Stability of Symplectic Leaves

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Paulette Libermann :: Héritage et Descendance
(Institut Henri Poincaré, December 2009)

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- Establish stability results for symplectic leaves of Poisson manifolds;
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Based on the paper:

• M. Crainic and RLF, Stability of symplectic leaves, Preprint arXiv:0810.4437 (to appear in *Inventiones Mathematicae*).

Flows: Stability of periodic orbits

Definition

A periodic orbit of a vector field $X \in \mathfrak{X}(M)$ is called stable if every nearby vector field also has a nearby periodic orbit.

- Basic Fact: Stability is controled by the Poincaré return map $h : T \to T$.

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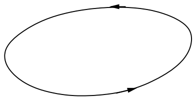

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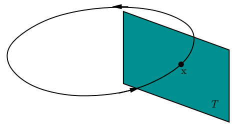

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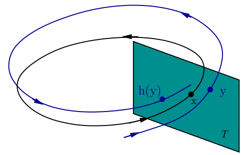

Flows: Stability of periodic orbits

Simple assumptions on $\mathrm{d}_x h$ lead to stability:

- If 1 is not an eigenvalue of $\mathrm{d}_x h$ then the orbit is stable.

Consider the representation $\rho : \mathbb{Z} \to \mathrm{GL}(\nu(\mathcal{O})_x)$, defined by

$$n \cdot v := (\mathrm{d}_x h)^n v.$$

$H^\bullet(\mathbb{Z}, \nu(\mathcal{O})_x) \equiv$ group cohomology with coefficients in $\nu(\mathcal{O})_x$.

Theorem

Let $\mathcal{O}$ be a periodic orbit of a vector field X and assume that

$$H^1(\mathbb{Z}, \nu(\mathcal{O})_x) = 0.$$

Then $\mathcal{O}$ is stable: every nearby vector field has a family of nearby periodic orbits smoothly parametrized by $H^0(\mathbb{Z}, \nu(\mathcal{O})_x)$.

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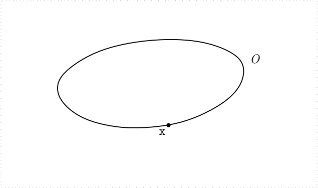
An orbit $\mathcal{O}$ of $\alpha \in \mathrm{Act}(G; M)$ is called stable if every nearby action in $\mathrm{Act}(G; M)$ has a nearby orbit diffeomorphic to $\mathcal{O}$.

Group actions: stability of orbits

The stability of an orbit $\mathcal{O}$ is controled by the isotropy representation:

- $G_x := \{g \in G : g \cdot x = x\}$ isotropy group at $x \in \mathcal{O}$.
- $g \in G_x$ induces a map $\alpha_g : M \to M$, $y \mapsto g \cdot y$ that fixes x and maps the orbit to the orbit.



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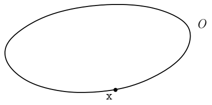

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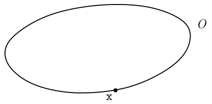

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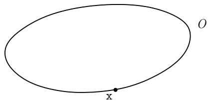

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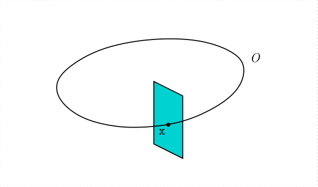

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$\Rightarrow \quad \rho(g) : \nu(\mathcal{O})_x \to \nu(\mathcal{O})_x$.

Group actions: stability of orbits

Linear normal isotropy representation:

$$\rho : G_x \to GL(\nu(\mathcal{O})_x)$$

$H^\bullet(G_x, \nu(\mathcal{O})_x)$ denotes the corresponding group cohomology.

Theorem (Hirsch,Stowe)

Let $\mathcal{O}$ be a compact orbit and assume that

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Foliations: stability of leaves

Fix a manifold M and denote by $\mathrm{Fol}_q(M)$ the set of codimension q foliations. Frobenius says:

$$\mathrm{Fol}_q(M) \longleftrightarrow \{D : M \to \mathrm{Gr}_q(TM) | D \text{ is involutive}\}$$

$\Longrightarrow$ $\mathrm{Fol}_q(M)$ has a natural C^r-topology

Definition

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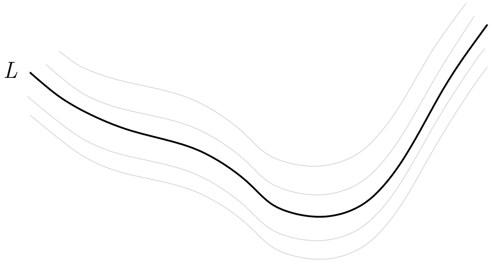

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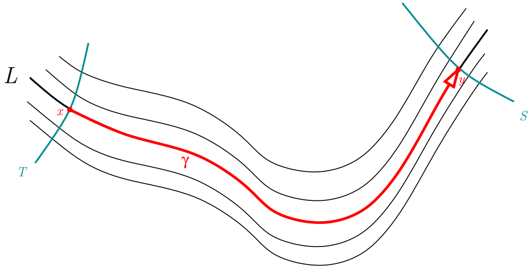

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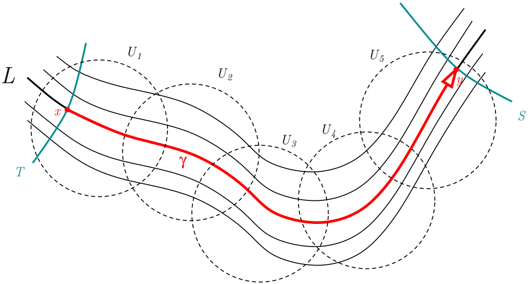

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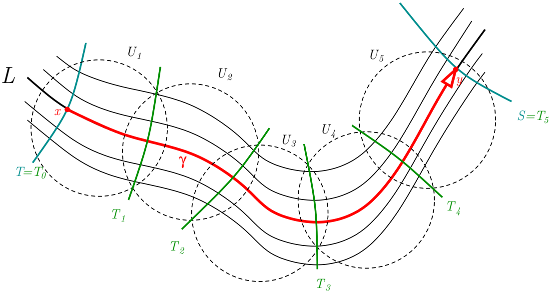

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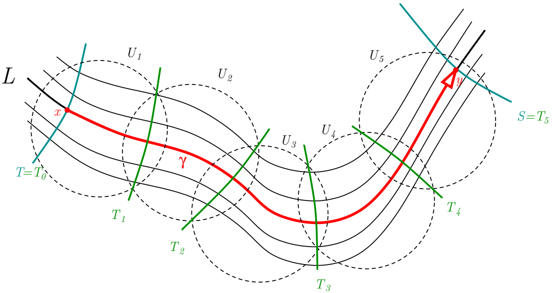

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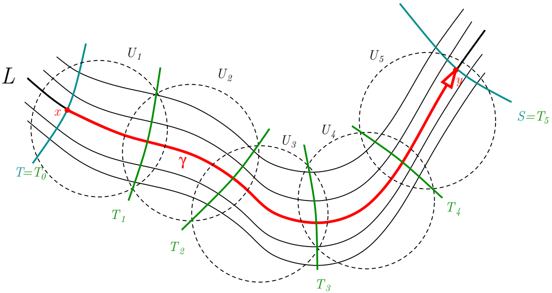

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Note: The Poincaré return map is a special case of this.

Foliations: stability of leaves

Differentiating gives the linear holonomy representation:

$$\rho : \pi_1(L, x) \to GL(\nu(L)_x), \quad \rho := \mathrm{d}_x \circ \mathrm{Hol}$$

$H^\bullet(\pi_1(L, x), \nu(L)_x)$ denotes corresponding group cohomology.

Theorem (Reeb, Thurston, Langevin & Rosenberg)

Let L be a compact leaf and assume that

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Stability of leaves versus stability of orbits

In general, the two theorems are quite different (e.g., dimension of orbits of actions can vary).

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Then the linear holonomy coincides with the linear isotropy representation, and the theorem for actions follows from the theorem for foliations.

Poisson structures

For Poisson bracket $\{\ ,\ \}$ we denote by π the associated Poisson bivector:

$$\pi(\mathrm{d}f, \mathrm{d}g) := \{f, g\}.$$

$$\mathrm{Poiss}(M) \quad \longleftrightarrow \quad \{\pi : M \to \wedge^2(TM) \mid [\pi, \pi] = 0\}.$$

$$\Longrightarrow \quad \mathrm{Poiss}(M) \text{ has a natural } C^r \text{ topology}$$

Definition

A symplectic leaf S of $\pi \in \mathrm{Poiss}(M)$ is called:

- stable if every nearby Poisson structure in $\mathrm{Poiss}(M)$ has a nearby leaf diffeomorphic to S.
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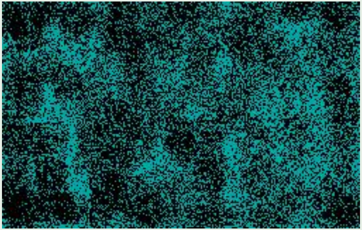

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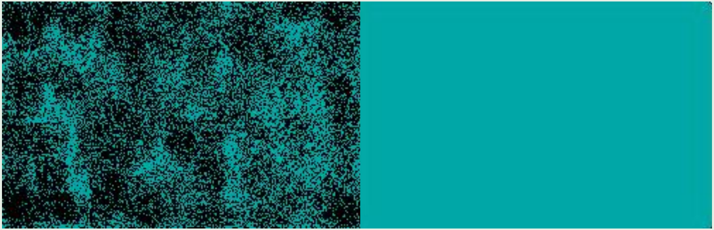

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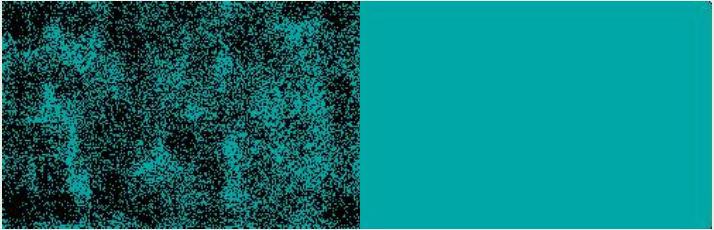

$\Longrightarrow$ no leaves are stable.

Examples (Constant and linear Poisson structures)

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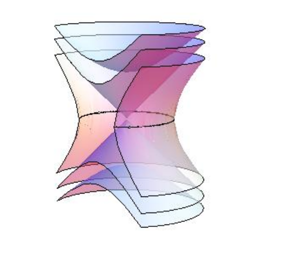

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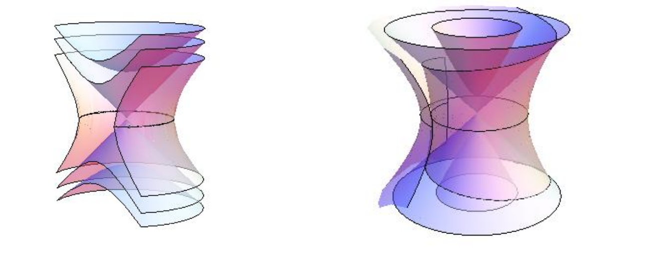

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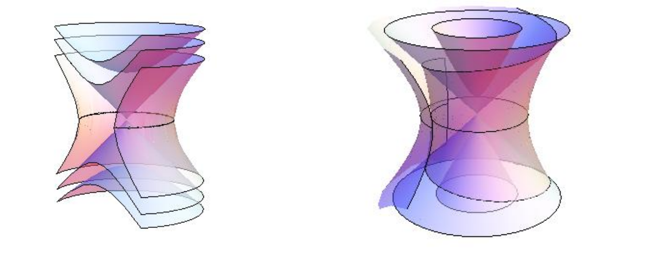

$\Longrightarrow$ some leaves are stable.

Examples (Constant and linear Poisson structures)

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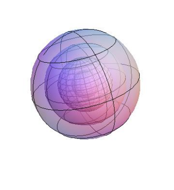

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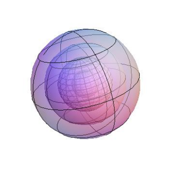

$\Longrightarrow$ all leaves are stable.

Poisson cohomologies

Stability of a symplectic leaf S of a Poisson manifod (M, π) is controled by certain cohomologies:

- Poisson cohomology: $H^\bullet_\pi(M)$ is the cohomology of $(\mathfrak{X}^k(M), \mathrm{d}_\pi)$, the complex of multivector fields with $\mathrm{d}_\pi := [\pi, \ ]$.
- Restricted Poisson cohomology: $H^\bullet_{\pi,S}(M)$ is the cohomology of $(\mathfrak{X}^\bullet_S(M), \mathrm{d}_\pi|_S)$, the complex of multivector fields along S.
- Relative Poisson cohomology: $H^\bullet_\pi(M, S)$ is the cohomology of the quotient complex:

$$\mathfrak{X}^\bullet(M, S) := \mathfrak{X}^\bullet_S(M)/\Omega^\bullet(S)$$

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Stability

Theorem (Crainic & RLF)

Let S be a compact symplectic leaf and assume that

$$H^2_\pi(M,S)=0.$$

Then S is stable: every nearby Poisson structure has a family of nearby diffeomorphic leaves smoothly parametrized by $H^1_\pi(M,S)$.

- The relative Poisson cohomology $H^\bullet_\pi(M,S)$ is typically finite dimensional.
- $H^1_\pi(M,S)$ coincides with the space of leaves of the first jet approximation $j^1_S\pi$ which project diffeomorphically to S.
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Theorem (Crainic & RLF)

Let S be a compact symplectic leaf and assume that

$$H^2_{\pi,S}(M) = 0.$$

Then S is strongly stable: every nearby Poisson structure has a family of nearby symplectomorphic leaves smoothly parametrized by by the image of $\Phi : H^1_{\pi,S}(M) \to H^1_\pi(M,S)$.

- The parameter space of strongly stable leaves is a subspace of the parameter space for stable leaves.
- It coincides with the space of leaves of $j^1_S\pi$ which project diffeomorphically to S and which have symplectic form isotopic to ω_S.

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Stability versus strong stability

The condition for strong stability (i.e., $H^2_{\pi,S}(M) = 0$) **does not** imply the condition for stability ($H^2_\pi(M, S) = 0$).

Conjecture

If S is a compact symplectic leaf and the map

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Necessary conditions for stability

A Poisson structure π is said to be of first order around a symplectic leaf S if π is Poisson diffeomorphic to $j^1_S\pi$ in some neighborhood of S.

Theorem (Crainic & RLF)

Let π be a Poisson structure which is of first order around a compact symplectic leaf S. Then:

(i) *If S is stable, then the map $\Phi : H^2_{\pi,S}(M) \to H^2_\pi(M,S)$ vanishes.*

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Example: dual of a Lie algebra

$M = \mathfrak{g}^*$ where $\mathfrak{g}$ is a compact semi-simple Lie algebra:

- Symplectic leaves are the coadjoint orbits: $S = \mathcal{O}_\xi$;
- All leaves satisfy criteria for stability and strong stability: $H^2_{\pi,S}(M) = H^2_\pi(M,S) = 0$.
- We have $H^1_{\pi,S}(M) = H^2(S)$ and $H^1(M,S) = 0$;

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Conclusion: Every nearby Poisson structure has a one nearby symplectic leaf symplectomorphic to S and a dim $Z(\mathfrak{g}_\xi)$-family of symplectic leaves diffeomorphic to S.

Basic problem

- Is there a general setup to deal with stability problems?

To answer this question one requires a setup where geometric objects such as flows, actions, foliations, Poisson structures, etc., are all on equal footing.

$\Longrightarrow$ Lie groupoids/algebroids

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Lie algebroids

Definition

A Lie algebroid is a vector bundle $A \to M$ with:

(i) a Lie bracket $[\ ,\ ]_A : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$;

(ii) a bundle map $\rho : A \to TM$ (the anchor);

such that:

$$[\alpha, f\beta]_A = f[\alpha\beta]_A + \rho(\alpha)(f)\beta, \quad (f \in C^\infty(M), \alpha, \beta \in \Gamma(A)).$$

$\mathrm{Im}\,\rho \subset TM$ is a integrable (singular) distribution

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Lie algebroids have a characteristic foliation

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- **Flows.** For $X \in \mathfrak{X}(M)$, the associated Lie algebroid is:
  $$A = M \times \mathbb{R}, \quad [f,g]_A := fX(g) - gX(f), \quad \rho(f) = fX.$$
  Leaves of A are the orbits of X.

- **Actions.** For $\alpha \in \mathrm{Act}(G;M)$, the associated Lie algebroid is:
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Examples

- **Foliations.** For $\mathcal{F} \in \mathrm{Fol}_k(M)$, the associated Lie algebroid is:
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Universal Stability Theorem

- For a fixed vector bundle A there is a natural C^r topology on the set **Algbrd**(A) of Lie algebroid structures on A.
- A leaf L of A is called stable if every nearby Lie algebroid structure in **Algbrd**(A) has a nearby leaf diffeomorphic to L.
- The normal bundle $\nu(L)$ carries a canonical Bott type $A|_L$-connection.
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From Lie algebr(oids) to Lie group(oids)

The fundamental group of A based at x:

$$G_x(A) = \frac{\{A\text{-loops based at } x\}}{A\text{-homotopies}}.$$

- $G_x(A)$ need not be smooth. If it is smooth, then it is a Lie group integrating the isotropy Lie algebra $\mathfrak{g}_x(A)$.
- Parell transport along A-paths applied to the Bott representation of $A|_L$ gives the linear holonomy representation:

$$\mathrm{hol} : G_x(A) \longrightarrow GL(\nu(L)_x).$$

Proposition

If $G_x(A)$ is smooth, then for any $x \in L$:

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Foliations: If $A = T\mathcal{F}$, then $G_x(A) = \pi_1(L, x)$, the Bott representation becomes the usual one on $\nu(L)$, and we have:

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We recover the classical result.

Actions: If $A = \mathfrak{g} \ltimes M$ is associated with a 1-connected Lie group G, then $G_x(A) = G_x$, the Bott representation becomes the linear isotropy Lie algebra on $\nu(\mathcal{O})_x$ and we have:

$$H^1(A; \nu(\mathcal{O})) \cong H^1(G_x; \nu(\mathcal{O})_x).$$

We recover the classical result under the assumption that G is 1-connected.

Recovering the stability theorems:

Foliations: If $A = T\mathcal{F}$, then $G_x(A) = \pi_1(L, x)$, the Bott representation becomes the usual one on $\nu(L)$, and we have:

$$H^1(A|_L; \nu(L)) \cong H^1(\pi_1(L, x); \nu(L)_x).$$

We recover the classical result.

Actions: If $A = \mathfrak{g} \ltimes M$ is associated with a 1-connected Lie group G, then $G_x(A) = G_x$, the Bott representation becomes the linear isotropy Lie algebra on $\nu(\mathcal{O})_x$ and we have:

$$H^1(A; \nu(\mathcal{O})) \cong H^1(G_x; \nu(\mathcal{O})_x).$$

We recover the classical result under the assumption that G is 1-connected.

Poisson vs algebroid stability

Examples

$M = \mathfrak{g}^*$ with $\mathfrak{g}$ the non-abelian 2-dimensional Lie algebra:

$$\pi = x\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$$

so that $A = T^*\mathbb{R}^2$ and

$$[\mathrm{d}x, \mathrm{d}y] = \mathrm{d}x, \quad \rho(\mathrm{d}x) = x\frac{\partial}{\partial y}, \quad \rho(\mathrm{d}y) = -x\frac{\partial}{\partial x}.$$

- $H^2_{\pi,0}(\mathfrak{g}^*) = H^2(\mathfrak{g}) = 0$, so the origin is Poisson stable;
- $H^1(A|_0, \nu(\{0\})) = H^1(\mathfrak{g}, \mathfrak{g}) = \mathbb{R}$;
- The origin is not algebroid stable: take $A_\varepsilon = T^*\mathbb{R}^2$ with

$$[\mathrm{d}x, \mathrm{d}y]_\varepsilon = \mathrm{d}x, \quad \rho_\varepsilon(\mathrm{d}x) = x\frac{\partial}{\partial y}, \quad \rho_\varepsilon(\mathrm{d}y) = -x\frac{\partial}{\partial x} + \varepsilon\frac{\partial}{\partial y}.$$

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Examples

$M = S \times \mathbb{R}$ with leaves $(S \times \{t\}, \omega_t)$. Look at stability of $S = S \times \{0\}$.

- Set $\sigma := \frac{\mathrm{d}}{\mathrm{d}t}\omega_t\big|_{t=0}$ and define:

$$C^\bullet_\sigma(S) = \Omega^\bullet(S) \oplus \Omega^{\bullet-1}(S), \quad \mathrm{d}_\sigma(\omega, \eta) = (\mathrm{d}\omega + \sigma \wedge \eta, \mathrm{d}\eta)$$

- One finds that:

$$\mathfrak{X}^\bullet_\pi(M, S) \cong \Omega^{\bullet-1}(S), \quad \mathfrak{X}^\bullet_{\pi,S}(M) \cong C^\bullet_\sigma(S) \cong \mathfrak{X}^\bullet(T^*_S M; \nu(S))$$

- This leads to:
  - $H^2_\pi(M, S) = 0 \Leftrightarrow H^1(S) = 0$.
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Conclusion: the criteria for stability of symplectic leaves in the Poisson setting is stronger than the criteria arising from the general algebroid setting.

- There are Poisson structures π such that $A = T^*M$ has many closeby Lie algebroids which are not associated with a Poisson structure.

- There is a chain map $i : \mathfrak{X}^\bullet(M, S) \to \mathfrak{X}^{\bullet-1}(T^*_S M; \nu(S))$ inducing an injection in degree 2:

$$H^2_\pi(M, S) \hookrightarrow H^1(T^*_S M; \nu(S)).$$

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