Hilbert's 5th Problem and NSA Part I

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Lie Groups

Definition

A Lie group is a topological group G for which inversion $x \mapsto x^{-1} : G \to G$ and multiplication $(x, y) \mapsto xy : G \times G \to G$ are smooth with respect to some compatible smooth manifold structure on its underlying space.

- A canonical example is $GL_n(\mathbb{R})$: identifying it with a subset of $\mathbb{R}^{n^2}$, it is open via the determinant map and thus inherits a smooth manifold structure from $\mathbb{R}^{n^2}$.
- We can think of a Lie group as a topological group with an additional smooth structure on top.
- Our goal, tantamount to Hilbert's 5th Problem, is to show that Lie groups can be characterized without actually mentioning this smooth structure. We will show that being locally Euclidean is the same as being a Lie group.

Definitions Toward the Main Theorem

Let G be a topological group.

Definition

1. G is locally Euclidean if some neighborhood of its identity is homeomorphic to some $\mathbb{R}^n$.
2. G has no small subgroups (NSS) if there is a neighborhood U of 1 in G that contains no subgroup of G other than $\{1\}$.
3. G has no small connected subgroups (NSCS) if there is a neighborhood U of 1 in G that contains no connected subgroup of G other than $\{1\}$.
4. G is bounded in dimension if for some n no subspace is homeomorphic to the unit cube $[0, 1]^n$.

Note: locally Euclidean groups are locally compact.

Main Theorem

Theorem

Given G a locally compact group, TFAE:

1. *G is a Lie group;*
2. *G has NSS;*
3. *G is locally Euclidean;*
4. *G is locally connected and has NSCS;*
5. *G is locally connected and bounded in dimension.*

Relevant History

These are "black box" facts we'll need:

- Pontrjagin solved all this for commutative G.
- von Neumann: if G is a locally compact topological group with an injective continuous homomorphism $\rho : G \to GL_n(\mathbb{R})$ for some n, then G can be given the structure of a Lie group. See `https://terrytao.wordpress.com/tag/cartans-theorem/` for more.
- Kuranishi: if G has a commutative closed normal subgroup N such that N and G/N are Lie groups, then G is a Lie group.

A broad theme of this proof is to reduce the problem to one already understood before 1950.

Important Definition: $L(G)$

Definition

A 1-parameter subgroup (or 1-ps) of G is a continuous group homomorphism $\mathbb{R} \to G$. Set

$$L(G) := \{X : X \text{ is a 1-ps subgroup of } G\}.$$

- The trivial 1-ps O is defined by $O(t) := 1$ for all $t \in \mathbb{R}$.
- For $r \in \mathbb{R}, X \in L(G)$ we define $rX \in L(G)$ by $(rX)(t) := X(rt)$ and refer to
  $$(r, X) \mapsto rX : \mathbb{R} \times L(G) \to L(G)$$
  as scalar multiplication.
- Denote $(-1)X = -X$. Then: $0X = O, 1X = X, -X = X^{-1}$, and $r(sX) = (rs)X$ for all $r, s \in \mathbb{R}, X \in L(G)$.

Why $L(G)$?

Suppose for the moment G is a Lie group.

- Each $X \in L(G)$ is a smooth function $\mathbb{R} \to G$ and

$$X \mapsto X'(0) : L(G) \to T_1(G)$$

  is a bijection.
- This bijection respects scalar multiplication, and there is an addition operation on $L(G)$ that makes this bijection into an isomorphism of $\mathbb{R}$-vector spaces.
- Thus we see that $L(G)$ acts like a tangent space at 1, suggesting a smooth structure compatible with G.
- For our situation, this is a hint that we should try to establish $L(G)$ as a substitute tangent space at 1 towards finding a compatible manifold structure on G.

Sketch of NSS $\Rightarrow$ Lie

Suppose G has NSS.

- Show for any $X, Y \in L(G)$ there is $X + Y \in L(G)$ given by

$$(X + Y)(t) = \lim_{n \to \infty} (X(1/n)Y(1/n))^{[nt]},$$

  and that this addition along with scalar multiplication make $L(G)$ a vector space over $\mathbb{R}$.
- Equip $L(G)$ with its compact-open topology and show that this makes $L(G)$ a topological vector space.
- Show that the "exponential" map $X \mapsto X(1) : L(G) \to G$ maps some neighborhood of O in $L(G)$ homeomorphically onto a neighborhood of 1 in G. Then the local compactness of G yields the local compactness of $L(G)$ and hence the finite-dimensionality of $L(G)$ as an $\mathbb{R}$-vector space. Then G is locally Euclidean.

Sketch of NSS $\Rightarrow$ Lie, cont.

- Replacing G by the connected component of 1, we may assume that G is connected. Then the adjoint representation $G \to L(G)$ (defined below) on the finite-dimensional vector space $L(G)$ has as its kernel a commutative closed normal subgroup N of G and yields a continuous injective group homomorphism $G/N \to GL_n(\mathbb{R})$. Since N has NSS it is locally Euclidean by the third bullet. But N is commutative and hence a Lie group by Pontrjagin. The injective group homomorphism $G/N \to GL_n(\mathbb{R})$ makes G/N a Lie group by von Neumann. Finally, applying Kuranishi we conclude G is a Lie group.

Sketch of Locally Euclidean $\Rightarrow$ NSS

- The first step in the last proof sketch, showing that $L(G)$ can be made into an $\mathbb{R}$-vector space, is done without assuming NSS.
- Establish that if G is locally connected and has NSCS, then G has NSS.
- Establish that if G does not have NSCS, then G contains a homeomorphic copy of $[0, 1]^n$ for all n.
- If G is locally Euclidean, then G is locally connected and bounded in dimension (minor black boxes). Thus G must have NSCS, and G is locally connected with NSCS meaning G has NSS.

Generalities On 1-ps

In the following, G is a locally compact group.

Lemma

Suppose $X \in L(G)$ and $X \neq O$. Then either $\ker X = \{0\}$ or $\ker X = \mathbb{Z}r$ for some $r \in \mathbb{R}^{>0}$. In the first case, X maps each bounded interval of the form $(-a, a)$ homeomorphically onto its image. In the second, X maps the interval $(-r/2, r/2)$ homeomorphically onto its image.

Proof:

- Notice the kernel K of X is a closed subgroup of $\mathbb{R}$. If K is not discrete, then it is dense in $\mathbb{R}$ [if x_0 is a limit point of K, use translations to show that any interval of arbitrarily small radius contains a point in K]. Therefore if K is not $\{0\}$ or $\mathbb{R}$ then it is discrete. The element $\inf\{x \in K : x > 0\}$ generates K.
- Now use that any continuous bijection from a compact space onto a Hausdorff space is a homeomorphism.

Generalities On 1-ps, cont.

Definition

For $X, Y \in L(G)$, suppose $\lim_{s\to\infty}(X(1/s)Y(1/s))^{[st]}$ exists for all $t \in \mathbb{R}$. Then

$$t \mapsto \lim_{s\to\infty}(X(1/s)Y(1/s))^{[st]} : \mathbb{R} \to G$$

is a 1-ps of G, which we define to be $X + Y$. In this case we say $X + Y$ exists.

Lemma

Let $X, Y \in L(G)$ and $p, q \in \mathbb{R}$.

1. *$X + O$ exists and equals X.*
2. *$pX + qX$ exists and equals $(p+q)X$.*
3. *If $X + Y$ exists then $Y + X$ exists and equals $X + Y$.*
4. *If $X + Y$ exists then $pX + pY$ exists and equals $p(X + Y)$.*

The proof is a useful exercise. See pg. 5 for some helpful hints.

Generalities On 1-ps, cont.

Moreover, we have the adjoint action of G on $L(G)$:

$$(a, X) \mapsto aXa^{-1} : G \times L(G) \to L(G), \qquad (aXa^{-1})(t) := aX(t)a^{-1}.$$

Since it is an action, each $a \in G$ induces a bijection

$$\mathrm{Ad}(a) : L(G) \to L(G), \qquad \mathrm{Ad}(a)(X) = aXa^{-1}.$$

What's more,

- for $r \in \mathbb{R}$ and $X \in L(G)$, $\mathrm{Ad}(a)(rX) = r\mathrm{Ad}(a)(X)$,
- for $X, Y \in L(G)$, if $X + Y$ exists, then $\mathrm{Ad}(a)(X) + \mathrm{Ad}(a)(Y)$ exists and equals $\mathrm{Ad}(a)(X + Y)$.

Generalities On 1-ps, cont.

Thus:

Theorem

Suppose $X + Y$ exists for all $X, Y \in L(G)$ and that the binary operation $+$ on $L(G)$ is associative. Then

- *$L(G)$ with $+$ as its addition and the usual scalar multiplication is a vector space over $\mathbb{R}$ with O as the zero element.*
- *We have a group homomorphism of G into the group of autmorphisms of the vector space $L(G)$ given by $a \mapsto \mathrm{Ad}(a) : G \to \mathrm{Aut}(L(G))$, called the adjoint representation of G.*

More on the Adjoint

- Suppose we've established $L(G) \cong \mathbb{R}^n$ as $\mathbb{R}$-vector spaces, and take an $\mathbb{R}$-linear isomorphism between them.
- This induces a group isomorphism

$$\mathrm{Aut}(L(G)) \cong GL_n(\mathbb{R}) \subseteq \mathbb{R}^{n^2}$$

  since an element of $\mathrm{Aut}(L(G))$ is determined by a matrix in $GL_n(\mathbb{R})$.
- Once we give $\mathrm{Aut}(L(G))$ the topology that makes this isomorphism into a homeomorphism, we have an isomorphism of topological groups. It turns out the adjoint representation $G \to \mathrm{Aut}(L(G))$ is continuous, its kernel is a commutative closed normal subgroup N of G, and we thus obtain the crucial injective group homomorphism $G/N \to GL_n(\mathbb{R})$ establishing G/N as a Lie group.

Brief Functorial Interlude

- Consider a continuous group homomorphism $\varphi : G \to H$. We have a map
$$L(\varphi) : L(G) \to L(H), \qquad L(\varphi)(X) := \varphi \circ X.$$
- It has nice properties: $L(\varphi)(rX) = rL(\varphi)(X)$ for all $r \in \mathbb{R}, X \in L(G)$, if $X, Y \in L(G)$ such that $X + Y$ exists then
$$L(\varphi)(X + Y) = L(\varphi)(X) + L(\varphi)(Y),$$
etc. See pg. 6 for more.
- It turns out that assigning to each G the set $L(G)$ and to each φ the map $L(\varphi)$ yields a functor L from the category of locally compact groups and continuous group morphisms to the category of sets.
- Significant? Possibly.