v_k) = \alpha(v_1, \dots, v_k)$$

form an isomorphism of chain complexes $\Omega^\bullet(M, \mathcal{F}) \rightarrow \Omega^{\bullet-1}(\mathcal{F}, \nu^)$.*

We now have a short exact sequence of complexes

$$0 \rightarrow \Omega^{\bullet-1}(\mathcal{F}, \nu^*) \rightarrow \Omega^\bullet(M) \rightarrow \Omega^\bullet(\mathcal{F}) \rightarrow 0.$$

It has an associated long exact sequence in cohomology, which near $H^k(M)$ looks like

$$\dots \rightarrow H^{k-1}(\mathcal{F}, \nu^*) \rightarrow H^k(M) \rightarrow H^k(\mathcal{F}) \xrightarrow{d_{\mathfrak{h}}} H^k(\mathcal{F}, \nu^*) \rightarrow \dots$$

The connecting morphism from $H^k(\mathcal{F})$ to $H^k(\mathcal{F}, \nu^*)$ is a type of transverse differentiation of the leafwise form, which is why we denote it by $d_{\mathfrak{h}}$. We have the following corollary.

Corollary 28. *A leafwise form ω on \mathcal{F} has a closed extension if and only if $d_{\mathfrak{h}}[\omega] = 0$.*

For every leaf L of the foliation, there is a map $H^k(\mathcal{F}, \nu^*) \rightarrow H^k(L, \nu^*|_L)$. Therefore, if a leafwise symplectic form is to have a closed extension, we must have $d_{\mathfrak{h}}[\omega]|_L = 0$. This is not a sufficient condition, however, as the following example shows.

Example. Consider the Reeb foliation of \mathbb{S}^3 . We denote the compact leaf by T . Pick a small tubular neighborhood $T \times (-1, 1)$ around T and a symplectic form ω_T on T . Equip the tubular neighborhood with the 2-form given by pulling back along the projection $T \times (-1, 1) \rightarrow T$. Restrict the resulting form to the leaves to get a leafwise form near T , and extend this leafwise form to get a leafwise symplectic form ω on all of \mathcal{F} .

We claim that the leafwise symplectic form has no closed extension, even though $d_{\mathfrak{h}}[\omega]|_L = 0$ for every leaf L . There is no closed extension because the Reeb foliation is not taut. The only leaf with $H^2(L, \nu^*) \neq 0$ is the compact leaf T (note that ν^* is a trivial bundle since there is no linear holonomy). But we can find an extension of ω that is closed near T (we pick the extension to equal the pullback we originally constructed on $T \times (-1, 1)$), which shows that $d_{\mathfrak{h}}[\omega]$ vanishes when restricted to L . (By the geometric interpretation of monodromy, we have shown that this Poisson manifold has trivial monodromy.) \blacktriangleleft

2.6 The cosymplectic case

In this section we discuss manifolds with cosymplectic structures. A cosymplectic structure is a special case of a calibrated structure.

Definition 29. A *cosymplectic structure* on M is a closed non-vanishing 1-form $\alpha \in \Omega^1(M)$ together with a closed 2-form $\omega \in \Omega^2(M)$ such that $(M, \ker(\alpha), \omega)$ is a calibrated foliation on M .

In other words, a cosymplectic structure is a calibrated foliation for which the foliation is determined by a *closed* 1-form. In particular, the foliation has no holonomy. Note that ω being non-degenerate on $\ker(\alpha)$ is equivalent to $\alpha \wedge \omega^n$ being non-vanishing.

Proposition 30. *Let (M, \mathcal{F}, ω) be a calibrated symplectic foliation. Write π for the induced Poisson bivector on M . Then the following are equivalent:*

1. *There is a closed 1-form α defining \mathcal{F} (i.e. the calibration comes from a cosymplectic structure).*
2. *There is a volume form on M invariant under all Hamiltonian vector fields (one says that the Poisson structure is unimodular).*
3. *There is a Poisson vector field E transverse to \mathcal{F} .*

Proof. (1) \Rightarrow (2) Take $\mu = \alpha \wedge \omega^n$, a volume form on M . Then μ is invariant under Hamiltonian vector fields:

$$\begin{aligned} \mathcal{L}_{X_f}(\mu) &= \mathcal{L}_{X_f}(\alpha \wedge \omega^n) \\ &= (\mathcal{L}_{X_f}\alpha) \wedge \omega^n + \alpha \wedge (\mathcal{L}_{X_f}\omega^n) \end{aligned}$$

The last of these terms vanishes because $\mathcal{L}_{X_f}\omega^n = 0$ along \mathcal{F} . We get

$$\begin{aligned}\mathcal{L}_{X_f}(\mu) &= (\mathcal{L}_{X_f}\alpha) \wedge \omega^n \\ &= (di_{X_f}\alpha + i_{X_f}d\alpha) \wedge \omega^n \\ &= (0 + 0) \wedge \omega^n \\ &= 0.\end{aligned}$$

(2) \Rightarrow (1) Let μ be a volume form invariant under Hamiltonian flows. Then let α be the unique 1-form on M defining \mathcal{F} that also satisfies $\alpha \wedge \omega^n = \mu$. By the calculation above, we have that $i_{X_f}d\alpha = 0$ for any Hamiltonian function f . This means that $d\alpha$ vanishes whenever an \mathcal{F} -tangential vector is plugged in, so it only takes nonzero values in the normal direction. But there is only one normal direction, and $d\alpha$ is a 2-form. This shows that $d\alpha = 0$.

(1) \Rightarrow (3) Let E be defined by $i_E\omega = 0$ and $\alpha(E) = 1$. Then clearly $\mathcal{L}_E(\omega) = 0$, so ω is preserved by E . Also, $\mathcal{L}_E(\alpha) = d(i_E\alpha) = 0$, so α is preserved by E . Therefore, \mathcal{F} is also preserved by E . But π is determined by ω and \mathcal{F} , and is therefore preserved by E . This shows that E is Poisson.

(3) \Rightarrow (1) If $\mathcal{L}_E\pi = 0$ then \mathcal{F} is preserved by E . Let α be the 1-form determining \mathcal{F} such that $\alpha(E) = 1$. Now $\mathcal{L}_E(E) = [E, E] = 0$, so the flow of E preserves E and \mathcal{F} and therefore also α . This means that $0 = \mathcal{L}_E\alpha = i_E(d\alpha)$. But then $i_E d\alpha = 0$ and $\alpha \wedge d\alpha = 0$, so we must have $d\alpha = 0$. This proves the result. \square

Cosymplectic structures have been completely characterized in [14]. They are quite restricted. For example, if a cosymplectic structure has a compact leaf, then all leaves are compact, and the manifold is a mapping torus of a compact leaf.

2.7 Transverse geometry in the calibrated case

In [32], the authors prove the following result using approximately holomorphic techniques following Donaldson. We refer the reader to [24, section 5.4] for the notions of essential equivalence and Morita equivalence between groupoids.

Theorem 31. *Let M be a closed smooth manifold and (\mathcal{F}, ω) a calibratable symplectic codimension-1 foliation. Then there exists a 3-dimensional closed submanifold W of*

M , transverse to \mathcal{F} , which inherits a taut foliation \mathcal{F}_W from \mathcal{F} , and such that the map between holonomy groupoids $\text{Hol}(\mathcal{F}_W) \rightarrow \text{Hol}(\mathcal{F})$ is an essential equivalence.

In other words, symplectic codimension-1 foliations have transverse geometry that is captured completely by the induced foliation on some 3-dimensional submanifold.

They also prove the following version for homotopy groupoids.

Theorem 32. *Let M be a closed smooth manifold and (\mathcal{F}, ω) a calibratable symplectic codimension-1 foliation. Then there exists a 5-dimensional closed submanifold W of M , transverse to \mathcal{F} , which inherits a taut foliation \mathcal{F}_W from \mathcal{F} , and such that the map between homotopy groupoids $\Pi_1(\mathcal{F}_W) \rightarrow \Pi_1(\mathcal{F})$ is an essential equivalence.*

These statements do not relate the symplectic geometry of the leaves of W to that of M . To gain insight into the symplectic geometry of this situation, one can wonder how the symplectic groupoids integrating M and W (which are both Poisson manifolds) are related.

The following proposition shows that a Morita equivalence on Π_1 is equivalent to a Morita equivalence on symplectic integrations.

Proposition 33. *Let M be a closed smooth manifold and (\mathcal{F}, ω) a symplectic codimension-1 foliation with a calibration. Let W be a closed submanifold of M , transverse to \mathcal{F} . It inherits a taut foliation \mathcal{F}_W from \mathcal{F} . The map between source-simply connected symplectic integrations*

$$\Sigma(W) \rightarrow \Sigma(M)$$

is a symplectic Morita equivalence if and only if the map between fundamental groupoids

$$\Pi_1(\mathcal{F}_W) \rightarrow \Pi_1(\mathcal{F})$$

is a Morita equivalence.

Proof. A map between Lie groupoids is a Morita equivalence if and only if it induces a homeomorphism on leaf spaces, isomorphisms on isotropy groups and isomorphisms of normal representations [10, theorem 4.3.1]. The maps $\Sigma(W) \rightarrow \Sigma(M)$ and $\Pi_1(\mathcal{F}_W) \rightarrow \Pi_1(\mathcal{F})$ induce the same maps on leaf spaces. The normal space at any point $x \in M$, for any of the groupoids, is the normal space to the foliation, and the action on it is given by holonomy. It therefore suffices to check that one map induces isomorphisms on isotropy groups if and only if the other does.

Suppose $x \in W$. Write L for the leaf of M that contains x , and L_W for the leaf of W that contains x . By proposition 3.21 in [8], there exists a diagram as follows, with exact rows:

$$\begin{array}{ccccccccc}
\pi_2(L_W, x) & \xrightarrow{\partial} & \nu_x^* & \longrightarrow & \Sigma(W)_x & \longrightarrow & \pi_1(L_W, x) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi_2(L, x) & \xrightarrow{\partial} & \nu_x^* & \longrightarrow & \Sigma(M)_x & \longrightarrow & \pi_1(L, x) & \longrightarrow & 1
\end{array}$$

The maps ∂ are the monodromy homomorphisms of the cotangent algebroids of W and M , and because ω has a closed extension, we have $\partial = 0$. We therefore have the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \nu_x^* & \longrightarrow & \Sigma(W)_x & \longrightarrow & \pi_1(L_W, x) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \nu_x^* & \longrightarrow & \Sigma(M)_x & \longrightarrow & \pi_1(L, x) & \longrightarrow & 1
\end{array}$$

The map $\nu_x^* \rightarrow \nu_x^*$ is the identity, and by the five lemma we see that the map $\Sigma(W)_x \rightarrow \Sigma(M)_x$ is an isomorphism if and only if the map $\pi_1(L_W, x) \rightarrow \pi_1(L, x)$ is. \square

If we have a symplectic foliation of codimension 1 with a calibration, we can essentially capture its symplectic integration using a 5-dimensional transverse submanifold. A 3-dimensional submanifold suffices to capture the transverse geometry of the foliation. As the following example shows, it is not generally possible to capture the Morita equivalence class of the symplectic integration using a 3-dimensional submanifold.

Example. Consider $M = \mathbb{T}^4 \times \mathbb{S}^1$ with the foliation with leaves $\mathbb{T}^4 \times \{*\}$. Equip M with a leafwise symplectic form, such as $d\theta_1 \wedge d\theta_2 + d\theta_3 \wedge d\theta_4$ (where the θ_i are angular coordinates on \mathbb{T}^4). Then M has no 3-dimensional closed transverse submanifold W for which $\Sigma(W) \rightarrow \Sigma(M)$ is a Morita equivalence. Indeed, by the lemma above this would mean that $L_W \hookrightarrow \mathbb{T}^4$ induces an isomorphism on π_1 , where L_W is a closed surface. But there is no closed surface with fundamental group isomorphic to $\pi_1(\mathbb{T}^4)$. \blacktriangleleft

Chapter 3

Currents

In this chapter, we first review some elements of the theory of currents as developed by de Rham [9]. Currents are generalizations of singular chains, and one particular class of them, the so-called *foliation currents*, have been used successfully in the study of tautness of foliations. Foliation currents are currents that “lie tangent to” a given foliation.

Foliation currents can be generalized to structure currents, which are required to “lie tangent to” a given cone structure. This setup is much more general. The goal of this chapter is to show that these structure currents are interesting in the study of calibrated symplectic foliations. For a limited class of symplectic foliations, they allow the study of calibratability to be done in a leafwise fashion. We unify two conjectures due to Sullivan and Ghys, and try to give the reader some intuition for why the new conjecture might hold.

Throughout this chapter, we will assume that M is a smooth manifold of dimension m . The codimension of the foliation under consideration will be q .

3.1 Definitions

We start by recalling the theory of currents, originally due to de Rham [9]. All of the material here can be found in chapter 10 of [6]. Write $\Omega^p(M)$ for the space of all p -forms on M , equipped with the C^∞ -topology. Write $\Omega_c^p(M)$ for the space of all compactly supported p -forms on M , equipped with the C^∞ -topology.

Definition 34. A p -current on M is a continuous linear map $\Omega_c^p(M) \rightarrow \mathbb{R}$ ($0 \leq p \leq m$). We write $\mathcal{C}_p(M) = \mathcal{C}_p$ for the space of all p -currents on M . A *compactly supported p -current* on M is a continuous linear map $\Omega^p(M) \rightarrow \mathbb{R}$ ($0 \leq p \leq m$). We write $\mathcal{C}_p^c(M)$ for the space of all compactly supported p -currents on M .

We will consider $\mathcal{C}_p^c(M)$ as a subset of $\mathcal{C}_p(M)$ by restriction.

Remark. The *support* of a current c is defined in the way that is usual for distributions (generalized functions): a point $x \in M$ is not in the support if it has a neighborhood U such that $c|_U = 0$. This definition agrees with our use of “compactly supported” above.

Example (Dirac currents). Let $v \in \Lambda^p TM$. Then we can consider v as a current

$$\Omega^p(M) \rightarrow \mathbb{R} : \alpha \mapsto \alpha(v).$$

Such currents are called *Dirac currents*. They are compactly supported. ◀

Example (Smooth currents). Assume that M is oriented, and let $\beta \in \Omega^q(M)$, where $p + q = m$. Then we can consider β as a current

$$\Omega_c^p(M) \rightarrow \mathbb{R} : \alpha \mapsto \int_M \beta \wedge \alpha.$$

Such currents are called *smooth currents*, or also *diffuse currents*. Note that β is compactly supported as a current precisely if it is compactly supported as a form. ◀

Example (Smooth singular chains). Every smooth p -simplex $c : \Delta_p \rightarrow M$ can be considered as a (compactly supported) p -current

$$\Omega^p(M) \rightarrow \mathbb{R} : \alpha \mapsto \int_{\Delta_p} c^* \alpha.$$

More generally, every smooth singular p -chain can be considered as a p -current. Intuitively, these currents are not as concentrated as a Dirac current, and not as spread out as a diffuse current. ◀

While currents are sometimes used as generalizations of differential forms (they are “distributional forms”), we will take the viewpoint that currents are generalizations of chains. Indeed, currents can be naturally pushed forward, but not pulled back. A Dirac current should be thought of as an infinitesimally small singular p -chain that is infinitely concentrated, where the “infinitesimally” and the “infinitely” are balanced out to get a finite number when integrating a p -form over it (much like the Dirac delta function is like a peak that is infinitesimally narrow and infinitely high, where the “infinitesimally” and the “infinitely” are balanced out to get finite answers when integrating). A smooth current can be thought of as a singular chain that has been spread out smoothly over the manifold (it has “diffused”).

We now focus our attention on the case where M is compact. In this case, we do not need to keep track of the compactness of supports. We equip the space $\mathcal{C}_p(M)$ with a topology. We will say that a subset of $\Omega^p(M)$ is bounded if it is bounded relative to each C^k -norm. The topology on \mathcal{C}_p is generated by sets of the form

$$\{c \in \mathcal{C}_p(M) \mid |c(\alpha)| < \varepsilon \text{ for all } \alpha \in B\},$$

where $\varepsilon > 0$ and $B \subset \Omega^p(M)$ is bounded. This topology on $\mathcal{C}_p(M)$ has several nice properties.

Proposition 35 (de Rham [9, theorem 13]). *The topology on $\mathcal{C}_p(M)$ as described above turns the space of p -currents into a locally convex topological vector space. It is Hausdorff. Moreover, the continuous linear functionals on $\mathcal{C}_p(M)$ are precisely of the form $\mathcal{C}_p(M) \rightarrow \mathbb{R} : c \mapsto c(\alpha)$ for some $\alpha \in \Omega^p(M)$. In other words, $\Omega^p(M)$ and $\mathcal{C}_p(M)$ are each other's continuous dual.*

We have seen that currents generalize singular chains. As in the case of chains, we can define the *boundary* of a current, as follows. Exterior differentiation $d : \Omega^p(M) \rightarrow \Omega^{p+1}(M)$ has a dual map $\partial : \mathcal{C}_{p+1}(M) \rightarrow \mathcal{C}_p(M)$ defined by

$$(\partial c)(\alpha) = c(d\alpha).$$

This map ∂ is the boundary operator on currents.

Definition 36. A current $c \in \mathcal{C}_p(M)$ is *closed* if $\partial c = 0$. A current $c \in \mathcal{C}_p(M)$ is *exact* if $c = \partial c'$ for some current $c' \in \mathcal{C}_{p+1}(M)$. We will write $\mathcal{Z}_p(M)$ for the space of closed p -currents on M , and $\mathcal{B}_p(M)$ for the space of exact p -currents on M .

Lemma 37 (see [6, lemma 10.1.24]). *The sets $\mathcal{Z}_p(M)$ and $\mathcal{B}_p(M)$ are closed subspaces of $\mathcal{C}_p(M)$.*

Under the inclusion of singular chains into the space of currents, this boundary operator corresponds precisely to the usual boundary operator (this is Stokes' theorem). Moreover, a differential form is closed iff it is closed as a current, and a differential form is exact iff it is exact as a current.

The following definition and result are due to de Rham [9].

Definition 38. The *de Rham homology* of M is defined as

$$H_p^{\text{dR}}(M) = \frac{\mathcal{Z}_p(M)}{\mathcal{B}_p(M)}.$$

Theorem 39 (de Rham). *The pairing $H_p^{\text{dR}}(M) \times H^p(M; \mathbb{R}) \rightarrow \mathbb{R} : ([c], [\alpha]) \mapsto c(\alpha)$ is well-defined and determines an isomorphism*

$$H_p^{\text{dR}}(M) \rightarrow H^p(M; \mathbb{R})' \cong H_p(M; \mathbb{R}).$$

The inclusion of ordinary p -cycles into $\mathcal{C}_p(M)$ descends to homology to give the inverse map

$$H_p(M; \mathbb{R}) \rightarrow H_p^{\text{dR}}(M).$$

(Here, $H^p(M; \mathbb{R})'$ is the dual of $H^p(M; \mathbb{R})$. Note that these are finite-dimensional because M is assumed to be compact.)

3.2 Foliation currents

Suppose that M is a compact manifold equipped with an oriented foliation \mathcal{F} of dimension p . There is a special class of currents associated to \mathcal{F} , the so-called *foliation currents*, first introduced by Sullivan [29]. The book [6] is also a good reference for this material.

Definition 40. A *Dirac foliation current* for \mathcal{F} (or *Dirac \mathcal{F} -current* for short) is a p -current of the form

$$\Omega^p(M) \rightarrow \mathbb{R} : \alpha \mapsto \alpha(v_x),$$

where $x \in M$ and $v \in \Lambda^p T_x \mathcal{F}$ are fixed, and v is positively oriented with respect to the orientation of \mathcal{F} .

Intuitively, a Dirac foliation current is a Dirac current that measures “along \mathcal{F} ”. To define what a foliation current is in general, we need to consider the cone generated by Dirac foliation currents.

Definition 41. A *cone* in a real vector space is a non-empty subset of the vector space that is invariant under multiplication by a non-negative number.

The cone generated by the Dirac foliation currents for \mathcal{F} (i.e. the smallest cone in $\mathcal{C}_p(M)$ that contains all Dirac foliation currents for \mathcal{F}) is not closed. Note also that this cone intersects $\mathcal{Z}_p(M)$ only at 0 (if $p \geq 1$). Once we take the closure of this cone, we obtain the cone of \mathcal{F} -currents.

Definition 42. A *foliation current* for \mathcal{F} (or \mathcal{F} -current for short) is a p -current in the closure of the cone in $\mathcal{C}_p(M)$ spanned by Dirac foliation currents. We write $\mathcal{C}_{\mathcal{F}}$ for the space of foliation currents. We also write $\mathcal{Z}_{\mathcal{F}} = \mathcal{C}_{\mathcal{F}} \cap \mathcal{Z}_p$ for the space of *foliation cycles* for \mathcal{F} , and $\mathcal{B}_{\mathcal{F}} = \mathcal{C}_{\mathcal{F}} \cap \mathcal{B}_p$ for the space of *foliation boundaries* for \mathcal{F} .

Example (Compact leaf). Suppose that \mathcal{F} is an oriented p -dimensional foliation on M , and suppose that \mathcal{F} has a compact leaf L . Then this leaf determines a current

$$c : \Omega^p(M) \rightarrow \mathbb{R} : \alpha \mapsto \int_L \alpha.$$

This current is closed. It is also an \mathcal{F} -current: the integral $\int_L \alpha$ can be approximated by Riemann sums, and this approximation shows how to approximate c by Dirac foliation currents.

If, instead, we take some open subset $U \subset L$ of the leaf, we can consider the current $\alpha \mapsto \int_U \alpha$. It is still a foliation current, but it is not, in general, closed. ◀

Example (Compact oriented manifold). Suppose that M is a compact oriented manifold. We can consider the codimension-0 foliation on M (its only leaf is M). One of the foliation currents for this foliation is the manifold itself:

$$\alpha \mapsto \int_M \alpha.$$

This is just a particular case of the example above, but for this (rather uninteresting) foliation, all foliation cycles are easily understood: they are precisely the multiples of this one (this statement is a variation of exercise 10.1.25 in [6]). One way to see this is as follows: suppose that c is a foliation cycle on M . Then there is a constant $\lambda \in \mathbb{R}$ such that $[c] = \lambda[M] \in H_m(M; \mathbb{R})$. Therefore, the foliation cycle $c - \lambda M$ is exact. But if a current is exact, it vanishes on all closed forms, and all top-degree forms are closed. Therefore $c - \lambda M$ must be the current 0, showing that $c = \lambda M$ as currents.

This observation agrees with the intuition that the only way to piece together infinitely many infinitesimally small top-dimensional singular simplices, and end up without boundary, is to end up with a multiple of the entire manifold. ◀

The importance of foliation currents is illustrated by the following theorem.

Theorem 43 (Sullivan, [29, theorem II.2]). *Let (M, \mathcal{F}) be a compact, connected manifold with an oriented and co-oriented codimension-1 foliation on it. Then the following are equivalent:*

1. *there is a closed (resp. exact) p -form on M that is non-vanishing on every leaf;*
2. *0 is the only foliation boundary (resp. cycle) for \mathcal{F} .*

This result shows the equivalence of items (c) and (d) in theorem 9. The next result characterizes the cohomology classes of closed forms that can be realized by closed forms as in theorem 43, in certain cases.

Theorem 44 (Sullivan, [29, theorem I.7]). *Let (M, \mathcal{F}) be a compact, connected manifold with an oriented and co-oriented codimension-1 foliation on it. Suppose that there exists a closed p -form on M that is non-vanishing on every leaf, but not an exact one. Let $C_{\mathcal{F}}$ be the image of $\mathcal{Z}_{\mathcal{F}}$ inside $H_p(M; \mathbb{R})$. Then $C_{\mathcal{F}}$ is a compact, convex cone. The dual cone*

$$C^{\mathcal{F}} = \{[\alpha] \in H^p(M; \mathbb{R}) \mid [\alpha](c) \geq 0 \text{ for all } c \in C_{\mathcal{F}}\} \subset H^p(M; \mathbb{R})$$

has non-empty interior consisting of precisely those cohomology classes that can be represented by a form that is non-vanishing on every leaf.

Example (Linear foliation of the 2-torus). We will use the linear foliation of the 2-torus (an example we gave on page 9). We orient the torus as $dx \wedge dy > 0$. Consider the vector field

$$V = \cos(\theta) \frac{\partial}{\partial x} + \sin(\theta) \frac{\partial}{\partial y}.$$

The foliation $T\mathcal{F}$ is spanned by V , and we will orient the foliation by prescribing that this vector field be positively oriented. (So changing θ to $\theta + \pi$ gives the same foliation, with opposite orientation.) Now $H^1(M; \mathbb{R})$ is 2-dimensional and spanned by $[dx]$ and $[dy]$. The dual basis in $H_1(M; \mathbb{R})$ is given by $\{[\gamma_x], [\gamma_y]\}$, circles running in the x - and y -directions respectively. We will calculate the cones $C_{\mathcal{F}}$ and $C^{\mathcal{F}}$.

Consider the foliation current

$$c : \Omega^1(M) \rightarrow \mathbb{R} : \beta \mapsto \int_0^1 \int_0^1 \beta(V) dx dy.$$

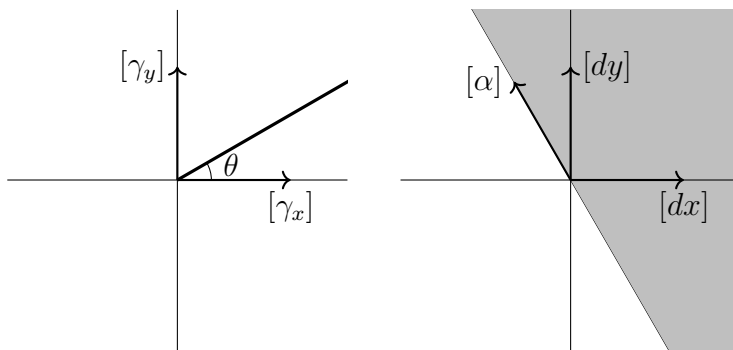


Figure 3.1: On the left a picture of $H_1(\mathbb{S}^1 \times \mathbb{S}^1; \mathbb{R})$. The thick ray is the cone $C_{\mathcal{F}}$. On the right a picture of $H^1(\mathbb{S}^1 \times \mathbb{S}^1; \mathbb{R})$. The shaded region, including its boundary, is the cone $C^{\mathcal{F}}$. Its interior consists of those classes that can be realized with a closed 1-form that is positive on each leaf.

It is closed. Moreover, $c(dx) = \cos(\theta)$ and $c(dy) = \sin(\theta)$ so that the class of c in $H_1(M; \mathbb{R})$ is $\cos(\theta)[\gamma_x] + \sin(\theta)[\gamma_y]$. We conclude that

$$C_{\mathcal{F}} \supset \{\lambda(\cos(\theta)[\gamma_x] + \sin(\theta)[\gamma_y]) \mid \lambda \geq 0\}. \quad (\dagger)$$

Now consider the closed form

$$a dx + b dy,$$

where $a, b \in \mathbb{R}$. Whenever $a \cos(\theta) + b \sin(\theta) > 0$, this is a closed form that is positive on every leaf. Because $C^{\mathcal{F}}$ is closed, this shows that

$$C^{\mathcal{F}} \supset \{a[dx] + b[dy] \mid a \cos(\theta) + b \sin(\theta) \geq 0\}. \quad (\ddagger)$$

Since $C^{\mathcal{F}}$ is a closed cone dual to $C_{\mathcal{F}}$, we can conclude that the inclusions (\dagger) and (\ddagger) are in fact equalities. These cones are shown in figure 3.1. \blacktriangleleft

We saw earlier that on a compact oriented manifold, all foliation cycles of the codimension-0 foliation are multiples of the manifold. While that is an easy consequence of de Rham homology, there is vast generalization of this statement that is much harder to prove. It is also due to Sullivan. We first need to introduce some language.

Suppose that \mathcal{F} is an oriented foliation. Fix a regular oriented foliated atlas for \mathcal{F} , indexed by $i \in I$. Pick a complete transversal T_i in each chart (so that T_i is the space of plaques in this chart).

Definition 45. A *transverse measure* for \mathcal{F} is a measure on $\bigsqcup_{i \in I} T_i$ that is invariant under holonomy.

Such transverse measures are often called *transverse invariant measures* to stress their invariance under holonomy. Transverse measures should be thought of as measures on the leaf space M/\mathcal{F} . Note that the choice of transversals does not matter: if we choose two different families of transversals, there is a bijection between holonomy-invariant measures for one family and the other.

If we have a transverse measure μ that is finite on compact sets, we can associate to it a foliation cycle c_μ as follows (see [6, section 10.1]). Choose a partition of unity $\{f_i\}_{i \in I}$ subordinate to the foliation atlas. The current c_μ is defined as

$$c_\mu : \Omega^p(M) \rightarrow \mathbb{R} : \alpha \mapsto \sum_{i \in I} \int_{T_i} \left(\int_{\text{plaque}} f_i \alpha \right) d\mu|_{T_i}.$$

That is, in each chart we integrate $f_i \alpha$ over each plaque. This results in a function on the transversal T_i , which we then integrate using the measure μ . We sum the contributions of all charts, giving us $c_\mu(\alpha)$. This current does not depend on the choice of partition of unity, and it is a foliation cycle.

We are now ready to state Sullivan's result.

Theorem 46 (Sullivan, [29]). *The correspondence $\mu \mapsto c_\mu$ is a bijection between*

$$\left\{ \begin{array}{l} \text{transverse measures for } \mathcal{F} \text{ that are} \\ \text{finite on compact sets} \end{array} \right\} \text{ and } \{ \text{foliation cycles for } \mathcal{F} \}.$$

Example (Foliation cycles for the Reeb foliation of \mathbb{S}^3). Let \mathcal{F} be the Reeb foliation of \mathbb{S}^3 . Write L for the torus leaf (this is the only compact leaf). Choose $x \in L$. Choose a short compact transversal T through x . This transversal intersects every leaf, so a transverse measure is completely determined by its restriction to T .

We claim that every transverse measure is supported on $\{x\}$. Choose y_0 in the interior of T close to (but different from) x . We will show that y_0 is not in the support of μ . One of the elements $[\gamma] \in \pi_1(L, x)$ has holonomy that maps y_0 to a point $y_1 \in T$ that is closer to x . Transport this point y_1 to obtain another point $y_2 \in T$, even closer to x . Continuing this way, we obtain a sequence of points y_0, y_1, y_2, \dots in T converging to x . By holonomy-invariance, each of the intervals (y_{2k}, y_{2k+2}) ($k \in \mathbb{Z}_{\geq 0}$) has the same measure. They are disjoint, and their union has finite measure. Therefore, each of them has measure 0. This shows that y_1 is not in the support of μ , and therefore y_0 is

not either. This proves that any transverse measure for \mathcal{F} is a multiple of the torus leaf (when considered as a transverse measure, it is a multiple of the Dirac measure supported on the torus leaf). ◀

3.3 Structure currents

Foliation currents are a very special class of currents. Sullivan developed a more general theory, that of *structure currents*. These structure currents can be used to study more general structures on manifolds than just foliations. We review the essential parts of this theory.

Let M be a manifold of dimension m , and let $0 \leq p \leq m$.

Definition 47. A *cone structure* on M is a subset C of $\Lambda^p TM$ that satisfies the following properties:

1. for each $x \in M$, the fiber $C_x = C \cap \Lambda^p T_x M$ is a convex cone in $\Lambda^p T_x M$;
2. there exists a $\beta \in \Omega^p(M)$ such that $\beta(v) > 0$ for every non-zero $v \in C$.

This definition is more general than Sullivan's [29, definition 1.2] in two ways:

- the continuity requirement on the cones is replaced by the second condition above; this is a weaker condition, yet it suffices in all the proofs;
- we do not require that each cone C_x be compact.

We will require the compactness separately when needed.

Definition 48. We say that a cone structure is *compact* if the image of $C \setminus \{0\}$ under the map

$$\Lambda^p TM \setminus \{0\} \rightarrow (\Lambda^p TM \setminus \{0\})/\mathbb{R}_{>0}$$

is compact.

In other words, a cone structure is compact if the set of all its directions is compact. Suppose that C is a cone structure on M . The following are generalizations of (Dirac) foliation currents.

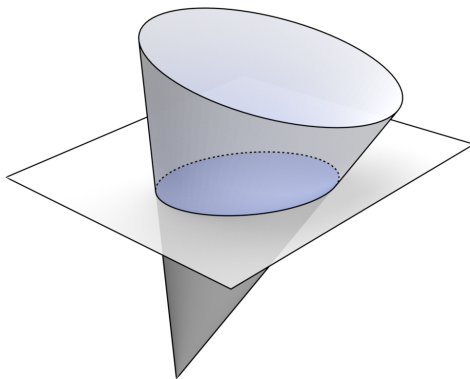


Figure 3.2: Low-dimensional analogue of the cone \mathcal{C}_C inside of \mathcal{C}_p . The cone has its tip at the origin. Intersecting it with the affine hyperplane $\beta^{-1}(1)$ gives us the base of the cone, shown in blue. This base is convex. If C is a compact cone structure, the base of \mathcal{C}_C is also compact.

Definition 49. A *Dirac structure current* for C is a current of the form

$$\Omega^p(M) \rightarrow \mathbb{R} : \alpha \mapsto \alpha(v)$$

where $v \in C$.

Definition 50. A *structure current* for C (or *C-current*) is a current that lies in the closure of the cone in $\mathcal{C}_p(M)$ spanned by Dirac structure currents for C . We write $\mathcal{C}_C = \mathcal{C}_C(M)$ for the space of structure currents. We also write $\mathcal{Z}_C = \mathcal{C}_C(M) \cap \mathcal{Z}_p$ for the *structure cycles* (or *C-cycles*), and $\mathcal{B}_C = \mathcal{C}_C(M) \cap \mathcal{B}_p$ for the *structure boundaries* (or *C-boundaries*).

If $\beta \in \Omega^p(M)$ is as in the definition of a cone structure, we will refer to $\mathcal{C}_C \cap \beta^{-1}(1)$ as a *base* of the cone \mathcal{C}_C (see figure 3.2). Here is a simple but important observation about compact cone structures.

Lemma 51 (see [29]). *If C is a compact cone structure, the base $\mathcal{C}_C \cap \beta^{-1}(1)$ is compact.*

Definition 52. Suppose that C is a cone structure on M . A p -form α on M is said to be *positive on C* if $\alpha(v) > 0$ for all non-zero $v \in C$.

Example. Suppose that M is a manifold equipped with an oriented codimension-1 foliation \mathcal{F} (so $p = m - 1$). Then $\Lambda^p TM$ is a line bundle over M . Taking C_x to be

the ray in $\Lambda^p T_x M$ determined by the chosen orientation of \mathcal{F} , we see that structure currents are precisely foliation currents in this case, and the existence of a closed form that is positive on C amounts to tautness of \mathcal{F} . ◀

Example. Let us give a toy example. We will use $M = \mathbb{R}^2$ and $p = 1$. Choose a non-vanishing vector field X on \mathbb{R}^2 . Set

$$C = \{\lambda X_x \mid \lambda \in \mathbb{R}_{\geq 0}, x \in \mathbb{R}^2\}.$$

In other words, C consists of the rays in $T\mathbb{R}^2$ determined by X . Note that C is really the structure cone for the foliation obtained as the flow of X .

We claim that there exists a closed form positive on C precisely if there is a dimension-1 foliation \mathcal{F} of the plane such that \mathcal{F} and X are transverse. If there exists a closed form that is positive on C , then its kernel defines such a foliation \mathcal{F} . Suppose, conversely, that there is such a foliation \mathcal{F} . Every foliation of the plane is the collection of (the connected components of) level sets of a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ with non-vanishing gradient [17] (see also [18, theorem 42] and [16, section 2] for more information about foliations of the plane). Then df (or $-df$) is a closed form that is positive on C . ◀

Example. Suppose that M is a manifold equipped with an almost complex structure $J : TM \rightarrow TM$, $J^2 = -\text{Id}$. Choose

$$C = \{v \wedge J(v) \mid v \in TM\}.$$

Then a form α is positive on C precisely if $\alpha(v, J(v)) > 0$ for all non-zero $v \in TM$. The existence of a closed form that is positive on C therefore amounts to the existence of a symplectic structure on M that tames J . This example appears in [29, §10]. ◀

Example. Generalizing the previous example, suppose that (M, \mathcal{F}) is a foliated manifold, and $J : T\mathcal{F} \rightarrow T\mathcal{F}$ is a leafwise almost complex structure. We choose

$$C = \{v \wedge J(v) \mid v \in T\mathcal{F}\}.$$

Then a closed form that is positive on C is precisely a calibrated leafwise symplectic form that tames J . ◀

Example. We can also specialize to (the closure of) one specific leaf. Suppose that (M, \mathcal{F}) is a foliated manifold and that L is a leaf of \mathcal{F} . Suppose moreover that

$J : T\mathcal{F} \rightarrow T\mathcal{F}$ is a leafwise almost complex structure. We choose

$$C = \{v \wedge J(v) \mid v \in T_x\mathcal{F}, x \in \bar{L}\}.$$

Then a closed form is positive on C precisely if its restriction to L is a symplectic form for which $\omega(v, J(v))$ is bounded away from 0 for unit-length v . \blacktriangleleft

The following results are the generalizations of theorem 43 and theorem 44 to structure currents. Their proofs can be found in [29, theorem I.7] (and one can check that their proofs go through under our slightly weaker assumptions).

Theorem 53. *Suppose that C is a compact cone structure on a compact manifold M . Then the following are equivalent:*

1. *there is a closed (resp. exact) p -form on M that is positive on C ;*
2. *0 is the only structure boundary (resp. cycle) for C .*

Theorem 54. *Let C be a compact cone structure on a compact manifold M . Suppose that there is a closed p -form that is positive on C , but not an exact one. Let H_C be the image of \mathcal{Z}_C inside $H_p(M; \mathbb{R})$. Then H_C is a convex cone with compact base. The dual cone*

$$H^C = \{[\alpha] \in H^p(M; \mathbb{R}) \mid \alpha(c) \geq 0 \text{ for all } c \in H_C\} \subset H^p(M; \mathbb{R})$$

has non-empty interior consisting of precisely those cohomology classes that can be represented by a form that is positive on C .

This theorem shows that structure cycles can be used in trying to understand forms that are positive on a chosen cone structure C . The technical tool underlying the theorems above is a separation theorem that follows from the Hahn-Banach theorem of functional analysis. Even though we will not need to know anything about the proof of theorem 53, we will give a sketch of what is going on behind the scenes. Here is the technical heart of the matter.

Theorem 55 (Hahn-Banach separation theorem). *Let W be a locally convex topological real vector space. Suppose that $X \subset W$ is a non-empty compact convex subset, and $V \subset W$ a non-empty closed convex subset. If X and V are disjoint, then there is a continuous linear functional $\alpha : W \rightarrow \mathbb{R}$ and a real number d such that $\alpha(V) < d < \alpha(X)$.*

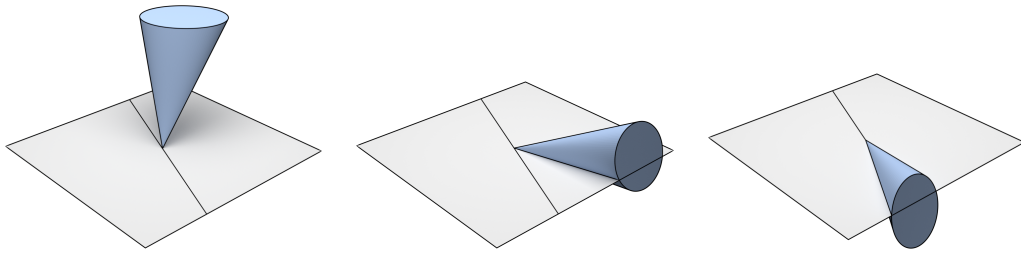


Figure 3.3: The possible ways that the cone \mathcal{C}_C can be positioned relative to \mathcal{Z}_p and \mathcal{B}_p . The blue cone represents \mathcal{C}_C . The space \mathcal{Z}_p is represented by the plane, and its subspace \mathcal{B}_p is in this case shown as a line. *Left:* the cone intersects \mathcal{Z}_p only at the tip; in this case there is an exact form positive on C . *Middle:* the cone intersects \mathcal{Z}_p non-trivially, but only touches \mathcal{B}_p at the tip; in this case there is a closed, but not an exact form that is positive on C . *Right:* the cone intersects \mathcal{B}_p non-trivially; there is no closed form that is positive on C .

Sketch of proof of theorem 53. Suppose there is a closed (resp. exact) p -form ω on M that is positive on C . Then ω takes positive values on all non-zero C -currents. However, it must vanish on exact (resp. closed) currents, and therefore 0 is the only structure boundary (resp. cycle) for C .

Conversely, suppose that 0 is the only structure boundary (resp. cycle) for C . This means that \mathcal{C}_C only touches the space \mathcal{B}_p (resp. \mathcal{Z}_p) at the origin. We can therefore, by Hahn-Banach, find a hyperplane containing \mathcal{B}_p (resp. \mathcal{Z}_p) that doesn't intersect the base of the cone. The corresponding linear functional $\mathcal{C}_p \rightarrow \mathbb{R}$ is precisely the closed (resp. exact) p -form we are looking for. \square

Figure 3.3 illustrates the possible ways that \mathcal{C}_C can be positioned relative to \mathcal{Z}_p and \mathcal{B}_p .

Let us list several interesting cone structures.

The following result gives some insight into the general form of structure currents.

Proposition 56 ([29, proposition I.8]). *Any structure current c can be represented as*

$$c = \int_M s d\mu,$$

(meaning $c(\alpha) = \int_M \alpha(s) d\mu$) where μ is a non-negative measure on M , and s is a μ -integrable section of $\Lambda^p TM$ taking values in C .

It is possible to multiply a structure current by a bounded measurable function, by multiplying the function s above by it.

3.4 Extreme currents

While foliation cycles can be understood using theorem 46, no such theorem is available for structure cycles in general. Nevertheless, some things can be said about them. Theorem 46 describes foliation cycles as “mixtures” of leaves. It is still true for structure cycles that they can be described as “mixtures” of certain structure currents. In fact, this is just a special case of a standard result in the analysis of locally convex topological vector spaces, the Krein-Milman theorem.

Definition 57. Let $A \subset V$ be a subset of a vector space. A point $a \in A$ is an *extreme point* of A if for all $b, c \in A$ and $t \in (0, 1)$ such that

$$a = tb + (1 - t)c$$

we must have $a = b = c$. In other words, a point is an extreme point if it is not a convex combination of two other points in A .

Theorem 58 (Krein-Milman). *A compact convex set in a locally convex topological vector space is the closure of the convex hull of its extreme points.*

If $A \subset V$ is a cone in a vector space, it has no non-zero extreme points. One can study *extreme rays* instead.

Definition 59. Let $A \subset V$ be a cone in a vector space. Let $a \in A \setminus \{0\}$. We say that the ray $[a] = \{\lambda a \mid \lambda \in \mathbb{R}_{>0}\}$ is an *extreme ray* of A if for all $b, c \in A$ and $t \in (0, 1)$ such that

$$a = tb + (1 - t)c$$

we must have $b, c \in [a]$.

This leads us to the following definition.

Definition 60. A structure cycle $c \in \mathcal{Z}_C(M)$ is called an *extreme cycle* if its ray $[c]$ is an extreme ray of $\mathcal{Z}_C(M)$. A structure boundary $c \in \mathcal{B}_C(M)$ is called an *extreme boundary* if its ray $[c]$ is an extreme ray of $\mathcal{B}_C(M)$.

From the Krein-Milman theorem, together with lemma 51, we then have the following result.

Proposition 61. *Suppose that C is a compact cone structure. Every structure cycle for C can be approximated arbitrarily closely by a sum of positive multiples of extreme cycles. Every structure boundary for C can be approximated arbitrarily closely by a sum of positive multiples of extreme boundaries.*

Proof. We prove the result for cycles. The result for boundaries is analogous. Suppose that c is a structure cycle for C . Let β be as in the definition of a cone structure. Applying the Krein-Milman theorem to the base $\beta^{-1}(1) \cap \mathcal{Z}_C$ of the cone \mathcal{Z}_C (this is a compact convex set) shows that $c/\beta(c)$ can be approximated arbitrarily closely by a sum of positive multiples of extreme cycles. Therefore, the same holds for c . \square

The following theorem allows us to be more precise about “mixtures” of extreme currents. It says, roughly speaking, that every point in a compact convex set can be obtained as a weighted barycenter of extreme points.

Theorem 62 (Choquet-Bishop-de Leeuw, [3, theorem 5.6]). *Let X be a compact convex subset of a real locally convex topological vector space. Let E be the set of extreme points of X . Let \mathcal{S} be the σ -algebra generated by E and by the Baire subsets of X . Then each $x \in X$ has a representation of the form*

$$\int y d\mu(y)$$

for some non-negative measure μ on \mathcal{S} that satisfies

$$\mu(E) = \mu(X) = 1.$$

Remark. The theorem requires the use of the σ -algebra \mathcal{S} because, in general, the set of extreme points of X need not be Borel. It suffices to take the σ -algebra generated by the Baire sets (recall that Baire sets are Borel), and the set of extreme points.

The theorem above immediately leads to the following variation of proposition 61.

Proposition 63. *Suppose that C is a compact cone structure. Write E for the set of extreme structure cycles for C . Let \mathcal{S} be the σ -algebra generated by E and by the Baire subsets of \mathcal{Z}_C . Every structure cycle for C can be written in the form*

$$\int c' d\mu(c'),$$

where μ is a measure such that

$$\mu(E) = \mu(\mathcal{Z}_C) = 1.$$

Let us now focus on extreme currents for cone structures that lie tangent to a foliation. There is a straightforward restriction on what the support of an extreme cycle can look like.

Definition 64. Two leaves L_1, L_2 of a foliation are called *estranged* if there are saturated opens $U_1 \supset L_1$ and $U_2 \supset L_2$ such that $U_1 \cap U_2 = \emptyset$.

Proposition 65. *Suppose that $C \subset \Lambda^p T\mathcal{F}$ is a compact cone structure on M . If c is an extreme C -cycle, no two estranged leaves intersect its support.*

Proof. Suppose that c is C -cycle whose support intersects the estranged leaves L_1 and L_2 . Let $U_1 \supset L_1$ and $U_2 \supset L_2$ be disjoint saturated opens. By the lemma below, the decomposition

$$c = \mathbb{1}_{U_1}c + (1 - \mathbb{1}_{U_1})c$$

shows that c can be written as a non-trivial sum of two C -cycles. Therefore, c is not extreme (here, $\mathbb{1}_{U_1}$ is the indicator function of the set U_1). \square

If we think of currents as generalizations of singular chains, the following should come as no surprise. The proof is similar to that on pages 246–249 of [6].

Lemma 66. *Suppose that (M, \mathcal{F}) is a compact foliated manifold, and let $C \subset \Lambda^p T\mathcal{F}$ be a compact cone structure (so that C “lies tangent to” \mathcal{F}). Suppose that $f : M \rightarrow \mathbb{R}$ is a bounded measurable function that is constant along leaves. If $c \in \mathcal{C}_C(M)$ is closed, then so is fc .*

Proof. Using proposition 56, let s and μ be such that

$$c : \Omega^p(M) \rightarrow \mathbb{R} : \alpha \mapsto \int_M \alpha(s) d\mu.$$

Pick a point in M . We will prove that it is not in the support of $\partial(fc)$. Let $x_1, \dots, x_p, y_1, \dots, y_q$ be local foliated coordinates near the chosen point (with, let us say, all variables between -1 and 1 and the point at the origin). We will write $x = (x_1, \dots, x_{m-q})$ and $y = (y_1, \dots, y_q)$. Write $X = (-1, 1)^p$ and $Y = (-1, 1)^q$ so that points in the charts are parameterized by $(x, y) \in X \times Y$.

Let ν be the push-forward of μ along the projection $\pi : X \times Y \rightarrow Y$. In other words, let ν be the measure on Y given by

$$\nu(S) = \mu(\pi^{-1}(S)).$$

By a standard result in measure theory [5, theorem 1, page 58], the measure μ can be “disintegrated”, and there is a family of measures μ_y on X , indexed by $y \in Y$ such that

$$\int_{\text{chart}} g d\mu = \int_Y \left(\int_X g(x, y) d\mu_y(x) \right) d\nu(y)$$

for every μ -integrable function g on the chart.

For $y \in Y$, define the elements $c_y \in \mathcal{C}_p(X)$ by

$$c_y : \Omega_c^p(X) \rightarrow \mathbb{R} : \alpha \mapsto \int_X \alpha(s(x, y)) d\mu_y(x).$$

We will show that c_y is closed for ν -almost every $y \in Y$, using essentially the argument from [6, lemma 10.2.17]. Let $\beta \in \Omega_c^{p-1}(X)$ and define

$$g_\beta : Y \rightarrow \mathbb{R} : y \mapsto (\partial c_y)(\beta) = c_y(d\beta).$$

For any $h \in C^\infty(Y)$, we have

$$c_y(dh \wedge \beta) = \int_X (dh \wedge \beta)(s(x, y)) d\mu_y(x) = 0$$

because s takes values in $\Lambda^p T\mathcal{F}$ and dh annihilates $T\mathcal{F}$ (we have used h also as the name for the function $\pi^*h \in C^\infty(X \times Y)$). Therefore

$$c_y(hd\beta) = c_y(dh \wedge \beta + hd\beta) = c_y(d(h\beta)).$$

We now integrate h against the measure $g_\beta\nu$. This results in

$$\begin{aligned}
\int_Y h d(g_\beta\nu) &= \int_Y h g_\beta d\nu \\
&= \int_Y h c_y(d\beta) d\nu \\
&= \int_Y c_y(d(h\beta)) d\nu \\
&= \int_Y \int_X d(h\beta)(s(x, y)) d\mu_y(x) d\nu \\
&= \int_{\text{chart}} d(h\beta)(s(x, y)) d\mu \\
&= c(d(h\beta)) \\
&= 0
\end{aligned}$$

because c is closed. This shows that integrating any smooth function against the measure $g_\beta\nu$ gives 0, proving that $g_\beta\nu$ is the zero measure. Therefore, $(\partial c_y)(\beta) = 0$ for ν -almost all $y \in Y$. Because the topological space $\Omega_c^{p-1}(X)$ is separable, we can apply this argument for each β in a countable dense subset of $\Omega_c^{p-1}(X)$, and conclude that for ν -almost all $y \in Y$, the current c_y vanishes on all these β 's simultaneously (and is thus closed). This shows that c_y is closed for ν -almost every $y \in Y$.

It is now easy to show that the current $\partial(fc)$ vanishes at our chosen point. Let $\beta \in \Omega_c^{p-1}(X \times Y)$. It suffices to show that $(fc)(d\beta)$ vanishes. But

$$\begin{aligned}
(fc)(d\beta) &= \int_{\text{chart}} (fd\beta)(s) d\mu \\
&= \int_Y \int_X (fd\beta)(s) d\mu_y(x) d\nu \\
&= \int_Y \left(f \int_X (d\beta)(s) d\mu_y(x) \right) d\nu \\
&= 0
\end{aligned}$$

because the inner integral vanishes for ν -almost every y . This proves the result. \square

Definition 67. We say that a foliation \mathcal{F} on M is *pleasant* if every closed saturated subset $S \subset M$ that contains no pair of estranged leaves, is contained in the closure of some leaf.

The following are examples of pleasant foliations, showing that plenty of them exist:

- simple foliations (i.e. those obtained from submersions),

- foliations with only compact leaves,
- the Reeb foliation,
- foliations that have a dense leaf,
- the foliation obtained by gluing together (at the boundary) two pleasant foliations of manifolds with boundary (with the foliation tangent to the boundary components),
- the turbulization of a pleasant foliation.

Recall that H_C is the set of classes that can be represented by a C -cycle.

Proposition 68. *Suppose that \mathcal{F} is a pleasant foliation of a manifold M . Let $C \subset \Lambda^p T\mathcal{F}$ be a compact cone structure. Then we have*

$$H_C = \overline{\text{ConvexHull} \left(\bigcup_{\text{leaf } L} H_{C|_L} \right)},$$

where for each leaf L , the cone $C|_L$ is the intersection of C with $\Lambda^p T\bar{L}$.

Proof. By the Krein-Milman theorem (theorem 58), the cone \mathcal{Z}_C is the closure of the convex hull of its extreme rays. Therefore the image H_C of \mathcal{Z}_C under the projection $\mathcal{Z}_p \rightarrow H_p$ is the convex hull of the images of these extreme rays. Suppose now that $c \in \mathcal{Z}_C$ is extreme. Because \mathcal{F} is pleasant, this means that the support of c is contained in the closure of a single leaf L . But then c is also a $(C|_{\bar{L}})$ -cycle, and so $[c] \in H_{C|_{\bar{L}}} \subset H_p$. This proves the result. \square

3.5 Regularization of currents

Recall that a smooth p -current on an oriented manifold is a current of the form

$$\alpha \mapsto \int_M \beta \wedge \alpha$$

for a fixed $\beta \in \Omega^q(M)$ ($p + q = m$). Because smooth currents are easier to handle than other currents, it is convenient to be able to approximate currents by smooth currents. In [9], de Rham proved that such approximation is always possible, and that every closed current is homologous to a smooth one.

De Rham's regularization

For the reader's convenience, we give here a variation of the statement from [9, theorem 12, page 80]. The original statement is stronger and a little more general, but this version will do for our purposes.

Theorem 69 (de Rham). *Let M be a smooth compact manifold. Then one can construct linear operators $R : \mathcal{C}_p \rightarrow \mathcal{C}_p$ and $A : \mathcal{C}_p \rightarrow \mathcal{C}_{p+1}$, depending on a positive parameters ε , with the following properties:*

1. *For any p -current, $R(c)$ is smooth and*

$$R(c) - c = \partial A(c) + A(\partial c).$$

2. *For any neighborhood of the support c , one can find a sufficiently small $\varepsilon > 0$ such that the supports of $R(c)$ and $A(c)$ are contained in this neighborhood.*

Corollary 70 (de Rham). *Every closed current on a smooth compact manifold is homologous to a smooth current.*

Let us outline the construction of the regularization operator R as it is done by de Rham. We refer the reader to de Rham's book [9] for the details.

- Suppose c is a compactly supported p -current on \mathbb{R}^m . Let ρ be a smooth bump function supported in the ball of radius ε at the origin of \mathbb{R}^m , of integral 1. Let t_y be the translation $\mathbb{R}^m \rightarrow \mathbb{R}^m : x \mapsto x + y$. Then the convolution of c with ρ ,

$$\alpha \mapsto c \left(\int_y \rho(y) t_y^* \alpha \right),$$

is a smooth p -current on \mathbb{R}^m .

- Let $B^m \subset \mathbb{R}^m$ be the unit ball. There is a suitable diffeomorphism $h : \mathbb{R}^m \rightarrow B^m$, such that the maps $ht_y h^{-1}$ extend to smooth maps $\varphi_y : \mathbb{R}^m \rightarrow \mathbb{R}^m$ that are the identity outside B^m . Then if c is a compactly supported p -current,

$$R_{B^m, \varepsilon}(c) : \alpha \mapsto c \left(\int_y \rho(y) \varphi_y^* \alpha \right) \quad (*)$$

is a compactly supported p -current. It is smooth in B^m . It equals c outside of $\overline{B^m}$.

Using $R_{B^m, \varepsilon}$ we can smoothen a current locally on a manifold: suppose $B \subset M$ is the unit ball in a chart for M . Let $f \in C^\infty(M, [0, 1])$ be supported in the chart, and $f = 1$ on B . Then we have an operator

$$R_{B, \varepsilon} : \mathcal{C}_p(M) \rightarrow \mathcal{C}_p(M) : c \mapsto (1 - f)c + R_{B^m, \varepsilon}(fc).$$

This operator gives currents that are smooth in B .

- Given our manifold M , we cover it using balls B_1, \dots, B_k . Then the smoothing operator R is given by

$$R = R_{B_k, \varepsilon_k} \circ \dots \circ R_{B_1, \varepsilon_1}.$$

We do not describe the construction of A , since the details are not important to us.

Regularization of structure currents

Suppose that C is a compact cone structure on M . It is not generally true that every C -cycle is homologous to a smooth C -cycle. However, the following result shows that C -cycles are homologous to smooth currents that are very close to C -currents.

Proposition 71 (see [29, Theorem I.9]). *Let M be a smooth compact manifold. Suppose that $C \subset \Lambda^p TM$ is a compact cone structure. Let D be another compact cone structure, such that $C \subset \text{int}(D)$. Then by choosing the parameters ε_i for R sufficiently small, $R(c)$ will be a smooth D -current for every C -current c .*

This proposition follows from the following local result.

Proposition 72. *Let $C \subset S\Lambda^p T\mathbb{R}^m$ be a compact cone structure, and let D be a cone structure whose interior contains C . Let $\varepsilon > 0$ be so small that $(\varphi_y)_*C \subset D$ for all y with $|y| \leq \varepsilon$. Then for any compactly supported C -current on \mathbb{R}^m , the current $R_{B, \varepsilon}(c)$ is smooth in B , and over B it is a D -current.*

Proof of proposition 71 from proposition 72. Cover M with balls B_1, \dots, B_k . Choose compact sets $C \subset C_1 \subset C_2 \subset \dots \subset C_k = D$ such that C_1 is a neighborhood of C and C_{i+1} is a neighborhood of C_i . By taking ε_1 small enough, we can ensure that R_{B_1, ε_1} maps C -currents to C_1 -currents. By taking ε_2 small enough, we can ensure that R_{B_2, ε_2} maps C_1 -currents to C_2 -currents. We continue this way, finally picking ε_k small enough

such that R_{B_k, ε_k} maps C_{k-1} -currents to D -currents. By construction, this means that R will map C -currents to D -currents. This proves the result. \square

Proof of proposition 72. The crucial observation is that $(\varphi_y)_*C_y \subset D_y$ guarantees that $R_{B^m, \varepsilon}(c)$, given by formula (*), is a D -current. \square

3.6 The local structure of structure cycles

In light of theorems 53 and 54, it is interesting to better understand structure cycles. For foliation cycles, Sullivan's result (theorem 46) completely describes what they look like. For other structures, no such result is available. There are, however, conjectures. We state two conjectures, and generalize them to a single conjecture about structure cycles.

Unifying existing conjectures

In [29], Sullivan discusses the case of a compact complex manifold M^{2n} . For each $k \in \{0, \dots, n\}$, there is a natural cone structure in dimension $p = 2k$. At a point $x \in M$, it is the compact convex cone in $\Lambda^p T_x M$ generated by complex subspaces of complex dimension k . The corresponding structure cycles are referred to as *complex cycles*.

Conjecture 73 (Sullivan, [29, III.9]). *Complex cycles can be locally represented near x in M as $c = \int c' d\nu$, where the c' are irreducible subvarieties passing near x and ν is a non-negative measure on the space of such (regarded as a subspace of the currents on some neighborhood of x).*

An analogous conjecture for the case of a leafwise (almost) complex structure was made by Étienne Ghys, at the *Workshop on topological aspects of symplectic foliations* that took place at Université de Lyon 1 (France) in September 2017. Ghys' conjecture was not stated in precise mathematical terms. The following formulation is an attempt at making the conjecture more precise, by phrasing it similarly to Sullivan's conjecture. Any mistakes in formulating this conjecture are solely the author's.

Conjecture 74 (Ghys). *Suppose \mathcal{F} is a foliation of a manifold M , with a leafwise (almost) complex structure J . Let C be the structure cone associated to J :*

$$C = \{v \wedge J(v) \mid v \in T\mathcal{F}\}.$$

Then every C -cycle can be locally represented near $x \in M$ as $c = \int c' d\nu$, where the c' are pseudoholomorphic curves passing near x and ν is a non-negative measure on the space of such (regarded as a subspace of the currents on some neighborhood of x).

The conjectures above have a natural variation for general structure cycles. We need the following definition.

Definition 75. Suppose that $C \subset \Lambda^p TM$ is a compact cone structure on M . An embedded p -dimensional submanifold $S \subset M$ is said to be *tangent to C* if for each point $x \in S$, we have $\Lambda^p T_x S \subset C$.

One straightforward variation for more general cone structures is the following.

Conjecture (first version). *Suppose that $C \subset \Lambda^p TM$ is a compact cone structure on M . Then every C -cycle can be locally represented near $x \in M$ as $c = \int c' d\nu$, where the c' are submanifolds of M , tangent to C , passing near x , and ν is a non-negative measure on the space of such (regarded as a subspace of the currents on some neighborhood of x).*

As stated above, the conjecture is false. Here is a counterexample. Let $M = \mathbb{R}^4$ with coordinates (x_1, y_1, x_2, y_2) , and let C be the cone structure with

$$C_x = \text{span} \left\{ \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y_2} \right\}.$$

There is no submanifold that is tangent to C . Nevertheless, there are C -cycles, like

$$\Omega_c^2(M) \rightarrow \mathbb{R} : \alpha \mapsto \int_{\mathbb{R}^4} \alpha \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1} \right) + \alpha \left(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2} \right).$$

This example shows that, if we want the conjecture to have any chance of being true, we need to require the existence of “sufficiently many” submanifolds tangent to C near any point $x \in M$.

In the example above, there were no submanifolds through x tangent to C . Indeed, if S^p is a submanifold of M through x , then $\Lambda^p T_x S$ only contains pure tensors, but C_x contains no non-zero pure tensors. (By a *pure tensor*, we mean a decomposable tensor, i.e. a tensor that can be written as a wedge of 1-vectors.) For any $x \in M$, let $C_x^s \subset C_x$ be the set of all $v \in C_x$ such that there is a submanifold S^p through x , tangent to C , such that $\Lambda^p T_x S = \text{span}\{v\}$. In other words, $C_x^s \subset C_x$ is the set of all directions at x

that can be realized by a submanifold tangent to C . The only elements of C_x that can be realized by a sum of (positive multiples of) submanifolds tangent to C are those in the convex hull of C_x^s . We amend our conjecture as follows.

Conjecture (second version). *Suppose that $C \subset \Lambda^p TM$ is a compact cone structure on M . Assume that for each $x \in M$, we have*

$$\text{ConvexHull}(C_x^s) = C_x.$$

Then every C -cycle can be locally represented near $x \in M$ as $c = \int c' d\nu$, where the c' are submanifolds of M , tangent to C , passing near x , and ν is a non-negative measure on the space of such (regarded as a subspace of the currents on some neighborhood of x).

Note that this extra condition is satisfied in Sullivan's conjecture 73 because the cone C is *defined* as the convex hull of C_x^s (which is generated by complex subspaces). Now even if $\text{ConvexHull}(C_x^s) = C_x$ for all $x \in M$, it is not clear that there are sufficiently many local submanifolds tangent to C : indeed, it is possible that the cones C_x vary from point to point in such a way that no submanifold through x can be found tangent to C , even though there are many pure tensors in C . We make the following definition.

Definition 76. Let $C \subset \Lambda^p TM$ be a compact cone structure. We say that C is *integrable* if for every point $x \in M$ there is a chart around x in which C becomes constant (meaning that, in this chart, C is invariant under translations).

Example. Suppose that D is a p -dimensional oriented distribution on M . Then $\Lambda^p D$ is a line bundle over M , and we can take C_x to be the ray in $\Lambda^p D$ corresponding to the chosen orientation. This results in a compact cone C . The cone structure is integrable precisely if D is integrable, in which case this example reduces to that of foliation currents. ◀

In the case of integrable C , the set C_x^s is precisely the intersection of C_x with the decomposable tensors, because in a chart that makes C translation-invariant, affine p -dimensional subspaces are suitable submanifolds. Therefore, the condition $\text{ConvexHull}(C_x^s) = C_x$ is also satisfied in Ghys' conjecture 74 if we require integrability of C , because the cone C_x is defined as the convex hull of a set of pure tensors. (For example, in Ghys' conjecture, one could ask that J be integrable.) The following conjecture, weaker than the second version, is our final version.

Conjecture 77 (final version). *Suppose that $C \subset \Lambda^p TM$ is an integrable compact cone structure on M . Assume that for each $x \in M$, we have $\text{ConvexHull}(C_x^s) = C_x$. Then every C -cycle can be locally represented near $x \in M$ as $c = \int c' d\nu$, where the c' are submanifolds of M , tangent to C , passing near x , and ν is a non-negative measure on the space of such (regarded as a subspace of the currents on some neighborhood of x).*

Example (Contact structure). Suppose $D = \ker(\alpha)$ is a contact structure on M^3 given by the contact form α . The associated cone C is not integrable. For every $x \in M$, we do have $\text{ConvexHull}(C_x^s) = C_x$. Note that there are *no* submanifolds of M tangent to C . Nevertheless, the conjecture happens to hold for this particular case, because there also don't exist any non-zero C -cycles. Indeed, there is an exact form that is positive on D , namely $d\alpha$, so by theorem 53, there are no non-zero C -cycles. \blacktriangleleft

The case of smooth currents

Let us now focus on *smooth* currents. Write $q = m - p$ (recall that m is the dimension of M , and that $C \subset \Lambda^p TM$). The following lemma says that smooth cycles are locally sums of foliation cycles.

Lemma 78. *Suppose that β is a closed q -form on M , regarded as a closed p -current. Let $x \in M$. Near x , we have*

$$\beta = \sum_{i=1}^k c_i,$$

where the c_i are foliation cycles of foliations of open subsets of M .

Proof. Locally, $\beta = d\gamma$. In local coordinates, γ is given by

$$\sum_{i=1}^k f^i dx_{j_1^i} \wedge \dots \wedge dx_{j_{q-1}^i}.$$

Let c_i be the differential of the i 'th term, meaning

$$c_i = df^i \wedge dx_{j_1^i} \wedge \dots \wedge dx_{j_{q-1}^i}.$$

Then $\beta = \sum_i c_i$. But each c_i is a foliation cycle of a foliation on an open subset of M : it is a foliation cycle for the foliation

$$\ker(df^i) \cap \ker(dx_{j_1^i}) \cap \dots \cap \ker(dx_{j_{q-1}^i})$$

on the set $\{x \in M \mid c_i(x) \neq 0\}$. □

One would like to be able to relate the foliations that appear to the cone structure C . Ideally, the foliations are tangent to the cone structure.

Definition 79. Suppose that $C \subset \Lambda^p TM$ is a compact cone structure on M . A p -dimensional foliation of M is said to be *tangent to C* if each of its leaves is tangent to C .

The lemma above says that every smooth cycle is locally the sum of foliation cycles. This raises the following question: is every smooth C -cycle locally the sum of foliation cycles tangent to C ? For example, if $q = 1$, the answer is “yes”, because a 1-form is already decomposed. If $q = 2$ and $m \leq 3$, the answer is also “yes”, because a 2-form on a manifold of dimension at most 3 is always decomposable.

Chapter 4

Local Lie groupoids

In this chapter, we discuss the basics of local Lie groupoids. First, we define them. Next, we discuss their associativity properties, which will be important in all that follows. In the third section, we go over regularity and connectedness assumptions for local Lie groupoids. Finally, we explain how to associate a Lie algebroid to every local Lie groupoid, in the same way as for global Lie groupoids.

4.1 Definitions

We start by defining local Lie groupoids. Instead of using the symbol \mathcal{G} as we did for (global) Lie groupoids, we will now use G to indicate that this is only a *local* Lie groupoid. There are several different definitions of local Lie groupoids. We will adopt the following.

Definition 80. A *local Lie groupoid* G over a manifold M is a manifold G , together with maps

- $s, t : G \rightarrow M$ submersions (the *source* and *target* maps),
- $u : M \rightarrow G$ a smooth map (the *unit* map),
- $m : \mathcal{U} \rightarrow G$ a submersion (the *multiplication*), where $\mathcal{U} \subset G \times_t G$ is an open neighborhood of

$$(G \times_t M) \cup (M \times_t G) = \bigcup_{g \in G} \{(g, u(s(g))), (u(t(g)), g)\},$$

- $i : \mathcal{V} \rightarrow \mathcal{V}$ a smooth map (the *inversion*), where $\mathcal{V} \subset G$ is an open neighborhood of $u(M)$ such that $\mathcal{V} \times_t \mathcal{V} \subset \mathcal{U}$,

such that the following axioms hold:

- $s(m(g, h)) = s(h)$ and $t(m(g, h)) = t(g)$ for all $(g, h) \in \mathcal{U}$,
- $m(m(g, h), k) = m(g, m(h, k))$ in a neighborhood of

$$(M \times_s M \times_t G) \cup (M \times_s G \times_t M) \cup (G \times_s M \times_t M) \quad (\dagger)$$

in $G \times_s G \times_t G$,

- $m(g, u(s(g))) = m(u(t(g)), g) = g$ for all $g \in G$,
- $s(i(g)) = t(g)$ and $t(i(g)) = s(g)$ for all $g \in \mathcal{V}$,
- $m(i(g), g) = u(s(g))$ and $m(g, i(g)) = u(t(g))$ for all $g \in \mathcal{V}$.

A local Lie groupoid where M is a singleton is a *local Lie group*.

Remark. It follows from the definition that $u : M \rightarrow G$ is an embedding, and we will consider M as a submanifold of G using this embedding. In particular, if $x \in M$, we will write the unit at x as just x , leaving out the u . For the multiplication we usually write gh instead of $m(g, h)$. For the inversion, we write g^{-1} instead of $i(g)$.

Remark. Our definition of local Lie groupoids is just one of many possible versions. Here are some variations.

- One could merely require that m be defined in a neighborhood of $M \times_s M \subset G \times_s G$. This makes it possible to have elements $g \in G$ that cannot be multiplied by *anything*, effectively excluding them completely from the groupoid structure. In our definition, for every $g \in G$ the products $gs(g)$ and $t(g)g$ are defined (and they equal g).
- It follows from our definition that $i : \mathcal{V} \rightarrow \mathcal{V}$ is an involutive diffeomorphism. An element $g \in \mathcal{V}$ will be called *invertible*. One could weaken the definition by asking for an inversion map $i : \mathcal{V} \rightarrow G$ instead of $i : \mathcal{V} \rightarrow \mathcal{V}$. It is then possible to have an invertible element $g \in \mathcal{V}$ such that g^{-1} is not invertible (i.e. is not in \mathcal{V}). This change does not change the theory of local Lie groupoids, because it is possible to recover an inversion $i : \mathcal{V} \rightarrow \mathcal{V}$ by restricting the inversion map.
- Some authors require $\mathcal{V} = G$, so that every groupoid element is invertible. We do not need this stronger requirement. We will, however, later require that our local

Lie groupoid be strongly connected. As we will see later, a consequence of strong connectedness is that every element is some product of invertible elements.

- Instead of requiring associativity in a neighborhood of (\dagger), one could ask that $(gh)k = g(hk)$ whenever both sides are defined. This is a stronger requirement. However, we will see later that this does not change the theory of local Lie groupoids because this stronger requirement is always satisfied after restricting m and i sufficiently.

Remark. We do not assume that the manifold G is Hausdorff. All other manifolds, including M and all source and target fibers, are assumed to be Hausdorff.

There are two different ways of obtaining another local Lie groupoid G' from a given local Lie groupoid G . Both of them are relevant for us.

- We say that G' is obtained by *restricting* G , if both local groupoids have the same manifolds of arrows and objects, the same source and target maps, and the multiplication and inversion in G' are obtained by restricting the ones of G to smaller domains.
- We say that G' is obtained by *shrinking* G if G' is an open neighborhood of M in G , the source and target maps are the restrictions of s and t to G' , multiplication is the restriction of m to $G'^{(2)} = \mathcal{U} \cap (G' \times_s G') \cap m^{-1}(G')$, and inversion is the restriction of i to $\mathcal{V}' = (\mathcal{V} \cap G') \cap i(\mathcal{V} \cap G')$.

When restricting a local Lie groupoid G , its space of arrows does not change. When shrinking a local Lie groupoid G , its space of arrows becomes smaller. It is important to clearly distinguish these two operations.

Example (Restriction of a Lie groupoid). Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. Any open $\mathcal{U} \subset \mathcal{G}^{(2)}$ containing $(\mathcal{G} \times_s M) \cup (M \times_t \mathcal{G})$ determines a restriction G of \mathcal{G} . On the other hand, any open neighborhood $G' \subset \mathcal{G}$ of the identity manifold M determines a local Lie groupoid G' shrinking \mathcal{G} . ◀

Example ($\mathbb{R} \cup \{\infty\}$). Consider $G = \mathbb{R} \cup \{\infty\}$, equipped with addition (where $r + \infty = \infty + r = \infty$ for all $r \in \mathbb{R}$). This is a local Lie group. The domain of multiplication is $G \times G \setminus \{(\infty, \infty)\}$. The domain of inversion is \mathbb{R} . Note that it is not possible to extend the multiplication to include $\infty + \infty = \infty$ because this would make the map m discontinuous.

More generally, one can turn the one-point compactification of a non-compact connected Lie group into a local Lie group by setting $g \cdot \infty = \infty \cdot g = \infty$. ◀

Morphisms between local Lie groupoids are defined in a more or less obvious way.

Definition 81. Suppose that G_1 and G_2 are local Lie groupoids over M_1 and M_2 , respectively. A *morphism of local Lie groupoids* is a pair (F, f) , where $F : G_1 \rightarrow G_2$ and $f : M_1 \rightarrow M_2$ are smooth maps such that

- $F \circ u_1 = u_2 \circ f$,
- $f \circ s_1 = s_2 \circ F$ and $f \circ t_1 = t_2 \circ F$,
- $(F \times F)(\mathcal{U}_1) \subset \mathcal{U}_2$ and $F(m_1(g, h)) = m_2(F(g), F(h))$ whenever $(g, h) \in \mathcal{U}_1$,
- $F(\mathcal{V}_1) \subset \mathcal{V}_2$ and $F \circ i_1 = i_2 \circ F$.

4.2 Degrees of associativity

In our definition of a local Lie groupoid, we required that $(gh)k = g(hk)$ for all (g, h, k) in some neighborhood of

$$(M \times_t M \times_t G) \cup (M \times_t G \times_t M) \cup (G \times_t M \times_t M)$$

in $G \times_t G \times_t G$. This condition is called *local associativity*, and it is the weakest associativity condition we will use. There are other notions of associativity for local Lie groupoids, and they are (perhaps surprisingly) not equivalent. We give an overview of all the notions we will need.

Definition 82. A local Lie groupoid is called *3-associative* if

$$(gh)k = g(hk)$$

whenever both sides of the equation are defined (in other words, whenever $(g, h), (h, k), (gh, k)$ and (g, hk) are all in the domain of multiplication \mathcal{U}).

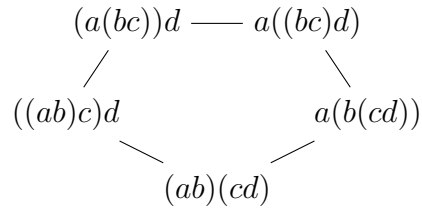
This definition is perhaps the most natural choice for an associativity requirement. Not every local Lie groupoid is 3-associative. We will give an example later in this section. A stronger notion than that of 3-associativity is the following. The following definition is taken from [27], with group replaced by groupoid.

Definition 83. A local Lie groupoid is called *associative to order n* (or *n -associative*) if for every $3 \leq m \leq n$ and every ordered m -tuple of groupoid elements $(x_1, \dots, x_m) \in G^m$, all corresponding defined m -fold products are equal (in other words, if an m -fold product can be evaluated by parenthesizing in multiple ways, all result in the same answer).

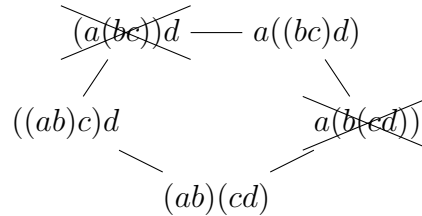
For every $n \geq 3$ there are n -associative local Lie groupoids that are not $(n + 1)$ -associative (this is already true in the case of local Lie groups, [27, section 3]). The following notion is the strongest version of associativity we will use.

Definition 84. A local Lie groupoid is called *globally associative* if it is associative to every order $n \geq 3$.

Remark. At this point, one can ask why 3-associativity does not imply n -associativity for higher n , the way it does for Lie groupoids. Let us illustrate why 3-associativity does not imply 4-associativity for local groups. There are five different ways of parenthesizing a 4-fold product, as shown in this diagram:



In the diagram, there is an edge between two parenthesizations if their equality follows from 3-associativity, and in a (global) groupoid, this proves 4-associativity. However, in a local groupoid, it is possible that some of these 4-fold products are not defined. The graph above can then become disconnected, like so:



In this case, the product may be different on different components of the graph (so that $a((bc)d) \neq ((ab)c)d = (ab)(cd)$ in this example).

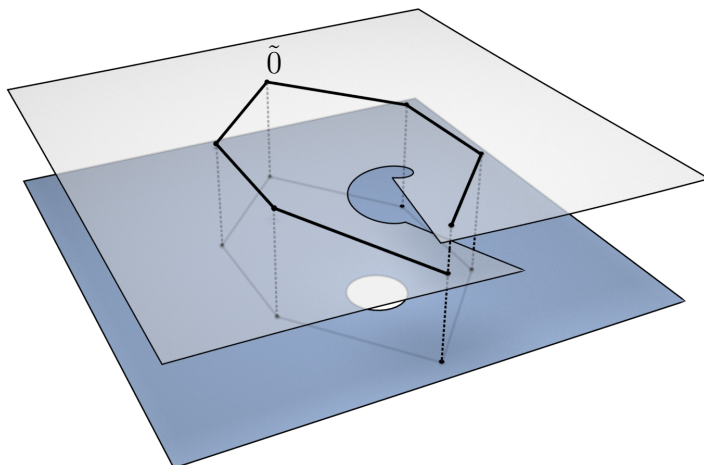


Figure 4.1: A locally associative local Lie groupoid that is not 3-associative. The manifold X is the complex plane with a disk removed (shown at the bottom). The local Lie group is the universal cover of X (shown on top). Starting at the neutral element $\tilde{0}$, we can evaluate a triple product in two different ways, ending up over the same point of X , but on different sheets. This local Lie group is not 3-associative, but it is locally associative.

Let us give an example of a locally associative local Lie group that is not 3-associative. The example is a slightly modified version of an example given in [27]. Let X be the complex plane \mathbb{C} , with a small ball centered at 1 taken out. Let G be the universal cover of X . Let $\tilde{0} \in G$ be a preimage of $0 \in X$ under the projection. This point $\tilde{0}$ will be the neutral element in our local Lie group. Let $B \subset X$ be the open ball of radius $\frac{3}{4}$ centered at 0, and let \tilde{B} be the ball above B containing $\tilde{0}$.

The multiplication will be defined on a subset of $(G \times \tilde{B}) \cup (\tilde{B} \times G)$. Suppose that $(\tilde{g}, \tilde{h}) \in G \times \tilde{B}$, and write g and h for the projections to M . The product $\tilde{g}\tilde{h}$ will be defined if the line segment (in \mathbb{C}) from g to $g+h$ lies completely in X . In that case, the product $\tilde{g}\tilde{h}$ is obtained as the endpoint of the lift of this line segment, starting at \tilde{g} . Similarly, for $(\tilde{g}, \tilde{h}) \in \tilde{B} \times G$, the product is defined as the endpoint of the lift of the line segment from h to $g+h$, starting at \tilde{h} (if that segment lies completely in X). Inverses are defined on \tilde{B} , in the obvious way.

This local Lie group is locally associative. However, it is not 3-associative. By picking $a, b, c \in \tilde{B}$ appropriately, one can ensure that $(ab)c \neq a(bc)$, as shown in figure 4.1.

While local associativity is not the same as 3-associativity, the following proposition shows that they are the same if we consider local groupoids up to restriction.

Proposition 85. *Suppose that G is a local Lie groupoid. Then there is a restriction of*

G that is 3-associative.

Proof. Case 1: if G is compact. Let G_0 be a neighborhood of M in G that is 3-associative. Let K be a neighborhood of M in G_0 such that the set

$$(G \times_t K \times_t K) \cup (K \times_t G \times_t K) \cup (K \times_t K \times_t G)$$

is contained in the domain of 3-associativity (this is the set $\{(a, b, c) \in G^3 \mid (ab)c = a(bc)\}$). Let $B \subset K$ be a neighborhood of M in G , consisting of invertible elements, small enough to ensure that if $x, y \in B$, then $x^{-1}y$ is defined and lies in K . Now restrict the multiplication to a domain \mathcal{U}' that is contained in

$$(G \times_t B) \cup (B \times_t G).$$

We claim that this makes the multiplication 3-associative. Suppose that $a, b, c \in G$ are such that $a(bc)$ and $(ab)c$ are both defined. We will show that $a(bc) = (ab)c$. If all three of a, b, c are in G_0 , this is immediate. We may therefore assume that one of a, b, c is not in G_0 :

- if $a \notin G_0$, we must have $b \in B$ because ab is defined. We must also have $bc \in B$ because $a(bc)$ is defined. By our choice of B , the product $b^{-1}(bc)$ must be defined and lie in K . But this product equals c , so we have $c \in K$. This shows that $(a, b, c) \in G \times_t K \times_t K$, and so (a, b, c) lies in the domain of 3-associativity.
- if $b \notin G_0$, we must have $a \in B \subset K$, because ab is defined. We must also have $c \in B \subset K$ because bc is defined. This shows that $(a, b, c) \in K \times_t G \times_t K$, and so (a, b, c) lies in the domain of 3-associativity.
- the case $c \notin G_0$ is analogous to the first one, by symmetry.

Case 2: if G is not compact. Let $G_1 \subset G_2 \subset \dots \subset G_k \subset \dots \subset G$ be an exhaustion of G by open, precompact local subgroupoids. The proof of case 1 shows that for each k there is a set $\mathcal{U}'_k \subset G_k \times_t G_k$ where 3-associativity holds. Then

$$\mathcal{U}' = \bigcup_{k=1}^{\infty} \mathcal{U}'_k$$

gives an appropriate restriction of G to ensure 3-associativity. □

Remark. If we are working with local Lie groupoids up to restriction, the result above says that we can always assume our local groupoid to be 3-associative. We will make use of this fact by assuming 3-associativity whenever we need it, keeping in mind that we merely need to restrict our groupoid to ensure this. Note that we do not need to shrink the groupoid to obtain 3-associativity.

Remark. In fact, the proof above generalizes to show that every local Lie groupoid has a restriction that is n -associative for any $n \geq 3$. This generalized proof quickly becomes unwieldy, though. We will not use the result for $n > 3$.

In the article [12], a (somewhat) different proof of the result for $n = 3$ is given. That proof also generalizes, and is perhaps less unwieldy for $n > 3$.

4.3 Regularity and connectedness

In this section, we study several basic properties of local Lie groupoids. Most of the material in this section is an adaptation of material in [27] to the groupoid case.

Consider $\mathbb{R} \cup \{\infty\}$ as before. The point at infinity plays a special role here: left (or right) multiplication by ∞ is a constant map. We want to exclude these special points in our study of local Lie groupoids. We write T^sG for the kernel of the map $ds : TG \rightarrow TM$, and T^tG for the kernel of $dt : TG \rightarrow TM$.

Definition 86. Let G be a local Lie groupoid over M . Let $g \in G$. We say that G is *right-regular at g* if right multiplication by g induces an isomorphism between $T_{t(g)}^sG$ and T_g^sG . We say that G is *left-regular at g* if left multiplication by g induces an isomorphism between $T_{s(g)}^tG$ and T_g^tG . We say that G is *bi-regular at g* if it is both left- and right-regular at g . The local groupoid G is called *bi-regular* if it is bi-regular at all $g \in G$.

In the above example, the groupoid is bi-regular at all points except ∞ .

Definition 87. An element g of a local Lie groupoid is called *inversional* if it can be written as a well-defined product of invertible elements. A local Lie groupoid is called *inversional* if all of its elements are inversional.

For $\mathbb{R} \cup \{\infty\}$ as above, all elements except ∞ are inversional. Every Lie groupoid is obviously inversional. Restricting a local groupoid (by restricting its multiplication and

inversion maps) can change it from inversional to not inversional. For example, we can consider the Lie group $\mathbb{R} \times \mathbb{Z}$ as a local Lie groupoid (with the usual addition). If we restrict its inversion map to a domain \mathcal{V}' , the resulting local Lie groupoid is inversional precisely if $\{n \in \mathbb{Z} \mid \mathbb{R} \times \{n\} \cap \mathcal{V}' \neq \emptyset\} \subset \mathbb{Z}$ generates \mathbb{Z} .

Definition 88. We say that a local Lie groupoid is *source-connected* (or *s-connected*) if all of its source fibers are connected. We say that a local Lie groupoid is *target-connected* (or *t-connected*) if all of its target fibers are connected.

To define connectedness of a local Lie groupoid, one could just ask that G be connected as a manifold. However, this is not the correct notion for our purposes. In [27], connectedness of a local Lie group requires that every neighborhood of the identity generate the local group (in the same way that in a connected Lie group, every neighborhood of the identity generates the group). We have the analogous criterion for groupoids.

Definition 89. Let U be a neighborhood of M in G . We say that U *generates* G if every element of g can be written as a well-defined product of elements in U .

The notion of connectedness that we require is the following. We call it *strong connectedness* to avoid any ambiguity.

Definition 90. A local Lie groupoid G over M is *strongly connected* if

- (a) M is connected
- (b) the domains \mathcal{U} and \mathcal{V} of the multiplication and inversion maps are connected,
- (c) the set $\{(g, h) \in \mathcal{U} \mid s(g) = t(h) = x\}$ is connected for all $x \in M$,
- (d) G is *s-connected* and *t-connected*, and
- (e) G is bi-regular.

In the case of a local Lie group, condition (a) is automatic, and conditions (c) and (d) both amount to connectedness of the manifold G .

We have the following result. It implies, in particular, that a strongly connected local Lie groupoid is inversional.

Proposition 91. *If G is right-regular and s-connected, then any neighborhood of M in G generates G . If G is left-regular and t-connected, the same conclusion holds.*

Proof. We will discuss the case where G is right-regular and s -connected. The other case is analogous. Let U be a neighborhood of M in G . Pick an element $x \in M$. We will show that the set

$$A = \{g \in s^{-1}(x) \mid g \text{ can be written as a product of elements in } U\}$$

is both open and closed in $s^{-1}(x)$ (and therefore, by s -connectedness, it is the entire source fiber). This suffices to prove the result.

By right-regularity of G , it is clear that A is open. Pick $g \in \partial A$. We will show that $g \in A$. There is a neighborhood V of $t(g)$ in $s^{-1}(t(g))$, consisting of invertible elements, contained in U , such that for all $h \in V$ we have

$$(h, g) \in \mathcal{U} \quad \text{and} \quad (h^{-1}, hg) \in \mathcal{U}.$$

By right-regularity of G , the set $\{hg \mid h \in V\}$ is a neighborhood of g in $s^{-1}(x)$, and so contains an element of A . Pick such an element $a \in A$. Then $a = hg$ for some $h \in V$, and by our choice of V , the product $g = h^{-1}a$ is defined. But since a can be written as a product of elements of U , this last equation shows that g also can. \square

The following lemma shows that, in some sense, little information is lost by restricting a strongly connected local Lie groupoid.

Lemma 92. *Suppose that we have two strongly connected local Lie groupoid structures on G over M , with the same source, target and unit maps, and the same domains for multiplication and inversion. Assume that both are 3-associative. In other words, we have two tuples $(s, t, u, \mathcal{U}, \mathcal{V}, m_j : \mathcal{U} \rightarrow G, i_j : \mathcal{V} \rightarrow G)$ for $j \in \{1, 2\}$, each making G into a strongly connected local Lie groupoid over M . If the two structures have a common restriction, so if there is some $\mathcal{U}' \subset \mathcal{U}$ and $\mathcal{V}' \subset \mathcal{V}$ such that $m_1 \upharpoonright_{\mathcal{U}'} = m_2 \upharpoonright_{\mathcal{U}'}$ and $i_1 \upharpoonright_{\mathcal{V}'} = i_2 \upharpoonright_{\mathcal{V}'}$ (neighborhoods of M and $M \times_s G \cup G \times_t M$), then $m_1 = m_2$ and $i_1 = i_2$.*

Proof. Let us first show that the multiplication maps agree. Pick a point $x \in M$. Let $F = \{(g, h) \in \mathcal{U} \mid s(g) = t(h) = x\}$. By strong connectedness of G , the set F is connected. We will show that $m_1 \upharpoonright_F = m_2 \upharpoonright_F$. Since x was arbitrary, this will prove that $m_1 = m_2$. We argue that the set

$$A = \{(g, h) \in F \mid m_1(g, h) = m_2(g, h)\}$$

is open and closed (so that it is all of F). Closedness of A is obvious, so we just have to show that A is open.

Pick $(g, h) \in A$. Let us write gh for $m_1(g, h) = m_2(g, h)$. Let N_h be a compact neighborhood of $s(h)$ in $t^{-1}(s(h))$, small enough such that for all $\bar{h} \in N_h$ we have

- $(h, \bar{h}) \in \mathcal{U}'$ (so that we can write $h\bar{h}$ without ambiguity),
- $(g, h\bar{h}) \in \mathcal{U}$,
- $(gh, \bar{h}) \in \mathcal{U}'$.

Then

$$\begin{aligned} m_1(g, h\bar{h}) &= m_1(gh, \bar{h}) \\ &= m_2(gh, \bar{h}) \\ &= m_2(g, h\bar{h}), \end{aligned}$$

showing that $(g, h\bar{h}) \in A$ (so that we can write $gh\bar{h}$ without ambiguity). Now let N_g be a neighborhood of $t(g)$ in $s^{-1}(t(g))$ small enough such that for all $\bar{h} \in N_h$ and $\bar{g} \in N_g$ we have

- $(\bar{g}, g) \in \mathcal{U}'$ (so that we can write $\bar{g}g = m_1(\bar{g}, g) = m_2(\bar{g}, g)$),
- $(\bar{g}, gh\bar{h}) \in \mathcal{U}'$,
- $(\bar{g}g, h\bar{h}) \in \mathcal{U}$.

Then

$$\begin{aligned} m_1(\bar{g}g, h\bar{h}) &= m_1(\bar{g}, gh\bar{h}) \\ &= m_2(\bar{g}, gh\bar{h}) \\ &= m_2(\bar{g}g, h\bar{h}), \end{aligned}$$

showing that $(\bar{g}g, h\bar{h}) \in A$. But by bi-regularity of m_1 and m_2 , elements of the form $(\bar{g}g, h\bar{h})$ form a neighborhood of (g, h) in F . This shows that A is open, and therefore that the multiplications agree. Because of 3-associativity, the multiplication map completely determines the inversion map (up to restriction). This shows that the inversion maps also agree. \square

Remark. The above lemma shows that we do not lose any information by restricting a local Lie groupoid, on the condition that it is sufficiently connected. This essentially means that we can identify a local Lie groupoid with any of its restrictions without changing the theory of local Lie groupoids.

The situation is different when talking about *shrinking* local Lie groupoids. If we identify G and G' whenever G' can be obtained by shrinking G , we are studying *germs* of local Lie groupoids. When doing this, one misses out on some of the subtler points of the theory of associativity for local Lie groupoids. As we will see, the associativity properties of a local Lie groupoid can change drastically upon shrinking.

4.4 The Lie algebroid of a local Lie groupoid

In the same way as for Lie groupoids, one can associate a Lie algebroid to each local Lie groupoid. However, there are some subtleties about left- and right-invariant vector fields that we will discuss explicitly.

Suppose that G is a local Lie groupoid over M . Just like for Lie groupoids, the vector bundle underlying the algebroid is

$$A = T_M^s G,$$

the restriction of $T^s G$ to M . The anchor is the map $\rho : A \rightarrow TM$ obtained by restricting $dt : TG \rightarrow TM$ to A .

In order to define the algebroid bracket, we must discuss invariant vector fields on G . To each section $\alpha \in \Gamma(A)$, we can associate a vector field

$$\tilde{\alpha}_g = d_{t(g)} R_g(\alpha_{t(g)}),$$

where R_g is right translation by $g \in G$. Note that $d_{t(g)} R_g$ maps $T_{t(g)}^s G$ to $T_g^s G$, so that $\tilde{\alpha}$ is a well-defined vector field that is tangent to the source fibers. A vector field $X \in \mathfrak{X}(G)$ is *right-invariant* if it is tangent to the source fibers and

$$dR_h(X_g) = X_{gh} \quad \text{for all } g, h \in \mathcal{U}.$$

Lemma 93. *Suppose that G is a 3-associative local Lie groupoid. Then the right-invariant vector fields are precisely of the form $\tilde{\alpha}$ for $\alpha \in \Gamma(A)$.*

Proof. It is clear that every right-invariant vector field must be of the form $\tilde{\alpha}$ for some $\alpha \in \Gamma(A)$. The converse follows from 3-associativity in the usual way. \square

The Lie bracket of two right-invariant vector fields is again right-invariant, and we define the algebroid bracket by requiring

$$[\widetilde{\alpha}, \widetilde{\beta}] = [\tilde{\alpha}, \tilde{\beta}].$$

Then $(A, \rho, [,])$ is the Lie algebroid of G . Note that for a local Lie groupoid that is not 3-associative, we can define its Lie algebroid by going to a 3-associative restriction.

Chapter 5

Associativity and globalizability

In this chapter, we discuss the *globalizability* of local Lie groupoids, i.e. whether they are part of a global Lie groupoid or not. The main result of this chapter is the generalization of Mal'cev's theorem, which answers this question for local Lie groups, to local Lie groupoids.

5.1 Mal'cev's theorem for local Lie groups

Perhaps the most natural examples of local Lie groupoids are open neighborhoods of the identity manifold in Lie groupoids, and restrictions thereof. We would like to characterize the local Lie groupoids that arise in this way.

Definition 94. We call a local Lie groupoid *globalizable* if it is a restriction of an open neighborhood of the identity manifold in a Lie groupoid.

A theorem by Mal'cev characterizes globalizable local Lie groups. In the language we will define in a moment, the theorem states the following.

Theorem 95 (Mal'cev). *A strongly connected local Lie group is globalizable if and only if it is globally associative.*

We will generalize this result to local Lie groupoids. Our proof will be analogous to the proof for local groups, though a little bit of extra effort is required in proving smoothness of the constructed groupoid.

5.2 Example of a non-globalizable local Lie groupoid

Let us illustrate the globalization problem and the various notions of associativity with some examples of local Lie groupoids.

A Lie groupoid

We start by describing a very simple global Lie groupoid. We take $M = \mathbb{S}^2$, equipped with its usual area form ω of total area 4π . We will write A for the map that calculates the area enclosed by a loop:

$$A : \Omega(M) \rightarrow \frac{\mathbb{R}}{4\pi} : \gamma \mapsto \int_{\Gamma} \omega,$$

where $\gamma : [0, 1] \rightarrow M$ is a loop and $\Gamma : [0, 1] \times [0, 1] \rightarrow M$ is any homotopy contracting γ to the trivial loop at $\gamma(0) = \gamma(1)$. Note that this area is well-defined up to 4π , because any two such homotopies will have areas differing by an element $\int_{[\alpha]} \omega \in 4\pi\mathbb{Z}$ where $[\alpha] \in \pi_2(\mathbb{S}^2)$.

Now let

$$P = \{\text{piecewise smooth paths in } M\} \times \frac{\mathbb{R}}{4\pi}.$$

We say that two elements $(\gamma_1, a_1), (\gamma_2, a_2) \in P$ are equivalent for \sim if $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$ and

$$a_2 = a_1 + A(\gamma_2^{-1} \cdot \gamma_1),$$

where \cdot represents concatenation of paths, and γ_2^{-1} is the reverse of γ_2 .

Let

$$\mathcal{G} = P/\sim.$$

This is a smooth manifold of dimension 5. It is a Lie groupoid over M with multiplication

$$[\gamma_1, a_1][\gamma_2, a_2] = [\gamma_1 \cdot \gamma_2, a_1 + a_2].$$

The Lie algebroid of \mathcal{G} is the one associated to the area form ω (example 2.26 in [8]). More precisely, as a bundle it is $TM \oplus \mathbb{R}$ (with anchor the projection to the first summand), and the bracket is given by

$$[(X, f), (Y, g)] = ([X, Y], \mathcal{L}_X(g) - \mathcal{L}_Y(f) + \omega(X, Y)).$$

A globalizable local Lie groupoid

We now focus our attention on an open part of the \mathcal{G} we just constructed. Note that any element of \mathcal{G} that does not have antipodal source and target has a unique representative

whose path is a geodesic for the usual round metric. We can restrict to the elements whose source and target are not antipodal, and obtain a (globalizable) local Lie groupoid. If we represent each such element using the geodesic representative, we can write down this local Lie groupoid explicitly as

$$G' = \{(y, x) \in M \times M \mid x + y \neq 0\} \times \frac{\mathbb{R}}{4\pi}.$$

The multiplication of $(z, y, a), (y, x, a') \in G'$ is then defined whenever $x + z \neq 0$, and is given by

$$(z, y, a) \cdot (y, x, a') = (z, x, a + a' + A(\Delta xyz)),$$

where $A(\Delta xyz) \in \mathbb{R}/4\pi$ is the signed area of the spherical triangle Δxyz . This local Lie groupoid G' is globalizable, because it is an open part of \mathcal{G} under the inclusion

$$(y, x, a) \mapsto [\text{geodesic from } x \text{ to } y, a].$$

A non-globalizable local Lie groupoid

By introducing a slight variation, we can make the above local Lie groupoid non-globalizable. We will no longer quotient out the areas by 4π . Explicitly, we still have $M = \mathbb{S}^2$ and we set

$$G'' = \{(x, y) \in M \times M \mid x + y \neq 0\} \times \mathbb{R}.$$

We will define the multiplication of (z, y, a) and (y, x, a') only if $x + z \neq 0$ and $-\pi < A(\Delta xyz) < \pi$. In that case, it is defined as above, by the formula

$$(z, y, a) \cdot (y, x, a') = (z, x, a + a' + A(\Delta xyz)).$$

To check that this local Lie groupoid is 3-associative, let $(z, y, a_1), (y, x, a_2), (x, w, a_3) \in G''$. Then

$$\begin{aligned} ((z, y, a_1) \cdot (y, x, a_2)) \cdot (x, w, a_3) &= (z, w, a_1 + a_2 + a_3 + A(\Delta xyz) + A(\Delta wxz)) \\ (z, y, a_1) \cdot ((y, x, a_2) \cdot (x, w, a_3)) &= (z, w, a_1 + a_2 + a_3 + A(\Delta wxy) + A(\Delta wyz)) \end{aligned}$$

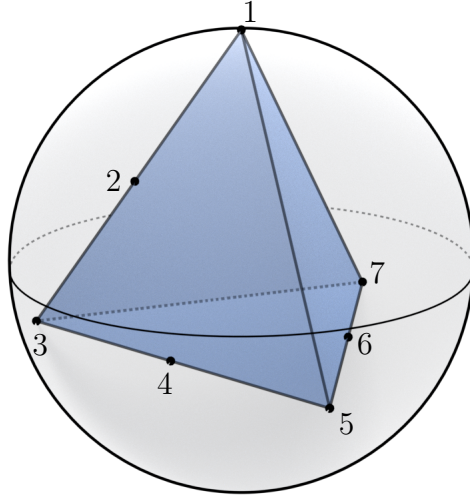


Figure 5.1: Demonstrating the lack of 6-associativity. The tetrahedron is inscribed in a sphere, and the even-numbered points are midpoints of the edges they lie on.

(if both are defined) and $A(\Delta xyz) + A(\Delta wxz) = A(\Delta wxy) + A(\Delta wyz) \pmod{4\pi}$, because both equal the area of the quadrangle $wxyz$ (which is defined up to 4π). But by the restriction that the areas have to be in $(-\pi, \pi)$, this implies $A(\Delta xyz) + A(\Delta wxz) = A(\Delta wxy) + A(\Delta wyz)$, proving 3-associativity.

This local Lie groupoid is not globally associative. Consider figure 5.1, which shows a regular tetrahedron inscribed in \mathbb{S}^2 . Let $x_j \in \mathbb{S}^2$ be the normalization of the j 'th marked point in the picture (i.e. the radial projection of the point to the circumscribed sphere). Let

$$\begin{aligned} A &= (x_2, x_1, 0) & D &= (x_5, x_4, 0) \\ B &= (x_3, x_2, 0) & E &= (x_6, x_5, 0) \\ C &= (x_4, x_3, 0) & F &= (x_7, x_6, 0). \end{aligned}$$

Then

$$(x_7, x_1, 2\pi) = F(E((D(CB))A)) \neq ((F((ED)C))B)A = (x_7, x_1, -2\pi).$$

Of course, global associativity is necessary for globalizability. Our groupoid G'' is therefore not globalizable. Note, however, that there is a neighborhood of the identity manifold in G'' that *is* globalizable: if $G''' = \{(y, x, a) \in G'' \mid -2\pi < a < 2\pi\}$, then G''' is globalizable (and in fact isomorphic to an open part of G' and hence of \mathcal{G}). As we will see later, this is always the case for local Lie groupoids whose algebroid is integrable.

5.3 Associative completions

In this section, we review a construction that associates to a local Lie groupoid a global groupoid. In [1, appendix A], this construction is called the *formal completion* of the local groupoid (and it is used only for globally associative structures). The construction is also used in [27], though it is not explicitly named. We will call this construction the *associative completion* of the local Lie groupoid, to highlight that it has the effect of

- making the structure globally associative,
- globalizing the structure to a (global) groupoid, so that the multiplication is complete.

Suppose that G is a local Lie groupoid. Then $\bigsqcup_{n \geq 1} G^n$ is the set of words on G . A word (w_1, \dots, w_n) is called *well-formed* if $s(w_i) = t(w_{i+1})$ for all $i \in \{1, \dots, n-1\}$. The set of well-formed words on G of length n is nothing but

$$\underbrace{G \times_t G \times_t \cdots \times_t G}_{n \text{ copies}}.$$

Write

$$W(G) = \{\text{well-formed words on } G\}.$$

Note that if G happens to be an open part of a global Lie groupoid, then a groupoid globalizing it may be obtained as a quotient of $W(G)$ by an appropriate equivalence relation, as follows.

Definition 96. If $w = (w_1, \dots, w_k, w_{k+1}, \dots, w_n)$ is a well-formed word and $(w_k, w_{k+1}) \in \mathcal{U}$, we will say that the word

$$w' = (w_1, \dots, w_k w_{k+1}, \dots, w_n)$$

is obtained from w by *contraction*. We also say that w is obtained from w' by *expansion*. If two words are related by an expansion or contraction, we will say that they are *elementarily equivalent*.

We say that two words are *equivalent* (written $w \sim w'$) if they are equivalent under the equivalence relation generated by elementary equivalences.

Definition 97. If G is a local Lie groupoid, then

$$\mathcal{AC}(G) = W(G)/\sim$$

is called the *associative completion* of G .

Proposition 98. *If G is an inversive local Lie groupoid, then $\mathcal{AC}(G)$ is a groupoid (in the algebraic sense).*

Proof. The source and target maps are given by

$$s([w_1, \dots, w_n]) = s(w_n) \quad \text{and} \quad t([w_1, \dots, w_n]) = t(w_1).$$

The multiplication map is juxtaposition. The unit at $x \in M$ is given by $[x]$. It remains to show that $[w_1, \dots, w_n]$ has an inverse. Because G is inversive, every w_i can be written as a product of invertible elements $w_i^1, \dots, w_i^{k_i}$. Then we have

$$[w_1, \dots, w_n]^{-1} = [(w_n^{k_n})^{-1}, \dots, (w_n^1)^{-1}, \dots, (w_1^{k_1})^{-1}, \dots, (w_1^1)^{-1}].$$

These operations turn $\mathcal{AC}(G)$ into a groupoid. □

If $f : G \rightarrow H$ is a morphism of local Lie groupoids, we have a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ \downarrow & & \downarrow \\ \mathcal{AC}(G) & \xrightarrow{\mathcal{AC}(f)} & \mathcal{AC}(H). \end{array}$$

It is interesting to study the kernel of the map $G \rightarrow \mathcal{AC}(G)$ (i.e. those elements that are mapped to a unit). The following elements are clearly in this kernel.

Definition 99. Suppose that G is a local Lie groupoid over M , and that $x \in M$. Then an element $g \in G_x = s^{-1}(x) \cap t^{-1}(x) \subset G$ is called an *associator at x* if there is a word

$$(w_1, \dots, w_k) \in G^k$$

that can be evaluated (in G) to both x and g (by choosing different ways of putting in brackets). We write $\text{Assoc}_x(G)$ for the set of all associators at x . We write $\text{Assoc}(G)$ for the set of all associators in G .

Definition 100. Suppose G is a local Lie groupoid. We say that G has products connected to the axes if for every $(g, h) \in \mathcal{U}$, there is a path γ from $t(h)$ to g in G such that $(\gamma(\tau), h) \in \mathcal{U}$ for all τ , or there is a path γ from $s(g)$ to h in G such that $(g, \gamma(\tau)) \in \mathcal{U}$ for all τ .

Note that every local Lie groupoid has a restriction with products connected to the axes.

The following result seems fairly innocent, but it is non-trivial and at the heart of many results about associativity for local groupoids.

Proposition 101. *Suppose G is a bi-regular local Lie groupoid that has products connected to the axes. Let $g, h \in G$. Then $(g) \sim (h)$ if and only if there is a sequence of expansions, followed by a sequence of contractions, starting at (g) and ending at (h) .*

We will prove this proposition in section 5.4. It has the following corollary.

Corollary 102. *For a bi-regular local Lie groupoid G with products connected to the axes, the kernel of $G \rightarrow \mathcal{AC}(G)$ is precisely $\text{Assoc}(G)$.*

In that case, we have the following relations:

$$\begin{aligned} G \rightarrow \mathcal{AC}(G) &\text{ is injective} \\ \iff G &\text{ is globally associative} \\ \implies \text{Assoc}(G) &\text{ is trivial (i.e. } \text{Assoc}(G) = M \subset G). \end{aligned}$$

Note that triviality of $\text{Assoc}(G)$ does not imply that G is globally associative. (A morphism of local Lie groupoids with trivial kernel is not necessarily injective. For a simple example, consider the morphism of Lie groups $\mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z}$. Its restriction to $(-0.6, 0.6)$ has trivial kernel, but it is not injective.)

Definition 103. We say that $\text{Assoc}(G) \subset G$ is *uniformly discrete* if there is an open neighborhood U of M in G such that $U \cap \text{Assoc}(G) = M$.

We can now state one of our main results. We will prove it in the next section.

Theorem A. *If G is a bi-regular local Lie groupoid with products connected to the axes, then $\mathcal{AC}(G)$ is smooth if and only if $\text{Assoc}(G)$ is uniformly discrete in G (and in that case, $G \rightarrow \mathcal{AC}(G)$ is a local diffeomorphism).*

(By “ $\mathcal{AC}(G)$ is smooth”, we mean, of course, that $\mathcal{AC}(G)$ has a smooth structure as a quotient of $W(G)$, where $W(G)$ is considered as a manifold with components of various dimensions.) The following corollary characterizes globalizable local Lie groupoids. The results is entirely analogous to the situation for local Lie groups.

Corollary 104 (Mal’cev’s theorem for local Lie groupoids). *A strongly connected local Lie groupoid is globalizable if and only if it is globally associative.*

Proof. Call the local Lie groupoid G . If G globalizable, it is clearly globally associative. If G is globally associative, then restrict it to get a local Lie groupoid G' with products connected to the axes. Then $G' \hookrightarrow \mathcal{AC}(G')$ is the inclusion of G' as a restriction of an open part U of a global Lie groupoid. (Injectivity follows from global associativity.) This means that G and U have a common restriction (namely G'), so that by strong connectedness and lemma 92 the inclusion $G \hookrightarrow \mathcal{AC}(G')$ is also the inclusion of a restriction of U . The Lie groupoid $\mathcal{AC}(G')$ therefore globalizes G . \square

Every lie algebroid can be integrated to a local Lie groupoid [7, corollary 5.1]. Because the globalizability of a local Lie groupoid is related to its higher associativity, this points to a link between associativity and integrability. The following corollary is a first indication.

Corollary 105. *A Lie algebroid is integrable if and only if there is a local Lie groupoid integrating it that has uniformly discrete associators.*

Proof. Suppose a Lie algebroid is integrable. Then by definition, there is a Lie groupoid integrating it, and this is a local Lie groupoid with uniformly discrete (indeed, trivial) associators.

Suppose that a Lie algebroid has a local Lie groupoid integrating it, with uniformly discrete associators. Then by taking an open neighborhood of the units in this local Lie groupoid, small enough to intersect the set of associators only in identities, we can get a strongly connected local Lie groupoid integrating the Lie algebroid with trivial associators. This local Lie groupoid is therefore globalizable, and we can conclude that there exists a Lie groupoid integrating our Lie algebroid. \square

5.4 Proof of Mal'cev's theorem

We now prove Mal'cev's theorem by establishing theorem A. Our first step is to prove proposition 101, which in turn requires some lemmas. They will be about well-formed words on G . Let us call any (sub)word of the form $(a_1, \dots, a_k, a_k^{-1}, \dots, a_1^{-1})$ a *block*.

Lemma 106. *Let G be a strongly connected local Lie groupoid with products connected to the axes. Let w and w' be well-formed words on G . Suppose that w' is a contraction of w . Then there is a series of expansions starting at w and ending at a word that equals w' with a block inserted at some position.*

Lemma 107. *Let G be a strongly connected local Lie groupoid. If $z \in G$, a well-formed word of the form (z, block) can be transformed using expansions only into a word of the form $(\text{some other block}, z)$.*

We postpone the proofs of these lemmas to the end of this section.

Proof of proposition 101. Suppose $g \sim h$. Then there is some sequence of elementary equivalences, that will transform the word (g) into the word (h) (expansions and contractions may show up mixed, in any order). Let $w^0 = (g)$, w^1, \dots, w^{n-1} , $w^n = (h)$ be the sequence of words connecting (g) to (h) . This means that each w^{i+1} is either an expansion or a contraction of w^i .

The plan is now as follows: we will build up a sequence u^0, \dots, u^n starting at $u^0 = (g)$, where each u^{i+1} is obtained from u^i by a series of expansions, and u^i equals w^i preceded by zero or more blocks. Then u^n is of the form $(\text{zero or more blocks}, h)$, and by a series of contractions we can get rid of the blocks and transform this into (h) . This shows that there is a sequence of expansions, followed by contractions, connecting (g) to (h) .

To build up the sequence u^i , set $u^0 = w^0$. Once u^i has been defined, we obtain u^{i+1} as follows. If w^{i+1} is an expansion of w^i , then u^{i+1} is obtained from u^i by the analogous expansion (leaving the blocks in front alone). If w^{i+1} is a contraction of w^i , the procedure is more complicated. We will make use of the lemmas above.

To find u^{i+1} , we first apply lemma 106 to the contraction of u^i corresponding to the contraction from w^i to w^{i+1} (leaving the blocks in front alone). We obtain a word of the form

$$(\text{blocks, letters of } w^{i+1}, \text{block, letters of } w^{i+1}).$$

Using lemma 107 we can move the block that has appeared in w^{i+1} to the front using expansions, obtaining a word of the form

$$(\text{blocks}, w^{i+1}).$$

This then is the word we take for u^{i+1} . This shows how to build up the required sequence u^0, \dots, u^n , proving the result. \square

We are now ready to prove the theorem.

Proof of theorem A. Suppose first that $\mathcal{AC}(G)$ is smooth (with smooth structure determined by the submersion $W(G) \rightarrow \mathcal{AC}(G)$). Note that the fibers of $G \rightarrow \mathcal{AC}(G)$ are countable, so that $\dim(\mathcal{AC}(G)) \geq \dim(G)$. Because $W(G)$ has a component of dimension $\dim(G)$ and $\mathcal{AC}(G)$ is a quotient of $W(G)$, we must have $\dim(\mathcal{AC}(G)) \leq \dim(G)$, and therefore $\dim(\mathcal{AC}(G)) = \dim(G)$ (this is of course the only reasonable dimension to expect). The map $G \rightarrow \mathcal{AC}(G)$ is therefore a local diffeomorphism. The inverse image of $M \subset \mathcal{AC}(G)$ under $G \rightarrow \mathcal{AC}(G)$, which is precisely $\text{Assoc}(G)$, is therefore an embedded submanifold of G . This shows that $\text{Assoc}(G)$ is uniformly discrete in G .

Suppose now that $\text{Assoc}(G)$ is uniformly discrete. We will prove that $\mathcal{AC}(G)$ is smooth. Given an element $h \in \mathcal{AC}(G)$, represented by a word (x_1, \dots, x_n) , we will construct a chart near h modeled on an open part of G . For each x_i , take a small submanifold $N_i \subset G$ of the same dimension as G through x_i that is transverse to both source and target fibers (i.e. a local bisection of the local Lie groupoid near x_i). Then s and t define diffeomorphisms between the N_i and open subsets of M . Write $s_i = s|_{N_i}$ and $t_i = t|_{N_i}$ for these diffeomorphisms.

Let $k \in \{1, \dots, n\}$. Let U be a small neighborhood of x_k . Then the chart we use near h is defined as

$$\varphi : U \rightarrow \mathcal{AC}(G) : y \mapsto [y_1, \dots, y_{k-1}, y, y_{k+1}, \dots, y_n]$$

where

$$\begin{aligned} y_{k+1} &= t_{k+1}^{-1}(s(y)) \\ y_{k+2} &= t_{k+2}^{-1}(s(y_{k+1})) \\ &\dots \\ y_n &= t_n^{-1}(s(y_{n-1})) \end{aligned}$$

and

$$\begin{aligned}
y_{k-1} &= s_{k-1}^{-1}(t(y)) \\
y_{k-2} &= s_{k-2}^{-1}(t(y_{k-1})) \\
&\dots \\
y_1 &= s_1^{-1}(t(y_2)).
\end{aligned}$$

If we pick U sufficiently small, this is well-defined. Note that it maps x_k to h . We claim that φ is injective, if we pick U small enough (this is where we need the uniform discreteness). Indeed, suppose that $\varphi(y) = \varphi(z)$. Write the representatives of $\varphi(y)$ and $\varphi(z)$ as given above as

$$(y_1, \dots, y_{k-1}, y, y_{k+1}, \dots, y_n) \quad \text{and} \quad (z_1, \dots, z_{k-1}, z, z_{k+1}, \dots, z_n).$$

Because $\varphi(y) = \varphi(z)$, we must have $s(\varphi(y)) = s(\varphi(z))$ and therefore $s(y_n) = s(z_n)$. But then y_n and z_n are both on N_n , and have the same source. Therefore, $y_n = z_n$. This implies that $s(y_{n-1}) = s(z_{n-1})$. But then y_{n-1} and z_{n-1} are both on N_{n-1} , and have the same source. Therefore, $y_{n-1} = z_{n-1}$. Continuing this way, we find that $y_i = z_i$ for $i > k$. The same argument applied starting from the left and using the target map shows that $y_i = z_i$ for $i < k$. We conclude that

$$[y_1, \dots, y_{k-1}, y, y_{k+1}, \dots, y_n] = \varphi(y) = \varphi(z) = [y_1, \dots, y_{k-1}, z, y_{k+1}, \dots, y_n].$$

Multiplying both sides by $[y_1, \dots, y_{k-1}]^{-1}$ (from the left) and by $[y_{k+1}, \dots, y_n]^{-1}$ (from the right) shows that $[y] = [z]$. If B is a neighborhood of M in G such that $B \cap \text{Assoc}(G) = M$, then for U small enough, we must have $z = y\delta$ for δ in B . But then $[\delta]$ must be trivial in $\mathcal{AC}(G)$, so that it is an associator, implying that $\delta \in M$. We conclude that $y = z$.

Let us check smoothness of the transition maps. Given a point in $\mathcal{AC}(G)$, a chart is determined by

- a choice of representative,
- a choice of k ,
- a choice of local bisections.

We will show smoothness of the following transitions:

- for a fixed representative and index k , the transition between charts coming from different choices of local bisections,
- for a fixed representative, the transition between charts for two different choices of k ,
- the transition between charts constructed using different representatives of the same element of $\mathcal{AC}(G)$.

Together, these show that all the transitions are smooth.

Let us start by checking the first kind of transition map. Suppose (x_1, \dots, x_n) represents $h \in \mathcal{AC}(G)$. Pick $k \in \{1, \dots, n\}$. Making one choice of local bisections N_i leads to a chart

$$\varphi : y \mapsto [y_1, \dots, y_{k-1}, y, y_{k+1}, \dots, y_n].$$

Note that the y_i depend smoothly on y . Now make a different choice of local bisections N'_i , leading to a second chart. We will calculate the transition map between the two charts. Suppose y is near x_k . Let z_n be the point on N'_n with source $s(y_n) = s(h)$. Then

$$z_n = \varepsilon_n y_n$$

for some invertible ε_n near $t(y_n)$ (on the condition that y is sufficiently close to x_k). Note that ε_n depends smoothly on y , and $\varepsilon_n = t(y_n)$ if $y = x_k$. The product $y_{n-1}\varepsilon_n^{-1}$ is defined for y close enough to x_k . Let z_{n-1} be the point on N'_{n-1} with source $t(z_n)$. Then

$$z_{n-1} = \varepsilon_{n-1}(y_{n-1}\varepsilon_n^{-1})$$

for some invertible ε_{n-1} near $t(y_{n-1})$ (on the condition that y is sufficiently close to x_k). Note that ε_{n-1} depends smoothly on y , and $\varepsilon_{n-1} = t(y_{n-1})$ if $y = x_k$. Continuing this way, we construct a sequence z_n, \dots, z_{k+1} on the N'_i and $\varepsilon_n, \dots, \varepsilon_{k+1}$ such that

$$z_i = \varepsilon_i(y_i\varepsilon_{i+1}^{-1}),$$

where we will set $\varepsilon_{n+1} = s(y_n)$ for ease of notation. All the ε_i depend smoothly on y (we may have to restrict y to be in a smaller and smaller neighborhood of x_k). Similarly, we start from the left and construct a sequence z_1, \dots, z_{k-1} on the N'_i and $\varepsilon_1, \dots, \varepsilon_{k-1}$

such that

$$z_i = (\varepsilon_{i-1}^{-1} y_i) \varepsilon_i$$

with all the ε_i depending smoothly on y . Note now that

$$\begin{aligned} \varphi(y) &= [y_1, \dots, y_{k-1}, y, y_{k+1}, \dots, y_n] \\ &= [y_1, \varepsilon_1, \dots, \varepsilon_{i-2}^{-1}, y_{k-1}, \varepsilon_{k-1}, y, \varepsilon_{k+1}, y_{k+1}, \varepsilon_{k+2}^{-1}, \dots, \varepsilon_n, y_n] \\ &= [z_1, \dots, z_{k-1}, (\varepsilon_{k-1} y \varepsilon_{k+1}), z_{k+1}, \dots, z_n]. \end{aligned}$$

where the product in the middle will be defined if y is sufficiently close to x_k . This shows that the transition map between our two charts is given by

$$y \mapsto \varepsilon_{k-1} y \varepsilon_{k+1},$$

which is smooth in y (because the ε_i are). This proves smoothness of a transition of the first kind.

Let us now check smoothness of a transition between charts for the same representative, but different choices of k . This is much easier. We may assume that the two choices for k differ by 1, so notation-wise we will look at the chart associated to the representative (x_1, \dots, x_n) for index k and index $k + 1$. Again, write

$$y \mapsto [y_1, \dots, y, y_{k+1}, \dots, y_n]$$

for the chart associated to index k . Suppose that y is near x_k . Let y_k be the point on N_k with target $t(y)$. This point depends smoothly on y . For y sufficiently close to x_k , we may write $y_k = y\varepsilon$ where ε is near $s(x_k)$ and depends smoothly on y . Then

$$\begin{aligned} [y_1, \dots, y, y_{k+1}, \dots, y_n] &= [y_1, \dots, y, \varepsilon, \varepsilon^{-1}, y_{k+1}, \dots, y_n] \\ &= [y_1, \dots, y_k, (\varepsilon^{-1} y_{k+1}), \dots, y_n] \end{aligned}$$

for y sufficiently close to x_k . This shows that the transition map is $y \mapsto \varepsilon^{-1} y_{k+1}$, which is smooth.

We now check the smoothness of the third and final type of transition, that between different representatives of the same point. This type is the easiest to check. Indeed, we only need to check smoothness for the transition for a single contraction or expansion, but that is clear from smoothness of \mathcal{G} 's multiplication.

Note, finally, that $\mathcal{AC}(G)$ is second-countable because $W(\mathcal{G})$ is. We have proved that $\mathcal{AC}(G)$ is smooth. \square

Proofs of the lemmas

Finally, we give the proofs of the lemmas.

Proof of lemma 106. Suppose that

$$w = (x_1, \dots, x_k, x_{k+1}, \dots, x_n) \quad \text{and} \quad w' = (x_1, \dots, x_k x_{k+1}, \dots, x_n).$$

Because G has products connected to the axes, there is either a curve in G from $t(x_{k+1})$ to x_k in $s^{-1}(t(x_{k+1}))$ that can be right-translated by x_{k+1} , or a curve from $s(x_k)$ to x_{k+1} in $t^{-1}(s(x_k))$ that can be left-translated by x_k . We will assume the former, and show that w can be expanded to a word of the form

$$(x_1, \dots, \text{block}, x_k x_{k+1}, \dots, x_n).$$

The argument for the latter case is entirely similar, ending with a word of the form

$$(x_1, \dots, x_k x_{k+1}, \text{block}, \dots, x_n).$$

Let γ be such a curve in $s^{-1}(t(x_{k+1}))$ from $t(x_{k+1})$ to x_k and let γ' be the curve obtained from γ by right-translating by x_{k+1} .

Let $T \subset [0, 1]$ be the set of all $\tau \in [0, 1]$ such that there is a sequence $a_1, \dots, a_l \in G$ such that

$$\gamma(\tau) = a_l(\cdots(a_1(s(x_k))))$$

and

$$x_{k+1} = a_1^{-1}(\cdots(a_l^{-1}(\gamma'(\tau))))$$

are defined and true. We claim that $T = [0, 1]$. We will prove this by showing that $T \subset [0, 1]$ is both open and closed (clearly it is nonempty, because $0 \in T$).

We first prove that T is open. Pick $\tau_0 \in T$. Let a_1, \dots, a_l be a sequence such that $\gamma(\tau_0) = a_1(\cdots(a_l(s(x_k))))$ and $x_{k+1} = a_1^{-1}(\cdots(a_l^{-1}(\gamma'(\tau_0))))$. Now $t(\gamma(\tau_0))$ has a neighborhood in $s^{-1}(t(\gamma(\tau_0)))$ of invertible elements such that for every g in this

neighborhood we have

$$(g, \gamma(\tau_0)) \in \mathcal{U} \text{ and } (g^{-1}, g\gamma'(\tau_0)) \in \mathcal{U}.$$

Then by bi-regularity of G , for all τ close enough to τ_0 we can write

$$\gamma(\tau) = g\gamma(\tau_0) = g(a_l(\cdots(a_1(s(x_k))))))$$

and

$$x_{k+1} = a_1^{-1}(\cdots(a_l^{-1}(g^{-1}(\gamma'(\tau))))).$$

This shows that T is open.

We now show that T is closed. Pick $\tau_0 \in \partial T$. Now $t(\gamma(\tau_0))$ has a neighborhood in $t^{-1}(t(\gamma(\tau_0)))$ of invertible elements such that for every g in this neighborhood we have

$$(g^{-1}, \gamma'(\tau_0)) \in \mathcal{U} \text{ and } (g^{-1}, \gamma(\tau_0)) \in \mathcal{U} \text{ and } (g, g^{-1}\gamma(\tau_0)) \in \mathcal{U}.$$

Using bi-regularity of G and picking $\tau \in T$ sufficiently close to τ_0 we can find a g in this neighborhood such that $\gamma(\tau) = g^{-1}\gamma(\tau_0)$. If a_1, \dots, a_l are such that

$$\gamma(\tau) = a_l(\cdots(a_1(s(x_k))))$$

and

$$x_{k+1} = a_1^{-1}(\cdots(a_l^{-1}(\gamma'(\tau))))),$$

then

$$\gamma(\tau_0) = g(a_l(\cdots(a_1(s(x_k))))))$$

and

$$x_{k+1} = a_1^{-1}(\cdots(a_l^{-1}(g^{-1}(\gamma'(\tau_0))))).$$

This shows that $\tau_0 \in T$, proving that T is closed. Therefore, $T = [0, 1]$.

We have proved that $1 \in T$, so we can find a sequence a_1, \dots, a_l such that

$$x_k = a_l(\cdots(a_1(s(x_k)))) = a_l(\cdots(a_1))$$

and

$$x_{k+1} = a_1^{-1}(\cdots(a_l^{-1}(\gamma'(1)))) = a_1^{-1}(\cdots(a_l^{-1}(x_k x_{k+1}))).$$

This shows that we can expand

$$(x_1, \dots, x_k, x_{k+1}, \dots, x_n)$$

into

$$(x_1, \dots, a_l, \dots, a_1, a_1^{-1}, \dots, a_l^{-1}, x_k x_{k+1}, \dots, x_n),$$

proving the lemma. □

Proof of lemma 107. Suppose the initial word is $(z, x_1, \dots, x_n, x_n^{-1}, \dots, x_1^{-1})$. Note that z and x_1^{-1} are in the same source fiber. By source-connectedness, we can take a path γ in $s^{-1}(s(z))$ from z to x_1^{-1} . Let $T \subset [0, 1]$ consist of all $\tau \in [0, 1]$ for which there is a sequence a_1, \dots, a_k such that

$$\gamma(\tau) = a_k^{-1}(\dots(a_1^{-1}(z))) \text{ and } z = a_1(\dots(a_k(\gamma(\tau))).$$

We will show that $T \subset [0, 1]$ is both open and closed, and so $T = [0, 1]$ (clearly it is non-empty, because $0 \in T$).

Let $\tau_0 \in T$. Let a_1, \dots, a_k be a sequence as above. Then there is an open neighborhood of $t(\gamma(\tau_0))$ in $s^{-1}(t(\gamma(\tau_0)))$ consisting of invertible elements such that for each g in this neighborhood we have

$$(g^{-1}, \gamma(\tau_0)) \in \mathcal{U} \text{ and } (g, g^{-1}\gamma(\tau)) \in \mathcal{U}.$$

Therefore, using bi-regularity of G , for τ sufficiently close to τ_0 , we can find an element g such that

$$\gamma(\tau) = g^{-1}(a_k^{-1}(\dots(a_1^{-1}(z)))) \text{ and } z = a_1(\dots(a_k(g(\gamma(\tau)))).$$

This proves that T is open.

Now let $\tau_0 \in \partial T$. Then there is a neighborhood of $t(\gamma(\tau_0))$ in $t^{-1}(t(\gamma(\tau_0)))$ of invertible elements such that for every g in this neighborhood we have

$$(g^{-1}, \gamma(\tau_0)) \in \mathcal{U} \text{ and } (g, g^{-1}\gamma(\tau_0)) \in \mathcal{U}.$$

Using bi-regularity of G and picking $\tau \in T$ sufficiently close to τ_0 we can find a g in

this neighborhood such that $\gamma(\tau) = g^{-1}\gamma(\tau_0)$. If a_1, \dots, a_k are such that

$$\gamma(\tau) = a_k^{-1}(\dots(a_1^{-1}(z))) \text{ and } z = a_1(\dots(a_k(\gamma(\tau)))),$$

then

$$\gamma(\tau_0) = g(a_k^{-1}(\dots(a_1^{-1}(z)))) \text{ and } z = a_1(\dots(a_k(g^{-1}(\gamma(\tau_0))))).$$

This shows that $\tau_0 \in T$. Therefore, T is also closed. We conclude that $T = [0, 1]$.

Since $1 \in T$, we can find a sequence $a_1, \dots, a_k \in G$ such that

$$x_1^{-1} = a_k^{-1}(\dots(a_1^{-1}(z)))$$

and

$$z = a_1(\dots(a_k(x_1^{-1}))).$$

This means we can expand the initial word into

$$(a_1, \dots, a_k, x_1^{-1}, x_1, x_2, \dots, x_n, x_n^{-1}, \dots, x_2^{-1}, a_k^{-1}, \dots, a_1^{-1}, z).$$

We can expand this further into

$$(a_1, \dots, a_k, x_1^{-1}, x_1, x_2, \dots, x_n, x_n^{-1}, \dots, x_2^{-1}, t(a_k^{-1}), a_k^{-1}, \dots, a_1^{-1}, z)$$

and finally

$$\underbrace{(a_1, \dots, a_k, x_1^{-1}, x_1, x_2, \dots, x_n, x_n^{-1}, \dots, x_2^{-1}, x_1^{-1}, x_1, a_k^{-1}, \dots, a_1^{-1}, z)}_{\text{block}},$$

which is of the required form. This proves the lemma. □

Chapter 6

Classification of local Lie groupoids with integrable algebroids

Olver [27] has classified local Lie groups in terms of Lie groups. In this chapter, we generalize this result to local Lie groupoids. Our results are direct generalizations of results by Olver to the groupoid case [27, theorems 19 and 20]. The result we obtain can only be used to classify those local Lie groupoids that have integrable algebroids, a restriction that does not show up in Olver's result because all Lie algebras are integrable. Before we can explain the classification result proper, we have to study coverings of local Lie groupoids.

6.1 Coverings of local Lie groupoids

Our first task is to understand coverings of local Lie groupoids. Just like in the theory of global Lie groupoids, coverings play an important role for local Lie groupoids. Recall that for groupoids, the notion of covering concerns not the space of arrows, but rather the source fibers (or target fibers). We will assume that all the local groupoids in this section have connected source and target fibers.

Definition 108. Suppose that G is a local Lie groupoid over M (with source map s , target map t , unit map u). A *covering* (resp. *generalized covering*) of G is a tuple (G', s', t', u') and a map $\varphi : G' \rightarrow G$, such that

- $u' : M \hookrightarrow G$ is an embedding,
- $s', t' : G' \rightarrow M$ are surjective submersions such that $s' \circ u' = t' \circ u' = \text{Id}$,
- $\varphi : G' \rightarrow G$ is a smooth map such that $s' = s \circ \varphi$, $t' = t \circ \varphi$, $u = \varphi \circ u'$,
- for each $m \in M$, the map $\varphi|_{(s')^{-1}(m)} : (s')^{-1}(m) \rightarrow s^{-1}(\varphi(m))$ is a covering map (resp. local diffeomorphism).

Our goal for this section is the following result.

Theorem 109. *A generalized covering $\varphi : G' \rightarrow G$ of a bi-regular local Lie groupoid has the structure of a local Lie groupoid making φ into a morphism of local Lie groupoids. The local Lie groupoid G' is also bi-regular.*

Remark. Every local Lie groupoid has a source-simply connected cover. It is constructed in the same way as for global Lie groupoids (details can be found in [8]). In short, the source-simply connected cover is obtained as $p^{-1}(M)$, where

$$p : \Pi_1(\mathcal{F}_s) \rightarrow G,$$

is the source map of the fundamental groupoid of the source foliation of G . The theorem above therefore shows that every local Lie groupoid is covered by a source-simply connected local Lie groupoid. As explained in [12], it is possible to give a (simpler) proof of the existence of source-simply connected covering groupoids in its own right. We refer the reader to that article for the details.

The key to understanding theorem 109 are Maurer-Cartan forms. For a Lie group, the left- and right-invariant Maurer-Cartan forms are usually considered as Lie algebra valued 1-forms, but we will formulate them as bundle maps $TG \rightarrow \pi^*(T_{\{e\}}G)$ where $\pi : G \rightarrow \{e\}$ (this is the same thing, of course). The following are the analogues of the left-invariant and right-invariant Maurer-Cartan forms, respectively.

Definition 110. An *s-framing* is an isomorphism of vector bundles $\omega_s : T^sG \rightarrow t^*(T_M^sG)$ over the identity such that $\omega_s|_M = \text{Id}$. A *t-framing* is an isomorphism of vector bundles $\omega_t : T^tG \rightarrow s^*(T_M^tG)$ over the identity such that $\omega_t|_M = \text{Id}$.

If G is a bi-regular local Lie groupoid, the left- and right-invariant Maurer Cartan forms are *s-* and *t-*framings, with

$$\omega_{MC}^R : T_g^sG \ni v \mapsto (d_{t(g)}\rho_g)^{-1}(v)$$

and

$$\omega_{MC}^L : T_g^tG \ni v \mapsto (d_{s(g)}\lambda_g)^{-1}(v),$$

where ρ_g is right multiplication by g , and λ_g is left multiplication by g . For a local Lie groupoid, the framings used will be these Maurer-Cartan forms. Given framings, we can talk about invariant vector fields.

Definition 111. Suppose we are given an s -framing ω_s . A section of $T^s G$ is called *right-invariant* if its image under the s -framing is of the form $t^*\sigma$ for some $\sigma \in \Gamma(T_M^s G)$.

Suppose we are given a t -framing. A section of $T^t G$ is called *left-invariant* if its image under the t -framing is of the form $s^*\sigma$ for some $\sigma \in \Gamma(T_M^t G)$.

Lemma 112. *If G is a local Lie groupoid, then*

- *the Lie bracket of right-invariant vector fields is right-invariant,*
- *the Lie bracket of left-invariant vector fields is left-invariant,*
- *the Lie bracket of a right-invariant and a left-invariant vector field is zero.*

Proof. We first prove that the bracket of right-invariant vector fields is right-invariant. Note that a vector field X on G is right-invariant if and only if its value at $g \in G$ is $(d\rho_g)(X_{t(g)})$. Let X_1 and X_2 be right-invariant vector fields on G . By the local associativity of G , the vector fields X_1 and X_2 are invariant under ρ_g near $t(g)$. Therefore, the value of $[X_1, X_2]$ at g is indeed equal to $d\rho_g([X_1, X_2](t(g)))$, and $[X_1, X_2]$ is right-invariant. Similarly, the bracket of left-invariant vector fields is left-invariant.

We now show that right-invariant vector fields commute with left-invariant vector fields. Suppose that X is a right-invariant vector field on G , and Y is a left-invariant vector field on G . Let $g \in G$. Let $\tau \mapsto \gamma_X(\tau)$ be the integral curve of X with $\gamma_X(0) = t(g)$, and let $\tau' \mapsto \gamma_Y(\tau')$ be the integral curve of Y with $\gamma_Y(0) = s(g)$. Then note that $\tau \mapsto \gamma_X(\tau) \cdot g$ is also an integral curve of X (defined for τ small), by local associativity. Also $\tau \mapsto \gamma_X(\tau) \cdot g \cdot \gamma_Y(\tau')$ is an integral curve of X (defined for τ and τ' small). Similarly, the curves $\tau' \mapsto g \cdot \gamma_Y(\tau')$ and $\tau' \mapsto \gamma_X(\tau) \cdot g \cdot \gamma_Y(\tau')$ are integral curves of Y (for τ, τ' small). It follows that the flows of X and Y commute for small times, because flowing g first by X for time τ and then by Y for time τ' gives the same result as first flowing by Y and then by X (namely $\gamma_X(\tau) \cdot g \cdot \gamma_Y(\tau')$). \square

Remark. For a 3-associative local Lie groupoid, right-invariant vector fields are indeed invariant under right multiplication (and similarly for left-invariant vector fields under left multiplication). This justifies the choice of terminology.

Theorem 109 will follow from the following characterization. It is a direct generalization of theorem 18 in [27].

Proposition 113. *Let G , M , s , t and u be as above. Then given s - and t -framings, there is a local Lie groupoid structure giving rise to these framings if and only if*

- the Lie bracket of left-invariant vector fields is left-invariant,
- the Lie bracket of right-invariant vector fields is right-invariant,
- the Lie bracket of a left-invariant vector field and a right-invariant vector field is zero.

If we demand that the local Lie groupoid be strongly connected, the structure is unique up to restriction.

Remark. Theorem 18 in [27] claims that giving a local Lie group structure is equivalent to prescribing right-invariant vector fields, without specifying the left-invariant vector fields. Explicit examples show, however, that this is not sufficient, and the left-invariant vector fields need to be specified as well. More precisely, if only right-invariant vector fields are specified, it may happen that no compatible left-invariant vector fields exist. The mistake in the proof is that the multiplication does not get defined near $G \times \{e\}$ (only near $\{e\} \times G$). For an explicit example, one can look at $M = \mathbb{R}_{>0} \times \mathbb{S}^1$, with coordinates (x, θ) . The vector fields $X = x \frac{\partial}{\partial x}$ and $Y = x \frac{\partial}{\partial \theta}$ satisfy $[X, Y] = Y$. If we try to prescribe X and Y as right-invariant vector fields on M , there is no way to find corresponding left-invariant vector fields to make M into a local Lie group: if Z is a left-invariant vector field that equals $\frac{\partial}{\partial x}$ at $(1, 1)$, we must have $[Y, Z] = 0$, but the flow of Y maps the point $(1, 1)$ to itself after time 2π , mapping $\frac{\partial}{\partial x}$ to $\frac{\partial}{\partial x} + 2\pi \frac{\partial}{\partial \theta}$. This shows that Z cannot be invariant under the flow of Y .

Proof of proposition 113. The “only if” part of the proof is lemma 112. We prove the “if” part. Suppose that we have framings that satisfy the above conditions. We will prescribe the multiplication and inversion maps. Let us start by defining the multiplication on a neighborhood of $G \times_t M$. Pick $g \in G$ and write $x = s(g)$. If $h \in t^{-1}(x)$ is sufficiently close to x , then

$$h = \phi_{\omega_t^{-1}(s^*\sigma)}^1(x)$$

for some local section σ of $T_M^t G$ near x (here ϕ^1 denotes the time-1 flow of a vector field). Note that $\omega_t^{-1}(s^*\sigma)$ is left-invariant. We set

$$g \cdot h = \phi_{\omega_t^{-1}(s^*\sigma)}^1(g)$$

if this flow exists. Note that this product $g \cdot h$ is well-defined for h small, in the sense that it doesn't depend on the choice of σ . This defines the multiplication near $G \times_t M$.

Similarly, one defines $g \cdot h = \phi_{\omega_s^{-1}(t^*\tau)}^1(h)$ if $g = \phi_{\omega_s^{-1}(t^*\tau)}^1(x)$ is small, and τ is a local section of $T_M^s G$.

One thing remains to check to ensure we have a well-defined multiplication: if $g = \phi_{\omega_s^{-1}(t^*\tau)}^1(x)$ and $h = \phi_{\omega_t^{-1}(s^*\sigma)}^1(x)$ for local sections τ and σ as above, we need to ensure that the two definitions of $g \cdot h$ agree. The first definition defines $g \cdot h$ as

$$\phi_{\omega_t^{-1}(s^*\sigma)}^1(g) = \phi_{\omega_t^{-1}(s^*\sigma)}^1(\phi_{\omega_s^{-1}(t^*\tau)}^1(x))$$

and the second as

$$\phi_{\omega_s^{-1}(t^*\tau)}^1(h) = \phi_{\omega_s^{-1}(t^*\tau)}^1(\phi_{\omega_t^{-1}(s^*\sigma)}^1(x)).$$

Because left- and right-invariant vector fields commute, these definitions coincide for g, h sufficiently close to x . Therefore, by restricting if necessary, we have defined a multiplication map on $G \times_s G$ near $G \times_s M \cup M \times_s G$.

We leave it to the reader to check that this multiplication is locally associative (possibly after restricting further), and to define an inversion map near M . The uniqueness up to restriction follows immediately from lemma 92. \square

6.2 Classification result

In [27], Olver shows roughly speaking that every local Lie group is covered by a cover of a globalizable local Lie group. We prove the analogous result for groupoids, and strengthen it slightly.

Theorem B. *Suppose G is a bi-regular s -connected local Lie groupoid over M with integrable Lie algebroid A . Write \tilde{G} for its source-simply connected cover, and write $\mathcal{G}(A)$ for the source-simply connected integration of A . Then we have the following commutative diagram.*

$$\begin{array}{ccc}
 & \tilde{G} & \\
 p_1 \swarrow & & \searrow p_2 \\
 G & & U \subseteq \mathcal{G}(A) \\
 & \searrow & \swarrow \\
 & \mathcal{AC}(G) &
 \end{array}$$

Here, U is an open neighborhood of M in $\mathcal{G}(A)$, p_1 is the covering map, and the map $G \rightarrow \mathcal{AC}(G)$ is the natural map from a local Lie group to its associative completion. The map p_2 is a generalized covering of local Lie groupoids, which sends a point $g \in \tilde{G}$ to the class of the A -path associated to a \tilde{G} -path from $s(g)$ to g .

Proof. Our first step is to show the existence of the upper half of the diagram. This is the statement that is proved in [27, theorem 21] for the case of local groups, and the proof is analogous. We use Cartan's method of the graph. The argument is similar to the usual argument for integrability of Lie algebroid morphisms to Lie groupoid morphisms [24, proposition 6.8].

Consider the fibered product $\tilde{G} \times_t \mathcal{G}(A)$. We equip it with a foliation \mathcal{F} , given by

$$\mathcal{F}_{(g_1, g_2)} = \left\{ (\xi \cdot g_1, \xi \cdot g_2) \mid \xi \in A_{t(g_1)} \right\}.$$

Integrability of this distribution follows from the fact that \tilde{G} and $\mathcal{G}(A)$ have the same Lie algebroid. Fix a point $x \in M$. Write N_x for the leaf of \mathcal{F} through (x, x) .

We claim that the projection $\pi_1 : N_x \rightarrow s^{-1}(x)$ is a covering map (we are taking the source fiber in \tilde{G}). Note that $\pi_1 : N_x \rightarrow s^{-1}(x)$ is a local diffeomorphism by bi-regularity of the local groupoid. Now if $(g_1, g_2) \in N_x$, and U_1 is a neighborhood of (g_1, g_2) in N_x that gets mapped diffeomorphically to a neighborhood U_2 of g_1 in $s^{-1}(x)$, then U_2 is uniformly covered. Indeed, if (g_1, g_3) is another point on N_x , then by invariance of \mathcal{F} under right multiplication in the $\mathcal{G}(A)$ -direction, the translated submanifold $U_1 \cdot g_2^{-1} \cdot g_3$ is a neighborhood of (g_1, g_3) in N_x that gets mapped diffeomorphically to U_1 . This shows that the projection $\pi_1 : N_x \rightarrow s^{-1}(x)$ is a covering map. Because $s^{-1}(x)$ is simply connected, it is a diffeomorphism. The inverse is a diffeomorphism $\phi_x : s^{-1}(x) \rightarrow N_x$.

Let $N = \bigcup_{x \in M} N_x$. Putting together the ϕ_x gives a map $\phi : \tilde{G} \rightarrow N$. This map is smooth, because it is the extension by holonomy of the map $x \mapsto (x, x)$. We define

$$p_2 = \pi_2 \circ \phi.$$

Letting U be the image of p_2 , we obtain the required diagram

$$\begin{array}{ccc} & \tilde{G} & \\ p_1 \swarrow & & \searrow p_2 \\ G & & U \subseteq \mathcal{G}(A) \end{array} \quad (\dagger)$$

We now establish the existence of the lower half of the diagram. We apply the functor \mathcal{AC} to p_1 , which results in

$$\begin{array}{ccc}
 & \tilde{G} & \\
 p_1 \swarrow & & \searrow \\
 G & & \mathcal{AC}(\tilde{G}) \\
 & \searrow & \swarrow \\
 & \mathcal{AC}(G) &
 \end{array}$$

$\mathcal{AC}(p_1)$

We claim that $\mathcal{AC}(\tilde{G}) \cong \mathcal{G}(A)$, and that under this identification the natural map $\tilde{G} \rightarrow \mathcal{AC}(\tilde{G})$ corresponds to p_2 .

Note that $\mathcal{AC}(\tilde{G})$ is smooth (the associators of \tilde{G} are discrete because they lie in the fiber of $\tilde{G} \rightarrow U$ over the identity). We have maps $\mathcal{AC}(\tilde{G}) \rightarrow \mathcal{AC}(U) \rightarrow \mathcal{AC}(\mathcal{G}(A)) \cong \mathcal{G}(A)$, and all three of these are Lie groupoids. At the level of Lie algebroids, these maps are isomorphisms, and because $\mathcal{G}(A)$ is source-simply connected, we must in fact have

$$\mathcal{AC}(\tilde{G}) \cong \mathcal{AC}(U) \cong \mathcal{AC}(\mathcal{G}(A)) \cong \mathcal{G}(A).$$

This is the identification we were looking for. Following along the maps, we see that an element $g \in \tilde{G}$ is mapped to $\mathcal{G}(A)$ as follows:

$$g \in \tilde{G} \mapsto [g] \in \mathcal{AC}(\tilde{G}) \mapsto [p_2(g)] \in \mathcal{AC}(U) \mapsto [p_2(g)] \in \mathcal{AC}(\mathcal{G}(A)) \mapsto p_2(g) \in \mathcal{G}(A).$$

The resulting diagram

$$\begin{array}{ccc}
 & \tilde{G} & \\
 p_1 \swarrow & & \searrow p_2 \\
 G & & \mathcal{G}(A) \\
 & \searrow & \swarrow \\
 & \mathcal{AC}(G) &
 \end{array}$$

is what we were looking for. □

Corollary 114. *If G is a s -simply connected local Lie groupoid with integrable algebroid, then $\text{Assoc}(G)$ is uniformly discrete (and, in particular, $\mathcal{AC}(G)$ is smooth).*

Proof. The above result shows that G is a generalized cover of a globalizable local Lie group. The associators of G are therefore contained in the fiber of this generalized covering over the identity, and this fiber is discrete. \square

Note that this corollary is false without the assumption of s -simply connectedness. For example, there are local Lie groups with non-discrete associators. An example appears in the next chapter, on section 7.1.

Chapter 7

Associativity and integrability

In this chapter, we discuss the relationship between the integrability of a Lie algebroid and the associativity of a local groupoid integrating it. After some basic observations in 7.1, we give an example in 7.2, in order to give the reader more intuition. In 7.3, 7.4 and 7.5, we briefly discuss some ideas about simplicial sets and about monodromy groups. Finally, in 7.6 and 7.7, we prove that associators and monodromy groups are intimately related.

7.1 Observations about associators

Let us study the set of associators of a local Lie groupoid more closely. The product of two associators (if defined) is again an associator. Note that in a globally associative local Lie group, all associators are trivial, meaning that for all $x \in M$, $\text{Assoc}_G(x) = \{x\}$. As it turns out, restricting a local Lie groupoid does not change its associators, in the following sense.

Lemma 115. *Let G be a local Lie groupoid over M , with multiplication $m_1 : \mathcal{U}_1 \rightarrow G$ and inversion map $i_1 : \mathcal{V}_1 \rightarrow G$. Write G_1 for G with this local groupoid structure. Suppose that we restrict G to get a local Lie groupoid over M with multiplication map $m_2 : \mathcal{U}_2 \rightarrow G$ and inversion map $i_2 : \mathcal{U}_2 \rightarrow G$. Write G_2 for G with this local groupoid structure. Assume that G_1 and G_2 have products connected to the axes. Then for all $x \in M$ we have*

$$\text{Assoc}_x(G_1) = \text{Assoc}_x(G_2).$$

Proof. Let W be the set of well-formed words on G . We consider two equivalence relations on W . We will write \sim_1 for the equivalence relation on W generated by contractions and expansions for the multiplication on G_1 , and \sim_2 for the equivalence relation on W generated by contractions and expansions for the multiplication on G_2 .

(These are precisely the equivalence relations that were considered in the construction of the associative completions of G_1 and G_2 .) Proposition 101 tells us that

$$g \in \text{Assoc}_x(G_1) \iff (g) \sim_1(x) \quad \text{and} \quad g \in \text{Assoc}_x(G_2) \iff (g) \sim_2(x).$$

It therefore suffices to show that \sim_1 and \sim_2 are the same equivalence relation.

Clearly, $w \sim_2 w' \Rightarrow w \sim_1 w'$. We claim that the converse implication also holds, so that the two equivalence relations are equal. It suffices to show that two words that are elementarily equivalent for \sim_1 are also equivalent for \sim_2 . Let $(w_1, \dots, w_k, w_{k+1}, \dots, w_n) \in W$ such that $(w_k, w_{k+1}) \in \mathcal{U}_1$. Following along with the proof of lemma 106 *for the local groupoid* G_2 , we find that we can expand (for the multiplication m_2) the subword $(\dots, w_k, w_{k+1}, \dots)$ into

$$(\dots, a_l, \dots, a_1, a_1^{-1}, \dots, a_l^{-1}, m_1(w_k, w_{k+1}), \dots),$$

where the inverses are inverses for G_2 (and thus also for G_1 , but that is not important). We can then contract this word (for the multiplication m_2) into

$$(\dots, m_1(w_k, w_{k+1}), \dots),$$

so that $(w_1, \dots, w_k, w_{k+1}, \dots, w_n) \sim_2 (w_1, \dots, w_k w_{k+1}, \dots, w_n)$. This shows that \sim_1 and \sim_2 are equal, proving the result. \square

Shrinking a local Lie groupoid (i.e. replacing it with a neighborhood of M in G) *can* change the associators drastically. To illustrate, we will give an example of a local Lie group G such that $\text{Assoc}(G)$ is not discrete. Because every local Lie group has a neighborhood of the identity that is globalizable (and therefore has trivial associators), this shows that the associators can change from non-discrete to discrete by shrinking the local groupoid.

Example (Local Lie group with non-discrete associators). Let M be a singleton. Let $B \subset \mathbb{R}^2$ be the ladder-shaped set

$$B = (\{0\} \times \mathbb{R}) \cup (\{1\} \times \mathbb{R}) \cup ([0, 1] \times \mathbb{Z})$$

and let G be its thickening

$$G = \left\{ p \in \mathbb{R}^2 \mid \exists q \in B \text{ such that } d(p, q) < \frac{1}{10} \right\}.$$

(Here, d is the Euclidean distance.) We will equip G with the structure of a local Lie group in such a way that $\text{Assoc}(G)$ is not discrete.

For each $n \in \mathbb{Z}_{>0}$, let $f_n : [\frac{1}{10}, \frac{9}{10}] \rightarrow \mathbb{R}_{>0}$ be a smooth function that is 1 outside of $[\frac{1}{3}, \frac{2}{3}]$ and such that the time- $\frac{8}{10}$ flow of the vector field $f_n \frac{\partial}{\partial x}$ on \mathbb{R} maps $\frac{1}{10}$ to $\frac{9}{10} + \frac{1}{100n}$. In other words, the flow of $f_n \frac{\partial}{\partial x}$ is just a little bit faster than that of $\frac{\partial}{\partial x}$. For each $n \in \mathbb{Z}_{\leq 0}$, let f_n be the function on $[\frac{1}{10}, \frac{9}{10}]$ that takes value 1 everywhere. Now consider the following vector fields on G :

$$X(x, y) = \begin{cases} f_n(x) \frac{\partial}{\partial x} & \text{if } x \in [\frac{1}{10}, \frac{9}{10}] \text{ and } y \in (n - \frac{1}{10}, n + \frac{1}{10}) \\ \frac{\partial}{\partial x} & \text{otherwise,} \end{cases}$$

$$Y(x, y) = \frac{\partial}{\partial y}.$$

There is a unique local Lie group structure on G over M such that X and Y are bi-invariant (more precisely, unique up to restriction if we ask that the local groupoid has with products connected to the axes). We claim that for this local Lie group structure, $\text{Assoc}(G)$ is not discrete.

Let $a = (\frac{1}{20}, 0) \in G$ and $b = (0, \frac{1}{20}) \in G$. Then consider the product

$$\underbrace{a^{-1} \cdots a^{-1}}_{20} \underbrace{b^{-1} \cdots b^{-1}}_{20n} \underbrace{b \cdots b}_{20n} \underbrace{a \cdots a}_{20},$$

where the numbers under the braces indicate the number of repetitions. Clearly, this product results in the neutral element $(0, 0)$ when evaluated from the inside out. However, evaluating first the second half $c = b \cdots ba \cdots a$ from left to right, and then evaluating the rest $a^{-1} \cdots a^{-1} b^{-1} \cdots b^{-1} c$ from right to left, we get $(\frac{1}{100n}, 0)$. This shows that $(\frac{1}{100n}, 0) \in \text{Assoc}(G)$ for all positive n , so that the associators do not form a discrete set. ◀

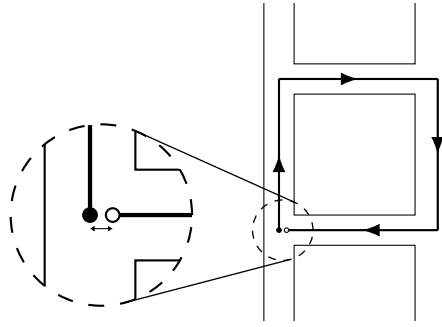


Figure 7.1: The vector field X is obtained by slightly modifying $\frac{\partial}{\partial x}$. The origin is the black dot. If we flow the origin along $Y (= \frac{\partial}{\partial y})$ for time 1, then along X for time 1, then along $-Y$ for time 1, and finally along $-X$ for time 1, we end up slightly away from the origin, at the white dot. This white dot is an associator. The gap is exaggerated in the picture for clarity.

7.2 Example

So far, we have not given an example of a local Lie groupoid with non-integrable algebroid. We do this now. The example will hopefully give the reader some intuition for associators.

Let $M = \mathbb{S}^2 \times \mathbb{S}^2$, where each copy of \mathbb{S}^2 is equipped with the usual round metric. Let

$$H = \{((y, y'), (x, x')) \in M \times M \mid x + y \neq 0 \neq x' + y'\} \times \mathbb{R}.$$

We prescribe the source and target maps as

$$s((y, y'), (x, x'), a) = (x, x') \in M$$

and

$$t((y, y'), (x, x'), a) = (y, y') \in M.$$

We introduce a multiplication on H by the formula

$$((z, z'), (y, y'), a_1) \cdot ((y, y'), (x, x'), a_2) = ((z, z'), (x, x'), a_1 + a_2 + A(\Delta xyz) + \lambda A(\Delta x'y'z')),$$

where $\lambda \in \mathbb{R}$ is a fixed parameter (recall that A is the area), and which is defined whenever

$$A(\Delta xyz) \in (-\pi, \pi) \quad \text{and} \quad A(\Delta x'y'z') \in \left(-\frac{\pi}{|\lambda|}, \frac{\pi}{|\lambda|}\right).$$

(The condition on $A(\Delta x'y'z')$ is void if $\lambda = 0$.) The algebroid of this local Lie groupoid is integrable precisely if λ is rational [8, example 3.1].

Let us check that H is 3-associative. Suppose that $((z, z'), (y, y'), a_1)$, $((y, y'), (x, x'), a_2)$ and $((x, x'), (w, w'), a_3)$ are in H . Then

$$\begin{aligned} & (((z, z'), (y, y'), a_1) \cdot ((y, y'), (x, x'), a_2)) \cdot ((x, x'), (w, w'), a_3) \\ &= ((z, z'), (x, x'), a_1 + a_2 + a_3 + A(\Delta xyz) + A(\Delta wxz) + \lambda A(\Delta x'y'z') + \lambda A(\Delta w'x'z')) \end{aligned}$$

and

$$\begin{aligned} & ((z, z'), (y, y'), a_1) \cdot (((y, y'), (x, x'), a_2) \cdot ((x, x'), (w, w'), a_3)) \\ &= ((z, z'), (x, x'), a_1 + a_2 + a_3 + A(\Delta wxy) + A(\Delta wyz) + \lambda A(\Delta w'x'y') + \lambda A(\Delta w'y'z')). \end{aligned}$$

Now

$$A(\Delta xyz) + A(\Delta wxz) = A(\Delta wxy) + A(\Delta wyz) \pmod{4\pi}$$

because both equal the area of the quadrangle $wxyz$ (which is defined up to π). By the restriction on the areas of these triangles, equality must hold in \mathbb{R} (not just in $\mathbb{R}/4\pi$). Similarly, we will have

$$\lambda A(\Delta x'y'z') + \lambda A(\Delta w'x'z') = \lambda A(\Delta w'x'y') + \lambda A(\Delta w'y'z').$$

This proves 3-associativity.

This local Lie groupoid is not globally associative. The counterexample to 6-associativity of G'' on section 5.2 also works here, by considering G'' as the local Lie subgroupoid $\{(y, y'), (N, N)\} \times \mathbb{R}$ of H , where $N \in \mathbb{S}^2$ is the north pole. Recall, however, that there was a neighborhood of $M \subset G''$ that was globally associative. This is not necessarily the case for H .

The following result will be a consequence of theorem C.

Proposition 116. *There is an open neighborhood of $M \subset H$ that is globally associative if and only if $\lambda \in \mathbb{Q}$ (i.e. iff the Lie algebroid is integrable).*

7.3 Review of simplicial complexes and sets

We will describe the precise link between associators and monodromy groups later. First, we quickly review the basics of simplicial complexes and simplicial sets.

An (*abstract*) *simplicial complex* is a collection S of finite non-empty sets, such that if A is an element of S , then so is every non-empty subset of A . The elements of S are called *simplices*. A simplex that contains $k + 1$ elements is called a *k-simplex*. We will use the terms *vertices*, *edges* and *faces* for the 0-, 1- and 2-simplices of a simplicial complex. A face is said to be *attached* to an edge if the edge is contained in the face. In our arguments, S will always be finite. The *dimension* of a simplicial complex is the largest k for which it has a k -simplex. The *standard n-simplex* is the power set of $\{0, \dots, n\}$, with the empty set removed. It is a simplicial complex of dimension n , and we will denote it by $[n]$.

An *ordered simplicial complex* is a simplicial complex S , together with a partial ordering of its vertices, such that the order is linear (i.e. total) on each simplex. The standard n -simplex comes with an obvious ordering of its vertices, making it into an ordered simplicial complex.

Let Δ be the simplex category (its objects are linearly ordered sets of the form $\{0, \dots, n\}$, and its morphisms are (non-strictly) order-preserving functions between such sets). A *simplicial set* is a contravariant functor from Δ to the category of small sets. A morphism of simplicial sets is a natural transformation between them. The *degeneracy maps* of a simplicial set are the images of all of the maps from $\{0, \dots, n + 1\}$ to $\{0, \dots, n\}$ (for any n). The *face maps* of a simplicial set are the images of all of the maps from $\{0, \dots, n\}$ to $\{0, \dots, n + 1\}$ (for any n).

Every ordered simplicial complex S can be turned into a simplicial set, by prescribing it on objects as

$$\{0, \dots, n\} \mapsto \{\text{order-preserving maps } \{0, \dots, n\} \rightarrow \text{Vert}(S) \text{ whose image is in } S\},$$

where $\text{Vert}(S)$ is the set of vertices of S , and on morphisms by composition.

Given a local Lie groupoid G over M , we can associate to it a simplicial set \mathcal{N}_2G of

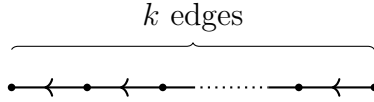


Figure 7.2: The simplicial complex W_k is just a sequence of k edges, head to tail.

dimension 2 by setting

$$\begin{aligned} \{0\text{-simplices}\} &= M \\ \{1\text{-simplices}\} &= G \\ \{2\text{-simplices}\} &= \mathcal{U} \subset G \times_s G. \end{aligned}$$

(Recall that \mathcal{U} is the domain of the multiplication map.) The higher-dimensional simplices are all degenerate. The degeneracy maps insert identity elements, the face maps $G \rightarrow M$ are s and t , the face maps $\mathcal{U} \rightarrow G$ are $(g, h) \mapsto h$, $(g, h) \mapsto gh$, $(g, h) \mapsto g$.

7.4 Expansion and contraction of simplicial complexes

Let W_k be the ordered simplicial complex

$$\{\{0\}, \dots, \{k\}\} \cup \{\{0, 1\}, \{1, 2\}, \dots, \{k-1, k\}\}.$$

It is shown in figure 7.2. A map $W_k \rightarrow \mathcal{N}_2 G$ is nothing but a well-formed word on G of length k . We now want to express equivalence of words (for the equivalence relation \sim of before) using simplicial terminology.

The simplicial complexes that we will work with will be particularly well-behaved: they will be 2-dimensional, and each edge will have at most two faces attached to it. We will say that an edge is a *boundary edge* if it has at most one face attached to it.

Suppose that S is an ordered simplicial complex. If $\{u, w\}$ is a boundary edge of S with $u < w$, we can obtain a new ordered simplicial complex S' by

1. adding a new vertex v to the simplicial complex with $u < v < w$,
2. adding edges $\{u, v\}$ and $\{v, w\}$, and a face $\{u, v, w\}$.

We will say that S' can be obtained from S by *expansion*.

Suppose that S is an ordered simplicial complex. If $\{u, v\}$ and $\{v, w\}$ are boundary edges of S , with $u < v < w$, and $\{u, w\}$ is not an edge of S , we can obtain a new ordered simplicial complex S' by

1. adding the edge $\{u, w\}$,
2. adding the face $\{u, v, w\}$.

We will say that S' can be obtained from S by *contraction*.

Definition 117. We say that an ordered simplicial complex S is a *good complex* if there is an integer $k \geq 1$ such that S can be obtained from W_k by repeated expansion or contraction.

Note that a good complex comes with a choice of two vertices: there is one vertex that is minimal among boundary vertices (we will call it the *source* of the good complex) and one vertex that is maximal among boundary vertices (we will call it the *target* of the good complex).

Definition 118. If S is a good complex, a *boundary path* of S is an ordered subcomplex S' of S that is contained in its boundary, and for which there is an isomorphism of ordered complexes $W_k \rightarrow S'$ that maps the source of W_k to the source of S , and the target of W_k to the target of S' .

Remark. A good complex can have many boundary paths. For example, the complex shown in figure 7.4 has eight of them. The good complexes that will be most relevant for us are those that are homeomorphic to a disk. These have precisely two boundary paths.

Every good complex is a bunch of disks and lines attached source-to-target (like in 7.4, where the complex consists of two disks, a line, and another disk). If the number of such disks is d , there are 2^d boundary paths. Whenever we ask for a good complex to be homeomorphic to a disk, our goal is to be able to speak unambiguously about its two boundary paths.

In other words, a boundary path of S is just a path from source to target along the boundary of S such that the vertices are increasing along the path.

Definition 119. If S is a good complex (which we can consider as a simplicial set) and $\phi : S \rightarrow \mathcal{N}_2G$ is a simplicial map, then the words corresponding to the restriction of ϕ to the boundary paths of S are called the *boundary words* of ϕ .

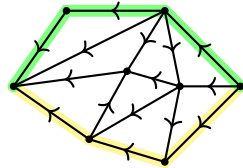


Figure 7.3: A good complex. This one can be obtained from the yellow-marked W_3 on the bottom by (for example) an expansion, a contraction, an expansion, a contraction, an expansion, two contractions and an expansion. The source is the rightmost vertex, the target is the leftmost vertex. This complex has two boundary paths (one on the top, one on the bottom). They are highlighted in green and yellow.

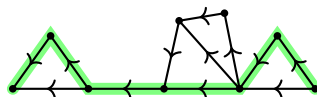


Figure 7.4: A good complex that is not homeomorphic to a disk. This one has eight boundary paths. One of them is highlighted.

The following result now holds more or less by construction.

Proposition 120. *Two well-formed words on G are equivalent (for the equivalence relation \sim defined earlier) if and only if there is a good complex S and a simplicial map $\phi : S \rightarrow \mathcal{N}_2G$ such that both words are boundary words of ϕ .*

Proof. If the two words w and w' are equivalent, then there is a sequence of expansions and contractions of words turning w into w' . Let $w = w_0, w_1, \dots, w_n = w'$ be the shortest sequence of expansions and contractions from w to w' . We can use this sequence to build a good complex S and a map $\phi : S \rightarrow \mathcal{N}_2G$, as we explain now.

Start with a map $\phi_0 : W_k \rightarrow \mathcal{N}_2G$ representing w_0 (so w_0 has k letters). If w_1 is an expansion of w_0 , we can expand W_k in the corresponding edge to get a map $\phi_1 : S_1 \rightarrow \mathcal{N}_2G$ with boundary words w_0 and w_1 . If w_1 is a contraction of w_1 , we can contract W_k in the corresponding edge to get a map $\phi_1 : S_1 \rightarrow \mathcal{N}_2G$ with boundary words w_0 and w_1 . (Note that the edge that we need to add to the complex for the contraction is not already present. This edge can only be present if we have an expansion followed by a contraction undoing that expansion, but we chose the shortest sequence of words linking w to w' .)

Conversely, if we have a good complex S and a map $\phi : S \rightarrow \mathcal{N}_2G$ with boundary words w_1 and w_2 , we can use this information to exhibit a sequence of expansions and contractions of words turning w_1 into w_2 . \square

7.5 Monodromy groups

The integrability of a Lie algebroid is controlled by its *monodromy groups* [7]. The monodromy group $\tilde{\mathcal{N}}_x$ at x is a subgroup of $\mathcal{G}(\mathfrak{g}_x)$ contained in the center $Z(\mathcal{G}(\mathfrak{g}_x))$. Its construction uses the language of A -paths, which we will assume the reader is familiar with. A detailed exposition can be found in [7] and [8, sections 2.2, 3.3 and 3.4]. Let us briefly recall the construction of $\tilde{\mathcal{N}}_x$. Let A be a Lie algebroid over M .

Definition 121. Suppose a and a' are two A -paths over the same base path. We say that a and a' are *A -homotopic along a trivial sphere* if there is an A -homotopy between a and a' whose base homotopy determines a trivial element in $\pi_2(\mathcal{O}_x)$ (where \mathcal{O}_x is the orbit of x).

Suppose a_0 is an A -path with base path γ_0 in the orbit \mathcal{O}_x . If $S : [0, 1]^2 \rightarrow \mathcal{O}_x$ is such that

- $S(0, t) = \gamma_0(t)$,
- $S(s, 0) = \gamma_0(0)$ and $S(s, 1) = \gamma_0(1)$ for all s ,

then we can lift $dS : TI \times TI \rightarrow T\mathcal{O}$ to a morphism of algebroids $a dt + b ds : TI \times TI \rightarrow A$ such that $a(0, t) = a_0(t)$ and such that $b(s, 0)$ and $b(s, 1)$ vanish. We obtain an A -path $a_1(t) = a(1, t)$. Its class modulo A -homotopies along trivial spheres only depends on the homotopy class (rel boundary) of S [8, proof of prop. 3.21]. Let us write $\tilde{\partial}(a_0, S)$ for the class of a_1 modulo A -homotopy along trivial spheres.

Now if in the above situation the path $t \mapsto S(1, t)$ is constant, say at $x \in M$, then $\tilde{\partial}(a_0, S)$ is (the class of) a \mathfrak{g}_x -path. By integration to a $\mathcal{G}(\mathfrak{g}_x)$ -path, it determines an element of $\mathcal{G}(\mathfrak{g}_x)$, which we call

$$\partial(a_0, S) = \text{endpoint of integration of } \tilde{\partial}(a_0, S).$$

We will also write 0_x for the trivial A -path at x . The monodromy group $\tilde{\mathcal{N}}_x$ is defined as the image of

$$\partial : \pi_2(\mathcal{O}_x) \rightarrow \mathcal{G}(\mathfrak{g}_x) : [\alpha] \mapsto \partial(0_x, \alpha),$$

where we consider an element $[\alpha] \in \pi_2(\mathcal{O}_x, x)$ as represented by a map $[0, 1]^2 \rightarrow \mathcal{O}_x$.

7.6 Associators are monodromy elements

In order to give a precise connection between monodromy groups and associators, we will need a condition on G that ensures that elements can only be multiplied if they are sufficiently small (i.e. we may need to restrict G 's multiplication).

We will need some auxiliary data. The goal of this data is to associate to every sufficiently small element $g \in G$ an A -path $P(g)$ representing it (up to homotopy along trivial spheres). Moreover, we will do this in such a way that if $g_1, g_2 \in G$ satisfy $s(g_1) = s(g_2)$ and $t(g_1) = t(g_2)$, then $P(g_1)$ and $P(g_2)$ lie over the same base path in M .

Choose a function $f : [0, 1] \rightarrow [0, 1]$ that is 0 near 0 and 1 near 1. We will use it to reparameterize curves to ensure smoothness of concatenations (but it has no further meaning). Choose a metric $\langle \cdot, \cdot \rangle$ on M . Pick an A -connection ∇ on A . This connection allows us to define an exponential map $\exp_{\nabla} : A \rightarrow G$, defined in a neighborhood of the zero section M ([8, section 4.4]). If $g \in G$ is sufficiently close to M , then it is in the image of \exp_{∇} , say $g = \exp(\xi)$. Write $x = s(g)$. Then we have a G -path $\tau \mapsto \exp(f(\tau)g)$. Associated to this G -path is an A -path (by differentiating and applying the Maurer-Cartan form), which we call $\tilde{P}(g)$. Moreover, if $g \in G$ is sufficiently close to M , the base path of $\tilde{P}(g)$ is contained completely in a uniformly normal subset of M . (An open subset of a Riemannian manifold is *uniformly normal* if there exists some $\delta > 0$ such that the subset is contained in a geodesic ball of radius δ around each of its points. See [20, lemma 5.12]. Important for us will be the fact that between any two points in a uniformly normal neighborhood, there is a unique minimizing geodesic, and it lies in this neighborhood.) Now we modify the A -path $\tilde{P}(g)$ into an A -path $P(g)$ that lies over the geodesic $\gamma_{s(g), t(g)}$ from $s(g)$ to $t(g)$. We do this by prescribing

$$P(g) = \partial(\tilde{P}(g), [\alpha]),$$

where $[\alpha] : [0, 1] \times [0, 1] \rightarrow M$ is the map such that $s \mapsto \alpha(t, s)$ is the geodesic from $(\rho \circ \tilde{P}(g))(t)$ to $\gamma_{s(g), t(g)}(t)$. This way, we have associated to each $g \in G$ sufficiently close to M an A -path $P(g)$ that lies over the geodesic from $s(g)$ to $t(g)$. Write $P(g_k, \dots, g_1)$ for the concatenation of $P(g_1), \dots, P(g_k)$. If $w = (w_k, \dots, w_1)$ is a well-formed word on G , write $P(w) = P(w_k, \dots, w_1)$.

Write $\Delta[k]$ for the standard k -simplex:

$$\Delta[k] = \left\{ (\lambda_0, \dots, \lambda_k) \in \mathbb{R}^{k+1} \mid \lambda_i \geq 0, \sum_i \lambda_i = 1 \right\}.$$

The boundary of $\Delta[2]$ consists of three line segments. If $(g, h) \in \mathcal{U}$, write $\beta_{(g,h)}$ for the map $\partial\Delta[2] \rightarrow M$ that sends the first edge of $\Delta[2]$ to the base path of $P(g)$, the second edge to the base path of $P(gh)$, and the third edge to the base path of $P(h)$.

Let S'' be a neighborhood of the diagonal in $M \times M$ such that for all $(x, y) \in S''$ there is a unique shortest geodesic from x to y . Let

$$S' = \{(x, y, z) \in M \times M \times M \mid (x, y), (y, z), (x, z) \in S''\}.$$

If $(x, y, z) \in S'$, we will write $\beta_{(x,y,z)} : \partial\Delta[2] \rightarrow M$ for the map that sends the edges of $\Delta[2]$ to the three geodesics between x, y and z . Consider the space

$$S = \left\{ (x, y, z, [\alpha]) \in S' \times \left(\frac{\text{maps } \Delta[2] \rightarrow M}{\text{homotopy rel } \partial} \right) \mid \alpha \upharpoonright_{\partial\Delta[2]} = \beta_{(x,y,z)} \right\}.$$

It is a smooth manifold, and the map $\pi : S \rightarrow M^3 : (x, y, z, [\alpha]) \mapsto (x, y, z)$ is a local diffeomorphism. Note that

$$T = \{(x, y, z) \in M \times M \times M \mid x = y \text{ or } y = z \text{ or } x = z\}$$

naturally sits inside S by

$$T \ni (x, y, z) \mapsto (x, y, z, [\text{maps whose image is contained in the image of } \beta_{(x,y,z)}]).$$

Choose a map $\varphi_2 : U \rightarrow S$ from a neighborhood U of T in M^3 such that

- $\pi \circ \varphi_2 = \text{Id}_U$,
- $\varphi_2 \upharpoonright_T = \text{Id}_T$.

Definition 122. Suppose that G is a local Lie groupoid over M , and that we have chosen auxiliary information $(f, \langle \cdot, \cdot \rangle, \nabla, \varphi_2)$ as above. We say that G is *shrunk* if

1. the path $P(g)$ is defined for every $g \in G$,
2. the isotropy groups G_x are simply connected (this implies that we have well-defined maps $G_x \rightarrow \mathcal{G}(\mathfrak{g}_x)$ as explained below),
3. for each $x \in M$, the map $G_x \rightarrow \mathcal{G}(\mathfrak{g}_x)$ is injective,
4. if $(g, h) \in \mathcal{U}$ then $\{s(h), t(h) = s(g), t(g)\} \in U$, so that we may write $\varphi_2(g, h) := \varphi_2(s(h), t(h), t(g))$,

5. for every $(g, h) \in \mathcal{U}$, the class of $P(gh)$ (as A -path modulo A -homotopy along trivial spheres) equals

$$\tilde{\partial}(P(g, h), \varphi_2(g, h)).$$

The map $G_x \rightarrow \mathcal{G}(\mathfrak{g}_x)$ works as follows. Let $g \in G_x$. Take a path in G_x from x to g . Differentiate this path to get a \mathfrak{g}_x -path. This path integrates to a $\mathcal{G}(\mathfrak{g}_x)$ -path. The endpoint of this path is the image of g under the map $G_x \rightarrow \mathcal{G}(\mathfrak{g}_x)$. If G_x is simply-connected, this map is well-defined, because any two paths in G_x from x to g are homotopic (and this homotopy induces an A -homotopy at the level of A -paths, which implies that the integrations to $\mathcal{G}(\mathfrak{g}_x)$ -paths have the same endpoint).

Every transitive local Lie groupoid has an open neighborhood of the identities that is shrunk (after restricting). The first three conditions can be satisfied by shrinking G . The fourth can be satisfied by restricting the multiplication. The last condition can be satisfied by shrinking and restricting: it is automatically satisfied for a global Lie groupoid, and over every contractible neighborhood U , the algebroid is integrable (and over U and near M , the local groupoid is isomorphic to a global integration of $A|_U$, so that the condition holds for these small groupoid elements).

Proposition 123. *Suppose that G is a shrunk local Lie groupoid over M . If $w = (w_1, \dots, w_k)$ and $w' = (w'_1, \dots, w'_{k'})$ are equivalent words, then the A -paths $P(w_1, \dots, w_k)$ and $P(w'_1, \dots, w'_{k'})$ are A -homotopic.*

Proof. It suffices to prove this if w and w' are elementarily equivalent, so we will assume that w' is obtained from w by multiplying the letters w_i and w_{i+1} . The A -path

$$\tilde{\partial}(P(w_i, w_{i+1}), \varphi_2(w_i, w_{i+1}))$$

is homotopic to $P(w_i, w_{i+1})$ by construction. Because G is shrunk, it is also A -homotopic to the A -path $P(w_i w_{i+1})$. We conclude that $P(w_i, w_{i+1})$ and $P(w_i w_{i+1})$ are A -homotopic, which suffices to prove the result. \square

The monodromy group $\tilde{\mathcal{N}}_x$ consists precisely of the endpoints of integrations of \mathfrak{g}_x -paths that are A -homotopic to the trivial path 0_x . This gives us the following corollary.

Corollary 124. *Suppose that G is a shrunk local Lie groupoid over M . Let $x \in M$. Consider G_x as a subset of $\mathcal{G}(\mathfrak{g}_x)$ using the natural map $G_x \rightarrow \mathcal{G}(\mathfrak{g}_x)$. Then $\text{Assoc}_x(G) \subset \tilde{\mathcal{N}}_x$.*

7.7 Monodromy elements are associators

Suppose that G is shrunk. If we have a simplicial map $[1] \rightarrow \mathcal{N}_2G$, we can associate to it a curve in M as before: we take the base path of the curve $P(g)$ where g is the image of the 1-cell. If we have a simplicial map $[2] \rightarrow \mathcal{N}_2G$, we get a homotopy class of maps $\Delta[2] \rightarrow M$ rel boundary, namely $\varphi_2(g, h)$ as before, where $(g, h) \in \mathcal{U}$ is the image of the 2-cell. More generally, if we have a good complex S and a simplicial map $\phi : S \rightarrow \mathcal{N}_2G$, we get an induced homotopy class of maps $|S| \rightarrow M$ rel boundary, where $|S|$ is the geometric realization of S , by mapping each 1-cell and 2-cell as above, which we call $[\phi]$.

Lemma 125. *Let G be a shrunk local Lie groupoid. Suppose S is a good complex homeomorphic to a disk, and $\phi : S \rightarrow \mathcal{N}_2G$ is a simplicial map. Let w and w' be the boundary words of ϕ . Then*

$$\tilde{\partial}(P(w), [\phi]) \text{ and } P(w')$$

are A -homotopic along a trivial sphere.

Proof. If S has only one face, this follows immediately from condition (5) in the definition of a shrunk local groupoid. By induction on the number of faces of S , the statement follows. \square

Proposition 126. *Suppose that G is a shrunk local Lie groupoid over M . Let $x \in M$. Consider G_x as a subset of $\mathcal{G}(\mathfrak{g}_x)$ using the natural map $G_x \rightarrow \mathcal{G}(\mathfrak{g}_x)$. Suppose that S is a good complex homeomorphic to a disk, and that*

$$\phi : S \rightarrow \mathcal{N}_2G$$

is a simplicial map sending the boundary to x . (In particular, ϕ induces a class $[\phi] \in \pi_2(M, x)$.) Suppose that one of the boundary words of ϕ is (x) , and that the other boundary word has all letters in G_x . Then $\partial([\phi]) \in \text{Assoc}_x(G)$.

Proof. Let $w = (w_1, \dots, w_k)$ be the other boundary word of ϕ . By lemma 125, $\partial([\phi])$ is the endpoint of the integration of $P(w) = P(w_1, \dots, w_k)$ to a $\mathcal{G}(\mathfrak{g}_x)$ -path. Now working in $\mathcal{G}(\mathfrak{g}_x)$, we see that

$$\partial([\phi]) = \prod_{i=1}^k w_i.$$

By lemma 127 below, applied to $G_x \subset \mathcal{G}(\mathfrak{g}_x)$, we have

$$(\partial([\phi])) \sim (w_1, \dots, w_k) \quad \text{in } W(G_x).$$

In particular, this equivalence holds in $W(G)$. However, the map ϕ shows that $(w_1, \dots, w_k) \sim (x)$ in $W(G)$ (by proposition 120). This implies that $(\partial([\phi])) \sim (x)$ in $W(G)$, and so $\partial([\phi]) \in \text{Assoc}_x(G)$. \square

Lemma 127. *Let \mathcal{G} be a source-simply connected Lie groupoid, and let $U \subset \mathcal{G}$ be an open neighborhood of the identities, considered as a local Lie groupoid. Suppose that $g_1, \dots, g_n \in U$ are such that their product $g = g_1 \cdots g_n$ lies in U . Then in $W(U)$ we have*

$$(g_1, \dots, g_n) \sim (g).$$

Proof. Apply the functor \mathcal{AC} to the inclusion $U \hookrightarrow \mathcal{G}$. We get $\mathcal{AC}(U) \rightarrow \mathcal{AC}(\mathcal{G}) \cong \mathcal{G}$, which is a morphism of Lie groupoids. It is an isomorphism at the level of Lie algebroids, and \mathcal{G} is source-simply connected, so this map is an isomorphism. In particular, it is an injection. Because the words $(g_1, \dots, g_n), (g) \in W(U)$ are mapped to the same element of \mathcal{G} , they must represent the same element of $\mathcal{AC}(U)$. This means that they are equivalent. \square

Note that this lemma is false if \mathcal{G} is not s -simply connected. For example, if we take $U = (-0.5, 0.5)/\mathbb{Z} \subset \mathbb{R}/\mathbb{Z} = \mathcal{G}$, then $0.25 + 0.25 + 0.25 + 0.25 = 0$ in \mathcal{G} , but $(0.25, 0.25, 0.25, 0.25) \not\sim (0)$ as words in $W(U)$.

We now state and prove the main theorem relating associators to monodromy.

Theorem C. *Suppose that G is a shrunk local Lie groupoid over M . Let $x \in M$. Consider G_x as a subset of $\mathcal{G}(\mathfrak{g}_x)$ using the natural map $G_x \rightarrow \mathcal{G}(\mathfrak{g}_x)$. Then*

$$\text{Assoc}_x(G) = \tilde{\mathcal{N}}_x \cap G_x.$$

Proof. The inclusion \subset was corollary 124. We now prove the inclusion \supset . By proposition

126, it suffices to show that for every $[\alpha] \in \pi_2(M, x)$, there is a good complex S , homeomorphic to a disk, and a simplicial map $\phi : S \rightarrow \mathcal{N}_2G$, such that

1. the induced map $|S| \rightarrow M$ maps the boundary of $|S|$ to x ,
2. one of the boundary words of ϕ is (x) ,
3. $[\phi] = [\alpha]$ as elements of $\pi_2(M, x)$.

Write $\Delta = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x+y \leq 1\}$. Represent $[\alpha]$ as a map $\alpha : \Delta \rightarrow M$, mapping the boundary of Δ to x . Fix a splitting $\sigma : TM \rightarrow A$ of the anchor. If e is a sufficiently short oriented line segment in Δ , from a point v_1 to a point v_2 , we will associate to it an element g_e of G , as follows. There is a tangent vector V_e in $T_{\alpha(v_1)}M$ such that $t(\exp(\sigma(V_e))) = \alpha(v_2)$ (if e is sufficiently short). Take $g_e = \exp(\sigma(V_e))$. The base path of $P(g_e)$ is then a geodesic from $\alpha(v_1)$ to $\alpha(v_2)$.

If k is a positive integer, we can subdivide Δ into k^2 triangles, as shown in figure 7.5. By choosing k large enough, we can ensure that

1. every edge e of the triangulation is short enough to ensure that g_e is defined,
2. if e_3, e_2, e_1 are the edges of a face, the products $(g_{e_3}g_{e_2})g_{e_1}$ and $g_{e_3}(g_{e_2}g_{e_1})$ are both defined,
3. if e_3, e_2, e_1 are the edges of a face, and f_m, \dots, f_1 are edges of the triangulation forming a path of total Euclidean length at most 10 (as measured in \mathbb{R}^2) with $t(e_3) = s(e_1) = t(f_m)$, then the product

$$(g_{f_1}^{-1} \cdots (g_{f_m}^{-1}(g_{e_3}g_{e_2}g_{e_1})g_{f_m}) \cdots g_{f_1}) \quad (\dagger)$$

is defined (the brackets that are not specified may be put in in any way).

If k is large enough to guarantee these properties, we get an induced element of $\pi_2(M, x)$, as follows: we can map the 1-skeleton of the triangulated Δ to M by mapping each vertex of the triangulation to its image under α , and mapping each edge e to the base path of $P(g_e)$. For each face of the triangulation, there is a preferred homotopy class of map into M (rel boundary), namely the one determined by φ_2 . All in all, we get a well-defined homotopy class of map from Δ to M , rel 1-skeleton. In particular, this

²this number should be squared

defines a homotopy class of map from Δ to M , rel boundary, and hence an element of $\pi_2(M, x)$. After choosing k larger if necessary, we may assume that this class is precisely $[\alpha]$.

Let us write $\varepsilon_{3k}, \dots, \varepsilon_1$ for the $3k$ edges at the circumference of Δ , in counterclockwise order starting at the origin $(0, 0)$, and with the corresponding orientation (so $\varepsilon_k, \dots, \varepsilon_1$ are oriented to the right, $\varepsilon_{2k}, \dots, \varepsilon_{k+1}$ to the upper left, and $\varepsilon_{3k}, \dots, \varepsilon_{2k+1}$ down). We will work with ordered sequences of oriented edges of the triangulation. Such a sequence of oriented edges (e_n, \dots, e_1) is *well-formed* if e_i ends where e_{i+1} starts (for all i). A *block* will be a well-formed sequence (possibly empty) of the form

$$(e_1^{-1}, \dots, e_n^{-1}, e_n, \dots, e_1),$$

where e_i^{-1} is the edge e_i with its opposite orientation. This name is chosen to be the same as in the proof of theorem A. If F is one of the faces of the triangulation, a *lace around F* will be a sequence of the form

$$(e_1^{-1}, \dots, e_n^{-1}, c, b, a, e_n, \dots, e_1),$$

where a, b, c are the three sides of F , oriented counterclockwise, and e_1 starts at $(0, 0)$. (So it loops around F once, and encloses nothing else.) The words (e_n, \dots, e_1) and $(e_1^{-1}, \dots, e_n^{-1})$ are the *ends* of the lace.

Now let (e_n, \dots, e_1) be a sequence of oriented edges of the triangulation that simultaneously satisfies the conditions

- (a) the sequence can be obtained from $(\varepsilon_{3k}, \dots, \varepsilon_2, \varepsilon_1)$ by repeated insertion of blocks,
- (b) the sequence is a concatenation of laces ℓ_N, \dots, ℓ_1 , one around each face of the triangulation, and each end of a lace has at most $2k$ edges (which certainly ensures that the total euclidean length of the edges of that end is less than 10).

Such a sequence always exists. Figure 7.6 shows an example for $k = 2$. The video at <http://math.uiuc.edu/~michiel2/edge-sequence/> illustrates an appropriate sequence of edges for $k = 5$, which can immediately be generalized to work for any value of k . Note that for each of the laces, the product of the elements associated to the edges is defined if we use the order as in (†). We will call this the *product* of the lace.

Now this sequence allows us to build up the required good complex S and the simplicial map $\phi : S \rightarrow \mathcal{N}_2G$. Start with the complex W_1 , mapped to the element x .

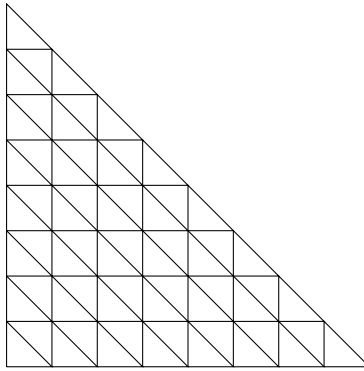


Figure 7.5: The triangulation of Δ , shown here for $k = 8$.

Expand $3k - 1$ times, to get a good complex with two boundary paths: one is the original W_1 , the other is a copy of W_{3k} . On the complex we have so far, ϕ will still map everything to x .

Recall that (e_n, \dots, e_1) can be obtained from $(\varepsilon_{3k}, \dots, \varepsilon_1)$ by insertion of blocks. This means that the word $(g_{e_n}, \dots, g_{e_1})$ can be obtained from (x, \dots, x) by expansion. Expanding our good complex further, we therefore get a complex and a map ϕ with boundary words (x) and $(g_{e_n}, \dots, g_{e_1})$.

Now we group the (e_n, \dots, e_1) by laces. Because the corresponding products are defined, we may now continue building our good complex using contractions. This results in a complex and map ϕ with boundary words (x) and (p_N, \dots, p_1) , where p_i is the product of the i 'th lace.

By construction, this simplicial map ϕ induces $[\alpha]$, proving the result. □

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Appendix: Computer code

The following code renders a video that illustrates the triangulation of figure 7.5 and a sequence of edges that satisfies the conditions in the proof of theorem C. To execute this code, Python 3 is required, with the numpy and matplotlib libraries installed.

```
1 import matplotlib.pyplot as plt
2 import matplotlib.animation as animation
3 import numpy
4 from math import ceil
5
6 k = 5
7 fps = 30
8 speed = 3.0
9
10 # lace around a lower triangle
11 def lower_lace(i, j):
12     for x in range(1,i+1):
13         yield (x,0)
14     for y in range(0,j+1):
15         yield (i+1,y)
16     yield (i,j+1)
17     yield (i,j)
18     yield (i+1,j)
19     for y in range(j,0,-1):
20         yield (i+1, y-1)
21     for x in range(i+1,0,-1):
22         yield (x-1,0)
23
```

```

24 # lace around an upper triangle
25 def upper_lace(i,j):
26     for x in range(1,i+1):
27         yield (x,0)
28     for y in range(0,j+1):
29         yield (i+1,y)
30     yield (i+1, j+1)
31     yield (i,j+1)
32     yield (i+1,j)
33     for y in range(j,0,-1):
34         yield (i+1, y-1)
35     for x in range(i+1,0,-1):
36         yield (x-1,0)
37
38 # laces for all triangles in a column
39 def column(i):
40     yield from lower_lace(i,k-i-1)
41     for j in range(k-i-2, -1, -1):
42         yield from upper_lace(i,j)
43         yield from lower_lace(i,j)
44
45 # the entire sequence
46 def everything():
47     yield (0,0)
48     for i in range(k-1,-1,-1):
49         yield from column(i)
50
51 path = list(everything())
52 xs = [p[0] for p in path]
53 ys = [p[1] for p in path]
54
55 def animate(n, point):
56     point.set_data([
57         numpy.interp(speed*n/fps, range(len(xs)), xs),

```

```

58         numpy.interp(speed*n/fps, range(len(ys)), ys)])
59
60 if __name__ == "__main__":
61     fig = plt.figure()
62     fig.set_facecolor('white')
63     plt.plot(xs, ys, color='#cccccc', linewidth=2.0)
64     plt.plot([i for i in range(k+1) for j in range(k+1-i)],
65             [j for i in range(k+1) for j in range(k+1-i)],
66             'o', color='#cccccc')
67     plt.axis([-1, k+1, -1, k+1])
68     plt.axis('off')
69     ax = plt.gca()
70     plt.tight_layout(pad=0.1)
71     point, = ax.plot([0],[0], 'o', color='#000000',
72                     markersize=10)
73     ani = animation.FuncAnimation(fig, animate,
74                                  list(range(ceil(fps*len(path)/speed))),
75                                  fargs=[point], interval=(1000/fps))
76
77     # save animation to a file
78     FFMpegWriter = animation.writers['ffmpeg']
79     metadata = dict(title='Edge sequence')
80     writer = FFMpegWriter(fps=fps, metadata=metadata)
81     ani.save('edge-sequence.mp4', writer=writer)
82     # show animation on screen
83     plt.show()

```
