MATH 415 G83 - WORKSHEET 02

The purpose of this worksheet is to help you get accustomed to doing linear algebra by hand, not just with *Mathematica*. Please note that it is NOT the purpose of the worksheet to explain what you are doing in your homework assignments. You are still expected to read and know the Basics and Tutorials in the courseware.

Notation & Terminology. The following chart shows the differences in the standard notation seen in regular textbooks and used by most people, versus that seen in the courseware and *Mathematica*. Try not to get confused between the two!

	Standard	*Mathematica*/Courseware
matrices	$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ or $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$	`{{a,b},{c,d}}`
matrix multiplication	AX or AB	`A.X` or `A.B`

Matrices and their Operations. An $m \times n$ *matrix* is an array of numbers, consisting of m rows and n columns. For example, $A = \begin{bmatrix} 3 & 1 \\ 0 & -1 \end{bmatrix}$ is a 2×2 matrix, $B = \begin{bmatrix} 1 & 1 & 7 \\ 8 & 0 & 0 \end{bmatrix}$ is a 2×3 matrix, and the vector $X = \begin{bmatrix} 17 \\ -1 \end{bmatrix}$ is a 2×1 matrix. Note that vectors are treated as *column matrices*, meaning that they only have one column.

Given the three matrices above, you can multiply A and X, denoted AX, by forming a new 2×1 column matrix whose first entry is the dot product of the first row of A with X, and whose second entry is the dot product of the second row of A with X:

$$AX = \begin{bmatrix} row1 \cdot X \\ row2 \cdot X \end{bmatrix} = \begin{bmatrix} 3 \cdot 17 + 1 \cdot (-1) \\ 0 \cdot 17 + (-1) \cdot (-1) \end{bmatrix} = \begin{bmatrix} 50 \\ 1 \end{bmatrix}.$$

Similarly, AB is defined to be the 2×3 matrix whose entry in the i^{th} row and j^{th} column is the dot product of the i^{th} row of A and the j^{th} column of B:

$$AB = \begin{bmatrix} 11 & 3 & 21 \\ -8 & 0 & 0 \end{bmatrix}.$$

Note, however, that we cannot take the product BX nor BA. Indeed, the corresponding dot product of rows and columns do not make sense, since two vectors must be of the same size in order to take their dot product. Thus, in practice, **the number of columns of the first matrix must equal the number of rows of the second matrix** in the product. You can confirm that this is true in the two computations above.

Some other matrix operations include *transposition* and *inversion*. The transpose of an $m \times n$ matrix A, denoted A^t, is the $n \times m$ matrix whose i^{th} row is equal to the i^{th} column of A; that is, you just switch the rows for the columns to get from A to A^t. One can view this as flipping the matrix over its *main diagonal*. An $m \times m$ (square) matrix A has an inverse A^{-1} if $AA^{-1} = I_m$ and

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$A^{-1}A = I_m$ where I_m is the $m \times m$ *identity matrix* which has 1's running down the main diagonal, and 0's everywhere else:

$$I_m = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 \end{bmatrix}.$$

This matrix is special because, for any $n \times m$ matrix A, $AI_m = A$; and for any $m \times n$ matrix B, $I_mB = B$. If A has an inverse, we say that A is *invertible*.

There are two more operations on matrices to discuss: *scalar multiplication* and *addition*. For any number $k \in \mathbb{R}$, and any $m \times n$ matrix A, the matrix kA is the same as A except each of its entries have been multiplied by k. So, for example, if $k = 2$ and B is the matrix defined above, then

$$kB = \begin{bmatrix} 2 & 2 & 14 \\ 16 & 0 & 0 \end{bmatrix}.$$

Also, if two matrices are of the same size (both are $m \times n$), then they can be added together by simply adding together the corresponding entries. For example, both B and AB are 2×3 matrices. Adding them together yields:

$$B + AB = \begin{bmatrix} 1+11 & 1+3 & 7+21 \\ 8+(-8) & 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 12 & 4 & 28 \\ 0 & 0 & 0 \end{bmatrix}.$$

Exercise 1 (Basic Matrix Operations)**.** Calculate the following where $X = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and A, B, and C are the following matrices:

$$A = \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix},\ B = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix},\ C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

(a) AX
(b) BX
(c) BC
(d) CB
(e) A^t
(f) C^t
(g) $(BC)^t - C^tB^t$
(h) A^{-1}
(i) B^{-1}
(j) $A + B$
(k) $3B - 2C$

Exercise 2 (Inverse of an Invertible 2×2 Matrix)**.** Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and assume that it is invertible. Show that

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Examine exercise 1 part (g) again. This in fact holds for all matrices:

$$(AB)^t = B^tA^t.$$

What about inverses?

Exercise 3. Given two invertible matrices A and B, which of the following is the inverse of AB?

(a) $A^{-1}B^{-1}$.
(b) $B^{-1}A^{-1}$.

Exercise 4. When you hit a circle (seen as a column matrix whose entries depend on a parameter) with a matrix you get:

(a) Another circle.
(b) An ellipse.
(c) An ellipse, a line, or just one point.
(d) An hyperbola, an ellipse, or a line.