Hilbert's 5th Problem and NSA Part II

Rocco Davino

March 2021

Review

- A Lie group is a topological group with an additional smooth structure. Hilbert's 5th Problem is about characterizing Lie groups without referring to this smooth structure.
- In particular, we want to show: locally Euclidean $\Rightarrow$ Lie.
- A few other relevant definitions are NSS, NSCS, etc. To achieve our implication above, we pass through the NSS condition, so

$$\text{locally Euclidean} \;\Rightarrow\; \text{NSS} \;\Rightarrow\; \text{Lie.}$$

Review, cont.

- The most important objects in this story are 1-parameter subgroups and $L(G)$, the set of 1-parameter subgroups of G. Motivated by our Lie group knowledge, we want to establish $L(G)$ as a substitute tangent space at 1 towards finding a compatible smooth manifold structure.
- We found a reasonable addition on $L(G)$ and showed that assuming it works, $L(G)$ is an $\mathbb{R}$-vector space, a first step towards the goal in the previous point.
- We introduced the adjoint representation $a \mapsto \mathrm{Ad}(a) : G \to \mathrm{Aut}(L(G))$ and discussed how $\mathrm{Aut}(L(G))$ will eventually act like $GL_n(\mathbb{R})$, allowing a continuous group homomorphism from G to $GL_n(\mathbb{R})$.

Note on Defining $X + Y$ in $L(G)$

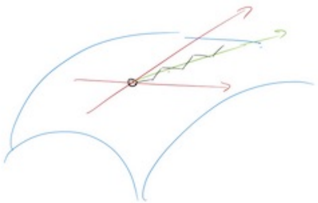
When G is a Lie group, $X(t)$ for $t \approx 0$ is like moving in the direction of $X'(0)$, similarly with $Y(t)$ and $Y'(0)$. Writing things additively in G, for large s and $t \approx 0$

$$[st](X(1/s) + Y(1/s))$$

is like alternating small steps in the $X'(0)$ direction and the $Y'(0)$ direction many times over, i.e. like moving in the $X'(0) + Y'(0)$ direction. Thus in defining

$$(X + Y)(t) = \lim_{s \to \infty} (X(1/s)Y(1/s))^{[st]}$$

we replicate this addition of tangent vectors.



Looking Forward

We want to use nonstandard analysis to show that $X + Y \in L(G)$ actually exists for all $X, Y \in L(G)$. We now make some definitions towards this goal, establish an important lemma, and discuss some of its consequences.

The Nonstandard Setting I

Let S be a Hausdorff space and $s \in S$.

Definition

The monad of s, denoted $\mu(s)$, is the intersection of all $U^* \subseteq S^*$ with U a neighborhood of s in S.

- These are the points that are "infinitely close" to s.
- A set $U \subseteq S$ is open if and only if for all $x \in U$, $\mu(x) \subseteq U^*$.

The Nonstandard Setting I, cont.

Definition

The points of S^* that are infinitely close to some $s \in S$ are called nearstandard. Let S_{ns} be the set of nearstandard points, i.e.

$$S_{\mathrm{ns}} := \bigcup_{s \in S} \mu(s).$$

- Since S is Hausdorff, $\mu(s) \cap \mu(s') = \varnothing$ if $s \neq s'$. Thus we can define a map $\mathrm{st} : S_{\mathrm{ns}} \to S$ by declaring $\mathrm{st}(x)$ to be the unique $s \in S$ such that $x \in \mu(s)$.
- We have an equivalence relation $\sim$ on S_{ns} whose equivalence classes are the monads:

$$x \sim y :\Leftrightarrow \mathrm{st}(x) = \mathrm{st}(y).$$

The Nonstandard Setting I, cont.

In the group setting,

- $G_{\mathrm{ns}} = \bigcup_{g \in G} \mu(g)$ is a subgroup of G^*.
- $\mathrm{st} : G_{\mathrm{ns}} \to G$ is a group homomorphism that is the identity on G with $\mu := \mu(1) = \ker \mathrm{st}$, the normal subgroup of infinitesimals of G_{ns}.
- $\sim$ on G_{ns} is given by

$$a \sim b \Leftrightarrow ab^{-1} \in \mu.$$

Useful Nonstandard Topological Facts

Let S, S' be Hausdorff spaces and $s, t \in S$.

- Robinson's characterization of compactness: $K \subseteq S$ is compact if and only if for every $t \in K^*$ there is $s \in K$ with $t \in \mu(s)$.
- Nonstandard characterization of closedness: $C \subseteq S$ is closed if and only if whenever $s \in S$ and $t \in C^*$ are such that $t \in \mu(s)$ then $s \in C$.
- Nonstandard characterization of continuity: if $f : S \to S'$ is continuous, then $t \in \mu(s) \Rightarrow f(t) \in \mu(f(s))$.

The Nonstandard Setting II

In the following everything ranges over its natural nonstandard counterpart, with the exception that $n \in \mathbb{N}$. We fix σ a positive infinite element of $\mathbb{R}^*$ and adopt "big O" and "little o" notations as follows.

- $\mathbb{R}^*$: for $x, y \in \mathbb{R}^*$ with $y > 0$, $x = o(y)$ means $|x| < y/n$ for all $n > 0$ and $x = O(y)$ means $|x| < ny$ for some $n > 0$.
- G^*:

$$
\begin{aligned}
O[\sigma] &:= \{a \in \mu : (\forall i = o(\sigma))\ a^i \in \mu\} \\
o[\sigma] &:= \{a \in \mu : (\forall i = O(\sigma))\ a^i \in \mu\} \\
&= \{a \in \mu : (\forall i \leq \sigma)\ a^i \in \mu\}
\end{aligned}
$$

We can think of $o[\sigma]$ as the "very small infinitesimals" and $O[\sigma]$ as the "small infinitesimals".

The Nonstandard Setting II, cont.

Notice:

- $o[\sigma] \subseteq O[\sigma] \subseteq \mu \subseteq G_{\mathrm{ns}}$.
- $o[\sigma]$ and $O[\sigma]$ are closed under $a \mapsto a^{\ell}$, for each $\ell \in \mathbb{Z}$. In particular, they are symmetric.
- Let $a \in G_{\mathrm{ns}}, b \in O[\sigma], c \in o[\sigma]$. Then $aba^{-1} \in O[\sigma]$ and $aca^{-1} \in o[\sigma]$, so they are almost normal subgroups of G_{ns}. Later we will see that they are.

The Nonstandard Setting II, cont.

Lemma

If $a \in O[\sigma]$ then $a^i \in G_{\mathrm{ns}}$ for all $i = O(\sigma)$.

Idea: suppose $a \in O[\sigma]$ and $a^j \in G_{\mathrm{ns}}$ for some j comparable to σ, i.e. $j = O(\sigma)$ and $\sigma = O(j)$. Then any $i = O(\sigma)$ is bounded by nj for some $n \in \mathbb{N}^{>0}$ and $a^i = (a^j)^n$ is again nearstandard.

The Nonstandard Setting II, cont.

Lemma

Let $a \in O[\sigma]$. Then the map $X_a : \mathbb{R} \to G$ defined by $X_a(t) := \mathrm{st}(a^{[\sigma t]})$ is a 1-ps of G. Moreover

1. $X_{a^\ell} = \ell X_a$ for all $\ell \in \mathbb{Z}$.
2. $b \in \mu \Rightarrow X_{bab^{-1}} = X_a$.
3. $X_a = O \Leftrightarrow a \in o[\sigma]$.
4. $L(G) = \{X_b : b \in O[\sigma]\}$.

Proof:

- The map is a group homomorphism since the standard part map is a group homomorphism.
- We now show continuity at 0, so let U be a neighborhood of 1 in G. Take a neighborhood V of 1 in G such that $\mathrm{cl}(V) \subseteq U$.

The Nonstandard Setting II, cont.

- Now $a^k \in \mu \subseteq V^*$ for all $k = o(\sigma)$ by definition, i.e. for any $n > 0$, $a^k \in V^* \subseteq \mathrm{cl}(V)^*$ for all $|k| < \sigma/n$. Also $a^k \in G_{\mathrm{ns}}$ for such k meaning $\mathrm{st}(a^k) \in \mathrm{cl}(V) \subseteq U$ whenever $|k| < \sigma/n$ by the nonstandard characterization of closedness.
- Therefore $X_a(t) = \mathrm{st}(a^{[\sigma t]}) \in U$ for $|t| < 1/n$.

The Nonstandard Setting II, cont.

- For (4), let $X \in L(G)$. Letting $b := X(1/\sigma)$, we want to show that $b \in O[\sigma]$ and $X = X_b$, i.e. $b^i \in \mu$ for $i = o(\sigma)$ and $X = X_b$.
- If $i = o(\sigma)$ then $|i| < \sigma/n$ for all $n > 0$ and $|i/\sigma| < 1/n$ for all $n > 0$, so that i/σ is infinitesimal. But

$$b^i = X(1/\sigma)^i = X(i/\sigma) \in \mu$$

  since $i/\sigma \in \mu(0) \Rightarrow X(i/\sigma) \in \mu(1)$ by continuity of X.
- Lastly,

$$X_b(t) = \mathrm{st}(X(1/\sigma)^{[\sigma t]}) = \mathrm{st}(X([\sigma t]/\sigma)) = X(t)$$

  since $\sigma t - 1 \leq [\sigma t] \leq \sigma t + 1$ yields $[\sigma t]/\sigma \in \mu(t)$, hence $X([\sigma t]/\sigma) \in \mu(X(t))$.

Applications of the Previous Lemma

- The previous lemma is crucial for showing $L(G)$ is a topological vector space. Recall $L(G)$ is an $\mathbb{R}$-vs *assuming* $X + Y$ *exists for all* $X, Y \in L(G)$.
- To show this we consider a map

  $$ao[\sigma] \mapsto X_a : O[\sigma]/o[\sigma] \to L(G).$$

  It takes some technical work in Section 5 of this paper to show that it is well-defined.
- $L(G) = \{X_a : a \in O[\sigma]\}$ from the last lemma shows that it is surjective.
- We need another technical result from Section 5: for $a, b \in O[\sigma]$, if $X_a = X_b$ then $ab^{-1} \in o[\sigma]$. This gives the injectivity of the map, so it is bijective.

Applications of the Previous Lemma, cont.

- From here, define an operation $+_\sigma$ on $L(G)$ making it into an abelian group (see Theorem 5.8) and this bijection $O[\sigma]/o[\sigma] \to L(G)$ into a group isomorphism, i.e.

$$X_a +_\sigma Y_b := X_{ab}$$

  for all $a, b \in O[\sigma]$.
- With this, the goal is the following:

Lemma

Let $X, Y \in L(G)$. Then $X + Y$ exists and equals $X +_\sigma Y$.

Why $O[\sigma]/o[\sigma]$?

- We are trying to turn $L(G)$ into a kind of tangent space, so we must deal with infinitesimal notions like limits when defining $X + Y$, for example. This is why we work with elements in $O[\sigma]$.
- We look at $O[\sigma]/o[\sigma]$ because it can be viewed as a standard object: take $G = \mathbb{R}$ and write things additively. If $a \in O[\sigma]$, then $\sigma a \in \mathbb{R}_{\mathrm{ns}}$ by the first lemma in this talk. Define a map

  $$a \mapsto \mathrm{st}(\sigma a) : O[\sigma] \to \mathbb{R},$$

  which is a homomorphism. Observe that a is in the kernel iff σa is infinitesimal iff $a \in o[\sigma]$. Therefore

  $$O[\sigma]/o[\sigma] \cong \mathbb{R}.$$

- Thus we can use the nice properties of $O[\sigma]$ for dealing with infinitesimals and "standardize" to make it behave like a tangent space.